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Uniform Output Regulation of Nonlinear Systems

A Convergent Dynamics Approach

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Preface

The problem of asymptotic regulation of the output of a dynamical system plays a central role in control theory. An important variant of this problem is the output regulation problem, which can be used in areas such as set-point control, tracking reference signals and rejecting disturbances generated by an external system, controlled synchronization of dynamical systems, and observer design for autonomous systems. At the moment this is a hot topic in nonlinear control.

This book is a result of a four-year research project conducted at the Eindhoven University of Technology. This project, entitled "Robust output regulation for complex dynamical systems," began with the observation that the problem of controlled synchronization of dynamical systems can be considered as a particular case of the output regulation problem. In the beginning of the project, known solutions to the controlled synchronization problem were global and dealt with nonlinear systems having complex ("chaotic") dynamics. At the same time, most of the existing solutions to the nonlinear output regulation problem were local and dealt mostly with exosystems being linear harmonic oscillators. Our initial idea was, using the results from the controlled synchronization problem as a starting point, to extend solutions of the nonlinear output regulation problem from the local case to the global case and to avoid restrictive assumptions on the exosystem.

As a first step, we started looking for points that were common to these two problems. In this way we encountered or, to be more precise, recalled the notion of convergent systems, which was overlooked in the West, but well known in Russia. It appeared to be *the* common point we were seeking. With this notion as a starting point, the local-to-global, simple-to-complex extension began. We started with improving some results on the local nonlinear output regulation problem. Then, we gradually managed to extend some controller design techniques to the global case. At some point, it appeared that the solvability theory—well developed for the case of the local nonlinear output regulation problem—can be extended, using the notion of convergence, to the global case. These achievements also led us to a new problem setting for the output regulation problem, which, again based on the notion of convergence, naturally extends the linear and local nonlinear output regulation problem to the global nonlinear case. Moreover, with this new problem setting, which we call the uniform output regulation problem, the results on solvability analysis and controller design became accurate and rather compact. This was a good sign. Both the controller design and the solvability analysis were founded on the concept of convergence and required further developments of the apparatus of convergent systems. After all, the developed techniques on convergent systems appeared to be very interesting and promising for application to other control problems as well. Now it is even difficult to say whether the machinery of convergent systems is a helpful tool for tackling the output regulation problem, or the output regulation problem serves as a good illustrating example for the power of convergent systems. Time will show whether it is one way or the other.

This four-year journey has been interesting and inspiring for us, and we hope that this book, as a result, will also be interesting and valuable for the reader.

In the end, we would like to thank all the people who helped us in this project and in the preparation of the book: Dr. Henri Huijberts, for initiating the project and providing very valuable comments on the manuscript (which, initially, was the PhD thesis of A. Pavlov); Dr. Sasha Pogromsky, for attracting our attention to convergent systems and for endless discussions on this subject; Bart Janssen—a master student at the Eindhoven University of Technology—for his invaluable help in building the experimental setup and conducting the experiments; Prof. Maarten Steinbuch and Prof. Okko Bosgra, for their valuable comments on the manuscript, and all our colleagues from Eindhoven University of Technology and from the St. Petersburg control community who directly or indirectly influenced (in a positive way) this work. This research was partially supported by the Netherlands Organization for Scientific Research (NWO).

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Uniform Output Regulation of Nonlinear Systems

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Introduction

1.1 The output regulation problem

The output regulation problem is one of the central problems in control theory. This problem deals with asymptotic tracking of prescribed reference signals and/or asymptotic rejection of undesired disturbances in the output of a dynamical system. The main feature that distinguishes the output regulation problem from conventional tracking and disturbance rejection problems is that, in the output regulation problem, the class of reference signals and disturbances consists of solutions of some autonomous system of differential equations. This system is called an *exosystem*. Reference signals and/or disturbances generated by the exosystem are called *exosignals*.

Many control problems can be formulated as a particular case of the output regulation problem. For example, in the set-point control problem the constant reference signals to be asymptotically tracked by the output of a system can be considered as outputs of an exosystem given by a differential equation with zero right-hand side. A particular value of the reference signal is, in this case, determined by the corresponding initial condition of the exosystem. In the same way, constant disturbances acting on a system can be considered as outputs of an exosystem with zero right-hand side. Therefore, the set-point control problem and the problem of asymptotic rejection of constant disturbances in the output of a system can be considered as particular cases of the output regulation problem. Similar to the case of constant exosignals, harmonic reference signals and disturbances can be considered as outputs of a linear harmonic oscillator. In this case, the parameters of the oscillator determine the frequency content of the exosignal, while the initial conditions of the oscillator determine particular amplitudes and phases of the exosignal. Here, we see that the problem of asymptotic tracking and disturbance rejection for the case of harmonic reference signals and disturbances can be considered as a particular case of the output regulation problem.

Examples of the output regulation problem with more complex exosystem dynamics can be found, for example, in the problem of controlled synchronization (see, e.g., [60, 61, 74]). In this problem one considers two systems of the same dimensions. The first system is autonomous and is called a "master" system. The master system usually has some complex dynamics, e.g., it may have a chaotic attractor. The second system can be controlled and is called a "slave" system. The controlled synchronization problem is to find a controller that, based on the measured signals from the master and slave systems, generates a control action such that the state of the slave system asymptotically tracks the state of the master system. In other words, the states of these two systems asymptotically synchronize. The fact that a controlled synchronization problem can be treated as a particular case of the output regulation problem was pointed out in [36]. From the formulation of the controlled synchronization problem, one can easily notice that this problem has a lot in common with the observer design problem for the autonomous master system. In fact, the slave system can be treated as an observer for the master system. Therefore, the problem of observer design for autonomous systems can also be considered as an output regulation problem.

For linear systems the output regulation problem was completely solved in the 1970s in the works of B.A. Francis, W.M. Wonham, E.J. Davison, and others [13, 21, 88]. This research resulted in the well-known *internal model principle* [21] and in the observation that solvability of the linear output regulation problem is related to the solvability of the so-called "regulator equations," which, in the linear case, are two linear matrix equations [20]. A different approach to the linear output regulation problem was pursued in the works of V.A. Yakubovich and his colleagues [56, 81, 90]. This approach is based on treating the output regulation problem as some kind of the linear-quadratic optimal control problem. Although controllers obtained within this approach do not guarantee that the regulated output converges to zero (it converges to small values depending on the chosen cost functional), they are less sensitive to variations in the exosystem parameters. The problem of output regulation for linear systems subject to constraints on the inputs and state variables was studied in a number of publications, see, e.g., [30, 77] and references therein.

Following the trend of developing nonlinear control systems theory (see, e.g., [38, 62] and references therein), in the 1980s several authors started studying the output regulation problem for nonlinear systems [16, 28, 29]. A breakthrough in the nonlinear output regulation problem was reported in the seminal paper [39] by A. Isidori and C.I. Byrnes. In that paper the authors showed that under the neutral stability assumption on the exosystem and some standard stabilizability/detectability assumptions on the system, the local output regulation problem is solvable if and only if certain mixed algebraic equations and partial differential equations are solvable. These equations are called the *regulator equations*. They are nonlinear counterparts of the regulator equations from the linear output regulation problem. An alternative solution to the local output regulation problem was proposed in [34]. These papers were followed by a number of publications dealing with various aspects of the local output regulation problem. For example, if it is difficult to solve the local output regulation problem (because it requires solving the regulator equations), then approximate (in some sense) solutions to the problem can be found, as reported in [8, 33, 35, 86]. The problem of structurally stable (i.e., when the system parameters are assumed to be close enough to their nominal values) output regulation was addressed in [8, 38]. The case when the system parameters are allowed to vary within a given compact set was considered in [8, 44, 50, 52]. The semiglobal output regulation problem with an adaptive internal model, which allowed for uncertainties in the exosystem, has been considered in [80]. Probably the most complete list of references to results on the output regulation problem can be found in [7, 8, 31, 42].

So far, the results on the output regulation problem mentioned above dealt either with the local or semiglobal (i.e., when initial conditions belong to some predefined compact set) case. Actually, the number of results on the global variant of the output regulation problem is very small compared to the number of results on the local and semiglobal cases. Only recently have more papers on the global output regulation problem started to appear. In [79] the global robust output regulation problem was solved for minimum-phase systems that are linear in the unmeasured variables. The same class of systems as in [79], but with unknown system and exosystem parameters, was considered in [17]. In that paper the global output regulation problem was solved using adaptive control techniques. In [12, 58] the global robust servomechanism problem for nonlinear systems in triangular form was considered. In [11] a problem formulation for the global robust output regulation problem was proposed and a possible conversion of this problem into a certain robust stabilization problem was suggested.

Careful examination of the global results mentioned above allows one to conclude two things. First, at the moment there is still no generally accepted problem statement for the global output regulation problem. Second, all these results start with the assumption that the regulator equations are solvable and that the corresponding solutions are defined either *globally* or in some predefined set. The only vague justification for this assumption is that in the *local* output regulation problem, the existence of *locally* defined solutions to the regulator equations is a necessary condition for the solvability of the problem. In fact, these two observations are, in a certain sense, coupled. Recall that in the local output regulation problem [8, 39], a properly chosen problem setting with a "right" set of standing assumptions allowed one to obtain necessary and sufficient conditions for the solvability of the problem and to build up a nice, complete theory for this problem. Our hypothesis, which is now confirmed by the results contained in this book, is that by choosing a proper problem setting for the *global* output regulation problem and a proper set of assumptions, one can build up a more or less complete theory for the *global* output regulation problem, just as was done for the local case in [8, 39]. Such a theory would include necessary and sufficient conditions for the solvability of the problem and would embrace the existing problem formulations and results on the global output regulation problem. Moreover, it would provide us with new solutions to the global output regulation problem for new classes of systems.

One possible way of defining such a new problem setting has been proposed in [40]. Although the approaches adopted in this book and in [40] are different, the corresponding final results are close to each other.

A cornerstone of such a new problem formulation for the global output regulation problem adopted in this book is the natural requirement that the closed-loop system must have some "convergence" property. Roughly speaking, this property means that all solutions of the closed-loop system "forget" their initial conditions and converge to some unique solution, which can be called a steady-state solution. This solution is determined only by the exosignal generated by the exosystem. This "convergence" property is discussed in the next section.

1.2 Convergent dynamics

In many control problems and, in particular, in the output regulation problem, it is required that controllers be designed in such a way that all solutions of the corresponding closed-loop system "forget" their initial conditions and converge to some steady-state solution, which is determined only by the input of the closed-loop system. This input can be, for example, a command signal or a signal generated by a feedforward part of the controller or, as in the output regulation problem, it can be the signal generated by the exosystem. For asymptotically stable linear systems excited by inputs, this is a natural property. Indeed, due to linearity of the system, every solution is globally asymptotically stable and, therefore, all solutions of such a system "forget" their initial conditions and converge to each other. After transients, the dynamics of the system are determined only by the input.

For nonlinear systems, in general, global asymptotic stability of a system with zero input does not guarantee that all solutions of this system with a *nonzero* input "forget" their initial conditions and converge to each other. There are many examples of nonlinear globally asymptotically stable systems that, being excited by a periodic input, have coexisting periodic solutions. These periodic solutions do not converge to each other. This fact indicates that for nonlinear systems the convergent dynamics property requires additional conditions.

The property that all solutions of a system "forget" their initial conditions and converge to some steady-state solution has been addressed in a number of papers. In [73] this property was investigated for systems of differential equations that are periodic in time. In that work systems with a unique periodic globally asymptotically stable solution were called *convergent*. Later, the definition of convergent systems given by V.A. Pliss in [73] was extended by B.P. Demidovich in [15] (see also [66]) to the case of systems that are not necessarily periodic in time. According to [15], a system is called convergent if there exists a unique globally asymptotically stable solution that is bounded on the whole time axis. Obviously, if such a solution does exist, all other solutions, regardless of their initial conditions, converge to it. This solution can be considered as a steady-state solution. In [14, 15] B.P. Demidovich presented a simple sufficient condition for such a convergence property (the English translation of this result can be found in [66]). With the development of absolute stability theory, V.A. Yakubovich showed in [89] that for a linear system with one scalar nonlinearity satisfying some incremental sector condition, the circle criterion guarantees the convergence property for this system with any nonlinearity satisfying this incremental sector condition.

In parallel with this Russian line of research, the property of solutions converging to each other was addressed in the works of T. Yoshizawa [91, 92] and J.P. LaSalle [54]. In [54] this property of a system was called *extreme stability.* In [91] T. Yoshizawa provided sufficient and, under certain assumptions, necessary conditions for this extreme stability. These conditions are formulated in terms of existence of a Lyapunov-type function satisfying certain conditions. Extremely stable systems with periodic and almost-periodic right-hand sides were studied in [92].

Several decades after these publications, the interest in stability properties of solutions with respect to one another revived. Incremental stability, incremental input-to-state stability, and contraction analysis are some of the terms related to such properties. In the mid-1990s, W. Lohmiller and J.-J.E. Slotine (see [57] and references therein) independently reobtained and extended the result of B.P. Demidovich. In particular, they pointed out that systems satisfying the (extended) Demidovich condition may enjoy certain properties of asymptotically stable linear systems that are not encountered in general asymptotically stable nonlinear systems. A different approach was pursued in the works by V. Fromion et al. [23–25]. In this approach a dynamical system is considered as an operator that maps some functional space of inputs to a functional space of outputs. If this operator is Lipschitz continuous (has a finite incremental gain or is incrementally stable), then, under certain observability and reachability conditions, all solutions of a state-space realization of this system converge to each other. The sufficient conditions for such Lipschitz continuity condition proposed in [25] are very close to the sufficient conditions for the convergence property obtained by Demidovich. In [2] D. Angeli developed a Lyapunov approach for studying both the global uniform asymptotic stability of all solutions of a system (in [2], this property is called incremental stability) and the so-called incremental input-to-state stability property, which is compatible with the input-to-state stability approach (see, e.g., [82]). As was pointed out in these recent papers, observer design and (controlled) synchronization problems are some of the possible applications of such stability properties.

In this book, for the property that all solutions of a system “forget” their initial conditions and converge to some steady-state solution, we will adopt the notion of *convergent systems* introduced by B.P. Demidovich. In comparison to the other notions mentioned above, the property of *convergence* has two main advantages: it is coordinate independent, while, for example, the notion of incremental stability and incremental input-to-state stability is not, and it allows us to define the steady-state solution in a *unique* way, which proves to be beneficial in further analysis and applications of convergent systems.

1.3 Book outline

In this book we systematically study the output regulation problem based on the notion of convergent systems. As a preliminary step, in Chapter 2 we extend the notion of convergent systems introduced by B.P. Demidovich, investigate various properties of such systems, and design certain tools for the analysis of convergent systems. All these results can be used not only in the context of the output regulation problem, but also in other problems in systems and control theory.

In Chapter 3 we formulate the so-called *uniform output regulation problem*. This is a new problem formulation for the output regulation problem based on the notion of convergent systems. We state global and local variants of the uniform output regulation problem as well as a robust variant of this problem for systems with uncertainties. This new problem formulation has several advantages over the existing problem formulations (see, e.g., [8, 11]). First, it allows one to deal with exosystems having complex dynamics, e.g., exosystems with a (chaotic) attractor. Up to now most of the results on the output regulation problem dealt only with exosystems having relatively simple dynamics, for example, with linear harmonic oscillators. The ability to deal with complex exosystem dynamics allows one to treat the problem of controlled synchronization (see, e.g., [36]) and the problem of observer design for autonomous systems with complex dynamics as particular cases of the uniform output regulation problem. The second advantage of this new problem setting is that, as will be discussed below, it allows one to treat the local and global variants of the uniform output regulation problem in a unified way regardless of the complexity of the exosystem dynamics. This new problem setting includes, as its particular cases, the output regulation problem for linear systems and the conventional local output regulation problem for nonlinear systems (see, e.g., [8]).

For the global, global robust, and local variants of the uniform output regulation problem, we provide necessary and sufficient conditions for the solvability of these problems as well as results on characterization of all controllers solving these problems. These results are presented in Chapter 4. For all these different variants of the problem, the obtained results on the solvability of the problem and controllers characterization look similar. Such a uniformity is a sign of the right choice of the problem setting. Moreover, we show that many of the existing controllers solving the global output regulation problem in other problem settings (see, e.g., [12, 58, 68, 69, 79]), which can be different from the global uniform output regulation problem, in fact solve the global uniform output regulation problem. The solvability analysis of the uniform output regulation problem is based on certain invariant manifold theorems. We demonstrate that these invariant manifold theorems can also be used for studying the so-called *generalized synchronization* of coupled systems, for the computation of periodic solutions of nonlinear systems excited by harmonic inputs and for the extension of frequency response functions and such a well-known analysis and design tool as the Bode magnitude plot from linear systems to nonlinear uniformly convergent systems. These nonlinear frequency response functions and the Bode plot can be used, for example, for nonlinear system performance analysis.

The solvability conditions for the global uniform output regulation problem do not provide direct recipes for finding controllers solving this problem. Therefore, in Chapter 5 we provide results on controller design for the global uniform output regulation problem for several classes of nonlinear systems. One of these controller designs is based on the notions of quadratic stabilizability and detectability. These notions extend the conventional notions of stabilizability and detectability from linear systems theory to the case of nonlinear systems. The controller design based on these notions extends the known controllers solving the linear and the local nonlinear output regulation problems to the case of the global uniform output regulation problem for nonlinear systems. For the case of a Lur'e system with a nonlinearity having a bounded derivative and an exosystem being a linear harmonic oscillator, feasibility conditions for this controller design are formulated in terms of linear matrix inequalities. Moreover, for this class of systems and exosystems we provide a robust controller design that copes not only with the uncertainties in the system parameters, but also with the uncertain nonlinearity from a class of nonlinearities with a given bound on their derivatives. All controller designs presented in Chapter 5 are based on certain general methods that allow us to design controllers making the corresponding closed-loop systems convergent. These methods can also be used for other control problems where the convergence property of the closed-loop system is required or desired.

If we cannot find a solution to the *global* uniform output regulation problem, it can still be possible to find a controller that solves the *local* output regulation problem. There are standard procedures for such controller designs (see, e.g., [8, 38]). The resulting controllers solve the output regulation problem for the initial conditions of the closed-loop system and the exosystem lying in *some* neighborhood of the origin. To enhance applicability of these controllers, in Chapter 6 we present estimation results that, given a controller solving the local output regulation problem, provide estimates of this neighborhood of initial conditions for which the controller works. Such estimation results are presented for both the exact and approximate variants of the local output regulation problem. These estimation results are also based on the notion of convergent systems.

The nonlinear output regulation problem has been studied from a theoretical point of view in a series of publications. At the same time, there are very few publications aiming at an experimental validation of solutions to the nonlinear output regulation problem [4, 55]. In Chapter 7 we address the nonlinear output regulation problem from an experimental point of view. We study a local output regulation problem for the so-called *TORA system*, which is a nonlinear mechanical benchmark system, see, e.g., [32, 45, 85]. A simple controller solving this problem is proposed. This controller is implemented in an experimental setup and its performance is investigated in experiments. The reason for this experimental study is twofold. The first reason is to check whether controllers from the nonlinear output regulation theory are applicable in an experimental setting in the presence of disturbances and modeling uncertainties, which are inevitable in practice. The second reason is to identify the factors that can deteriorate the controller performance and therefore require specific attention already at the stage of controller design. Successful results of this experimental study, which are presented in Chapter 7, show the applicability of the nonlinear output regulation theory in experiments and give new data for analysis and further developments in the field of nonlinear output regulation.

Finally, concluding remarks are presented in Chapter 8.

2

Convergent systems

In many control problems and, in particular, in the output regulation problem, the closed-loop system must have the following internal stability property: every solution of the closed-loop system "forgets" its initial condition and converges to a (unique) steady-state solution determined only by the input. This input can represent, for example, the feedforward part of the controller or a disturbance. This property is conveniently formalized in the notion of convergent systems. This notion forms the foundation of all results on the output regulation problem presented in this book. In this chapter we present definitions, properties, and sufficient conditions for various notions of convergent systems. Since these notions and results can be applied not only in the context of the output regulation problem, but also for other control problems, this chapter can be used separately from the rest of the material of the book.

2.1 Stability concepts

We begin by giving definitions of some stability concepts for nonautonomous systems which will be used as building blocks for the notion of convergent systems.

2.1.1 Lyapunov stability

Consider the system

$$\dot{z} = F(z,t), \quad z \in \mathbb{R}^d, \quad t \in \mathbb{R}, \tag{2.1}$$

where $F(z,t)$ is piecewise continuous in t and locally Lipschitz in z. First we give some standard definitions of Lyapunov stability of a solution of system (2.1).

Definition 2.1. *A solution $\bar{z}(t)$ of system (2.1), which is defined for $t \in (t_*, +\infty)$, is said to be*

- stable *if for any* $t_0 \in (t_*, +\infty)$ *and* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon, t_0) > 0$ *such that* $|z(t_0) - \bar{z}(t_0)| < \delta$ *implies* $|z(t) - \bar{z}(t)| < \varepsilon$ *for all* $t \geq t_0$.
- uniformly stable *if it is stable and the number* δ *in the definition of stability is independent of* t_0.
- asymptotically stable *if it is stable and for any* $t_0 \in (t_*, +\infty)$ *there exists* $\bar{\delta} = \bar{\delta}(t_0) > 0$ *such that* $|z(t_0) - \bar{z}(t_0)| < \bar{\delta}$ *implies* $\lim_{t\to+\infty} |z(t) - \bar{z}(t)| = 0$.
- uniformly asymptotically stable *if it is uniformly stable and there exists* $\bar{\delta} > 0$ *(independent of* t_0*) such that for any* $\varepsilon > 0$ *there exists* $T = T(\varepsilon) > 0$ *such that* $|z(t_0) - \bar{z}(t_0)| < \bar{\delta}$ *for* $t_0 \in (t_*, +\infty)$ *implies* $|z(t) - \bar{z}(t)| < \varepsilon$ *for all* $t \geq t_0 + T$.
- exponentially stable *if there exist* $\bar{\delta} > 0$, $C > 0$, *and* $\beta > 0$ *such that* $|z(t_0) - \bar{z}(t_0)| < \bar{\delta}$ *for* $t_0 \in (t_*, +\infty)$ *implies*

$$|z(t) - \bar{z}(t)| \leq C\mathrm{e}^{-\beta(t-t_0)}|z(t_0) - \bar{z}(t_0)|, \quad \forall\, t \geq t_0.$$

Asymptotic stability of $\bar{z}(t)$ implies that all solutions starting in *some* neighborhood of $\bar{z}(t)$ are attracted to $\bar{z}(t)$. If we are interested in asymptotic stability of the solution $\bar{z}(t)$ for a *predefined* set of initial conditions $\mathcal{Z} \subset \mathbb{R}^d$, we need the following definitions.

Definition 2.2. *A solution* $\bar{z}(t)$ *of system (2.1), which is defined for* $t \in (t_*, +\infty)$, *is said to be*

- asymptotically stable in a set $\mathcal{Z} \subset \mathbb{R}^d$ *if it is asymptotically stable and any solution of system (2.1) starting in* $z(t_0) \in \mathcal{Z}$, $t_0 \in (t_*, +\infty)$ *satisfies* $|z(t) - \bar{z}(t)| \to 0$ *as* $t \to +\infty$.
- uniformly asymptotically stable in $\mathcal{Z} \subset \mathbb{R}^d$ *if it is uniformly asymptotically stable and it attracts solutions of system (2.1) starting in* $z(t_0) \in \mathcal{Z}$, $t_0 \in (t_*, +\infty)$ *uniformly over* t_0, *i.e., for any compact set* $K \subset \mathcal{Z}$ *and any* $\varepsilon > 0$ *there exists* $T(\varepsilon, K) > 0$ *such that if* $z(t_0) \in K$, $t_0 \in (t_*, +\infty)$, *then* $|z(t) - \bar{z}(t)| < \varepsilon$ *for all* $t \geq t_0 + T(\varepsilon, K)$.
- exponentially stable in $\mathcal{Z} \subset \mathbb{R}^d$ *if it is exponentially stable and there exist constants* $C > 0$ *and* $\beta > 0$ *such that any solution starting in* $z(t_0) \in \mathcal{Z}$, $t_0 \in (t_*, +\infty)$ *satisfies*

$$|z(t) - \bar{z}(t)| \leq C\mathrm{e}^{-\beta(t-t_0)}|z(t_0) - \bar{z}(t_0)|. \tag{2.2}$$

Recall that the domain of attraction of an asymptotically stable solution $\bar{z}(t)$ is defined as a family of sets $D(t_0) \subset \mathbb{R}^d$, $t_0 \in (t_*, +\infty)$ such that if $z(t_0) \in D(t_0)$ then $|z(t) - \bar{z}(t)| \to 0$ as $t \to +\infty$. In general, a domain of attraction $D(t_0)$ depends on t_0. Thus, the requirement of asymptotic stability in a set $\mathcal{Z}$ means that $\mathcal{Z} \subset D(t_0)$ for all $t_0 \in (t_*, +\infty)$. In this case, the set $\mathcal{Z}$ can be called a uniform domain of attraction.

Definition 2.3. *A solution* $\bar{z}(t)$, *which is defined for* $t \in (t_*, +\infty)$, *is called globally (uniformly, exponentially) asymptotically stable if it is (uniformly, exponentially) asymptotically stable in* $\mathcal{Z} = \mathbb{R}^d$.

Remark. In the literature, stability notions are usually defined only with respect to the zero solution $\bar{z}(t) \equiv 0$ (see, e.g., [51]). This is due to the fact that if $\bar{z}(t)$ is not identically zero, one can make the coordinate transformation $x := z - \bar{z}(t)$. After such a coordinate transformation, system (2.1) takes the form

$$\dot{x} = F(x + \bar{z}(t), t) - F(\bar{z}(t), t), \tag{2.3}$$

with the solution $\bar{x}(t) \equiv 0$ corresponding to the solution $\bar{z}(t)$ in the original coordinates. After such a transformation, (uniform) stability and (uniform) asymptotic stability of solutions $\bar{z}(t)$ and $\bar{x}(t) \equiv 0$ are equivalent. At the same time, if we deal with (uniform) asymptotic stability of $\bar{z}(t)$ with some uniform domain of attraction $\mathcal{Z}$, this coordinate transformation not only transforms $\bar{z}(t)$ into $\bar{x}(t) \equiv 0$, but it also transforms the set $\mathcal{Z}$ into some time-dependent set $\mathcal{X}(t)$. Analysis of a solution with a time-dependent domain of attraction is rather complicated. Therefore, we provide the definitions of (uniform) asymptotic stability in a set $\mathcal{Z}$ with respect to the solution $\bar{z}(t)$ in the original coordinates. Notice that in the case of $\mathcal{Z} = \mathbb{R}^d$, global (uniform, exponential) asymptotic stability of a *bounded* solution $\bar{z}(t)$ is equivalent to global (uniform, exponential) asymptotic stability of the solution $\bar{x}(t) \equiv 0$ of system (2.3). In this case, Definition 2.3 is equivalent to conventional definitions of (uniform, exponential) global asymptotic stability, see, e.g., [51].$\lhd$

In the analysis of the output regulation problem, we will need the following two properties of uniformly asymptotically stable solutions.

Property 2.4. *Suppose $\bar{z}(t)$ is a solution of system (2.1) defined for all $t \in \mathbb{R}$ and uniformly asymptotically stable in $\mathcal{Z}$. If there exists a solution $\tilde{z}(t)$ that is defined for all $t \in \mathbb{R}$ and lies in some compact set $K \subset \mathcal{Z}$ for all $t \in \mathbb{R}$, then $\tilde{z}(t) \equiv \bar{z}(t)$.*

Proof: Suppose at some instant $t_* \in \mathbb{R}$ the solutions $\bar{z}(t)$ and $\tilde{z}(t)$ satisfy $|\tilde{z}(t_*) - \bar{z}(t_*)| \geq \varepsilon > 0$ for some $\varepsilon > 0$. Since $\bar{z}(t)$ is uniformly asymptotically stable in $\mathcal{Z}$, there exists a number $T(\varepsilon, K)$ such that if $\tilde{z}(t_0) \in K$ for some $t_0 \in \mathbb{R}$ then

$$|\tilde{z}(t) - \bar{z}(t)| < \varepsilon, \quad \forall\, t \geq t_0 + T(\varepsilon, K). \tag{2.4}$$

Set $t_0 := t_* - T(\varepsilon, K)$. Then for $t = t_*$ inequality (2.4) implies $|\tilde{z}(t_*) - \bar{z}(t_*)| < \varepsilon$. Thus, we obtain a contradiction. Since t_* has been chosen arbitrarily, this implies $\tilde{z}(t) \equiv \bar{z}(t)$. $\square$

In the case of $\mathcal{Z} = \mathbb{R}^d$ Property 2.4 means the following. If a solution $\bar{z}(t)$ defined for all $t \in \mathbb{R}$ is globally uniformly asymptotically stable, and there exists some solution $\tilde{z}(t)$ that is also defined and bounded for all $t \in \mathbb{R}$, then $\bar{z}(t)$ and $\tilde{z}(t)$ coincide. The next property states uniqueness of a solution that is uniformly asymptotically stable in a set $\mathcal{Z}$.

Property 2.5. *If there exists a solution $\bar{z}(t)$ of system (2.1) such that it is defined for all $t \in \mathbb{R}$ and uniformly asymptotically stable in $\mathcal{Z}$, then such a solution is unique.*

Proof: Suppose $\tilde{z}(t)$ is another solution that is defined for all $t \in \mathbb{R}$ and uniformly asymptotically stable in $\mathcal{Z}$. Suppose at some instant $t_* \in \mathbb{R}$ it holds that $|\tilde{z}(t_*) - \bar{z}(t_*)| \geq 2\varepsilon > 0$ for some $\varepsilon > 0$. Let z_* be some point in the set $\mathcal{Z}$. Since both $\tilde{z}(t)$ and $\bar{z}(t)$ are uniformly asymptotically stable in $\mathcal{Z}$, there exists $T(\varepsilon, z_*) > 0$ such that if $z(t_0) = z_*$ for some $t_0 \in \mathbb{R}$ then

$$|z(t) - \bar{z}(t)| < \varepsilon, \quad |z(t) - \tilde{z}(t)| < \varepsilon \quad \forall \ t \geq t_0 + T(\varepsilon, z_*). \tag{2.5}$$

Set $t_0 := t_* - T(\varepsilon, z_*)$. By the triangle inequality and inequality (2.5), it holds that for $t := t_* = t_0 + T(\varepsilon, z_*)$

$$|\tilde{z}(t_*) - \bar{z}(t_*)| \leq |\tilde{z}(t_*) - z(t_*)| + |z(t_*) - \bar{z}(t_*)| < 2\varepsilon.$$

Thus, we obtain a contradiction. Hence, $\tilde{z}(t_*) = \bar{z}(t_*)$. Due to the arbitrary choice of t_*, we conclude that $\tilde{z}(t) \equiv \bar{z}(t)$ for all $t \in \mathbb{R}$. □

2.1.2 Input-to-state stability

In this section we recall the notion of input-to-state stability (ISS) and review some related results. Prior to introducing the notion of ISS, we recall the following definitions of class $\mathcal{K}$ and class $\mathcal{KL}$ functions [51].

Definition 2.6. *A continuous function $\alpha : [0, a) \to [0, +\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\alpha(r) \to +\infty$ as $r \to +\infty$.*

Definition 2.7. *A continuous function $\beta : [0, a) \times [0, +\infty) \to [0, +\infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to +\infty$.*

With these definitions at hand, we can formulate the property of ISS. Consider the system

$$\dot{z} = F(z, w, t), \quad z \in \mathbb{R}^d, \quad w \in \mathbb{R}^m, \quad t \in \mathbb{R}, \tag{2.6}$$

where $F(z, w, t)$ is locally Lipschitz in z, continuous in w, and piecewise continuous in t. The input $w(t)$ is a piecewise continuous function of t. Suppose for $w(t) \equiv 0$ system (2.6) has an equilibrium point $z = 0$.

Definition 2.8 ([51]). *System (2.6) is said to be locally ISS if there exist a class $\mathcal{KL}$ function $\beta(r, s)$, a class $\mathcal{K}$ function $\gamma(r)$, and positive constants k_z and k_w such that for any initial state $z(t_0)$ with $|z(t_0)| \leq k_z$ and any input $w(t)$ with $\sup_{t \geq t_0} |w(t)| \leq k_w$, the solution $z(t)$ exists and satisfies*

$$|z(t)| \leq \beta(|z(t_0)|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} |w(\tau)|\right) \tag{2.7}$$

for all $t \geq t_0$. It is said to be ISS if inequality (2.7) is satisfied for any initial state $z(t_0)$ and any bounded input $w(t)$.

Remark. Local ISS implies, in particular, that for any input $w(t)$ satisfying $|w(t)| \leq k_w$ for all $t \geq t_0$ and $w(t) \to 0$ as $t \to +\infty$, any solution $z(t)$ of system (2.6) starting in $|z(t_0)| \leq k_z$ tends to zero, i.e., $z(t) \to 0$ as $t \to +\infty$.$\lhd$

Below we review some results related to the ISS that will be used further in the book. The following theorem gives a sufficient condition for ISS.

Theorem 2.9 ([51]). *Consider system (2.6). Let $V(z,t)$ be a continuously differentiable function such that*

$$\alpha_1(|z|) \leq V(z,t) \leq \alpha_2(|z|), \tag{2.8}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial z} F(z,w,t) \leq -\alpha_3(|z|), \quad \forall |z| \geq \rho(|w|) > 0, \tag{2.9}$$

for all $(z,w,t) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$, where $\alpha_1(r)$ is a class $\mathcal{K}_\infty$ function and $\alpha_2(r)$, $\alpha_3(r)$, and $\rho(r)$ are class $\mathcal{K}$ functions. Then system (2.6) is ISS with $\gamma = \alpha^{-1} \circ \alpha_2 \circ \rho$.[1]

The following lemma establishes a link between the uniform asymptotic stability of the equilibrium $z = 0$ of the unforced system (2.6) and the local ISS of system (2.6).

Lemma 2.10 ([51]). *Consider system (2.6). Suppose that in some neighborhood of the origin (z,w)=(0,0), the function $F(z,w,t)$ is continuously differentiable and the Jacobian matrices $\partial F/\partial z$ and $\partial F/\partial w$ are bounded, uniformly in t. If the equilibrium $z = 0$ of system (2.6) with $w(t) \equiv 0$ is uniformly asymptotically stable, then system (2.6) is locally ISS.*

The next lemma allows one to establish existence of a solution of a (locally) ISS system that is defined and bounded for all $t \in \mathbb{R}$.

Lemma 2.11. *Suppose system (2.6) is locally ISS. Then there exists a number $\tilde{k}_w > 0$ such that for any input $w(t)$ defined for all $t \in \mathbb{R}$ and satisfying $\sup_{t\in\mathbb{R}} |w(t)| \leq \tilde{k}_w$ there exists a solution $z_w(t)$ that is defined for all $t \in \mathbb{R}$ and satisfies*

$$\sup_{t\in\mathbb{R}} |z_w(t)| \leq \gamma \left(\sup_{t\in\mathbb{R}} |w(t)| \right), \tag{2.10}$$

where $\gamma(r)$ is the class $\mathcal{K}$ function from the definition of the ISS. If system (2.6) is ISS, then a solution $z_w(t)$ satisfying (2.10) exists for any input $w(t)$ bounded on $\mathbb{R}$.

Proof: See Appendix 9.1.

The ISS property is very useful for studying interconnected systems. Consider the systems

[1] Here $\circ$ denotes the composition of two functions, i.e., $(\alpha \circ \rho)(r) = \alpha(\rho(r))$.

$$\dot{z} = F(z, y, w, t), \tag{2.11}$$
$$\dot{y} = G(y, w, t), \tag{2.12}$$

with the functions $F(z, y, w, t)$ and $G(y, w, t)$ being locally Lipschitz in z and y, continuous in w, and piecewise continuous in t.

Theorem 2.12 ([82]). *Consider systems (2.11) and (2.12). Suppose the system (2.11) with (y, w) as input is ISS and the system (2.12) with w as input is ISS. Then the interconnection of systems (2.11) and (2.12) is also ISS.*

The theorem presented above deals only with series interconnection of ISS systems. The next theorem, which is known as the small gain theorem for ISS systems, allows one to establish ISS for arbitrarily interconnected ISS systems. Consider the systems

$$\dot{z} = F(z, y, w, t), \tag{2.13}$$
$$\dot{y} = G(z, y, w, t), \tag{2.14}$$

with functions $F(z, y, w, t)$ and $G(z, y, w, t)$ being locally Lipschitz in z and y, continuous in w, and piecewise continuous in t.

Theorem 2.13 ([48]). *Consider systems (2.13) and (2.14). Suppose system (2.13) with (y, w) as input and system (2.14) with (z, w) as input are ISS and, for some class $\mathcal{K}$ functions $\gamma_y(r)$, $\gamma_z(r)$, $\gamma_{wz}(r)$, and $\gamma_{wy}(r)$, and for some class $\mathcal{KL}$ functions $\beta_z(r, s)$ and $\beta_y(r, s)$, any solution of system (2.13) with bounded inputs $y(t)$ and $w(t)$ satisfies*

$$|z(t)| \leq \beta_z(|z(t_0)|, t - t_0) + \gamma_y\left(\sup_{t_0 \leq \tau \leq t} |y(\tau)|\right) + \gamma_{wz}\left(\sup_{t_0 \leq \tau \leq t} |w(\tau)|\right)$$

and any solution of system (2.14) with bounded inputs $z(t)$ and $w(t)$ satisfies

$$|y(t)| \leq \beta_y(|y(t_0)|, t - t_0) + \gamma_z\left(\sup_{t_0 \leq \tau \leq t} |z(\tau)|\right) + \gamma_{wy}\left(\sup_{t_0 \leq \tau \leq t} |w(\tau)|\right).$$

Suppose for some class $\mathcal{K}$ function $\rho(r)$ the following small gain relation is satisfied:

$$(\gamma_z + \rho) \circ (\gamma_y + \rho)(r) \leq r.$$

Then the interconnected system (2.13), (2.14) is ISS.

2.2 Convergent systems

2.2.1 Basic definitions

In this subsection we give definitions of convergent systems. These definitions extend the original definition of convergent systems given by B.P. Demidovich in [15]; see also [66]. Consider the system

$$\dot{z} = F(z, t), \tag{2.18}$$

where $z \in \mathbb{R}^d$ and $t \in \mathbb{R}$; $F(z,t)$ is locally Lipschitz in z and piecewise continuous in t.

Definition 2.14. *System (2.18) is said to be*

- convergent in a set $\mathcal{Z} \subset \mathbb{R}^d$ *if there exists a solution* $\bar{z}(t)$ *satisfying the following conditions:*
  (i) $\bar{z}(t)$ *is defined and bounded for all* $t \in \mathbb{R}$,
  (ii) $\bar{z}(t)$ *is asymptotically stable in* $\mathcal{Z}$.
- uniformly convergent in $\mathcal{Z}$ *if it is convergent in* $\mathcal{Z}$ *and* $\bar{z}(t)$ *is uniformly asymptotically stable in* $\mathcal{Z}$.
- exponentially convergent in $\mathcal{Z}$ *if it is convergent in* $\mathcal{Z}$ *and* $\bar{z}(t)$ *is exponentially stable in* $\mathcal{Z}$.

If system (2.18) is (uniformly, exponentially) convergent in $\mathcal{Z} = \mathbb{R}^d$*, then it is called globally (uniformly, exponentially) convergent.*

The solution $\bar{z}(t)$ will be called a *steady-state solution* and the set $\mathcal{Z}$ will be referred to as a *convergence region*. As follows from the definition of convergence, any solution of a convergent system starting in a convergence region "forgets" its initial condition and converges to some steady-state solution. In general, the steady-state solution $\bar{z}(t)$ may be nonunique. But for any two steady-state solutions $\bar{z}_1(t)$ and $\bar{z}_2(t)$, it holds that $|\bar{z}_1(t) - \bar{z}_2(t)| \to 0$ as $t \to +\infty$. This statement follows from the triangle inequality

$$|\bar{z}_1(t) - \bar{z}_2(t)| \leq |\bar{z}_1(t) - z(t)| + |z(t) - \bar{z}_2(t)|,$$

where $z(t)$ is some solution of system (2.18) starting in $z(t_0) \in \mathcal{Z}$, $t_0 \in \mathbb{R}$, and from the fact that $\bar{z}_1(t)$ and $\bar{z}_2(t)$ attract solutions starting in $\mathcal{Z}$.

In the original definition of convergent systems given by B.P. Demidovich in [15], it is required that there exists only one solution $\bar{z}_w(t)$ bounded on $\mathbb{R}$; i.e., the steady-state solution is unique. Therefore, the class of convergent systems defined above is larger than the class of convergent systems defined by B.P. Demidovich. At the same time, for the practically important case of *uniformly* convergent systems, we see that Property 2.5 yields uniqueness of the steady-state solution.

Property 2.15. *If system (2.18) is uniformly convergent in* $\mathcal{Z}$*, then the steady-state solution* $\bar{z}(t)$ *is unique.*

The convergence property is an extension of stability properties of asymptotically stable linear time-invariant (LTI) systems. Recall that for a piecewise continuous vector-function $f(t)$, which is bounded on $t \in \mathbb{R}$, the system $\dot{z} = Az + f(t)$ with a Hurwitz matrix A has a unique solution $\bar{z}(t)$ which is defined and bounded on $t \in (-\infty, +\infty)$. It is given by the formula

$\bar{z}(t) := \int_{-\infty}^{t} \exp(A(t-s))f(s)ds$. This solution is globally exponentially stable with the rate of convergence depending only on the matrix A. Thus, an asymptotically stable LTI system excited by a bounded piecewise-continuous function $f(t)$ is globally exponentially convergent.

2.2.2 Convergent systems with inputs

In the scope of control problems, time dependency of the right-hand side of system (2.18) is usually due to some input. This input may represent, for example, a disturbance or a feedforward control signal. In this section we will consider convergence properties for systems with inputs. So, instead of systems of the form (2.18), we consider systems

$$\dot{z} = F(z, w), \tag{2.19}$$

with state $z \in \mathbb{R}^d$ and input $w \in \mathbb{R}^m$. The function $F(z, w)$ is locally Lipschitz in z and continuous in w. The inputs $w(t)$ are assumed to be piecewise continuous functions of time defined for all $t \in \mathbb{R}$.

Classes of inputs
In this book we will deal with several important classes of inputs. The first class consists of piecewise continuous vector functions $w(t) \in \mathbb{R}^m$ that are defined and bounded on $t \in \mathbb{R}$. This class of inputs is denoted by $\overline{\mathbb{PC}}_m$. The second class $\overline{\mathbb{PC}}(\mathcal{W})$ is defined in the following way. Let $\mathcal{W}$ be some subset of $\mathbb{R}^m$. A function $w(\cdot) : \mathbb{R} \to \mathcal{W}$ belongs to the class $\overline{\mathbb{PC}}(\mathcal{W})$ if it is piecewise continuous and if there exists a compact set $\mathcal{K}_w \subset \mathcal{W}$ such that $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$. In particular, we obtain that $\overline{\mathbb{PC}}(\mathbb{R}^m) = \overline{\mathbb{PC}}_m$. Another important class of inputs considered in the book is related to solutions of the system

$$\dot{w} = s(w), \quad w \in \mathbb{R}^m, \tag{2.20}$$

with a locally Lipschitz function $s(w)$. Let the set $\mathcal{W} \subset \mathbb{R}^m$ be invariant with respect to system (2.20).[2] The class of inputs $\mathcal{I}_s(\mathcal{W})$ consists of solutions $w(t) = w(t, w_0)$ of system (2.20) starting in $w(0) = w_0 \in \mathcal{W}$. Note that since the set $\mathcal{W}$ is invariant, we have for $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ that $w(t) \in \mathcal{W}$ for all $t \in \mathbb{R}$.

Below we define the convergence property for systems with inputs.

Definition 2.16. *System (2.19) is said to be* (uniformly, exponentially) convergent in a set $\mathcal{Z} \subset \mathbb{R}^m$ for a class of inputs $\mathcal{N} \subset \overline{\mathbb{PC}}_m$ *if it is (uniformly, exponentially) convergent in $\mathcal{Z}$ for every input $w(\cdot) \in \mathcal{N}$. In order to emphasize the dependency of the steady-state solution on the input $w(t)$, it is denoted by $\bar{z}_w(t)$.*

[2] A set $\mathcal{W} \subset \mathbb{R}^m$ is called invariant (positively invariant) with respect to system (2.20) if $w_0 \in \mathcal{W}$ implies $w(t, w_0) \in \mathcal{W}$ for all $t \in \mathbb{R}$ (for all $t \geq 0$).

As follows from the previous section, a simple example of a system that is globally exponentially convergent for the class of inputs $\overline{\mathbb{PC}}_m$ is the system

$$\dot{z} = Az + Bw, \tag{2.21}$$

with a Hurwitz matrix A. The corresponding steady-state solution $\bar{z}_w(t)$ equals $\bar{z}_w(t) := \int_{-\infty}^{t} \exp(A(t-s))Bw(s)ds$. Moreover, this solution is bounded by

$$|\bar{z}_w(t)| \leq \int_{-\infty}^{0} \|\exp(-As)B\|ds \sup_{\tau\in\mathbb{R}} |w(\tau)|.$$

Thus, for all inputs $w(t)$ satisfying $|w(t)| \leq \rho$, for some $\rho > 0$ and all $t \in \mathbb{R}$, the corresponding steady-state solutions satisfy $|\bar{z}_w(t)| \leq \mathcal{R}$ for all $t \in \mathbb{R}$, where $\mathcal{R} := \rho \int_{-\infty}^{0} \|\exp(-As)B\|ds$. Therefore, all steady-state solutions corresponding to the inputs satisfying $|w(t)| \leq \rho$ for all $t \in \mathbb{R}$ are bounded by $\mathcal{R}$ uniformly with respect to all inputs from this set. This motivates the introduction of the uniformly bounded steady-state (UBSS) property. For a set $\mathcal{W} \subset \mathbb{R}^m$, consider some class of inputs $\mathcal{N}(\mathcal{W}) \subset \overline{\mathbb{PC}}(\mathcal{W})$.

Definition 2.17. *The system (2.19) that is convergent in $\mathcal{Z}$ for a class of inputs $\mathcal{N}(\mathcal{W})$ is said to have the UBSS property if for any compact set $\mathcal{K}_w \subset \mathcal{W}$ there exists a compact set $\mathcal{K}_z \subset \mathbb{R}^d$ such that for any input $w(\cdot) \in \mathcal{N}(\mathcal{W})$ the following implication holds:*

$$w(t) \in \mathcal{K}_w \;\; \forall\, t \in \mathbb{R} \quad \Rightarrow \quad \bar{z}_w(t) \in \mathcal{K}_z \;\forall\, t \in \mathbb{R}. \tag{2.22}$$

Remark. For $\mathcal{Z} = \mathbb{R}^d$, $\mathcal{W} = \mathbb{R}^m$, and $\mathcal{N}(\mathcal{W}) = \overline{\mathbb{PC}}_m$ this definition is equivalent to the following statement. For every $\rho > 0$ there exists $\mathcal{R} > 0$ such that if a piecewise continuous input $w(t)$ satisfies $|w(t)| \leq \rho$ for all $t \in \mathbb{R}$, then the corresponding steady-state solution satisfies $|\bar{z}_w(t)| \leq \mathcal{R}$ for all $t \in \mathbb{R}$. This UBSS property will prove to be useful in Chapter 4.$\lhd$

A property that is even stronger than the UBSS property and also holds for asymptotically stable LTI systems is presented in the following definition.

Definition 2.18. *System (2.19) is said to be input-to-state convergent if it is globally uniformly convergent for the class of inputs $\overline{\mathbb{PC}}_m$ and for every input $w(\cdot) \in \overline{\mathbb{PC}}_m$ system (2.19) is ISS with respect to the steady-state solution $\bar{z}_w(t)$, i.e., there exist a $\mathcal{KL}$-function $\beta(r,s)$ and a $\mathcal{K}_\infty$-function $\gamma(r)$ such that any solution $z(t)$ of system (2.19) corresponding to some input $\hat{w}(t) := w(t) + \Delta w(t)$ satisfies*

$$|z(t) - \bar{z}_w(t)| \leq \beta(|z(t_0) - \bar{z}_w(t_0)|, t - t_0) + \gamma\left(\sup_{t_0\leq\tau\leq t} |\Delta w(\tau)|\right). \tag{2.23}$$

In general, the functions $\beta(r,s)$ and $\gamma(r)$ may depend on the particular input $w(\cdot)$.

The following property establishes a link between the input-to-state convergence and the global uniform convergence with the UBSS property.

Property 2.19. *If system (2.19) is input-to-state convergent, then it is globally uniformly convergent with the UBSS property for the class of inputs* $\overline{\mathbb{PC}}_m$.

Proof: We only need to show that an input-to-state convergent system has the UBSS property for the class of inputs $\overline{\mathbb{PC}}_m$. Consider some input $\hat{w}(\cdot) \in \overline{\mathbb{PC}}_m$. Since the system is input-to-state convergent, there exists a steady-state solution $\bar{z}_{\hat{w}}(t)$, which is bounded on $t \in \mathbb{R}$. For any input $w(\cdot) \in \overline{\mathbb{PC}}_m$ satisfying $|w(t)| \leq \rho$ for some $\rho > 0$ and all $t \in \mathbb{R}$, the corresponding steady-state solution $\bar{z}_w(t)$ satisfies

$$\begin{aligned} |\bar{z}_w(t)| &\leq |\bar{z}_{\hat{w}}(t)| + |\bar{z}_w(t) - \bar{z}_{\hat{w}}(t)| \\ &\leq \sup_{t\in\mathbb{R}} |\bar{z}_{\hat{w}}(t)| + \beta(|\bar{z}_w(t_0) - \bar{z}_{\hat{w}}(t_0)|, t - t_0) + \gamma\left(\sup_{t\in\mathbb{R}} |w(t) - \hat{w}(t)|\right) \\ &\leq \sup_{t\in\mathbb{R}} |\bar{z}_{\hat{w}}(t)| + \gamma\left(\sup_{t\in\mathbb{R},\ |w|\leq\rho} |w - \hat{w}(t)|\right) =: \mathcal{R}. \end{aligned} \tag{2.24}$$

In the last inequality we have used the fact that $|\bar{z}_w(t_0) - \bar{z}_{\hat{w}}(t_0)|$ remains bounded as $t_0 \to -\infty$ and, therefore, $\beta(|\bar{z}_w(t_0) - \bar{z}_{\hat{w}}(t_0)|, t - t_0) \to 0$ as $t_0 \to -\infty$. Thus, we have shown that for any input $w(t)$ satisfying $|w(t)| \leq \rho$ for all $t \in \mathbb{R}$, the corresponding steady-state solution $\bar{z}_w(t)$ satisfies $|\bar{z}_w(t)| \leq \mathcal{R}$ for all $t \in \mathbb{R}$. Notice that $\mathcal{R}$ does not depend on $w(t)$. This proves the UBSS property. □

Similarly to the conventional ISS property, the property of input-to-state convergence is especially useful for studying convergence properties of interconnected systems. For this purpose the input-to-state convergence property will be used in Chapter 5. For local analysis we introduce the notion of local convergence.

Definition 2.20. *System (2.19) with* $F(0,0) = 0$ *is said to be* locally (uniformly, exponentially) convergent *for some class of inputs* $\mathcal{N} \subset \overline{\mathbb{PC}}_m$ *if there exists a neighborhood of the origin* $\mathcal{Z} \subset \mathbb{R}^d$ *and a number* $\rho > 0$ *such that system (2.19) is (uniformly, exponentially) convergent in* $\mathcal{Z}$ *for all inputs* $w(\cdot) \in \mathcal{N}$ *satisfying the condition* $|w(t)| < \rho$ *for all* $t \in \mathbb{R}$.

Roughly speaking, if system (2.19) is locally convergent, then for any sufficiently small input $w(t)$ from the class $\mathcal{N}$ all solutions of system (2.19) starting close enough to the origin converge to the same steady-state solution $\bar{z}_w(t)$.

2.2.3 Basic properties of convergent systems

As follows from the previous section, the (uniform) convergence property and the input-to-state convergence property are extensions of stability properties of asymptotically stable LTI systems. In this section we present certain properties of convergent systems that are inherited from asymptotically stable LTI systems. The results presented in this section will be used in subsequent chapters in the analysis and controller design for the output regulation problem.

Since all ingredients of the (uniform) convergence, the UBSS property, and the input-to-state convergence are invariant under smooth coordinate transformations (see Definitions 2.14, 2.17, 2.18), we can formulate the following statement.

Property 2.21. *The uniform convergence property is preserved under smooth coordinate transformations in the following sense. If system (2.19) is (uniformly) convergent in $\mathcal{Z}$ for some class of inputs $\mathcal{N} \subset \overline{\mathbb{PC}}_m$, then after a smooth coordinate transformation $\tilde{z} = \psi(z)$ the system in the new coordinates is (uniformly) convergent in $\tilde{\mathcal{Z}}$, where $\tilde{\mathcal{Z}} = \psi(\mathcal{Z})$ is the image of $\mathcal{Z}$ under the mapping ψ. Moreover, the UBSS property and the input-to-state convergence property are preserved under smooth coordinate transformations.*

Certain properties of convergent systems can be concluded if the input $w(t)$ is defined only on the positive half axis $t \in [t_0, +\infty)$ rather than on the whole time axis as in the definition of the convergence property.

Property 2.22. *Suppose system (2.19) is globally (uniformly) convergent for the class of inputs $\overline{\mathbb{PC}}(\mathcal{W})$. Then for every piecewise continuous input $w(t)$ defined for $t \geq t_0$ and lying in some compact subset of $\mathcal{W}$ for all $t \geq t_0$, there exists a solution $\tilde{z}_w(t)$ that is defined and bounded for all $t \geq t_0$ and that is (uniformly) globally asymptotically stable.*

Proof: Define $\tilde{w}(t)$ such that $\tilde{w}(t) = w(t)$ for all $t \geq t_0$ and $\tilde{w}(t) \equiv w(t_0)$ for all $t < t_0$. This $\tilde{w}(t)$ belongs to the class $\overline{\mathbb{PC}}(\mathcal{W})$. Since system (2.19) is globally (uniformly) convergent for the class of inputs $\overline{\mathbb{PC}}(\mathcal{W})$, there exists a steady-state solution $\bar{z}_{\tilde{w}}(t)$ corresponding to the input $\tilde{w}(t)$. This solution $\bar{z}_{\tilde{w}}(t)$ is (uniformly) globally asymptotically stable. By definition, for $t \geq t_0$ this $\bar{z}_{\tilde{w}}(t)$ is a solution of system (2.19) with the input $w(t)$. This establishes our claim. □

The next statement summarizes some properties of uniformly convergent systems excited by periodic or constant inputs.

Property 2.23. *Suppose system (2.19) with a given input $w(t)$ is uniformly convergent in $\mathcal{Z}$. If the input $w(t)$ is constant, then the corresponding steady-state solution $\bar{z}_w(t)$ is also constant; if the input $w(t)$ is periodic with period T, then the corresponding steady-state solution $\bar{z}_w(t)$ is also periodic with the same period T.*

Proof: Suppose the input $w(t)$ is periodic with period $T > 0$. Denote $\tilde{z}_w(t) := \bar{z}_w(t + T)$. Notice that $\tilde{z}_w(t)$ is a solution of system (2.19). Namely, by the definition of $\tilde{z}_w(t)$, it is a solution of the system

$$\dot{z} = F(z, w(t+T)) \equiv F(z, w(t)).$$

Here we have used the periodicity of $w(t)$, i.e., $w(t+T) \equiv w(t)$. Since $\bar{z}_w(t)$ is bounded on $t \in \mathbb{R}$ and uniformly asymptotically stable in $\mathcal{Z}$, so is $\tilde{z}_w(t)$, because $\tilde{z}_w(t)$ is a time-shifted version of $\bar{z}_w(t)$. Thus, $\tilde{z}_w(t)$ is also a steady-state solution of system (2.19). But as follows from Property 2.15, the steady-state solution of a uniformly convergent system is unique. Thus, $\bar{z}_w(t+T) = \tilde{z}_w(t) \equiv \bar{z}_w(t)$. This proves T-periodicity of the steady-state solution $\bar{z}_w(t)$. A constant input $w(t) \equiv w_*$ is periodic for any period $T > 0$. Hence, the corresponding steady-state solution $\bar{z}_w(t)$ is also periodic with any period $T > 0$. This implies that $\bar{z}_w(t)$ is constant. □

If a system is locally uniformly convergent for some class of inputs $\mathcal{N} \subset \overline{\mathbb{PC}}_m$ containing the zero input $w(t) \equiv 0$, then it has a certain continuity property that guarantees that for small inputs the corresponding steady-state solutions are also small. This is stated in the following property.

Property 2.24. *Consider system (2.19) with $F(0,0) = 0$ and $F(z,w)$ being C^1 in some neighborhood of the origin $(z,w) = (0,0)$. Suppose system (2.19) is locally uniformly convergent for some class of inputs $\mathcal{N} \subset \overline{\mathbb{PC}}_m$ containing the zero input $w(t) \equiv 0$. Then there exists a neighborhood of the origin $\mathcal{Z} \subset \mathbb{R}^d$, a number $k_w > 0$ and a class $\mathcal{K}$ function $\gamma(r)$ such that system (2.19) is uniformly convergent in $\mathcal{Z}$ for any $w(\cdot) \in \mathcal{N}$ satisfying $\sup_{t\in\mathbb{R}} |w(t)| \leq k_w$ and the corresponding steady-state solution $\bar{z}_w(t)$ satisfies*

$$\sup_{t\in\mathbb{R}} |\bar{z}_w(t)| \leq \gamma\left(\sup_{t\in\mathbb{R}} |w(t)|\right). \tag{2.25}$$

Proof: See Appendix 9.2.

If two inputs converge to each other, so do the corresponding steady-state solutions, as follows from the next property.

Property 2.25. *Suppose system (2.19) is globally uniformly convergent for the class of inputs $\overline{\mathbb{PC}}_m$ and $F(z,w)$ is C^1. Then for any two inputs $w_1(\cdot)$ and $w_2(\cdot) \in \overline{\mathbb{PC}}_m$ satisfying $w_1(t) - w_2(t) \to 0$ as $t \to +\infty$, the corresponding steady-state solutions $\bar{z}_{w_1}(t)$ and $\bar{z}_{w_2}(t)$ satisfy $\bar{z}_{w_1}(t) - \bar{z}_{w_2}(t) \to 0$ as $t \to +\infty$.*

Proof: See Appendix 9.3.

The next two properties relate to parallel and series connections of uniformly convergent systems, as shown in Figures 2.1 and 2.2.

Property 2.26 (Parallel connection). *Consider the system*

$$\begin{cases} \dot{z} = F(z,w), & z \in \mathbb{R}^d, \\ \dot{y} = G(y,w), & y \in \mathbb{R}^q. \end{cases} \tag{2.26}$$

Suppose the z- and y-subsystems are globally uniformly convergent for some class of inputs $\mathcal{N} \subset \overline{\mathbb{PC}}_m$ (input-to-state convergent). Then the system (2.26) is globally uniformly convergent for the class of inputs $\mathcal{N}$ (input-to-state convergent).

Proof: The proof directly follows from the definitions of uniformly convergent and input-to-state convergent systems. □

Property 2.27 (Series connection). *Consider the system*

$$\begin{cases} \dot{z} = F(z, y, w), & z \in \mathbb{R}^d, \\ \dot{y} = G(y, w), & y \in \mathbb{R}^q. \end{cases} \tag{2.27}$$

Suppose the z-subsystem with (y, w) as input is input-to-state convergent, and the y-subsystem with w as input is input-to-state convergent. Then system (2.27) is input-to-state convergent.

Proof: See Appendix 9.4.

The next property deals with bidirectionally interconnected input-to-state convergent systems, as shown in Figure 2.3.

Property 2.28. *Consider the system*

$$\begin{cases} \dot{z} = F(z, y, w), & z \in \mathbb{R}^d, \\ \dot{y} = G(z, y, w), & y \in \mathbb{R}^q. \end{cases} \tag{2.28}$$

Suppose the z-subsystem with (y, w) as input is input-to-state convergent. Assume that there exists a class $\mathcal{KL}$ function $\beta_y(r, s)$ such that for any input $(z(\cdot), w(\cdot)) \in \overline{\mathbb{PC}}_{d+m}$ any solution of the y-subsystem satisfies

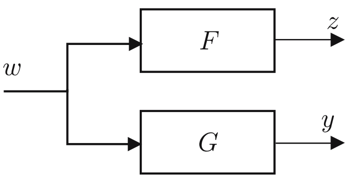

Fig. 2.1. Parallel connection of two systems with inputs.

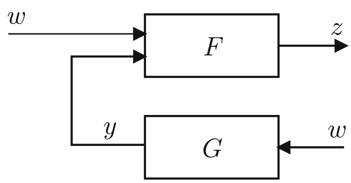

Fig. 2.2. Series connection of two systems with inputs.

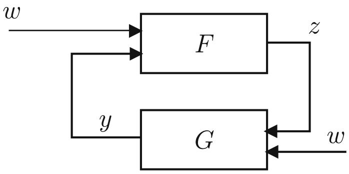

Fig. 2.3. Bidirectionally interconnected systems with inputs.

$$|y(t)| \le \beta_y(|y(t_0)|, t - t_0).$$

Then the interconnected system (2.28) is input-to-state convergent.

Proof: Denote $\bar{z}_w(t)$ to be the steady-state solution of the z-subsystem corresponding to $y = 0$ and to some $w(\cdot) \in \overline{\mathbb{PC}}$. Then $(\bar{z}_w(t), 0)$ is a solution of the interconnected system (2.28) that is defined and bounded for all $t \in \mathbb{R}$. We establish the property by performing the change of coordinates $\tilde{z} = z - \bar{z}_w(t)$ and applying the small gain theorem for ISS systems (Theorem 2.13). □

Remark. We will use Property 2.28 in Chapter 5 to prove the separation principle for input-to-state convergent systems. In that context system (2.28) represents a system in closed loop with a state-feedback controller and an observer generating state estimates for this controller. The y-subsystem corresponds to the state estimation error dynamics of the observer.◁

2.2.4 Sufficient conditions for convergence

In the previous sections we presented the definitions and basic properties of convergent systems. The next question is how to check whether a system is convergent. In this section we provide sufficient conditions for convergence for smooth and nonsmooth systems.

For smooth systems, a simple sufficient condition for the exponential convergence property was proposed in [14] (see also [66]). Here we give a different formulation of the result from [14] adapted for systems with inputs and extended to include the input-to-state convergence property.

Theorem 2.29. *Consider system (2.19) with the function $F(z,w)$ being C^1 with respect to $z \in \mathbb{R}^d$ and continuous with respect to $w \in \mathcal{W} \subset \mathbb{R}^m$. Suppose there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that*

$$P\frac{\partial F}{\partial z}(z,w) + \frac{\partial F}{\partial z}^T(z,w)P \le -Q, \quad \forall z \in \mathbb{R}^d, \quad w \in \mathcal{W}. \tag{2.29}$$

Then system (2.19) is globally exponentially convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\mathcal{W})$. If $\mathcal{W} = \mathbb{R}^m$, then system (2.19) is input-to-state convergent.

We will refer to condition (2.29) as the Demidovich condition, after B.P. Demidovich, who applied this condition for studying convergence properties of dynamical systems [14, 15, 66]. We will say that a system satisfies the Demidovich condition if the right-hand side of this system satisfies condition (2.29) for some matrices $P = P^T > 0$ and $Q = Q^T > 0$.

A complete proof of Theorem 2.29 is provided in Appendix 9.5. It is based on the two technical lemmas formulated below. These lemmas will also be used in subsequent results on the convergence properties.

Lemma 2.30 ([14, 66]). *Suppose $F(z,w)$ is C^1 with respect to z and continuous with respect to w. Let $\mathcal{C} \subset \mathbb{R}^d$ and $\mathcal{W} \subset \mathbb{R}^m$ be such that*

$$P\frac{\partial F}{\partial z}(z,w) + \frac{\partial F}{\partial z}^T(z,w)P \le -Q, \quad \forall z \in \mathcal{C}, \quad w \in \mathcal{W}, \tag{2.30}$$

for some positive definite matrices $P = P^T$ and $Q = Q^T$. Then there exists $\beta > 0$ such that for any $w \in \mathcal{W}$ and for any two points $z_1, z_2 \in \mathbb{R}^d$ such that the open line segment (z_1, z_2) connecting these two points lies in $\mathcal{C}$, it holds that

$$(z_1 - z_2)^T P(F(z_1,w) - F(z_2,w)) \le -\beta(z_1 - z_2)^T P(z_1 - z_2). \tag{2.31}$$

The number $\beta > 0$ depends only on the matrices P and Q. If the set $\mathcal{C}$ is convex, then relation (2.31) holds for any two points $z_1, z_2 \in \mathcal{C}$ and for any $w \in \mathcal{W}$.

Remark. The number β can be chosen equal to $\beta := \lambda_{min}(Q)/\lambda_{max}(P)$, where $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ denote the minimal and the maximal eigenvalues of a symmetric matrix. If $Q := aI$ for some scalar $a > 0$, then $\beta = a/\|P\|$, where $\|\cdot\| = \lambda_{max}(\cdot)$ is the matrix norm induced by the vector norm $|z| = (z^T z)^{1/2}$. This expression for β will be used in Chapter 6.$\lhd$

For the case of $\mathcal{C} = \mathbb{R}^d$, Lemma 2.30 allows one to establish global exponential stability of every solution of system (2.19) corresponding to any input $w(\cdot) \in \overline{\mathbb{PC}}(\mathcal{W})$ (see details in the proof of Theorem 2.29 in Appendix 9.5). The second lemma allows one to establish the existence of a solution $\bar{z}_w(t)$ that is defined and bounded on $\mathbb{R}$.

Lemma 2.31 ([14, 89]). *Consider system (2.19) with a given continuous input $w(t)$ defined for all $t \in \mathbb{R}$. Suppose $D \subset \mathbb{R}^d$ is a compact set that is positively invariant with respect to system (2.19) with the input $w(t)$. Then there exists a solution $\bar{z}_w(t)$ of system (2.19) satisfying $\bar{z}_w(t) \in D$ for all $t \in \mathbb{R}$.*

Theorem 2.29 is proved in the following way. The Demidovich condition (2.29) guarantees that for any input $w(\cdot) \in \overline{\mathbb{PC}}(\mathcal{W})$ any solution of system (2.19) is globally exponentially stable. Moreover, this condition guarantees that for any input $w(\cdot) \in \overline{\mathbb{PC}}(\mathcal{W})$ system (2.19) has a compact positively invariant set D_w (see details in the proof of Theorem 2.29 in Appendix 9.5). By Lemma 2.31 this implies the existence of a solution $\bar{z}_w(t)$ which is defined and bounded for all $t \in \mathbb{R}$. Therefore, system (2.19) is globally exponentially convergent. The UBSS property and the input-to-state convergence property require a rather technical proof (see Appendix 9.5). The reasoning used in the proof of Theorem 2.29 and Lemmas 2.30 and 2.31 will be used in the proofs of subsequent results on the convergence property.

Example 2.32. Let us illustrate the application of Theorem 2.29 with a simple example. Consider the system

$$\begin{aligned} \dot{z}_1 &= -z_1 + w z_2 + w, \\ \dot{z}_2 &= -w z_1 - z_2. \end{aligned} \tag{2.32}$$

The Jacobian of the right-hand side of system (2.32) equals

$$J(z_1, z_2, w) = \begin{pmatrix} -1 & w \\ -w & -1 \end{pmatrix}.$$

Obviously, $J + J^T = -2I < 0$. Thus, the Demidovich condition (2.29) is satisfied for all z_1, z_2, and w (with $P = I$ and $Q = 2I$). By Theorem 2.29, system (2.32) is input-to-state convergent.◁

If the right-hand side of system (2.19) is not smooth with respect to z (therefore, the Jacobian $\partial F/\partial z(z, w)$ may be undefined in certain points of the state space), after some adjustments we can still apply the Demidovich condition (2.29) for checking the exponential convergence property. The next theorem extends Theorem 2.29 to the case where the function $F(z, w)$ may lose continuous differentiability on certain low-dimensional sets.

Theorem 2.33. *Consider system (2.19). Let $F(z, w)$ be continuous with respect to $w \in \mathcal{W} \subset \mathbb{R}^m$ and locally Lipschitz with respect to $z \in \mathbb{R}^d$. Moreover, let $F(z, w)$ be C^1 with respect to z in $(z, w) \in (\mathbb{R}^d \setminus \Gamma) \times \mathcal{W}$, where $\Gamma \subset \mathbb{R}^d$ is a set consisting of a finite number of hyperplanes given by equations of the form $H_j^T z + h_j = 0$, for some $H_j \in \mathbb{R}^d$ and $h_j \in \mathbb{R}$, $j = 1, \ldots, k$. Suppose there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that*

$$P \frac{\partial F}{\partial z}(z, w) + \frac{\partial F}{\partial z}^T(z, w) P \leq -Q, \quad \forall z \in \mathbb{R}^d \setminus \Gamma, \quad w \in \mathcal{W}. \tag{2.33}$$

Then system (2.19) is globally exponentially convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\mathcal{W})$. If $\mathcal{W} = \mathbb{R}^m$, then system (2.19) is input-to-state convergent.

Proof: See Appendix 9.6.

Remark. It can be proved that the statement of Theorem 2.33 holds also for the case when the set Γ consists of a finite number of smooth manifolds given by equations of the form $h_j(z) = 0$, $j = 1, \ldots k$. The proof is based on the same ideas as the proof of Theorem 2.33, but contains much more technical details.◁

As a particular case of Theorem 2.33 we obtain the following result for piecewise affine systems.

Theorem 2.34. *Consider the state space* $\mathbb{R}^d$ *which is divided into non-intersecting cells* Λ_i, $i = 1, \ldots, l$, *by hyperplanes given by equations of the form* $H_j^T z + h_j = 0$, *for some* $H_j \in \mathbb{R}^d$ *and* $h_j \in \mathbb{R}$, $j = 1, \ldots, k$. *Consider the piecewise affine system*

$$\dot{z} = A_i z + b_i + Dw, \quad \text{for} \;\; z \in \Lambda_i, \quad i = 1, \ldots, l. \tag{2.34}$$

Suppose the right-hand side of (2.34) is continuous and there exists a positive definite matrix $P = P^T > 0$ *such that*

$$PA_i + A_i^T P < 0, \quad i = 1, \ldots, l. \tag{2.35}$$

Then system (2.34) is globally exponentially convergent for the class of inputs $\overline{\mathbb{PC}}_m$ *and input-to-state convergent.*

The continuity requirement on the right-hand side of system (2.34) can be checked with the following lemma.

Lemma 2.35. *Consider system (2.34). The right-hand side of system (2.34) is continuous if and only if the following condition is satisfied: for any two cells* Λ_i *and* Λ_j *having a common boundary* $H^T z + h = 0$ *the corresponding matrices* A_i *and* A_j *and the vectors* b_i *and* b_j *satisfy the equalities*

$$\begin{aligned} G_H H^T &= A_i - A_j, \\ G_H h &= b_i - b_j, \end{aligned} \tag{2.36}$$

for some vector $G_H \in \mathbb{R}^d$.

Proof: See Appendix 9.7.

Recall that by Property 2.27 a series connection of input-to-state convergent systems is again an input-to-state convergent system. Therefore we obtain the following corollary of Property 2.27 and Theorems 2.29, 2.33, and 2.34.

Corollary 2.36. *The series connection of systems satisfying the Demidovich condition for* $\mathcal{W} = \mathbb{R}^m$ *is an input-to-state convergent system.*

Taking into account the existence of powerful solvers for linear matrix inequalities (LMIs), condition (2.35) can be efficiently checked with a computer. In certain cases, feasibility of the Demidovich condition (2.29) or (2.33) can also be concluded from feasibility of LMIs of the form (2.35). Namely, suppose there exist matrices $A_1, \ldots, A_k$ such that

$$\frac{\partial F}{\partial z}(z, w) \in \mathrm{co}\{A_1, \ldots, A_k\}, \quad \forall \;\; z \in \mathbb{R}^d \setminus \Gamma, \; w \in \mathcal{W},$$

where

$$\mathrm{co}\{A_1,\ldots,A_k\} := \left\{A \in \mathbb{R}^{d\times d}:\ A = \sum_{i=1}^{k}\lambda_i A_i,\ \sum_{i=1}^{k}\lambda_i = 1,\ \lambda_i \geq 0\right\}$$

is the convex hull of matrices $A_1,\ldots,A_k$. If the LMIs

$$PA_i + A_i^T P < 0, \quad i = 1,\ldots,k \tag{2.37}$$

admit a common positive definite solution $P = P^T > 0$, then condition (2.29) or (2.33) is satisfied with this matrix P.

In some cases, feasibility of the LMI (2.37) can be checked using frequency domain methods following from the Kalman–Yakubovich lemma (see [51, 89]). For example, one can use the circle criterion [51, 78, 89], as follows from the next lemma.

Lemma 2.37. *Consider a matrix $A \in \mathbb{R}^{d\times d}$, matrices $B \in \mathbb{R}^{d\times 1}$, $C \in \mathbb{R}^{1\times d}$, and some number $\gamma > 0$. Denote $A_\gamma^- := A - \gamma BC$ and $A_\gamma^+ := A + \gamma BC$. The following conditions are equivalent:*

(i) *There exists $P = P^T > 0$ such that*

$$PA_\gamma^- + (A_\gamma^-)^T P < 0, \quad PA_\gamma^+ + (A_\gamma^+)^T P < 0. \tag{2.38}$$

(ii) *The matrix A is Hurwitz and $\left|C(i\omega I - A)^{-1}B\right| < \frac{1}{\gamma}$ for all $\omega \in \mathbb{R}$.*

This lemma allows one to check input-to-state convergence for the so-called Lur'e systems, as follows from the example below. Exponential convergence of Lur'e systems with nonlinearities satisfying some incremental sector condition has been studied in [89]. In that work the nonlinearities may even be discontinuous.

Example 2.38. Consider the system

$$\begin{aligned} \dot{z} &= Az + B\varphi(y) + Ew, \\ y &= Cz + Hw, \end{aligned} \tag{2.39}$$

with the Hurwitz matrix A, scalar output y, and scalar nonlinearity $\varphi(y) \in \mathbb{R}$. Suppose the nonlinearity $\varphi(y)$ is C^1 and it satisfies the condition $\left|\frac{\partial\varphi}{\partial y}(y)\right| \leq \gamma$ for all $y \in \mathbb{R}$. Then the Jacobian of the right-hand side of system (2.39), which is equal to $\frac{\partial F}{\partial z} := A + BC\frac{\partial\varphi}{\partial y}(y)$, satisfies $\frac{\partial F}{\partial z} \in \mathrm{co}\{A_\gamma^-, A_\gamma^+\}$ for all $y \in \mathbb{R}$. By Lemma 2.37, if the condition

$$\left|C(i\omega I - A)^{-1}B\right| < \frac{1}{\gamma}, \quad \forall\, \omega \in \mathbb{R}, \tag{2.40}$$

is satisfied, then LMI (2.38) admits a common positive definite solution. Therefore, system (2.39) satisfies the Demidovich condition (2.29) for all $z \in \mathbb{R}^d$ and all $w \in \mathbb{R}^m$. By Theorem 2.29, this system is globally exponentially convergent for the class of inputs $\overline{\mathbb{PC}}_m$ and it is input-to-state convergent.$\triangleleft$

If the nonlinearity $\varphi(y)$ is not C^1, in some cases one can still conclude the input-to-state convergence of the system, as follows from the next example.

Example 2.39. Consider system (2.39) with $H = 0$ and the nonlinearity $\varphi(y)$ given by the formula

$$\varphi(y) := \begin{cases} k_1 y, & |y| \leq \delta, \\ k_2 y + (k_1 - k_2)\delta, & y > \delta, \\ k_2 y - (k_1 - k_2)\delta, & y < \delta, \end{cases} \tag{2.41}$$

for some $\delta > 0$ and $k_2 > k_1 > 0$. The nonlinearity $\varphi(y)$ is shown in Figure 2.4.

The system (2.39), (2.41) represents a linear system in closed loop with a variable gain controller. In [26] such controllers are proposed to enhance the performance of DVD drives. One of the questions addressed in [26] is: under what conditions does system (2.39), (2.41), being excited by a periodic input $w(t)$, have a unique globally asymptotically stable periodic solution? This question can be easily answered using Lemma 2.37 and Theorem 2.34, which guarantee that system (2.39), (2.41) is globally exponentially convergent. Notice that system (2.39), (2.41) is an example of a continuous piecewise affine system (2.34) with two modes separated by the switching surfaces $Cz = \delta$ and $Cz = -\delta$. The matrices A_1 and A_2 corresponding to different modes can be written as $A_1 = \tilde{A} + \gamma BC$ and $A_2 = \tilde{A} - \gamma BC$, where $\tilde{A} = A + \frac{k_1 + k_2}{2} BC$ and $\gamma = (k_2 - k_1)/2$. Therefore, if $\tilde{A}$ is Hurwitz and $\left| C(i\omega I - \tilde{A})^{-1} B \right| < \frac{1}{\gamma}$ for all $\omega \in \mathbb{R}$, then, by Lemma 2.37, there exists a matrix $P = P^T > 0$ such that (2.35) is satisfied. By Theorem 2.34 system (2.39), (2.41) is globally exponentially convergent. This implies that all solutions of system (2.39), (2.41) exponentially converge to a unique steady-state solution $\bar{z}_w(t)$. By Property 2.23, for any periodic input $w(t)$ this steady-state solution $\bar{z}_w(t)$ is periodic with the same period T as the period of $w(t)$.$\lhd$

Theorem 2.29 is based on quadratic Lyapunov functions. It may happen that there are no matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that system

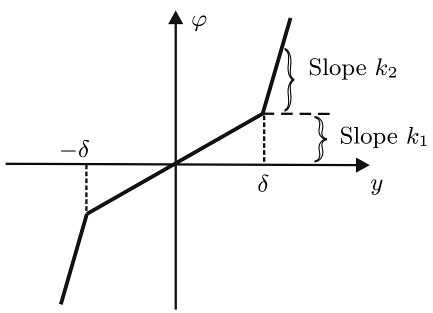

Fig. 2.4. Nonlinearity $\varphi(y)$.

(2.19) satisfies the Demidovich condition (2.29). Yet, this system can be uniformly convergent if it satisfies the conditions of the theorem presented below. This theorem is a generalization of Theorem 2.29.

Theorem 2.40. *Consider system (2.19). Suppose there exist C^1 functions $V_1(z_1, z_2)$ and $V_2(z)$, $\mathcal{K}$-functions $\alpha_2(s)$, $\alpha_3(s)$, $\alpha_5(s)$, $\gamma(s)$, and $\mathcal{K}_\infty$-functions $\alpha_1(s)$, $\alpha_4(s)$ satisfying the conditions*

$$\alpha_1(|z_1 - z_2|) \le V_1(z_1, z_2) \le \alpha_2(|z_1 - z_2|), \tag{2.42}$$

$$\frac{\partial V_1}{\partial z_1}(z_1, z_2)F(z_1, w) + \frac{\partial V_1}{\partial z_2}(z_1, z_2)F(z_2, w) \le -\alpha_3(|z_1 - z_2|), \tag{2.43}$$

$$\alpha_4(|z|) \le V_2(z) \le \alpha_5(|z|), \tag{2.44}$$

$$\frac{\partial V_2}{\partial z}(z)F(z, w) \le 0 \quad \text{for} \quad |z| \ge \gamma(|w|) \tag{2.45}$$

for all $z_1, z_2, z \in \mathbb{R}^d$ and all $w \in \mathbb{R}^m$. Then system (2.19) is globally uniformly convergent and has the UBSS property for the class of inputs $\overline{\mathbb{PC}}_m$.

Proof: See Appendix 9.8.

In order to establish the *local* exponential convergence property of system (2.19), one needs to check the linearization of system (2.19) at the origin, as follows from the next theorem.

Theorem 2.41. *Consider system (2.19) with $F(0, 0) = 0$ and $F(z, w)$ being C^1 with respect to z and continuous with respect to w in some neighborhood of $(z, w) = (0, 0)$. Let $\mathcal{N} \subset \overline{\mathbb{PC}}_m$ be some class of inputs $w(\cdot)$ containing the zero input $w(t) \equiv 0$. The following statements are equivalent:*

(i) *System (2.19) is locally exponentially convergent for the class of inputs $\mathcal{N}$.*
(ii) *System (2.19) is locally exponentially convergent for the class of inputs $\overline{\mathbb{PC}}_m$.*
(iii) *The matrix $\partial F/\partial z(0, 0)$ is Hurwitz.*

Proof: See Appendix 9.9.

2.3 Summary

In this chapter we have presented several notions of convergent systems, which will play a central role in the analysis and design problems for the output regulation problem. The notion of convergence represents a convenient formalization of the property that all solutions of a system with an input “forget” their initial conditions and converge to some steady-state solution, which is determined only by the input. The notion of convergence is more convenient than the other existing formalizations of such property (contraction property [57], incremental stability [2, 23], and incremental ISS [2]), because the convergence property is a rigorously defined topological property of solutions of a system with inputs; it is coordinate independent and it does not require an operator description of the system. The convergence property is an extension of stability properties of asymptotically stable LTI systems to the nonlinear case. Therefore, convergent systems have a number of stability properties that are inherited from asymptotically stable linear systems. Some of these properties have been presented in this chapter. We have provided sufficient conditions for various convergence properties. These conditions apply to systems with smooth right-hand sides and nonsmooth, but continuous right-hand sides. As a particular case, we have presented sufficient conditions for input-to-state convergence for continuous piecewise affine systems. The results on convergent systems presented in this chapter will be subsequently used in the analysis of and controller design for the output regulation problem. At the same time, they can be used for analysis and synthesis purposes in other control problems, where it is important that solutions of a system "forget" their initial conditions and converge to some steady-state solution determined by the input. These problems include tracking, synchronization, observer design, disturbance rejection, and nonlinear performance analysis.

3

The uniform output regulation problem

The output regulation problem is, roughly speaking, either a disturbance rejection problem or a tracking problem or a combination of these two problems. The key feature that distinguishes the output regulation problem from the conventional disturbance rejection and tracking problem is that disturbances and/or reference signals are generated by an external autonomous system, which is called an *exosystem*. Disturbances and reference signals generated by the exosystem are called *exosignals*. The exosystem together with the set of possible initial conditions determines the class of exosignals that affect the system. The output that we want to regulate (e.g., the tracking error in the tracking problem) is called the *regulated output*. The output that is available for measurement is called the *measured output*. The output regulation problem is, in general, to find a measured output feedback controller such that the closed-loop system is internally stable and the regulated output tends to zero along solutions of the closed-loop system regardless of the exosignals affecting the system. The internal stability requirement roughly means that all solutions of the closed-loop system "forget" their initial conditions and converge to some steady-state solution, which is determined only by the exosignal. As we have seen in the previous chapter, such an internal stability requirement can be formalized as the requirement of *(uniform) convergence* of the closed-loop system, which has been defined in Chapter 2.

In this chapter we state several variants of the uniform output regulation problem based on the notion of uniform convergence. First, in Section 3.1 we introduce the equations of the systems under consideration and make basic assumptions on these systems. After that, in Section 3.2 we state the global and local variants of the uniform output regulation problem as well as the problem of robust uniform output regulation. Relations between the uniform output regulation problem and the conventional formulations of the output regulation problem are discussed in Section 3.3. Relations between the observer design and controlled synchronization problems on the one hand and the uniform output regulation problem on the other hand are studied in Section 3.4.

3.1 System equations and basic assumptions

Consider systems modeled by equations of the form

$$\dot{x} = f(x, u, w), \tag{3.1}$$

$$e = h_r(x, w), \tag{3.2}$$

$$y = h_m(x, w), \tag{3.3}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^k$, regulated output $e \in \mathbb{R}^{l_r}$, and measured output $y \in \mathbb{R}^{l_m}$. The exogenous signal $w(t) \in \mathbb{R}^m$, which can be viewed as a disturbance in (3.1) or as a reference signal in (3.2), is generated by an external autonomous system

$$\dot{w} = s(w), \tag{3.4}$$

with some set of initial conditions $\mathcal{W} \subset \mathbb{R}^m$. System (3.4) is called an exosystem. The functions $f(x, u, w)$, $h_r(x, w)$, $h_m(x, w)$ and $s(w)$ are assumed to be continuous and, where necessary, locally Lipschitz in order to guarantee existence and uniqueness of solutions of the corresponding differential equations. Recall that if a set $\mathcal{W} \subset \mathbb{R}^m$ is invariant with respect to system (3.4), then $\mathcal{I}_s(\mathcal{W})$ denotes the class of solutions of the exosystem (3.4) starting in $w(0) \in \mathcal{W}$ (see Section 2.2.2). This notation will be widely used in the problem statement and in the analysis of the uniform output regulation problem.

In the context of the output regulation problem, we will consider several classes of exosystems. Exosystems of the first class satisfy the following boundedness assumption in a set of initial conditions $\mathcal{W}$:

A1 The set $\mathcal{W}$ is invariant with respect to exosystem (3.4) and for any compact set $\mathcal{K}_0 \subset \mathcal{W}$ there exists a compact set $\mathcal{K}_w \subset \mathcal{W}$ such that any solution $w(t)$ starting in $w(0) \in \mathcal{K}_0$ satisfies $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$.

For the case of $\mathcal{W} = \mathbb{R}^m$, Assumption **A1** can be reformulated in the following equivalent way: *for any $a > 0$ there exists $b > 0$ such that $|w(0)| \leq a$ implies $|w(t)| \leq b$ for all $t \in (-\infty, +\infty)$*. Using the definitions of Yoshizawa [91], such a property of solutions of exosystem (3.4) can be called *equi-boundedness* of solutions in forward and backward time.

A simple and practically important example of an exosystem satisfying the boundedness assumption **A1** is a linear system

$$\dot{w} = Sw, \tag{3.5}$$

with the matrix S such that its spectrum consists of simple eigenvalues on the imaginary axis with, possibly, multiple eigenvalues at zero. This system is a linear harmonic oscillator. Without loss of generality, we assume that the matrix S is skew-symmetric. In this case, the exosystem (3.5) satisfies **A1** in any ball $\mathcal{W}_r := \{w : |w| < r\}$, $0 < r \leq +\infty$. Indeed, for every solution of exosystem (3.5) it holds that $|w(t)| \equiv Const$. Thus, for any compact set $\mathcal{K}_0 \subset \mathcal{W}_r$ we can choose $\mathcal{K}_w$ to be a closed ball $\mathcal{K}_w := \{w : |w| \leq \bar{r}\}$, where

$\bar{r} > 0$ is such that $\bar{r} < r$ and $\mathcal{K}_0 \subset \mathcal{K}_w$. With this choice of $\mathcal{K}_w$ it holds that $\mathcal{K}_w$ is a compact subset of $\mathcal{W}_r$ and any solution $w(t)$ starting in $w(0) \in \mathcal{K}_0$ satisfies $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$.

Assumption **A1** is rather restrictive since a wide variety of practically and theoretically important exosystems do not satisfy this assumption. For example, exosystems having a limit cycle or any other bounded attractor with an unbounded domain of attraction do not satisfy this assumption because these exosystems have trajectories leaving any compact set as $t \to -\infty$. Such exosystems are encountered, for example, in (controlled) synchronization problems, see, e.g., [36, 37, 61]. As will be shown in Section 3.4, the controlled synchronization problem is a particular case of the output regulation problem. In practice we are interested not in the dynamics for negative time, but only in the asymptotic dynamics for positive time, i.e., for $t \to +\infty$. Thus, we come to the second class of exosystems to be considered in this book. Exosystems of this class satisfy the following assumption in a *compact* set of initial conditions $\mathcal{W}_+$:

A2 The set $\mathcal{W}_+$ is compact and positively invariant with respect to exosystem (3.4).

This set $\mathcal{W}_+$ may consist, for example, of a bounded attractor and some compact positively invariant subset of its domain of attraction.

For local variants of the output regulation problem, we will consider so-called *neutrally stable* exosystems.

Definition 3.1. *The exosystem (3.4) with an equilibrium $w = 0$ is called neutrally stable if $w = 0$ is stable in forward and backward time and for any solution of the exosystem $w(t, w_0)$ starting in a point $w(0, w_0) = w_0$ close enough to the origin, there exists a sequence $\{t_k\}_{k=0}^{+\infty}$ such that $t_k \to +\infty$ and $w(t_k, w_0) \to w_0$ as $k \to +\infty$.*

From this definition we see that any point w_0 close enough to the origin belongs to the ω-limit set of the trajectory $w(t, w_0)$. Similar to the case of exosystems satisfying the boundedness assumption **A1**, the class of neutrally stable exosystems contains linear harmonic oscillators. Indeed, any linear harmonic oscillator (3.5) has an equilibrium $w = 0$, which is stable in forward and backward time. Moreover, every trajectory $w(t, w_0)$ of exosystem (3.5) returns to any neighborhood of its initial state w_0.

3.2 The uniform output regulation problem

In this section we formulate several settings for the uniform output regulation problem. In every problem setting one has to find, if possible, a controller of the form

$$\dot{\xi} = \eta(\xi, y), \quad \xi \in \mathbb{R}^q, \tag{3.6}$$

$$u = \theta(\xi, y), \tag{3.7}$$

for some $q \geq 0$ such that the closed-loop system

$$\dot{x} = f(x, \theta(\xi, h_m(x, w)), w), \tag{3.8}$$
$$\dot{\xi} = \eta(\xi, h_m(x, w)) \tag{3.9}$$

satisfies three conditions: *regularity, uniform convergence,* and *asymptotic output zeroing.*

The *regularity* condition means that the closed-loop system satisfies conditions for existence and uniqueness of solutions. Throughout the book we will require that the closed-loop system is locally Lipschitz with respect to (x, ξ) and continuous with respect to w.

The *uniform convergence* condition means that the closed-loop system with w as input is uniformly convergent in some set $\mathcal{Z} \subset \mathbb{R}^{n+q}$ for every $w(t)$ being a solution of the exosystem starting in some predefined set of initial conditions $\mathcal{W} \subset \mathbb{R}^m$. The sets $\mathcal{Z}$ and $\mathcal{W}$ will be determined in the specific problem settings introduced in this section. The uniform convergence requirement guarantees that every solution of the closed-loop system starting in $(x(0), \xi(0)) \in \mathcal{Z}$ and corresponding to a solution of the exosystem $w(t)$ starting in $w(0) \in \mathcal{W}$ "forgets" its initial conditions and converges to a unique steady-state solution $\bar{z}_w(t)$, which is determined only by $w(t)$. Moreover, in some problem settings it will be required that the closed-loop system has the UBSS property (see Definition 2.17), which means that for inputs taking their values in some bounded set, the corresponding steady-state solutions are bounded uniformly with respect to these inputs.

The *asymptotic output zeroing* condition means that for every solution of the closed-loop system starting in $(x(0), \xi(0)) \in \mathcal{Z}$ and every solution of the exosystem starting in $w(0) \in \mathcal{W}$, it holds that the regulated output tends to zero:

$$e(t) = h_r(x(t), w(t)) \to 0 \quad \text{as} \quad t \to +\infty.$$

In the following subsections we formulate precise problem statements for the global and local variants of the uniform output regulation problem.

3.2.1 The global uniform output regulation problem

In this section we consider global variants of the uniform output regulation problem for two types of exosystems. The first variant is formulated for exosystems with trajectories starting in some open invariant set $\mathcal{W} \subset \mathbb{R}^m$ for which the exosystem satisfies the boundedness assumption **A1**. This variant is called the *global uniform output regulation problem.* The second variant is formulated for the case of exosystems for which the set of initial conditions $\mathcal{W}_+$ is compact and positively invariant, i.e., for exosystems that satisfy Assumption **A2** in the set $\mathcal{W}_+$. This variant is called the *global forward time uniform output regulation problem.* Both variants of the global uniform output regulation problem are formulated in the following definition. Requirements that are different for the forward time variant of the problem are given in brackets.

The global (forward time) uniform output regulation problem: *given exosystem (3.4) satisfying the boundedness assumption* **A1** *in an open set of initial conditions* $\mathcal{W} \subset \mathbb{R}^m$ [satisfying the assumption **A2** in a compact set of initial conditions $\mathcal{W}_+ \subset \mathbb{R}^m$]*, find, if possible, a controller of the form (3.6), (3.7) such that the closed-loop system (3.8), (3.9) satisfies the following conditions:*

a) *the right-hand side of the closed-loop system is locally Lipschitz with respect to* (x, ξ) *and continuous with respect to* w*;*
b) *the closed-loop system is globally uniformly convergent with the UBSS property for the class of inputs* $\mathcal{I}_s(\mathcal{W})$ [for the class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$, where $\widetilde{\mathcal{W}}$ is some neighborhood of $\mathcal{W}_+$]*;*
c) *for all solutions of the closed-loop system and the exosystem starting in* $(x(0), \xi(0)) \in \mathbb{R}^{n+q}$ *and* $w(0) \in \mathcal{W}$ $[w(0) \in \mathcal{W}_+]$ *it holds that* $e(t) = h_r(x(t), w(t)) \to 0$ *as* $t \to +\infty$*.*

The main difference between the forward time variant and the regular variant of the global uniform output regulation problem is in the uniform convergence condition b). In the forward time variant of the problem, the set $\mathcal{W}_+$ is compact and only *positively* invariant with respect to the exosystem. Therefore, certain solutions of the exosystem starting in $\mathcal{W}_+$ may be not defined for all $t \in \mathbb{R}$. This fact does not allow us to require the uniform convergence property in b) for the class of inputs consisting of solutions of the exosystem starting in $w(0) \in \mathcal{W}_+$ because in the definition of convergence the inputs must be defined for the whole time axis $\mathbb{R}$. In order to cope with this difficulty, the uniform convergence, in this case, is required for a larger class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$.

Both variants of the global uniform output regulation problem will be similarly treated at the stage of controller design in Chapter 5. We will encounter the differences between these two variants only at the analysis stage in Chapter 4, when the question of solvability of the problems is answered.

3.2.2 The local uniform output regulation problem

For the local variant of the uniform output regulation problem, it is assumed that $f(0, 0, 0) = 0$, $h_r(0, 0) = 0$, $h_m(0, 0) = 0$, $s(0) = 0$, and the functions $f(x, u, w)$ and $h_m(x, w)$ are C^1. Also, it is assumed that the exosystem (3.4) is *neutrally stable* (see Definition 3.1). Notice that neutral stability of the exosystem implies that arbitrarily close to the origin $w = 0$ there exists a neighborhood of the origin $\mathcal{W}$ that is invariant with respect to the exosystem dynamics. We will formulate two variants of the local output regulation problem: one, which is more general, using the uniform convergence property and another one using the exponential convergence property.

The local uniform (exponential) output regulation problem: *find, if possible, a controller of the form (3.6), (3.7) with* $\eta(0, 0) = 0$, $\theta(0, 0) = 0$ *such that the closed-loop system (3.8), (3.9) satisfies the following conditions:*

a) *the right-hand side of the closed-loop system is* C^1 *with respect to* (x, ξ) *and continuous with respect to* w*;*
b) *the closed-loop system is locally uniformly (exponentially) convergent for the class of inputs* $\mathcal{I}_s(\mathcal{W})$*, where* $\mathcal{W} \subset \mathbb{R}^m$ *is some invariant neighborhood of the origin;*
c) *for all solutions of the closed-loop system and the exosystem starting close enough to the origin* $(x, \xi, w) = (0, 0, 0)$ *it holds that*

$$e(t) = h_r(x(t), w(t)) \to 0 \quad \text{as} \quad t \to +\infty.$$

Remark. In the case of the local exponential output regulation problem, requirement b) is equivalent to the requirement that the Jacobian matrix of the right-hand side of the closed-loop system (3.8), (3.9) evaluated at $(x, \xi, w) = (0, 0, 0)$ is Hurwitz. This statement follows from Theorem 2.41 and from the fact that $w(t) \equiv 0$ is a solution of the exosystem and this solution belongs to the class $\mathcal{I}_s(\mathcal{W})$ for any neighborhood of the origin $\mathcal{W}$.$\triangleleft$

3.2.3 Types of controllers

Depending on the information available for feedback, one can distinguish different types of controllers. If $y = (x, w)$, i.e., the states of the system and the exosystem are available for feedback, controller (3.6), (3.7) is called a *state feedback*. If only the measured output y is available for feedback, then controller (3.6), (3.7) is called an *output feedback*. Notice that the controller (3.6), (3.7) may consist only of the static block (3.7). In this case, $\xi = \emptyset$ and the controller is called *static*. Otherwise, it is called *dynamic*.

3.2.4 Robust output regulation

In practice certain parameters of the system and the exosystem may not be known exactly. In this case, these parameters may be included in the system model as an unknown constant vector $p \in \mathbb{R}^r$, with a nominal value p°:

$$\dot{x} = f(x, u, w, p), \tag{3.10}$$
$$e = h_r(x, w, p), \tag{3.11}$$
$$y = h_m(x, w, p), \tag{3.12}$$
$$\dot{w} = s(w). \tag{3.13}$$

Suppose controller (3.6), (3.7) solves the (global, local) uniform output regulation problem for the nominal values of parameters p°. We say that controller (3.6), (3.7) is *structurally stable at* p° if it solves the (global, local) uniform output regulation problem for all parameters p from some neighborhood of the nominal vector p°. If controller (3.6), (3.7) solves the (global, local) uniform output regulation problem for all p from some predefined set $\mathcal{P} \subset \mathbb{R}^r$, it is called *robust* with respect to $p \in \mathcal{P}$. In other words, such a controller solves the *robust (global, local) uniform output regulation problem with respect to* $p \in \mathcal{P}$.

It may occur that not only certain constant parameters of the system are unknown, but also certain functional characteristics are not known exactly. Such an uncertainty may be represented as an unknown function φ in the right-hand side of the system equations:

$$\dot{x} = f(x, u, w, \varphi(x, w)). \tag{3.14}$$

The only information that is known about φ is that it belongs to a certain class $\mathcal{F}$. If controller (3.6), (3.7) solves the (global, local) uniform output regulation problem for all functions φ from the class $\mathcal{F}$, it is called robust with respect to the functional uncertainty $\varphi \in \mathcal{F}$.

3.2.5 Properties of the closed-loop system

Notice that, due to the requirement of uniform convergence, every solution of the closed-loop system (3.8), (3.9) corresponding to a controller solving one of the variants of the uniform output regulation problem is bounded for $t \geq 0$.

If the system and the exosystem satisfy $f(0,0,0) = 0$, $h_m(0,0) = 0$, $s(0) = 0$ and a controller solving the (local, global) uniform output regulation problem satisfies the conditions $\eta(0,0) = 0$ and $\theta(0,0) = 0$, then for $w(t) \equiv 0$ the closed-loop system (3.8), (3.9) has a (locally, globally) asymptotically stable equilibrium at the origin $(x, \xi) = (0, 0)$. This property guarantees that if there are no disturbances or reference signals ($w(t) \equiv 0$) then controller (3.6), (3.7) (locally, globally) stabilizes system (3.1)–(3.3) at the origin.

3.3 Relations to conventional problem settings

The uniform output regulation problem formulated in the previous sections is somewhat different from conventional variants of the output regulation problem, see, e.g., [8, 21, 34, 39]. Usually, instead of the uniform convergence condition some other internal stability condition is required. Yet, in some cases the conventional problem settings are particular cases of the uniform output regulation problem. We will illustrate this statement with two examples.

In the linear output regulation problem (see, e.g., [8, 13]), the system is given by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew, \\ e &= C_r x + H_r w, \\ y &= C_m x + H_m w, \end{aligned} \tag{3.15}$$

where x is the state, u is the control, e and y are the regulated and measured outputs respectively, and w is the external signal generated by the exosystem

$$\dot{w} = Sw,$$

which is a linear harmonic oscillator.

The linear output regulation problem is to find a controller of the form

$$\begin{aligned} \dot{\xi} &= K\xi + Ny, \\ u &= M\xi + Ly \end{aligned} \tag{3.17}$$

such that the closed-loop system

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} &= \begin{pmatrix} A + BLC_m & BM \\ NC_m & K \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} E + BLH_m \\ NH_m \end{pmatrix} w \\ &=: F \begin{pmatrix} x \\ \xi \end{pmatrix} + Rw \end{aligned}$$

satisfies the following two properties:

a) *Internal stability:* for $w(t) \equiv 0$ the closed-loop system is asymptotically stable, i.e., the matrix F is Hurwitz;
b) *Asymptotic output zeroing:* for all solutions of the closed-loop system and the exosystem it holds that $e(t) = C_r x(t) + H_r w(t) \to 0$ as $t \to +\infty$.

Notice that the closed-loop system is linear with a Hurwitz matrix F. This implies (see Section 2.2.2) that the closed-loop system is globally exponentially convergent and has the UBSS property for the class of inputs $\overline{\mathbb{PC}}_m$. Thus, the linear output regulation problem is a particular case of the global uniform output regulation problem.

For nonlinear systems, the output regulation problem has been most thoroughly investigated for the local problem setting. In this problem setting, which we will call the *conventional local output regulation problem*, one considers system (3.1)–(3.3) and exosystem (3.4) with $f(0,0,0) = 0$, $h_m(0,0) = 0$, $h_r(0,0) = 0$, $s(0) = 0$ such that the functions $f(x,u,w)$, $h_m(x,w)$, $h_r(x,w)$, and $s(w)$ have continuous partial derivatives of sufficiently high order. The exosystem is assumed to be neutrally stable.

The conventional local output regulation problem is to find a controller of the form (3.6), (3.7) with sufficiently smooth mappings $\eta(\xi,w)$ and $\theta(\xi,w)$ satisfying $\eta(0,0) = 0$ and $\theta(0,0) = 0$ such that the closed-loop system (3.8), (3.9) satisfies the following conditions:

a) *Local internal stability:* for $w(t) \equiv 0$ the closed-loop system has an asymptotically stable linearization at the origin;
b) *Local asymptotic output zeroing:* for every solution of the closed-loop system and the exosystem starting close enough to the origin $(x,\xi,w) = (0,0,0)$ it holds that $e(t) = h_r(x(t),w(t)) \to 0$ as $t \to +\infty$.

As follows from Theorem 2.41 (see also the remark to the formulation of the local uniform (exponential) output regulation problem in Section 3.2.2), such a local internal stability requirement is equivalent to local exponential convergence of the closed-loop system from the local exponential output regulation problem presented in Section 3.2.2. From this fact we conclude that the conventional local output regulation problem is, in fact, a particular case of the local exponential output regulation problem.

Both the linear output regulation problem and the conventional local output regulation problem for nonlinear systems admit complete solutions in the form of some necessary and sufficient conditions for the solvability of these problems [8]. As will become clear in Chapter 4, the global uniform output regulation problem admits similar necessary and sufficient conditions for its solvability.

The global output regulation problem for nonlinear systems, which remained a tough and quite unaddressed problem for a long time, has received attention in a number of recent publications [11, 17, 40, 58, 79]. In these papers the authors use problem settings for the global output regulation problem with various internal stability requirements. For example, in [79] all solutions of the closed-loop system must be bounded; in [58], in addition to the requirement of boundedness of solutions of the closed-loop system, for $w(t) \equiv 0$ the closed-loop system must have a globally asymptotically stable equilibrium at the origin. In [40] all solutions of the closed-loop system and the exosystem starting in any compact set of initial conditions must lie in a set with a compact closure for all $t \geq 0$. Due to the novelty of the problem, at the moment there are no conventional formulations for the global output regulation problem. At the same time, as we will see in Chapter 4, many of the existing controllers solving the global output regulation problem for various classes of systems in fact solve the global uniform output regulation problem. This observation indicates that the uniform output regulation problem may be a convenient problem setting for nonlinear nonlocal variants of the output regulation problem.

3.4 Observer design and controlled synchronization

Some classical control problems can be placed in the framework of the output regulation problem. In this section we show how the problem of observer design for autonomous systems and the controlled synchronization problem can be formulated as variants of the uniform output regulation problem.

3.4.1 Observer design problem

In the observer design problem for autonomous nonlinear systems, we consider a system of the form

$$\begin{aligned} \dot{w} &= s(w), \\ r &= h_o(w), \end{aligned} \tag{3.18}$$

with state w and measured output r. The problem is to find a system that asymptotically reconstructs the state of system (3.18) from the measured output r, i.e., we need to find a system of the form

$$\begin{aligned} \dot{z} &= F(z, r), \\ \hat{w} &= g(z, r), \end{aligned} \tag{3.19}$$

such that $\hat{w}(t) - w(t) \to 0$ as $t \to +\infty$ and all solutions of system (3.19) are bounded for $t \geq 0$. Such system (3.19) is called an observer for system (3.18).

In many observer design methods, the dimension of the observer state z equals the dimension of w and the output $\hat{w}$ equals z. In this case, especially if system (3.18) exhibits oscillatory behavior, the problem of finding such a system (3.19) is called a synchronization problem, because we find a system whose state $z(t)$ synchronizes with $w(t)$, in the sense that they converge to each other. Links between synchronization, which is extensively studied in the physics community (see, e.g., [9, 49, 65]), and observer design problems, which are studied in the field of control, have been revealed in [61]. In general, system (3.19) can have a higher dimension than system (3.18). It is assumed that solutions of system (3.18) start in some compact positively invariant set $\mathcal{W}_+ \subset \mathbb{R}^m$. In many cases this is a natural assumption since in most real life systems the variables describing the state of a system to be observed lie in some bounded set for all $t \geq 0$.

The problem of finding an observer for system (3.18) can be formulated as a global forward time uniform output regulation problem. To show this, consider the system

$$\dot{x} = u, \qquad x, u \in \mathbb{R}^m, \tag{3.20}$$

$$e = x - w, \tag{3.21}$$

$$y = \begin{pmatrix} x \\ h_o(w) \end{pmatrix}. \tag{3.22}$$

Suppose a controller of the form (3.6), (3.7) solves the global forward time uniform output regulation problem for system (3.20)–(3.22) and exosystem (3.18). By the formulation of the global forward time uniform output regulation problem, for every solution of system (3.18) starting in $w(0) \in \mathcal{W}_+$ and for every solution of the closed-loop system

$$\dot{x} = \theta(\xi, x, h_o(w)), \tag{3.23}$$

$$\dot{\xi} = \eta(\xi, x, h_o(w)), \tag{3.24}$$

starting in $(x(0), \xi(0)) \in \mathbb{R}^m \times \mathbb{R}^q$, it holds that $e(t) = x(t) - w(t) \to 0$ as $t \to +\infty$. At the same time, system (3.23), (3.24) is globally uniformly convergent for the class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$, where $\widetilde{\mathcal{W}}$ is some neighborhood of $\mathcal{W}_+$. This implies that every solution of system (3.23), (3.24) is defined and bounded for all $t \geq 0$. Hence, system (3.23), (3.24) is an observer for system (3.18).

3.4.2 Controlled synchronization problem

In the scope of the controlled synchronization problem, we consider two systems: a so-called master system and a slave system. The master system is given by an equation of the form

$$\begin{aligned} \dot{w} &= s(w), \\ r_m &= h_o(w), \end{aligned} \tag{3.25}$$

with state $w \in \mathbb{R}^m$ and output r_m. It is assumed that solutions of system (3.25) start in some compact positively invariant set $\mathcal{W}_+ \subset \mathbb{R}^m$. The slave system is given by the equation

$$\begin{aligned} \dot{x} &= f(x, u), \\ r_s &= g(x), \end{aligned} \tag{3.26}$$

with state $x \in \mathbb{R}^m$, control u and output r_s. The controlled synchronization problem is to find a controller that, based on the measured signals $r_m(t)$ and $r_s(t)$, generates a control action $u(t)$ such that $x(t) - w(t) \to 0$ as $t \to +\infty$ for all solutions of the slave system starting in $x(0) \in \mathbb{R}^m$ and all solutions of the master system starting in $w(0) \in \mathcal{W}_+$. One can easily see that this problem can be formulated as the global forward time uniform output regulation problem for exosystem (3.25) and system (3.26) with the regulated output $e = x - w$ and the measured output $y = (g(x), h_o(w))$. The observer design problem discussed in the previous section is a particular case of this controlled synchronization problem with the right-hand side $f(x, u) \equiv u$ and the output $r_s = x$.

3.5 Summary

In this chapter we have presented the global and local variants of the uniform output regulation problem. In all variants of the problem there are three basic requirements: *regularity*, *uniform convergence*, and *asymptotic output zeroing*. The regularity requirement guarantees existence and uniqueness of solutions of the closed-loop system. The uniform convergence requirement guarantees that solutions of the closed-loop system "forget" their initial conditions and converge to some steady-state solution determined only by the input. The asymptotic output zeroing condition means that along solutions of the closed-loop system and the exosystem, the regulated output tends to zero. The problem statements are presented for three classes of exosystems. The first class consists of exosystems satisfying the boundedness assumption **A1**. An important representative of this class is a linear harmonic oscillator. The second class consists of exosystems with initial conditions in a compact positively invariant set. This class includes exosystems with limit cycles and (chaotic) attractors. The third class of exosystems, which is considered in the local uniform output regulation problem, consists of neutrally stable exosystems. An important representative of this class is a linear harmonic oscillator. The presented global variants of the uniform output regulation problem extend the output regulation problem for *linear* systems and the conventional *local* output regulation problem for nonlinear systems to the case of *global* output regulation for *nonlinear* systems. It is shown that the problem of observer design for autonomous systems and the controlled synchronization problem can be considered as particular cases of the global uniform output regulation problem. The key ingredient of the uniform output regulation problem, which distinguishes it from other variants of the output regulation problem known in literature, is the requirement of uniform convergence. As will be shown in Chapter 4, this new problem setting with the uniform convergence requirement allows one to obtain necessary and sufficient conditions for solvability of the global uniform output regulation problem.

4

Solvability of the uniform output regulation problem

In this chapter we establish general conditions for solvability of the global and local uniform output regulation problem. First, we review some known ideas and results related to the conventional local output regulation problem. These results are based on the center manifold theorem. In order to extend these results to the uniform output regulation problem for both the local and global case, we present invariant manifold theorems, which serve as non-local counterparts of the center manifold theorem. In the formulation of these invariant manifold theorems, the notion of convergent systems, developed in Chapter 2, plays a central role. Based on these invariant manifold theorems, general necessary and sufficient conditions for the solvability of the global and local uniform output regulation problems are derived. These conditions also indicate what kind of properties a controller must have to solve the uniform output regulation problem. This information will be exploited at the stage of controller design in Chapter 5.

4.1 Analysis of the conventional local output regulation problem

The conventional local output regulation problem, which can also be called the local exponential output regulation problem, has been solved in [39] (see also [8, 38]). In that paper necessary and sufficient conditions for the solvability of this problem are obtained. We will review one of these results in order to motivate its extensions to the global uniform output regulation problem.

To understand the ideas and techniques used in the analysis of the conventional local output regulation problem, we investigate the dynamics of the closed-loop system (3.8), (3.9) corresponding to a controller (3.6), (3.7) solving the conventional local output regulation problem.

By $z := (x^T, \xi^T)^T$ denote the state of the closed-loop system (3.8), (3.9) and by $F(z, w)$, its right-hand side. With these new notations, the regulated output e equals $e = \bar{h}_r(z,w) := h_r(x,w)$. Therefore, the combination of the closed-loop system and the exosystem can be written as

$$\dot{z} = F(z,w), \tag{4.1}$$
$$\dot{w} = s(w), \tag{4.2}$$
$$e = \bar{h}_r(z,w).$$

As follows from the formulation of the conventional local output regulation problem, for any controller solving this problem the corresponding closed-loop system is such that $F(0,0) = 0$ and the function $F(z,w)$ has continuous partial derivatives of some high order. Moreover, the fact that for $w(t) \equiv 0$ the closed-loop system has an asymptotically stable linearization at the origin is equivalent to the Jacobian matrix $\partial F/\partial z(0,0)$ being Hurwitz. At the same time, the fact that the zero solution $w(t) \equiv 0$ of the exosystem is Lyapunov stable in forward and backward time (this is a consequence of the neutral stability assumption on the exosystem) implies that $\partial s/\partial w(0)$ has all its eigenvalues on the imaginary axis. These conditions allow us to apply the center manifold theorem (see, e.g., [10]), a particular case of which is formulated below.

Theorem 4.1. *Consider systems (4.1) and (4.2). Suppose $F(z,w)$ and $s(w)$ are C^2 vector-functions with $F(0,0) = 0$, $s(0) = 0$, and all eigenvalues of $\partial F/\partial z(0,0)$ have negative real parts, while all eigenvalues of $\partial s/\partial w(0)$ have zero real parts. Then there exist $\delta > 0$ and a C^1 function $\alpha(w)$ defined for all $|w| < \delta$ such that $\alpha(0) = 0$ and the graph $z = \alpha(w)$ is a locally invariant and locally exponentially attractive manifold for systems (4.1) and (4.2). The mapping $\alpha(w)$ satisfies the partial differential equation*

$$\frac{\partial \alpha}{\partial w}(w)s(w) = F(\alpha(w),w). \tag{4.3}$$

If a set $\mathcal{W} \subset \{w : \ |w| < \delta\}$ is (positively) invariant with respect to system (4.2), then the graph

$$\mathcal{M}(\mathcal{W}) := \{(z,w) : \ z = \alpha(w), \ w \in \mathcal{W}\}$$

is (positively) invariant with respect to systems (4.1) and (4.2), and for all solutions $(z(t), w(t))$ starting close enough to the origin $(0,0)$ it holds that

$$|z(t) - \alpha(w(t))| \leq C\mathrm{e}^{-\beta t}|z(0) - \alpha(w(0))| \tag{4.4}$$

for some $C > 0$ and $\beta > 0$.

The manifold $\mathcal{M}(\mathcal{W})$ is called *the center manifold*. As follows from (4.3), if $w(t)$ is a solution of system (4.2) satisfying $|w(t)| < \delta$ for all $t \in \mathbb{R}$, then $\bar{z}_w(t) := \alpha(w(t))$ is a solution of system (4.1) defined for all $t \in \mathbb{R}$. In general, the center manifold theorem is formulated for bidirectionally coupled systems, i.e., when the right-hand side of system (4.2) also depends on z. For the output regulation problem it is sufficient to formulate the center manifold theorem only for unidirectionally coupled systems (4.1) and (4.2).

Applying the center manifold theorem (Theorem 4.1) to systems (4.1) and (4.2), we conclude that there exists $\delta > 0$ and a C^1 mapping $\alpha(w)$ defined for all $|w| < \delta$ such that $\alpha(0) = 0$ and the graph $z = \alpha(w)$ is locally invariant and locally exponentially attractive with respect to systems (4.1) and (4.2). The mapping $\alpha(w)$ satisfies the partial differential equation

$$\frac{\partial \alpha}{\partial w}(w)s(w) = F(\alpha(w), w) \tag{4.5}$$

for all $w \in \mathcal{W}$. Moreover, since the zero solution $w(t) \equiv 0$ of the exosystem is Lyapunov stable in forward and backward time, there exists a neighborhood of the origin $\mathcal{W} \subset \{w : \ |w| < \delta\}$ that is invariant with respect to (4.2). Hence, the graph $\mathcal{M}(\mathcal{W}) := \{(z, w) : \ z = \alpha(w), \ w \in \mathcal{W}\}$ is invariant with respect to systems (4.1), (4.2) and for all solutions $z(t), w(t)$ starting close enough to the origin $(0, 0)$ it holds that

$$z(t) - \alpha(w(t)) \to 0 \quad \text{as} \ \ t \to +\infty. \tag{4.6}$$

This fact shows that in some neighborhood of the origin the dynamics of the closed-loop system (4.1) coupled with the exosystem (4.2) reduce, after transients, to the dynamics on the center manifold $\mathcal{M}(\mathcal{W})$. Hence, the properties of this center manifold determine whether the regulated output $e(t)$ tends to zero along solutions of the closed-loop system or not. In particular, it can be shown (see, e.g., [8, 39]) that, under the neutral stability assumption on the exosystem, the fact that $e(t) = \bar{h}_r(z(t), w(t)) \to 0$ as $t \to +\infty$ for all solutions of the closed-loop system (4.1) and the exosystem (4.2) starting close enough to the origin is equivalent to

$$\bar{h}_r(\alpha(w), w) = 0 \tag{4.7}$$

for all w in some neighborhood of the origin $\widehat{\mathcal{W}} \subset \mathbb{R}^m$.

As follows from the analysis presented above, the question of whether a controller solves the conventional local output regulation problem reduces to the questions of whether for $w(t) \equiv 0$ the corresponding closed-loop system has an asymptotically stable linearization at the origin and whether there exists a locally defined C^1 mapping $\alpha(w)$, with $\alpha(0) = 0$, satisfying (4.5) and (4.7). If we denote $(\pi(w), \sigma(w)) := \alpha(w)$, where $\pi(w)$ and $\sigma(w)$ are the components of the mapping $\alpha(w)$ corresponding to the x- and ξ-coordinates of the closed-loop system, respectively, this statement can be summarized in the following theorem.

Theorem 4.2 ([8]). *Under the neutral stability assumption on the exosystem (3.4), a controller of the form (3.6), (3.7) solves the conventional local output regulation problem if and only if the following two conditions hold:*

(i) *For* $w(t) \equiv 0$ *the corresponding closed-loop system (3.8), (3.9) has an asymptotically stable linearization at the origin.*
(ii) *There exist* C^1 *mappings* $\pi(w)$ *and* $\sigma(w)$ *defined in some neighborhood of the origin* $\widehat{\mathcal{W}}$ *and satisfying* $\pi(0) = 0$, $\sigma(0) = 0$ *and*

$$\begin{aligned} \frac{\partial \pi}{\partial w}(w)s(w) &= f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w)), w), \\ \frac{\partial \sigma}{\partial w}(w)s(w) &= \eta(\sigma(w), h_m(\pi(w), w)), \\ 0 &= h_r(\pi(w), w) \quad \forall \ \ w \in \widehat{\mathcal{W}}. \end{aligned}$$

This theorem provides a characterization of all controllers solving the conventional local output regulation problem. It also forms a foundation for further results related to solvability and controller design for the conventional local output regulation problem, which can be found, for example, in [8].

Since in this book we also study global variants of the output regulation problem, we need to extend the result of Theorem 4.2 to the global case. An essential obstacle for such an extension is that the analysis in Theorem 4.2 is based on the center manifold theorem (Theorem 4.1), which is a local result. Existing extensions of this theorem to nonlocal cases (see, e.g., [22, 49, 53, 87]) are based on certain *quantitative* conditions on the dynamics of the coupled systems (the closed-loop system and the exosystem in the case of the output regulation problem). We would like to avoid such quantitative conditions and find nonlocal counterparts of the center manifold theorem based on certain *qualitative* conditions on the coupled systems. As a preliminary observation, notice that in the center manifold theorem the Jacobian $\partial F/\partial z(0,0)$ must be a Hurwitz matrix. As we know from Theorem 2.41, this condition is equivalent to the requirement that system (4.1) be locally exponentially convergent for the class of inputs $\overline{\mathbb{PC}}_m$. This observation shows that the requirement of some convergence property on system (4.1) may serve as a nonlocal counterpart of the condition on $\partial F/\partial z(0,0)$. In fact, as we will see in the next section, existence of a continuous invariant manifold of the form $z = \alpha(w)$ for systems (4.1) and (4.2) is, under certain assumptions, equivalent to some form of the uniform convergence property of system (4.1). The invariant manifold theorems presented in the next section will naturally lead us to necessary and sufficient conditions for the solvability of the global and local variants of the uniform output regulation problem. This fact, in turn, explains why we have based the uniform output regulation problem studied in this book on the notion of uniform convergence.

4.2 Invariant manifold theorems

In this section we present certain invariant manifold theorems that serve as counterparts of the center manifold theorem for studying the solvability of the global and local variants of the uniform output regulation problem. To this end, we consider coupled systems of the form

$$\dot{z} = F(z, w), \tag{4.8}$$
$$\dot{w} = s(w), \tag{4.9}$$

where $z \in \mathbb{R}^d$, $w \in \mathbb{R}^m$. The function $F(z, w)$ is locally Lipschitz in z and continuous in w; $s(w)$ is locally Lipschitz. In the analysis of the uniform output regulation problem, system (4.8) corresponds to a closed-loop system and system (4.9) corresponds to an exosystem.

First, we consider the case of system (4.9) with some open invariant set of initial conditions $\mathcal{W} \subset \mathbb{R}^m$. Recall that $\mathcal{I}_s(\mathcal{W})$ denotes the class of all solutions of system (4.9) starting in $\mathcal{W}$. The next technical lemma provides conditions for the existence of a continuous asymptotically stable invariant manifold of the form $z = \alpha(w)$. This lemma will serve as a foundation for further results on invariant manifolds presented in this section.

Lemma 4.3. *Consider system (4.8) and system (4.9) with an open invariant set of initial conditions $\mathcal{W} \subset \mathbb{R}^m$. Suppose*

(i) *System (4.8) is uniformly convergent in a set $\mathcal{Z} \subset \mathbb{R}^d$ for the class of inputs $\mathcal{I}_s(\mathcal{W})$, and for any compact set $\mathcal{K}_0 \subset \mathcal{W}$ there exists a compact set $\mathcal{K}_z \subset \mathcal{Z}$ such that for any $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ satisfying $w(0) \in \mathcal{K}_0$ the corresponding steady-state solution satisfies $\bar{z}_w(t) \in \mathcal{K}_z$ for all $t \in \mathbb{R}$.*

Then

(ii) *There exists a continuous mapping $\alpha : \mathcal{W} \to \mathcal{Z}$ such that the graph*

$$\mathcal{M}(\mathcal{W}) := \{(z, w) : \ z = \alpha(w), w \in \mathcal{W}\}$$

is invariant with respect to systems (4.8) and (4.9). Moreover, for every $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ the corresponding solution of system (4.8) $\bar{z}_w(t) = \alpha(w(t))$ is uniformly asymptotically stable in $\mathcal{Z}$.

In general, the mapping $\alpha(w)$ is not unique. But for any two such mappings $\alpha_1(w)$ and $\alpha_2(w)$ and for any $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$, it holds that

$$\alpha_1(w(t)) - \alpha_2(w(t)) \to 0 \quad \text{as} \ \ t \to +\infty \tag{4.10}$$

and $\alpha_1(w(t)) \equiv \alpha_2(w(t))$ for any $w(t)$ lying in some compact subset of $\mathcal{W}$ for all $t \in \mathbb{R}$.
If system (4.9) satisfies the boundedness assumption **A1** *in the set $\mathcal{W}$, then statements* (i) *and* (ii) *are equivalent and the mapping $\alpha(w)$ defined in* (ii) *is unique.*

Proof: See Appendix 9.10.

This lemma is a preliminary technical result that allows us to obtain further global and local results related to the existence of continuous invariant manifolds of the form $z = \alpha(w)$. The conditions in Lemma 4.3 seem rather complicated because this lemma covers the general case. In particular cases of this lemma, which are formulated below, the conditions will simplify significantly. In particular, under the boundedness assumption **A1** on system (4.9) for $\mathcal{Z} = \mathbb{R}^d$ we obtain the following global result.

Theorem 4.4. *Consider system (4.8) and system (4.9) satisfying the boundedness assumption* **A1** *in some open invariant set* $\mathcal{W} \subset \mathbb{R}^m$. *The following statements are equivalent:*

(ig) *System (4.8) is globally uniformly convergent with the UBSS property for the class of inputs* $\mathcal{I}_s(\mathcal{W})$.

(iig) *There exists a unique continuous mapping* $\alpha : \mathcal{W} \to \mathbb{R}^d$ *such that the graph*
$$\mathcal{M}(\mathcal{W}) := \{(z, w) : \ z = \alpha(w), w \in \mathcal{W}\}$$
is invariant with respect to systems (4.8) and (4.9). Moreover, for every $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ *the corresponding solution of system (4.8)* $\bar{z}_w(t) = \alpha(w(t))$ *is uniformly globally asymptotically stable.*

Proof: We only need to show that the conditions given in (ig) are equivalent to the conditions (i) in Lemma 4.3 for $\mathcal{Z} := \mathbb{R}^d$.
(ig)$\Rightarrow$(i). Consider a compact set $\mathcal{K}_0 \subset \mathcal{W}$. By the boundedness assumption **A1**, there exists a compact set $\mathcal{K}_w \subset \mathcal{W}$ such that if a solution $w(t)$ of system (4.9) starts in $w(0) \in \mathcal{K}_0$ then $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$. At the same time, by the UBSS property, there exists a compact set $\mathcal{K}_z \subset \mathbb{R}^d$ such that the fact that $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$ implies $\bar{z}_w(t) \in \mathcal{K}_z$ for all $t \in \mathbb{R}$. This implies (i).
(i)$\Rightarrow$(ig). Consider a compact set $\mathcal{K}_w \subset \mathcal{W}$ and a solution of system (4.9) satisfying $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$. In particular, this solution satisfies $w(0) \in \mathcal{K}_0 := \mathcal{K}_w$. By the conditions given in (i), there exists a compact set $\mathcal{K}_z \subset \mathbb{R}^d$ such that for any solution $w(t)$ starting in $w(0) \in \mathcal{K}_0$ (hence, for any $w(t)$ satisfying $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$) the corresponding steady-state solution $\bar{z}_w(t)$ lies in $\mathcal{K}_z$. Thus, we have shown that system (4.8) has the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$, i.e. we have shown (ig). □

Under the boundedness assumption **A1** the class of inputs $\mathcal{I}_s(\mathcal{W})$ is contained in $\overline{\mathbb{PC}}(\mathcal{W})$, so we therefore obtain the following corollary to Theorem 4.4.

Corollary 4.5. *Consider system (4.8) and system (4.9) satisfying the boundedness assumption* **A1** *in some open invariant set* $\mathcal{W}$. *Suppose system (4.8) is globally uniformly convergent with the UBSS property for the class of inputs* $\overline{\mathbb{PC}}(\mathcal{W})$. *Then statement* (iig) *of Theorem 4.4 holds.*

In the global forward time uniform output regulation problem we deal with exosystems that do not need to satisfy the boundedness assumption **A1**, but they satisfy the assumption **A2**, i.e., their solutions start in some compact positively invariant set of initial conditions $\mathcal{W}_+ \subset \mathbb{R}^m$. For such systems we formulate the following result.

Theorem 4.6. *Consider systems (4.8) and (4.9). Let $\mathcal{W}_+$ be a compact positively invariant set of system (4.9) and $\mathcal{W}_\pm \subset \mathcal{W}_+$ be an invariant subset of $\mathcal{W}_+$. Suppose system (4.8) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$, where $\widetilde{\mathcal{W}}$ is some neighborhood of $\mathcal{W}_+$. Then there exists a continuous mapping $\alpha : \widetilde{\mathcal{W}} \to \mathbb{R}^d$ such that the set*

$$\mathcal{M}(\mathcal{W}_+) := \{(z, w) : \ z = \alpha(w), \ w \in \mathcal{W}_+\}$$

is positively invariant with respect to (4.8), (4.9), and for any solution of system (4.9) $w(t)$ starting in $w(0) \in \mathcal{W}_+$ the corresponding solution of system (4.8) $\bar{z}_w(t) = \alpha(w(t))$ is uniformly globally asymptotically stable. In general, the mapping $\alpha(w)$ is not unique. But for any two such mappings $\alpha_1(w)$ and $\alpha_2(w)$ and for any $w(t)$ starting in $w(0) \in \mathcal{W}_+$ it holds that

$$|\alpha_1(w(t)) - \alpha_2(w(t))| \to 0 \quad \text{as} \quad t \to +\infty, \tag{4.11}$$

and $\alpha_1(w) = \alpha_2(w)$ for all $w \in \mathcal{W}_\pm$.

Proof: See Appendix 9.11.

In Theorem 4.6 the mapping $\alpha(w)$ may be nonunique as can be seen from the following example, which is a modified example from [78].

Example 4.7. Consider two scalar systems

$$\dot{z} = -z, \tag{4.12}$$

$$\dot{w} = -\frac{w^3}{2}. \tag{4.13}$$

System (4.12) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_1$, since for every input $w(t)$ the steady-state solution equals $\bar{z}_w(t) \equiv 0$ and it is globally exponentially stable. For every $r > 0$ the set $\mathcal{W}_+(r) := \{w : |w| \le r\}$ is compact and positively invariant with respect to (4.13). The set $\mathcal{W}_\pm$ contains only the origin, $\mathcal{W}_\pm = \{0\}$. It can be easily checked that for any constant c the mapping

$$\alpha_c(w) = \begin{cases} c\mathrm{e}^{-1/w^2}, & w \neq 0, \\ 0, & w = 0, \end{cases}$$

is continuous and the graph $z = \alpha_c(w)$ is invariant with respect to (4.12) and (4.13). The mappings $\alpha_c(w)$ for all parameters c coincide in the origin, which belongs to $\mathcal{W}_{\pm}$. For any initial condition $w(0) \in \mathbb{R}$ the solution $w(t)$ of system (4.13) tends to zero, which implies $\alpha_c(w(t)) \to 0$ as $t \to +\infty$. Thus for any c_1 and c_2 it holds that

$$\alpha_{c_1}(w(t)) - \alpha_{c_2}(w(t)) \to 0, \quad \text{as} \;\; t \to +\infty. \lhd$$

The next theorem provides a local variant of the invariant manifold theorems presented above.

Theorem 4.8. *Consider systems (4.8) and (4.9) with $F(0,0) = 0$, $s(0) = 0$ and with $F(z,w)$ being C^1 with respect to z and continuous with respect to w. Let the equilibrium $w = 0$ of system (4.9) be stable in forward and backward time. Then the following statements are equivalent:*

(il) *System (4.8) is locally uniformly convergent for the class of inputs $\mathcal{I}_s(\mathcal{W}_*)$, where $\mathcal{W}_* \subset \mathbb{R}^m$ is some invariant neighborhood of the origin.*
(iil) *There exist an invariant neighborhood of the origin $\mathcal{W}$ and a unique continuous mapping $\alpha : \mathcal{W} \to \mathbb{R}^d$ such that $\alpha(0) = 0$ and the graph*

$$\mathcal{M}(\mathcal{W}) := \{(z,w) : \;\; z = \alpha(w), \;\; w \in \mathcal{W}\}$$

is invariant with respect to systems (4.8) and (4.9). Moreover, there exists a neighborhood of the origin $\mathcal{Z} \subset \mathbb{R}^d$ such that for every $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ the solution $\bar{z}_w(t) := \alpha(w(t))$ is uniformly asymptotically stable in $\mathcal{Z}$.

Proof: See Appendix 9.12.

In general, it is not a simple task to find an invariant manifold even if its existence is guaranteed by the invariant manifold theorems presented above. Yet, in some simple cases such a manifold can be found analytically. We will show this with a few examples.

Example 4.9. Consider a linear system

$$\dot{w} = Sw, \quad w \in \mathbb{R}^m, \tag{4.14}$$

with the matrix S having all its eigenvalues simple and lying on the imaginary axis. This system satisfies the boundedness assumption **A1** in the whole state space. Consider a system given by the equation

$$\dot{z} = Az + q(w), \tag{4.15}$$

where A is a Hurwitz matrix and $q(w)$ is a polynomial in w of some finite degree n. Notice that this system is globally exponentially convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_m$ (see, for example, Theorem 2.29). By Corollary 4.5, there exists a unique continuous function $\alpha(w)$ such that the graph $\mathcal{M} := \{(z,w) : \;\; z = \alpha(w), w \in \mathbb{R}^m\}$ is invariant with respect to systems (4.15) and (4.14). As follows from [8] (Lemma 1.2), the mapping $\alpha(w)$

is a polynomial in w of the same degree as the degree of $q(w)$. It is a unique solution of the equation

$$\frac{\partial \alpha}{\partial w}(w)Sw = A\alpha(w) + q(w).$$

The right- and left-hand sides of this equation are polynomials in w. Thus, by equating the corresponding components of these polynomials, we find the unique coefficients of the polynomial $\alpha(w)$.$\lhd$

Using ideas from [8], this example can be extended in the following way.

Example 4.10. Consider the nonlinear system

$$\begin{aligned}
\dot{z}_1 &= A_1 z_1 + q_1(z_2, \dots z_k, w), \quad && z_1 \in \mathbb{R}^{d_1}, \\
\dot{z}_2 &= A_2 z_2 + q_2(z_3, \dots z_k, w), \quad && z_2 \in \mathbb{R}^{d_2}, \\
&\dots \\
\dot{z}_k &= A_k z_k + q_k(w), && z_k \in \mathbb{R}^{d_k},
\end{aligned} \tag{4.16}$$

where the matrices A_i, $i = 1, \dots, k$, are Hurwitz and $q_i(\cdot)$, $i = 1, \dots, k$, are polynomials of their arguments. Every ith subsystem of system (4.16) with $z_{i+1}, \dots, z_k$ and w as inputs is input-to-state convergent (see Theorem 2.29). Therefore, system (4.16) is a series connection of input-to-state convergent systems. By Property 2.27, this system is input-to-state convergent. By Property 2.19, input-to-state convergence, in turn, implies that system (4.16) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_m$. By Corollary 4.5, there exists a unique continuous mapping $\alpha(w)$ such that the manifold $\mathcal{M} := \{(z, w) : \ z = \alpha(w), w \in \mathbb{R}^m\}$, where $z := (z_1^T, \dots, z_k^T)^T$, is invariant with respect to systems (4.16) and (4.14). Applying the results obtained for system (4.15) to the last equation in (4.16), we find the component of $\alpha(w)$ corresponding to z_k. This component $\alpha_k(w)$ is a polynomial. Substituting this $\alpha_k(w)$ in the $(k-1)$th equation, we again obtain an equation of the form (4.15), from which we can find $\alpha_{k-1}(w)$. Repeating this process, we find the remaining components of the mapping $\alpha(w)$.$\lhd$

These examples indicate that in some cases it is possible to find the invariant manifold, whose existence is guaranteed by the invariant manifold theorems presented in this section, analytically.

The invariant manifold theorems presented in this section state equivalence between the existence of a (globally) uniformly asymptotically stable invariant manifold of the form $z = \alpha(w)$ with a continuous function $\alpha(w)$ on the one hand, and certain convergence properties of system (4.8) on the other hand (under Assumptions **A1**, **A2**, or under the neutral stability assumption on system (4.9)). The sufficient conditions for various convergence properties presented in Section 2.2.4 allow us to determine whether systems (4.8) and (4.9) have such an invariant manifold.

As will be seen from the next sections, the invariant manifold theorems will naturally lead us to certain necessary and sufficient conditions for the solvability of different variants of the uniform output regulation problem.

4.3 ω-limit sets

Prior to deriving the conditions for solvability of the uniform output regulation problem, we recall the notion of ω-limit sets. This notion appears to be an important ingredient of the solvability analysis. Consider the system

$$\dot{w} = s(w), \quad w \in \mathbb{R}^m, \tag{4.17}$$

with a locally Lipschitz function $s(w)$. Let $w(t, w_0)$ denote the solution of system (4.17) starting in $w(0, w_0) = w_0$.

Definition 4.11 ([3]). *A point $w_* \in \mathbb{R}^m$ is called an ω-limit point of the trajectory $w(t, w_0)$ if for any $T > 0$ and any $\varepsilon > 0$ there exists $t_* > T$ such that $|w(t_*, w_0) - w_*| < \varepsilon$. The set of all ω-limit points of the trajectory $w(t, w_0)$ is called the ω-limit set and denoted by $\Omega(w_0)$. For trajectories starting in some set $\mathcal{W} \subset \mathbb{R}^m$, the notation $\Omega(\mathcal{W})$ denotes $\Omega(\mathcal{W}) := \bigcup_{w_0 \in \mathcal{W}} \Omega(w_0)$.*

The following statements reflect some standard facts on ω-limit sets, see, e.g., [3]. For a trajectory $w(t, w_0)$ that is bounded for $t \geq 0$ the ω-limit set $\Omega(w_0)$ is a bounded invariant set. If $\mathcal{W} \subset \mathbb{R}^m$ is a bounded positively invariant set, then $\Omega(\mathcal{W})$ is a bounded invariant set that attracts all trajectories $w(t, w_0)$ starting in $w_0 \in \mathcal{W}$, i.e., for any $w_0 \in \mathcal{W}$ it holds that $\text{dist}(w(t, w_0), \Omega(\mathcal{W})) \to 0$ as $t \to +\infty$. Here, the distance $\text{dist}(w, \mathcal{W})$ between a point $w \in \mathbb{R}^m$ and a set $\mathcal{W} \subset \mathbb{R}^m$ is defined as $\text{dist}(w, \mathcal{W}) := \inf_{w_* \in \mathcal{W}} |w - w_*|$. If $\mathcal{W}$ is a compact positively invariant set, then $\Omega(\mathcal{W}) \subset \mathcal{W}$.

With these facts at hand, we can proceed with the solvability analysis of the uniform output regulation problem.

4.4 Solvability of the global (forward time) uniform output regulation problem

In this section we apply the invariant manifold theorems to study solvability of the global uniform output regulation problem. Since there are two variants of the global uniform output regulation problem, we will obtain solvability results for both. Moreover, we will present solvability results for the robust global uniform output regulation problem.

4.4.1 Solvability of the global uniform output regulation problem

The next theorem, which is based on Theorem 4.4, establishes necessary and sufficient conditions for a controller (3.6), (3.7) to solve the global uniform output regulation problem.

Theorem 4.12. *Consider system (3.1)–(3.3) and exosystem (3.4) satisfying the boundedness assumption* **A1** *in an open invariant set of initial conditions $\mathcal{W}$. The following statements are equivalent:*

(i) *Controller (3.6), (3.7) solves the global uniform output regulation problem.*
(ii) *The closed-loop system is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$ and there exist continuous mappings $\pi : \mathcal{W} \to \mathbb{R}^n$ and $\sigma : \mathcal{W} \to \mathbb{R}^q$ satisfying the equations*

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w)), w),$$
$$\frac{d}{dt}\sigma(w(t)) = \eta(\sigma(w), h_m(\pi(w), w)), \qquad (4.18)$$
$$\forall \;\; w(t) = w(t, w_0) \in \mathcal{W},$$
$$0 = h_r(\pi(w), w) \quad \forall \;\; w \in \Omega(\mathcal{W}). \qquad (4.19)$$

(iii) *There exist continuous mappings $\pi : \mathcal{W} \to \mathbb{R}^n$ and $\sigma : \mathcal{W} \to \mathbb{R}^q$ satisfying equations (4.18) and (4.19) and for every $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ the solution of the closed-loop system $(\bar{x}_w(t), \bar{\xi}_w(t)) = (\pi(w(t)), \sigma(w(t)))$ is globally uniformly asymptotically stable.*

Proof: We will prove the equivalence of (i), (ii) and (iii) in the following sequence: (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii). Suppose controller (3.6), (3.7) solves the global uniform output regulation problem. Then the closed-loop system (3.8), (3.9) is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$. By Theorem 4.4, this implies the existence of a continuous mapping $\alpha(w)$ such that the graph of this mapping

$$\mathcal{M}(\mathcal{W}) := \{(x, \xi, w)) : (x, \xi) = \alpha(w), w \in \mathcal{W}\}$$

is invariant with respect to the closed-loop system (3.8), (3.9) and the exosystem (3.4). Denote by $\pi(w)$ and $\sigma(w)$ the x- and ξ-components of the mapping $\alpha(w)$. Since the graph $\mathcal{M}(\mathcal{W})$ is invariant, for any solution of the exosystem $w(t)$ starting in $w(0) \in \mathcal{W}$, the pair of functions $(\pi(w(t)), \sigma(w(t)))$ represents a solution of the closed-loop system (3.8), (3.9). This implies that the functions $\pi(w(t))$ and $\sigma(w(t)))$ satisfy (4.18). Since the regulated output $e(t)$ tends to zero along any solution of the closed-loop system and the exosystem starting in $(x(0), \xi(0)) \in \mathbb{R}^{n+q}$ and $w(0) \in \mathcal{W}$, respectively, $e(t)$ also tends to zero along the solution $(\pi(w(t)), \sigma(w(t)), w(t))$, i.e.,

$$h_r(\pi(w(t)), w(t)) \to 0 \quad \text{as} \;\; t \to +\infty. \qquad (4.20)$$

Let us show that this fact implies (4.19). Suppose there exists $w_* \in \Omega(\mathcal{W})$ such that $h_r(\pi(w_*), w_*) \neq 0$. By the definition of the ω-limit set $\Omega(\mathcal{W})$, there exists a solution $w(t)$ starting in $w(0) \in \mathcal{W}$ and a sequence $\{t_k\}_{k=1}^{+\infty}$ such that $t_k \to +\infty$ and $w(t_k) \to w_*$ as $k \to +\infty$. Since $h_r(\pi(w), w)$ is continuous in $\mathcal{W}$, we obtain

$$h_r(\pi(w(t_k)), w(t_k)) \to h_r(\pi(w_*), w_*) \neq 0, \quad \text{as} \;\; k \to +\infty.$$

This contradicts (4.20). Thus, indeed, the equality (4.19) holds. This completes the proof of this implication.

(ii)⇒(iii). Since the closed-loop system (3.8), (3.9) is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$, by Theorem 4.4 for every solution of the exosystem $w(t)$ starting in $\mathcal{W}$, the solution of the closed-loop system $(\bar{x}_w(t), \bar{\xi}_w(t)) := (\pi(w(t), \sigma(w(t)))$ lying on this manifold is uniformly globally asymptotically stable.

(iii)⇒(i). By Theorem 4.4, the existence of the continuous mappings $\pi(w)$ and $\sigma(w)$ given in (iii) implies that the closed-loop system (3.8), (3.9) is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$. We only need to show that for any solution of the closed-loop system and the exosystem starting in $(x(0), \xi(0)) \in \mathbb{R}^{n+q}$ and $w(0) \in \mathcal{W}$, the regulated output $e(t)$ tends to zero. Consider a solution of the exosystem $w(t)$ starting in $w(0) \in \mathcal{W}$ and the solution of the closed-loop system $(\bar{x}_w(t), \bar{\xi}_w(t)) := (\pi(w(t)), \sigma(w(t))$. Since the solution $(\bar{x}_w(t), \bar{\xi}_w(t))$ is globally uniformly asymptotically stable, for any other solution of the closed-loop system $(x(t), \xi(t))$ it holds that $x(t) - \pi(w(t)) \to 0$ and $\xi(t) - \sigma(w(t)) \to 0$ as $t \to +\infty$. Thus,

$$e(t) = h_r(x(t), w(t)) \to h_r(\pi(w(t)), w(t)) \quad \text{as} \quad t \to +\infty. \tag{4.21}$$

At the same time, $\operatorname{dist}(w(t), \Omega(\mathcal{W})) \to 0$ as $t \to +\infty$ (see Section 4.3). Since $w(t)$ is bounded, this implies

$$h_r(\pi(w(t)), w(t)) \to h_r(\pi(\Omega(\mathcal{W})), \Omega(\mathcal{W})) = \{0\} \quad \text{as} \quad t \to +\infty.$$

Together with (4.21), this implies $e(t) = h_r(x(t), w(t)) \to 0$ as $t \to +\infty$. This completes the proof of the theorem. □

Remark. In the literature, global variants of the output regulation problem are considered mostly for the case of exosystems being linear harmonic oscillators. Such exosystems satisfy the boundedness assumption **A1**. Many of the proposed controllers solving such variants of the global output regulation problem (see, e.g., [12, 58, 69, 79]) are designed in such a way that they guarantee existence and global uniform asymptotic stability of a sufficiently smooth invariant manifold $(x, \xi) = (\pi(w), \sigma(w))$, with $\pi(w)$ and $\sigma(w)$ satisfying (4.18), (4.19). As follows from Theorem 4.12, such controllers solve the global uniform output regulation problem.◁

Theorem 4.12 provides a criterion for checking whether a particular controller solves the global uniform output regulation problem. It can be used directly for controller design (we will address this problem in Chapter 5) in the following way: given *some* controller such that the corresponding closed-loop system satisfies the conditions (ii) or (iii) in Theorem 4.12, this theorem guarantees that this controller solves the global uniform output regulation problem. Alternatively, we can specifically design a controller such that the corresponding closed-loop system satisfies conditions (ii) or (iii). At the same time, Theorem 4.12 allows one to obtain certain controller-independent necessary conditions for the solvability of the global uniform output regulation problem as follows from the next lemma.

Lemma 4.13. *The global uniform output regulation problem is solvable* only if *there exist continuous mappings $\pi(w)$ and $c(w)$ defined in some neighborhood of $\Omega(\mathcal{W})$ satisfying the equations*

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \tag{4.22}$$

$$0 = h_r(\pi(w(t)), w(t)), \tag{4.23}$$

for all solutions of the exosystem $w(t)$ satisfying $w(t) \in \Omega(\mathcal{W})$, $t \in \mathbb{R}$.

Proof: The statement of the lemma is obtained from (4.18) and (4.19) by denoting $c(w) := \theta(\sigma(w), h_m(\pi(w), w))$. □

Equations (4.22) and (4.23) are the so-called *regulator equations*, see, e.g., [8, 38, 39]. Solvability of the regulator equations guarantees that for every solution of the exosystem lying in the ω-limit set $\Omega(\mathcal{W})$ there exists a control input $\bar{u}_w(t) := c(w(t))$ for which system (3.1) has the solution $\bar{x}_w(t) := \pi(w(t))$, and along this solution the regulated output equals zero. Notice that the ω-limit set $\Omega(\mathcal{W})$ can be treated, in a certain sense, as the steady-state dynamics of the exosystem, because this set is invariant and attracts all solutions of the exosystem starting in $\mathcal{W}$. From this point of view, solvability of the regulator equations can be interpreted in the following way: for any solution $w(t)$ of the exosystem from the steady-state dynamics set $\Omega(\mathcal{W})$, there exists at least one control input $\bar{u}_w(t)$ such that system (3.1) with these $w(t)$ and $\bar{u}_w(t)$ has a solution $\bar{x}_w(t)$ along which the regulated output $e(t)$ is identically zero.

Originally, solvability of the regulator equations in some neighborhood of the origin was obtained as a necessary condition for the solvability of the conventional local output regulation problem under the assumption that exosystem (3.4) is neutrally stable. Lemma 4.13 shows that solvability of the regulator equations (4.22) and (4.23) is also necessary for the solvability of the *global* uniform output regulation problem.

With the regulator equations at hand, we can obtain further necessary conditions for the solvability of the global uniform output regulation problem. As follows from (4.18), controller (3.6), (3.7) is such that if we excite it with the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$, for some solution of the exosystem $w(t) \in \Omega(\mathcal{W})$, it has a solution $\bar{\xi}_w(t) = \sigma(w(t))$, which is bounded on $\mathbb{R}$, and along this solution the output of the controller equals $\bar{u}_w(t) = c(w(t))$, where $\pi(w)$ and $c(w)$ are solutions of the regulator equations defined above. This property motivates the introduction of the following definition.

Definition 4.14. *Consider controller (3.6), (3.7). Let $\bar{y}(t)$ and $\bar{u}(t)$ be defined and bounded for all $t \in \mathbb{R}$. We say that the input $\bar{y}(t)$ induces the output $\bar{u}(t)$*

in controller (3.6), (3.7), if for this $\bar{y}(t)$ system (3.6), (3.7) has a solution $\bar{\xi}(t)$ defined and bounded on $\mathbb{R}$ and satisfying the equality $\bar{u}(t) = \theta(\bar{\xi}(t), \bar{y}(t))$ for all $t \in \mathbb{R}$.

We will say that controller (3.6), (3.7) has a *generalized internal model property* if for any solution of the exosystem $w(t)$ lying in the ω-limit set $\Omega(\mathcal{W})$ the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in controller (3.6), (3.7). The generalized internal model property closely relates to the notions of immersion and internal models used in the output regulation theory (see [8, 40, 42] for further details on immersion and internal models).

With these definitions at hand, we obtain the following necessary condition for the solvability of the global uniform output regulation problem.

Lemma 4.15. *Suppose the global uniform output regulation problem is solvable. Then there exists a controller of the form (3.6), (3.7) such that for any solution of the exosystem $w(t)$ lying in the ω-limit set $\Omega(\mathcal{W})$ the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in the controller (3.6), (3.7), where $c(w)$ and $\pi(w)$ are solutions to the regulator equations (4.22) and (4.23). In other words, there exists a controller with the generalized internal model property. Moreover, the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$.*

The requirement that the controller makes the corresponding closed-loop system globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$ is natural, since it comes from the problem statement. The generalized internal model property guarantees that controller (3.6), (3.7) is capable of generating the steady-state control $\bar{u}_w(t) = c(w(t))$ (see Lemma 4.13) based on the measured signal $y(t)$.

Lemmas 4.13 and 4.15 provide necessary conditions for the solvability of the global uniform output regulation problem. In fact, as follows from the next theorem, these conditions are not only necessary, but also sufficient for the solvability of the problem.

Theorem 4.16. *Consider system (3.1)–(3.3) and exosystem (3.4) satisfying the boundedness assumption* **A1** *in an open invariant set of initial conditions $\mathcal{W}$. The global uniform output regulation problem is solvable if and only if the following conditions are satisfied:*

(i) *There exist continuous mappings $\pi(w)$ and $c(w)$ defined in some neighborhood of $\Omega(\mathcal{W})$ and satisfying the regulator equations (4.22) and (4.23) for all solutions $w(t)$ of exosystem (3.4) satisfying $w(t) \in \Omega(\mathcal{W})$ for all $t \in \mathbb{R}$.*
(ii) *There exists a controller of the form (3.6), (3.7) such that for any solution of the exosystem $w(t)$ lying in the set $\Omega(\mathcal{W})$ the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in controller (3.6),*

(3.7), and the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$.

Under these conditions, a controller solves the global uniform output regulation problem if and only if it satisfies the conditions given in (ii).

Proof: The *only if* part of the theorem follows from Lemmas 4.13 and 4.15. We only need to show that if the condition (i) is satisfied then a controller satisfying the conditions given in (ii) solves the global uniform output regulation problem. We will do this by showing that if a controller satisfies the conditions given in (ii), then the corresponding closed-loop system satisfies condition (ii) in Theorem 4.12. Thus, by Theorem 4.12 this controller solves the global uniform output regulation problem.

Suppose controller (3.6), (3.7) satisfies the conditions given in (ii). Then by Theorem 4.4 there exist continuous functions $\tilde{\pi}(w)$ and $\tilde{\sigma}(w)$ such that the graph $(x, \xi) = (\tilde{\pi}(w), \tilde{\sigma}(w))$ for $w \in \mathcal{W}$ is invariant with respect to the closed-loop system (3.8), (3.9) and the exosystem (3.4). This implies that $\tilde{\pi}(w)$ and $\tilde{\sigma}(w)$ satisfy the following equations:

$$\begin{aligned} \frac{d}{dt}\tilde{\pi}(w(t)) &= f(\tilde{\pi}(w), \theta(\tilde{\sigma}(w), h_m(\tilde{\pi}(w), w)), w), \\ \frac{d}{dt}\tilde{\sigma}(w(t)) &= \eta(\tilde{\sigma}(w), h_m(\tilde{\pi}(w), w)), \end{aligned} \tag{4.24}$$

for all solutions of the exosystem $w(t)$ lying in the set $\mathcal{W}$. Moreover, for every $w(t)$ lying in $\mathcal{W}$, the solution of the closed-loop system $(\tilde{x}_w(t), \tilde{\xi}_w(t)) := (\tilde{\pi}(w(t)), \tilde{\sigma}(w(t)))$ is globally uniformly asymptotically stable. Let us show that the mapping $\tilde{\pi}(w)$ also satisfies the equation

$$h_r(\tilde{\pi}(w), w) = 0 \quad \forall \ \ w \in \Omega(\mathcal{W}). \tag{4.25}$$

Once this equality is proved, by Theorem 4.12 we obtain that controller (3.6), (3.7) solves the global uniform output regulation problem.

In order to prove (4.25), we will show that

$$\pi(w(t)) \equiv \tilde{\pi}(w(t)) \tag{4.26}$$

for any solution of the exosystem lying in $\Omega(\mathcal{W})$. Then equality (4.25) will follow from (4.23) and from the fact that $\Omega(\mathcal{W})$ is an invariant set with respect to system (3.1) (i.e., for any $w_* \in \Omega(\mathcal{W})$ there exists a solution $w(t)$ lying in $\Omega(\mathcal{W})$ for all $t \in \mathbb{R}$ and satisfying $w(0) = w_*$).

Let us first show that for every solution $w(t)$ lying in $\Omega(\mathcal{W})$ the closed-loop system (3.8), (3.9) has a solution $(\bar{x}_w(t), \bar{\xi}_w(t))$ which is defined and bounded for all $t \in \mathbb{R}$. This fact follows from the regulator equations (4.22) and from the property of the controller that for the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ it has a solution $\bar{\xi}_w(t)$ which is defined and bounded for all $t \in \mathbb{R}$ and for which $\theta(\bar{\xi}_w(t), h_m(\pi(w(t)), w(t))) \equiv c(w(t))$ for all $t \in \mathbb{R}$. Substituting this $(\bar{x}_w(t), \bar{\xi}_w(t)) := (\pi(w(t)), \bar{\xi}_w(t))$ in the equations of the closed-loop system (3.8), (3.9), one can easily check that such a pair $(\bar{x}_w(t), \bar{\xi}_w(t))$ is indeed a solution of the closed-loop system. Since $w(t)$ lies in a compact subset of $\Omega(\mathcal{W})$ (due to assumption **A1**) and since $\pi(w)$ is continuous in some neighborhood of $\Omega(\mathcal{W})$, the function $\pi(w(t))$ and hence $(\bar{x}_w(t), \bar{\xi}_w(t))$ are bounded for all $t \in \mathbb{R}$.

Recall that the solution $(\tilde{x}_w(t), \tilde{\xi}_w(t)) := (\tilde{\pi}(w(t)), \tilde{\sigma}(w(t)))$ is defined and bounded for all $t \in \mathbb{R}$ and it is globally uniformly asymptotically stable. By Property 2.4, this implies that $(\tilde{x}_w(t), \tilde{\xi}_w(t)) \equiv (\bar{x}_w(t), \bar{\xi}_w(t))$ for $t \in \mathbb{R}$. This, in turn, implies (4.26), which completes the proof of the theorem. □

Theorem 4.16 provides a way to solve the global uniform output regulation problem. First, one needs to solve the regulator equations (4.22) and (4.23) (or show that they are not solvable, which implies that the problem cannot be solved) and then to find a controller satisfying the conditions given in (ii). Particular ways of finding such controllers will be discussed in Chapter 5.

4.4.2 Solvability of the robust global uniform output regulation problem

In this section we provide solvability conditions for the robust global uniform output regulation problem. In this problem we consider systems of the form (3.10)–(3.12) depending on a vector of unknown, but constant, parameters p taken from an open set $\mathcal{P}$. The problem is to find a controller (independent of p) that solves the global uniform output regulation problem for all $p \in \mathcal{P}$. This problem can be reduced to a regular variant of the global uniform output regulation problem by extending the exosystem in the following way:

$$\begin{pmatrix} \dot{w} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} s(w) \\ 0 \end{pmatrix} =: \hat{s}(w,p). \tag{4.27}$$

After such an extension the parameter p is considered to be a part of the exosignal. Notice that if the original exosystem (3.4) satisfies the boundedness assumption **A1** in a certain open set $\mathcal{W} \subset \mathbb{R}^m$, then the extended exosystem (4.27) satisfies assumption **A1** in the set $\mathcal{W} \times \mathcal{P}$. Therefore, controller (3.6), (3.7) solves the global uniform output regulation problem for all parameters p taken from the set $\mathcal{P}$ if it solves the global uniform output regulation problem for the extended exosystem (4.27), with (w,p) being a new state of the exosystem. The converse statement is not true, in general, because the UBSS property of the closed-loop system for the class of inputs $\mathcal{I}_s(\mathcal{W})$ for every parameter $p \in \mathcal{P}$, which is required in the problem formulation of the robust global uniform output regulation problem, does not imply the UBSS property of the closed-loop system for the class of extended inputs $\mathcal{I}_{\hat{s}}(\mathcal{W} \times \mathcal{P})$, where $\mathcal{I}_{\hat{s}}(\mathcal{W} \times \mathcal{P})$ denotes all solutions of the extended exosystem (4.27) starting in the open invariant set $\mathcal{W} \times \mathcal{P}$. In fact, solvability of the global uniform output regulation problem for the extended exosystem (4.27) is necessary for the solvability of the so-called *strong robust global uniform output regulation problem*, which is formulated in the following way.

> Controller (3.6), (3.7) solves the **strong robust global uniform output regulation problem** if it solves the global uniform output regulation problem for all $p \in \mathcal{P}$, and for any compact subsets $\mathcal{K}_w \subset \mathcal{W}$ and $\mathcal{K}_p \subset \mathcal{P}$ there exists a compact set $\mathcal{K}_z \subset \mathbb{R}^d$ such that for any solution of the exosystem $w(t)$ starting in $w(0) \in \mathcal{K}_w$ and any parameter $p \in \mathcal{K}_p$ the corresponding steady-state solution $\bar{z}_{wp}(t)$ of the closed-loop system lies in the set $\mathcal{K}_z$ for all $t \in \mathbb{R}$.

One can easily check that this strong robust global uniform output regulation problem is equivalent to the global uniform output regulation problem for system (3.10)–(3.12) and exosystem (4.27). Using this fact, we can apply the results obtained in the previous section to study solvability of the strong robust global uniform output regulation problem. Consequently, we can formulate the following results, which are counterparts of Theorems 4.12 and 4.16.

Theorem 4.17. *Consider system (3.10)–(3.12) with the parameter p taken from an open set $\mathcal{P}$ and exosystem (3.4) satisfying the boundedness assumption* **A1** *in an open invariant set of initial conditions $\mathcal{W}$. The following statements are equivalent:*

(i) *Controller (3.6), (3.7) solves the strong robust global uniform output regulation problem.*
(ii) *The closed-loop system is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_{\hat{s}}(\mathcal{W} \times \mathcal{P})$ and there exist continuous mappings $\pi(\cdot,\cdot) : \mathcal{W} \times \mathcal{P} \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) : \mathcal{W} \times \mathcal{P} \to \mathbb{R}^q$ satisfying the equations*

$$\begin{aligned}
\frac{d}{dt}\pi(w(t),p) &= f(\pi(w,p), \theta(\sigma(w,p), h_m(\pi(w,p),w,p)), w, p), \\
\frac{d}{dt}\sigma(w(t),p) &= \eta(\sigma(w,p), h_m(\pi(w,p),w,p)), \qquad (4.28) \\
&\qquad \forall \;\; w(t) = w(t,w_0) \in \mathcal{W}, \quad p \in \mathcal{P}, \\
0 &= h_r(\pi(w,p),w,p) \quad \forall \;\; w \in \Omega(\mathcal{W}), \quad p \in \mathcal{P}. \qquad (4.29)
\end{aligned}$$

(iii) *There exist continuous mappings $\pi(\cdot,\cdot) : \mathcal{W} \times \mathcal{P} \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) : \mathcal{W} \times \mathcal{P} \to \mathbb{R}^q$ satisfying (4.28) and (4.29) and for every $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ and every $p \in \mathcal{P}$ the solution $(\bar{x}_{wp}(t), \bar{\xi}_{wp}(t)) = (\pi(w(t),p), \sigma(w(t),p))$ is globally uniformly asymptotically stable.*

The next theorem provides solvability conditions for the strong robust global uniform output regulation problem. It directly follows from Theorem 4.16.

Theorem 4.18. *Consider system (3.10)–(3.12) with the parameter p taken from an open set $\mathcal{P}$ and exosystem (3.4) satisfying the boundedness assumption* **A1** *in an open invariant set of initial conditions $\mathcal{W}$. The strong robust global uniform output regulation problem is solvable if and only if the following conditions are satisfied:*

(i) *There exist continuous mappings $\pi(w,p)$ and $c(w,p)$ defined in some neighborhood of $\Omega(\mathcal{W}) \times \mathcal{P}$, satisfying the regulator equations*

$$\frac{d}{dt}\pi(w(t),p) = f(\pi(w(t),p), c(w(t),p), w(t), p), \tag{4.30}$$

$$0 = h_r(\pi(w(t),p), w(t), p), \tag{4.31}$$

for all solutions $w(t)$ of the exosystem (3.4) lying in the set $\Omega(\mathcal{W})$ and for all $p \in \mathcal{P}$.

(ii) *There exists a controller of the form (3.6), (3.7) such that for any solution of the exosystem $w(t)$ lying in the set $\Omega(\mathcal{W})$ and for any $p \in \mathcal{P}$ the input $\bar{y}_w(t) := h_m(\pi(w(t),p), w(t), p)$ induces the output $\bar{u}(t) = c(w(t),p)$ in controller (3.6), (3.7) and the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_{\hat{s}}(\mathcal{W} \times \mathcal{P})$.*

Under these conditions, a controller solves the strong robust global uniform output regulation problem if and only if it satisfies the conditions given in (ii).

4.4.3 Solvability of the global forward time uniform output regulation problem

Solvability of the global forward time uniform output regulation problem can be studied in a similar way as solvability of the global uniform output regulation problem. The main difference is that instead of Theorem 4.4, which forms the foundation for the analysis in the previous sections, the results in this section are based on Theorem 4.6. The proofs are identical to the proofs of Theorems 4.12 and 4.16 and are omitted here. The first theorem, which is a counterpart of Theorem 4.12, provides necessary and sufficient conditions under which a controller solves the global forward time uniform output regulation problem.

Theorem 4.19. *Consider system (3.1)–(3.3) and exosystem (3.4) with a compact positively invariant set of initial conditions $\mathcal{W}_+ \subset \mathbb{R}^m$. The following statements are equivalent:*

(i) *Controller (3.6), (3.7) solves the global forward time uniform output regulation problem.*

(ii) *There exist continuous mappings* $\pi : \widetilde{\mathcal{W}} \to \mathbb{R}^n$ *and* $\sigma : \widetilde{\mathcal{W}} \to \mathbb{R}^q$, *where* $\widetilde{\mathcal{W}} \subset \mathbb{R}^m$ *is some neighborhood of* $\mathcal{W}_+$, *satisfying*

$$\begin{aligned}
\frac{d}{dt}\pi(w(t)) &= f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w)), w), \\
\frac{d}{dt}\sigma(w(t)) &= \eta(\sigma(w), h_m(\pi(w), w)), \qquad (4.32) \\
&\forall \;\; w(t) = w(t, w_0) \in \mathcal{W}_+, \quad \text{for } t \geq 0, \\
0 &= h_r(\pi(w), w) \quad \forall \;\; w \in \Omega(\mathcal{W}_+), \qquad (4.33)
\end{aligned}$$

and the closed-loop system (3.8), (3.9) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$.

The next theorem is a counterpart of Theorem 4.16. It provides necessary and sufficient conditions for solvability of the global forward time uniform output regulation problem.

Theorem 4.20. *Consider system (3.1)–(3.3) and exosystem (3.4) with a compact positively invariant set of initial conditions* $\mathcal{W}_+ \subset \mathbb{R}^m$. *The global forward time uniform output regulation problem is solvable if and only if the following conditions are satisfied:*

(i) *There exist continuous mappings* $\pi(w)$ *and* $c(w)$ *defined in some neighborhood of* $\Omega(\mathcal{W}_+)$ *and satisfying the regulator equations*

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \qquad (4.34)$$

$$0 = h_r(\pi(w(t)), w(t)), \qquad (4.35)$$

for all solutions of exosystem (3.4) satisfying $w(t) \in \Omega(\mathcal{W}_+)$ *for* $t \in \mathbb{R}$.

(ii) *There exists a controller of the form (3.6), (3.7) such that for any solution of the exosystem* $w(t)$ *lying in the set* $\Omega(\mathcal{W}_+)$ *the input* $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ *induces the output* $\bar{u}_w(t) = c(w(t))$ *in controller (3.6), (3.7), and the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs* $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$, *where* $\widetilde{\mathcal{W}}$ *is some neighborhood of* $\mathcal{W}_+$.

Under these conditions, a controller solves the global forward time uniform output regulation problem if and only if it satisfies the conditions given in (ii).

Results related to solvability of the robust variant of the global forward time uniform output regulation problem can be obtained in the same way as in Section 4.4.2.

4.5 Solvability of the local uniform output regulation problem

Analysis of the solvability of the local uniform output regulation problem is very close to the analysis in the global case (see Section 4.4.1). Analysis in the local case is based on the local invariant manifold theorem (Theorem 4.8). We omit the proofs of the results presented in this section since they are nearly identical to the proofs of the results from Section 4.4.1.

The following theorem provides necessary and sufficient conditions for a controller of the form (3.6), (3.7) to solve the local uniform output regulation problem.

Theorem 4.21. *Consider system (3.1)–(3.3) and exosystem (3.4) satisfying the neutral stability assumption. The following statements are equivalent:*

(i) *Controller (3.6), (3.7) solves the local uniform output regulation problem.*
(ii) *There exist continuous mappings $\pi(w)$ and $\sigma(w)$ defined in some invariant neighborhood of the origin $\mathcal{W} \subset \mathbb{R}^m$, satisfying $\pi(0) = 0$, $\sigma(0) = 0$, and*

$$\begin{aligned} \frac{d}{dt}\pi(w(t)) &= f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w)), w), \\ \frac{d}{dt}\sigma(w(t)) &= \eta(\sigma(w), h_m(\pi(w), w)), \\ &\forall \;\; w(t) = w(t, w_0) \in \mathcal{W}, \end{aligned} \tag{4.36}$$

$$0 = h_r(\pi(w), w), \quad \forall \; w \in \mathcal{W}, \tag{4.37}$$

for all $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$, and the closed-loop system (3.8), (3.9) corresponding to this controller is locally uniformly convergent for the class of inputs $\mathcal{I}_s(\mathcal{W})$.

The main difference between Theorem 4.21 and Theorem 4.12 (if we do not take into account that in the first case the analysis is local and in the second it is global) is in (4.37) and (4.19). In (4.37), the equality

$$h_r(\pi(w), w) = 0 \tag{4.38}$$

is required for all $w \in \mathcal{W}$, while in (4.19) this equality is required only for the set $\Omega(\mathcal{W})$. This difference is explained by the fact that the exosystem is neutrally stable. By the definition (see Definition 3.1), neutral stability implies that for some neighborhood of the origin $\widehat{\mathcal{W}}$ it holds that $\widehat{\mathcal{W}} \subset \Omega(\widehat{\mathcal{W}})$. Thus, for a sufficiently small neighborhood $\mathcal{W}$ of the origin the equality $h_r(\pi(w), w) = 0$ for all $w \in \Omega(\mathcal{W})$ implies that this equality is satisfied for all $w \in \mathcal{W}$. The opposite is also true. If equality (4.38) is satisfied for all w in some invariant neighborhood of the origin $\mathcal{W}$, one can choose another invariant neighborhood of the origin $\widetilde{\mathcal{W}}$ such that equality (4.38) holds for all $w \in \Omega(\widetilde{\mathcal{W}})$. The proof of this statement is as follows. From the definition of the set $\Omega(\widetilde{\mathcal{W}})$ one can conclude that $\Omega(\widetilde{\mathcal{W}}) \subset \mathrm{clos}(\widetilde{\mathcal{W}})$, where $\mathrm{clos}(\widetilde{\mathcal{W}})$ is the closure of the set $\widetilde{\mathcal{W}}$. Hence, if we find an invariant neighborhood of the origin $\widetilde{\mathcal{W}}$ such that $\mathrm{clos}(\widetilde{\mathcal{W}}) \subset \mathcal{W}$, then equality (4.38) is satisfied for all $w \in \Omega(\widetilde{\mathcal{W}})$. Such a neighborhood $\widetilde{\mathcal{W}}$ exists, because the trivial solution $w(t) \equiv 0$ is stable in forward and backward time (see the proof of Theorem 4.8, where this statement is proved and used several times).

The next theorem provides a local counterpart of Theorem 4.16.

Theorem 4.22. *Consider system (3.1)–(3.3) and exosystem (3.4) satisfying the neutral stability assumption. The local uniform output regulation problem is solvable if and only if the following conditions are satisfied:*

(i) *There exist continuous mappings $\pi(w)$ and $c(w)$ defined in some invariant neighborhood of the origin $\mathcal{W} \subset \mathbb{R}^m$, such that $\pi(0) = 0$, $c(0) = 0$, and*

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \tag{4.39}$$

$$0 = h_r(\pi(w(t)), w(t)), \tag{4.40}$$

for all solutions of exosystem (3.4) satisfying $w(t) \in \mathcal{W}$ for all $t \in \mathbb{R}$.

(ii) *There exists a controller of the form (3.6), (3.7) satisfying the following conditions:* **a)** *there exists a continuous mapping $\sigma : \mathcal{W} \to \mathbb{R}^q$ satisfying $\sigma(0) = 0$ and*

$$\begin{aligned} \frac{d}{dt}\sigma(w(t)) &= \theta(\sigma(w), h_m(\pi(w), w)), \\ c(w(t)) &= \theta(\sigma(w(t)), h_m(\pi(w(t)), w(t))), \end{aligned} \tag{4.41}$$

for all $w(t) \in \mathcal{W}$, and **b)** *the closed-loop system corresponding to this controller is locally uniformly convergent for the class of inputs $\mathcal{I}_s(\mathcal{W})$.*

Under these conditions, a controller satisfying the conditions given in (ii) *solves the local uniform output regulation problem.*

Remark. The requirement that the controller satisfy (4.41) for some continuous $\sigma(w)$ guarantees that for any solution of the exosystem $w(t)$ lying in the set $\mathcal{W}$ for all $t \in \mathbb{R}$ the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in controller (3.6) (3.7).$\triangleleft$

4.6 Applications of the invariant manifold theorems

All solvability results presented in this chapter are based on the invariant manifold theorems (Theorems 4.4, 4.6, and 4.8). Although these theorems were derived for studying the output regulation problem, they are interesting in their own respect. In this section we discuss how these invariant manifold theorems can be applied in the scope of so-called *generalized synchronization* and for the analysis of nonlinear systems excited by harmonic signals.

4.6.1 Generalized synchronization

In the field of master-slave synchronization one considers coupled systems of the form

$$\dot{z} = F(z, w), \tag{4.42}$$
$$\dot{w} = s(w). \tag{4.43}$$

System (4.43) can be treated as a *master* system that generates a driving signal for the *slave* system (4.42). One of the phenomena studied in the context of the master-slave synchronization is the so-called *generalized synchronization* [64, 65, 76]. Roughly speaking, generalized synchronization occurs if for some continuous function $\alpha(w)$ all solutions $z(t)$ of system (4.42) converge to the manifold $z = \alpha(w)$, i.e., $\lim_{t\to+\infty}(z(t) - \alpha(w(t))) = 0$. As follows from Theorem 4.6, if all solutions of system (4.43) start in a compact positively invariant set $\mathcal{W}_+$ and system (4.42) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$, where $\widetilde{\mathcal{W}}$ is some neighborhood of $\mathcal{W}_+$, then there exists a continuous function $\alpha(w)$ defined in $\widetilde{\mathcal{W}}$ such that for all initial conditions $z(0) \in \mathbb{R}^d$ and $w(0) \in \mathcal{W}_+$ it holds that $\lim_{t\to+\infty}(z(t) - \alpha(w(t))) = 0$. Since the ω-limit set $\Omega(\mathcal{W}_+)$ is an invariant set inside $\mathcal{W}_+$, Theorem 4.6 implies that the mapping $\alpha(w)$ is uniquely defined for all $w \in \Omega(\mathcal{W}_+)$. Therefore, we see that the result of Theorem 4.6 can be applied for studying generalized synchronization phenomena.

4.6.2 Nonlinear frequency response functions

A common way to analyze the behavior of a dynamical system is to investigate its responses to harmonic excitations at different frequencies. For linear systems, the information on responses to harmonic excitations, which is contained in frequency response functions and usually represented in the form of Bode plots, allows us to identify the system and analyze its properties such as performance and robustness. Moreover, it serves as a powerful tool for controller design. There exists a vast literature on frequency domain identification, analysis, and controller design methods for linear systems. Most (high-performance) industrial controllers, especially for motion systems, are designed and tuned based on these methods. The lack of such methods for nonlinear systems is one of the reasons why nonlinear systems and controllers are not popular in industry. Even if a (nonlinear) controller achieves a certain control goal (e.g., tracking or disturbance attenuation), which can be proved, for example, using Lyapunov stability methods, it is very difficult to say something about performance of the corresponding nonlinear closed-loop system, while performance is critical in many industrial applications. So, there is a need to extend the linear analysis and controller design methods based on harmonic excitations to nonlinear systems.

One of the first difficulties in such an extension is that a general nonlinear system being excited by a periodic (e.g., harmonic) signal can have none, one, or multiple periodic solutions and, if a periodic solution does exist, its period can differ from the period of the excitation signal. Moreover, if such periodic solutions exist, they essentially depend not only on the excitation frequency, but also on the amplitude of the excitation. As follows from Property 2.23, uniformly convergent systems, although nonlinear, have relatively simple dynamics and for any periodic excitation there exists a unique periodic solution that has the same period as the excitation. Such periodic solutions can be found numerically using, for example, shooting and path following methods [63]. These methods require significant computational efforts, since they are based on the integration of the corresponding differential equations. At the same time, if in addition to the uniform convergence property a system has the UBSS property for the class of bounded piecewise continuous inputs, periodic solutions corresponding to *all* harmonic excitations of the form $u(t) = \mathcal{A}\sin(\omega t)$ for *all* frequencies ω and *all* amplitudes $\mathcal{A}$ can be found from only *one* function. This statement follows from the next theorem.

Theorem 4.23. *Consider the system*

$$\dot{z} = F(z, u), \tag{4.44}$$
$$y = h(z), \tag{4.45}$$

with state $z \in \mathbb{R}^d$, input $u \in \mathbb{R}$ and output $y \in \mathbb{R}$; the function $F(z, u)$ is assumed to be locally Lipschitz with respect to z and continuous with respect to u. Suppose system (4.44) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_1$. Then there exists a unique continuous mapping $\alpha : \mathbb{R}^3 \to \mathbb{R}^d$ such that $\bar{z}_u(t) = \alpha(\mathcal{A}\sin(\omega t), \mathcal{A}\cos(\omega t), \omega)$ is a unique periodic solution of system (4.44) corresponding to the excitation input $u(t) = \mathcal{A}\sin(\omega t)$. Moreover, $\bar{z}_u(t)$ is uniformly globally asymptotically stable.

Proof: The proof of this theorem follows from the fact that harmonic signals of the form $u(t) = \mathcal{A}\sin(\omega t)$ for various amplitudes and frequencies are generated by the following system:

$$\begin{aligned} \dot{w}_1 &= \omega w_2, \\ \dot{w}_2 &= -\omega w_1, \\ \dot{\omega} &= 0, \\ u &= w_1. \end{aligned} \tag{4.46}$$

The initial conditions of this system determine the excitation amplitude $\mathcal{A}$ and frequency ω. Thus, we can treat system (4.44) excited by the input $u(t) = \mathcal{A}\sin(\omega t)$ as a system being coupled with exosystem (4.46). One can easily check that system (4.46) satisfies the boundedness assumption **A1**. Thus, by Corollary 4.5 there exists a unique continuous function $\alpha : \mathbb{R}^3 \to \mathbb{R}^d$ such that the steady-state solution corresponding to the solution of the exosystem $(w_1(t), w_2(t), \omega(t)) = (\mathcal{A}\sin(\omega t), \mathcal{A}\cos(\omega t), \omega)$ equals

$$\bar{z}_u(t) = \alpha(\mathcal{A}\sin(\omega t), \mathcal{A}\cos(\omega t), \omega).$$

Since system (4.44) is globally uniformly convergent for the class of inputs $\overline{\mathbb{PC}}_1$, by Property 2.23 we obtain that $\bar{z}_u(t)$ is a unique periodic solution of system (4.44) and, in addition, it is uniformly globally asymptotically stable. $\square$

As follows from Theorem 4.23, the function $\alpha(w_1, w_2, \omega)$ contains *all* information related to periodic solutions of system (4.44) corresponding to harmonic excitations, and the function $h(\alpha(w_1, w_2, \omega))$ contains all information on the periodic outputs corresponding to harmonic excitations. So, the function $h(\alpha(w_1, w_2, \omega))$ can be considered as a *nonlinear frequency response function.* Notice that this frequency response function depends, in the nonlinear case, not only on the frequency of the excitation, but also on its amplitude and phase. For the analysis of nonlinear systems it can be useful to introduce some kind of a magnitude plot for $h(\alpha(w_1, w_2, \omega))$. This can be done in the following way. Suppose we are interested in responses of system (4.44) to harmonic excitations at all frequencies $\omega \geq 0$ and all amplitudes not exceeding some $\mathcal{A}^* > 0$. Define

$$\Upsilon_{\mathcal{A}^*}(\omega) := \sup_{\mathcal{A}\in(0,\mathcal{A}^*]} \left(\sup_{w_1^2+w_2^2=\mathcal{A}^2} \frac{|h(\alpha(w_1, w_2, \omega))|}{\mathcal{A}} \right).$$

This function is a nonlinear analog of the Bode magnitude plot. The meaning of this function is the following. First, we take some $\mathcal{A} \in (0, \mathcal{A}^*]$ and compute the maximal absolute value of the periodic output corresponding to the excitation $u(t) = \mathcal{A}\sin(\omega t)$. Then we divide it by $\mathcal{A}$. Such normalized maximal value is a gain $k(\omega, \mathcal{A})$ with the following meaning: if the harmonic excitation with frequency ω has amplitude $\mathcal{A}$, then the maximal absolute value of the periodic output corresponding to this excitation equals $k(\omega, \mathcal{A})\mathcal{A}$. Finally, $\Upsilon_{\mathcal{A}_*}(\omega)$ is the maximal value of the gain $k(\omega, \mathcal{A})$ over all amplitudes from the set $\mathcal{A} \in (0, \mathcal{A}^*]$. For linear systems of the form $\dot{z} = Az + Bu$ with a Hurwitz matrix A and output $y = Cz$, the gain $k(\omega, \mathcal{A})$ is independent of the amplitude $\mathcal{A}$ and it equals $k(\omega) = |C(i\omega I - A)^{-1}B|$. Hence, $\Upsilon_{\mathcal{A}^*}(\omega)$ is independent of $\mathcal{A}^*$ and it equals $\Upsilon_{\mathcal{A}_*}(\omega) = |C(i\omega I - A)^{-1}B|$. Therefore, we see that for linear systems the graph of $\Upsilon_{\mathcal{A}_*}(\omega)$ versus the excitation frequency ω coincides with the Bode magnitude plot. The function $\Upsilon_{\mathcal{A}_*}(\omega)$ can be used further to study dynamical properties of uniformly convergent systems. Depending on the inputs and outputs that we choose for the nonlinear system (4.44), we can define nonlinear variants of the sensitivity and complementary sensitivity functions of controlled convergent systems.

As has been mentioned in Section 4.2, the problem of finding the mapping $\alpha(w_1, w_2, \omega)$ is, in general, not an easy task. But in certain cases it is possible to find this mapping analytically. Let us find $\alpha(w_1, w_2, \omega)$ for a particular example.

Example 4.24. Consider the system

$$\dot{z}_1 = -z_1 + z_2^2, \tag{4.47}$$

$$\dot{z}_2 = -z_2 + u, \tag{4.48}$$

$$y = z_1. \tag{4.49}$$

This system is a series connection of input-to-state convergent systems. Therefore, by Property 2.27, system (4.47), (4.48) is input-to-state convergent. This, by Property 2.19, implies that system (4.47), (4.48) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_1$. Consequently, by Theorem 4.23 the mapping $\alpha(w_1, w_2, \omega)$ exists and it is unique. Using the method described in Example 4.10, we will first find $\alpha_2(w_1, w_2, \omega)$ (the second component of α) from (4.48). In our case, $\alpha_2(w_1, w_2, \omega)$ is a polynomial function of degree 1 in the variables w_1 and w_2. So, we will seek α_2 in the form:

$$\alpha_2(w_1, w_2, \omega) = a_1(\omega)w_1 + a_2(\omega)w_2.$$

Substituting this expression in (4.48), we find

$$a_1(\omega) = \frac{1}{1+\omega^2}, \quad a_2(\omega) = \frac{-\omega}{1+\omega^2}.$$

Then, substituting the obtained α_2 for z_2 in (4.47), we compute $\alpha_1(w_1, w_2, \omega)$. In our case, it is a polynomial of w_1 and w_2 of the same degree as the polynomial $(\alpha_2(w_1, w_2, \omega))^2$. Thus, we will seek $\alpha_1(w_1, w_2, \omega)$ in the form

$$\alpha_1(w_1, w_2, \omega) = b_1(\omega)w_1^2 + 2b_2(\omega)w_1w_2 + b_3(\omega)w_2^2. \tag{4.50}$$

After the corresponding computations, we obtain

$$b_1(\omega) = \frac{2\omega^4 + 1}{(1+4\omega^2)(1+\omega^2)^2}, \quad b_2(\omega) = \frac{\omega^3 - 2\omega}{(1+4\omega^2)(1+\omega^2)^2},$$

$$b_3(\omega) = \frac{2\omega^4 + 5\omega^2}{(1+4\omega^2)(1+\omega^2)^2}.$$

After the function $\alpha(w_1, w_2, \omega)$ is found, one can numerically, though very efficiently, compute the magnitude characteristics $\Upsilon_{\mathcal{A}^*}(\omega)$ for some maximal excitation amplitude $\mathcal{A}^*$ and all frequencies over the band of interest. In Figure 4.1, $\Upsilon_{\mathcal{A}^*}(\omega)$ is computed for $\mathcal{A}^* = 1$. Since $\alpha_1(w_1, w_2, \omega)$ is a uniform polynomial function of degree 2 with respect to the variables w_1 and w_2 (see formula (4.50)), one can easily check that for arbitrary $\mathcal{A}^* > 0$ it holds that $\Upsilon_{\mathcal{A}^*}(\omega) = \mathcal{A}^*\Upsilon_1(\omega)$. Here we see the dependency of the amplification gain on the amplitude of the excitation. This is an essentially nonlinear phenomenon.$\lhd$

It is common knowledge that nonlinear systems may have very complex dynamics and that, in general, it is not possible to apply linear analysis and

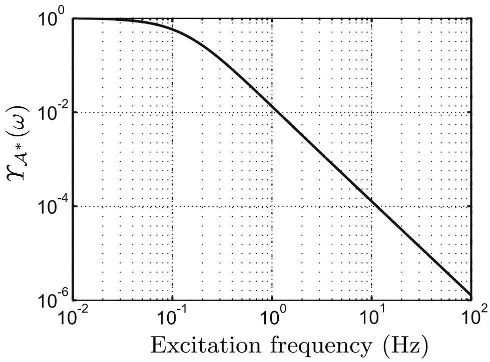

Fig. 4.1. The function $\Upsilon_{\mathcal{A}^*}(\omega)$ computed for $\mathcal{A}^* = 1$.

design methods to investigate nonlocal dynamical properties of nonlinear systems. At the same time, uniformly convergent systems, even when nonlinear, exhibit relatively simple dynamics. Moreover, for uniformly convergent systems with the UBSS property we can define a frequency response function and an analog of a well-known linear analysis tool such as the Bode plot, which can be used, for example, for studying attenuation properties at different excitation frequencies. It is still an open question whether such a nonlinear Bode plot contains enough information to fully identify the system or to design controllers based on this plot. Another open question is how to compute the function $\alpha(w_1, w_2, \omega)$. A standard solution would be to find it numerically. Yet, such numerical methods still need to be developed. As we have shown with an example, in certain cases $\alpha(w_1, w_2, \omega)$ can be found analytically. The results and open questions discussed in this section open an interesting direction in nonlinear systems and control analysis.

4.7 Summary

In this chapter we have presented several results related to solvability of the global, global robust, global forward time, and local uniform output regulation problems. Theorems 4.12, 4.17, 4.19, and 4.21 provide characterizations of all controllers solving the above-mentioned variants of the uniform output regulation problem. Theorems 4.16, 4.18, 4.20, and 4.22 provide necessary and sufficient conditions for the solvability of these problems. These solvability conditions consist of two ingredients: solvability of the regulator equations and existence of a controller which has the generalized internal model property and makes the closed-loop system uniformly convergent. Solvability of the regulator equations guarantees that for every solution of the exosystem lying in a certain ω-limit set it is possible to find at least one control input for which the controlled system has a solution along which the regulated output equals zero. The generalized internal model property of the controller guarantees that this controller is capable of generating this control input based on the information available from the measurements. The uniform convergence property guarantees that the above-mentioned solution, along which the regulated output equals zero, is (globally, locally) asymptotically stable.

All solvability results presented in this chapter are based on the invariant manifold theorems (Theorems 4.4, 4.6, and 4.8), which, in the context of the output regulation problem, serve as counterparts of the center manifold theorem. Although the invariant manifold theorems are derived in order to study solvability of the uniform output regulation problem, they are interesting in their own respect. As follows from the discussion in Section 4.6, these invariant manifold theorems can be used for checking the generalized synchronization property for coupled systems and for the computation of periodic solutions of uniformly convergent systems excited by harmonic inputs. Moreover, these theorems allow us to define nonlinear frequency response functions and a variant of the Bode plot for uniformly convergent nonlinear systems. This opens a new direction in the analysis of nonlinear systems.

5

Controller design for the global uniform output regulation problem

In the previous chapter we presented necessary and sufficient conditions for solvability of different variants of the uniform output regulation problem. Even if the problem is solvable, these conditions do not answer the question of how to find a particular controller to solve this problem. For the *local* exponential output regulation problem, such design methods are well known and can be found, for example, in [8]. In this chapter we discuss and present methods on controller design for the *global* uniform output regulation problem.

5.1 Controller design strategy

In this chapter we consider the system

$$\dot{x} = f(x, u, w), \tag{5.1}$$

$$e = h_r(x, w), \tag{5.2}$$

$$y = h_m(x, w), \tag{5.3}$$

with state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^k$, regulated output $e \in \mathbb{R}^{l_r}$, and measured output $y \in \mathbb{R}^{l_m}$. The external input $w \in \mathbb{R}^m$ is generated by the exosystem

$$\dot{w} = s(w). \tag{5.4}$$

We will consider both the regular and the forward time variants of the global uniform output regulation problem. In the first case, all solutions of the exosystem start in an open invariant set of initial conditions $\mathcal{W} \subset \mathbb{R}^m$ such that exosystem (5.4) satisfies the boundedness assumption **A1** in the set $\mathcal{W}$. A controller solving this variant of the problem makes the closed-loop system globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$ and guarantees that for all solutions of the closed-loop system starting in $\mathbb{R}^{n+q}$ (here q is the dimension of the controller) and all solutions of the exosystem starting in $\mathcal{W}$, the regulated output $e(t)$ tends to zero. In the forward time variant of the global uniform output regulation problem, all solutions of the exosystem start in a positively invariant compact set $\mathcal{W}_+ \subset \mathbb{R}^m$. A controller solves this problem if it makes the corresponding closed-loop system globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$, where $\widetilde{\mathcal{W}}$ is some neighborhood of $\mathcal{W}_+$, and it guarantees that for all solutions of the closed-loop system starting in $\mathbb{R}^{n+q}$ and all solutions of the exosystem starting in $\mathcal{W}_+$, the regulated output $e(t)$ tends to zero. Controller designs presented in this chapter are suitable for both the regular and the forward time variants of the global uniform output regulation problem.

As we have seen in Chapter 4 (see Theorem 4.16), if the global uniform output regulation problem is solvable, then there exist continuous mappings $\pi(w)$ and $c(w)$, defined in some neighborhood of the ω-limit set $\Omega(\mathcal{W})$, such that they satisfy the regulator equations

$$\begin{aligned} \frac{d}{dt}\pi(w(t)) &= f(\pi(w(t)), c(w(t)), w(t)), \\ 0 &= h_r(\pi(w(t)), w(t)), \end{aligned} \tag{5.5}$$

for any solution of the exosystem $w(t)$ lying in the ω-limit set $\Omega(\mathcal{W})$. In case of the forward time variant of the problem, a necessary condition for the solvability of the problem (see Theorem 4.20) is the existence of continuous mappings $\pi(w)$ and $c(w)$ defined in some neighborhood of the ω-limit set $\Omega(\mathcal{W}_+)$ such that they satisfy the regulator equations (5.5) for all solutions of the exosystem $w(t)$ lying in the ω-limit set $\Omega(\mathcal{W}_+)$. In the following we will denote the ω-limit sets $\Omega(\mathcal{W})$ and $\Omega(\mathcal{W}_+)$ by Ω omitting $\mathcal{W}$ or $\mathcal{W}_+$.

Since solvability of the regulator equations (5.5) is a necessary condition for the solvability of the global (forward time) uniform output regulation problem, we assume that this condition is satisfied and that the continuous mappings $\pi(w)$ and $c(w)$ satisfying these equations for all solutions of the exosystem $w(t)$ lying in the ω-limit set Ω are known. Moreover, we assume that these mappings $\pi(w)$ and $c(w)$ are continuously extended to the whole state space $\mathbb{R}^m$. This means that $\pi(w)$ and $c(w)$ are globally defined continuous mappings that satisfy the regulator equations (5.5) for all solutions of the exosystem $w(t)$ lying in the ω-limit set Ω.

In general, it can be rather difficult to check solvability of the regulator equations and find its solutions. Yet, for some particular systems this can be done easily, as illustrated by the following example.

Example 5.1. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3 - x_2 + \sin(x_2), \\ \dot{x}_3 &= u, \\ e &= x_1 - w_1, \end{aligned} \tag{5.6}$$

and the exosystem

$$\begin{aligned} \dot{w}_1 &= w_2, \\ \dot{w}_2 &= -w_1. \end{aligned} \tag{5.7}$$

Notice that exosystem (5.7) satisfies the boundedness assumption **A1** in the whole state space $\mathbb{R}^2$. Moreover, since all solutions of (5.7) are periodic, $\Omega(\mathbb{R}^2) = \mathbb{R}^2$. For system (5.6), the regulator equations have the form

$$\begin{aligned} \frac{d}{dt}\pi_1(w(t)) &= \pi_2(w), \\ \frac{d}{dt}\pi_2(w(t)) &= \pi_3(w) - \pi_2(w) + \sin(\pi_2(w)), \\ \frac{d}{dt}\pi_3(w(t)) &= c(w), \\ 0 &= \pi_1(w) - w_1. \end{aligned} \tag{5.8}$$

Here we adopt the notation $\pi(w) = [\pi_1(w), \pi_2(w), \pi_3(w)]^T$. The last equation in (5.8) gives us $\pi_1(w) = w_1$. Substituting this $\pi_1(w)$ into the first equation in (5.8), we obtain $\pi_2(w) = w_2$. Repeating this procedure for the second and the third equations, we obtain $\pi_3(w) = -w_1 + w_2 - \sin(w_2)$ and $c(w) = -w_1 - w_2 + w_1 \cos(w_2)$.$\lhd$

In subsequent sections, we will design controllers of the form

$$\begin{aligned} \dot{\xi} &= \eta(\xi, y), \\ u &= \theta(\xi, y), \end{aligned} \tag{5.9}$$

having the following two properties:

a) system (5.1)–(5.3) in closed loop with controller (5.9) is input-to-state convergent;
b) for any solution $w(t)$ of exosystem (5.4) lying in the ω-limit set Ω, the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in the controller (5.9) (see Definition 4.14).

Notice that a controller having these two properties solves the global uniform output regulation problem, and, in the forward time case, it solves the global forward time uniform output regulation problem. Namely, the input-to-state convergence property implies that the corresponding closed-loop system is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_m$. Since in the regular variant of the global uniform output regulation problem the exosystem satisfies the boundedness assumption **A1** in the set $\mathcal{W}$, the class of inputs $\mathcal{I}_s(\mathcal{W})$ is contained in $\overline{\mathbb{PC}}_m$. Therefore, the closed-loop system is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$. This fact, together with the generalized internal model property b) implies, by Theorem 4.16, that such a controller solves the global uniform output regulation problem. In the forward time case, Theorem 4.20 guarantees that such a controller solves the global forward time uniform output regulation problem.

One can try to find a controller with properties a) and b) stated above directly or try to tackle this problem by decomposing the desired controller and finding its parts separately. The last approach is discussed in the next section.

5.2 Controller decomposition

In practice the generalized internal model property and the input-to-state convergence property can be achieved with two different parts of the controller. Thus the problem of finding a controller solving the global uniform (forward time) output regulation problem can be tackled in two steps. First, we find a controller of the form

$$\begin{aligned} \dot{\xi}_1 &= \eta_1(\xi_1, y), \\ u_1 &= \theta_1(\xi_1, y), \end{aligned} \tag{5.10}$$

that has the generalized internal model property, i.e., that for any solution of exosystem (5.4) lying in the ω-limit set Ω, the input $\bar{y}_w(t) = h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in system (5.10). Once such a controller is found, we extend system (5.1) with this controller in the following way:

$$\begin{aligned} \dot{x} &= f(x, \theta_1(\xi_1, y) + u_2, w), \\ \dot{\xi}_1 &= \eta_1(\xi_1, y) + v, \\ y_e &= (\xi_1, y), \end{aligned} \tag{5.11}$$

where (u_2, v) is a new input and y_e is an extended output. After such an extension the problem reduces to finding a controller

$$\begin{aligned} \dot{\xi}_2 &= \eta_2(\xi_2, y_e), \\ u_2 &= \theta_2(\xi_2, y_e), \\ v &= \psi(\xi_2, y_e), \end{aligned} \tag{5.12}$$

such that the *extended* system (5.11) in closed loop with controller (5.12) is input-to-state convergent and that in the steady-state operation both u_2 and v are zero. Finding such a controller is the second step in the controller design problem. If we find it, then the overall controller takes the form

$$\begin{aligned} \dot{\xi}_1 &= \eta_1(\xi_1, y) + \psi(\xi_2, \xi_1, y), \\ \dot{\xi}_2 &= \eta_2(\xi_2, \xi_1, y), \\ u &= \theta_1(\xi_1, y) + \theta_2(\xi_2, \xi_1, y). \end{aligned} \tag{5.13}$$

Controller (5.13) solves the global uniform (forward time) output regulation problem. Namely, it makes the corresponding closed-loop system input-to-state convergent (this implies that the closed-loop system is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_m$), and for any solution of exosystem (5.4) lying in the ω-limit set Ω, the input $\bar{y}_w(t) = h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in the controller. Hence, by Theorem 4.16 (Theorem 4.20 for the forward time case) this controller solves the global uniform (forward time) output regulation problem.

After such a controller decomposition, the question is how to find controllers (5.10) and (5.12) with the properties described above. If the external signal $w(t)$ is measured, controller (5.10) can be set to $u_1(w) = c(w)$. If this is not the case, one can use the next obvious choice:

$$\begin{aligned} \dot{\xi}_1 &= s(\xi_1), \\ u_1 &= c(\xi_1). \end{aligned} \tag{5.14}$$

This controller does not use y, but, despite this fact, for any solution of the exosystem $w(t)$ lying in the set Ω, system (5.14) has a solution $\xi_1(t) \equiv w(t)$ along which its output u_1 equals $c(w(t))$. So, indeed, system (5.14) has the generalized internal model property. Being a part of the overall controller (5.13), ξ_1-system (5.14) serves as an observer for the exosystem. The next possible choice for the ξ_1-subsystem is a linear system, as stated in the next lemma.

Lemma 5.2. *Suppose $y \in \mathbb{R}$, $u \in \mathbb{R}$ and there exist numbers $a_0, \dots a_r$, $a_r \neq 0$ and $b_0, \dots b_r$ such that for any solution of exosystem (5.4) $w(t)$ lying in the set Ω, the functions $\bar{y}_w(t) = h_m(\pi(w(t)), w(t))$ and $\bar{u}_w(t) = c(w(t))$ satisfy the relation*

$$\sum_{i=0}^{r} a_i \frac{d^i}{(dt)^i} \bar{u}_w(t) = \sum_{i=0}^{r} b_i \frac{d^i}{(dt)^i} \bar{y}_w(t). \tag{5.15}$$

Then there exist matrices $\Phi \in \mathbb{R}^{(r-1)\times(r-1)}$, $N \in \mathbb{R}^{(r-1)\times 1}$, $M \in \mathbb{R}^{1\times(r-1)}$, and $\kappa \in \mathbb{R}$ such that the input $\bar{y}_w(t)$ induces the output $\bar{u}_w(t)$ in the system

$$\begin{aligned} \dot{\xi}_1 &= \Phi \xi_1 + N y, \\ u_1 &= M \xi_1 + \kappa y. \end{aligned} \tag{5.16}$$

Proof: Choose Φ, N, M, and κ such that system (5.16) is the state space realization of the linear system $u = W(s)y$ with the transfer function $W(s) := b(s)/a(s)$, where $a(s) = \sum_{i=0}^{r} a_i s^i$ and $b(s) = \sum_{i=0}^{r} b_i s^i$. Relation (5.15) shows that $\bar{y}_w(t)$ and $\bar{u}_w(t)$ satisfy the relation $u = W(s)y$. Therefore, system (5.16) with the input $\bar{y}_w(t)$ has a bounded solution along which the output u_1 equals $\bar{u}_w(t)$. Hence, system (5.16) has the required generalized internal model property. □

A different way of finding a ξ_1-subsystem with the generalized internal model property can be found, for example, in [11]. Why may we need several different implementations of the ξ_1-subsystem? It may happen that for a certain implementation of the ξ_1-subsystem it is not possible to design a ξ_2-subsystem that guarantees global uniform convergence of the closed-loop system. At the same time, for another implementation of the ξ_1-subsystem such a ξ_2-subsystem can be found. For example, in the local exponential output regulation problem, the internal model, which is a counterpart of the ξ_1-subsystem, is required to have certain detectability properties. Without these properties it is not possible to find a ξ_2-subsystem of the controller that makes the closed-loop system locally exponentially convergent (see, e.g., [8]).

Having found controller (5.10) with the generalized internal model property, we need to find controller (5.12) that makes the overall closed-loop system input-to-state convergent. This problem is discussed in the next section.

5.3 How to make a system input-to-state convergent?

In this section we present different methods for designing controllers that make the corresponding closed-loop system input-to-state convergent. These methods are based on basic results on convergent systems from Chapter 2.

5.3.1 Backstepping design

In this section we present an analog of the backstepping method for designing a feedback that makes a system input-to-state convergent. Consider the system

$$\dot{z}_1 = F(z_1, z_2, w), \tag{5.17}$$

$$\dot{z}_2 = u, \tag{5.18}$$

with states $z_1 \in \mathbb{R}^d$, $z_2 \in \mathbb{R}^k$, control $u \in \mathbb{R}^k$, and external input $w \in \mathbb{R}^m$. The function $F(z_1, z_2, w)$ is locally Lipschitz with respect to z_1 and z_2 and continuous with respect to w.

Theorem 5.3. *Consider the system (5.17), (5.18). Suppose there exists a C^1 function $\psi(z_1)$ such that the system*

$$\dot{z}_1 = F(z_1, \psi(z_1) + \bar{v}, w), \tag{5.19}$$

with inputs $\bar{v}$ and w is input-to-state convergent. Then for any scalar $b > 0$ the controller

$$u = -b(z_2 - \psi(z_1)) + \frac{\partial \psi}{\partial z_1}(z_1) F(z_1, z_2, w) + v \tag{5.20}$$

is such that the closed-loop system (5.17), (5.18), (5.20) with v and w as inputs is input-to-state convergent.

Proof: Consider the coordinate transformation $\xi_1 := z_1$, $\xi_2 = z_2 - \psi(z_1)$. In the new coordinates the system equations are

$$\dot{\xi}_1 = F(\xi_1, \psi(\xi_1) + \xi_2, w),$$
$$\dot{\xi}_2 = u - \frac{\partial \psi}{\partial z_1}(\xi_1) F(\xi_1, \psi(\xi_1) + \xi_2, w).$$

After applying the feedback

$$u = -b\xi_2 + v + \frac{\partial \psi}{\partial z_1}(\xi_1) F(\xi_1, \psi(\xi_1) + \xi_2, w), \tag{5.21}$$

the equations of the closed-loop system become

$$\dot{\xi}_1 = F(\xi_1, \psi(\xi_1) + \xi_2, w), \tag{5.22}$$
$$\dot{\xi}_2 = -b\xi_2 + v. \tag{5.23}$$

Due to the choice of $\psi(\xi_1)$, the ξ_1-subsystem with inputs (ξ_2, w) is input-to-state convergent. At the same time, the ξ_2-subsystem is input-to-state convergent because it is linear with the Hurwitz matrix $-bI$ (this system satisfies the Demidovich condition with the matrices $P = I$, $Q = 2bI$, see Theorem 2.29). By Property 2.27, the series connection of systems (5.22) and (5.23) is an input-to-state convergent system. Finally, notice that in the original coordinates (z_1, z_2) controller (5.21) equals the controller given in (5.20). □

Remark 1. The input-to-state convergence property of system (5.19) can be established, for example, using Theorem 2.29. Some methods for finding the function $\psi(z_1)$ with the required properties will be discussed later in this chapter.$\triangleleft$

Remark 2. The parameter $b > 0$ can be used to influence the rate of convergence in the closed-loop system, while the additional input v can be used to shape steady-state solutions of the closed-loop system (for example, in order to guarantee certain steady-state behavior of the closed-loop system). Actually, instead of the controller (5.20) we can use any controller of the form

$$u = -\kappa(z_2 - \psi(z_1)) + \frac{\partial \psi}{\partial z_1}(z_1) F(z_1, z_2, w) + v,$$

where the function $\kappa(\cdot)$ is such that the system $\dot{\xi}_2 = -\kappa(\xi_2) + v$ is input-to-state convergent.$\triangleleft$

Remark 3. The result of Theorem 5.3 can be extended to the case of arbitrary number of integrators in the system:

$$\begin{aligned} \dot{z}_1 &= F(z_1, z_2, w), \\ \dot{z}_2 &= z_3, \\ &\cdots \\ \dot{z}_r &= u. \end{aligned}$$

In this case, $F(z_1, z_2, w)$ and $\psi(z_1)$ must be sufficiently differentiable.$\lhd$

To illustrate the backstepping controller design method, consider the following example.

Example 5.4. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 - x_2^3 + x_3,\\ \dot{x}_3 &= u.\end{aligned} \tag{5.25}$$

It can be easily checked that for $\psi(x_1, x_2) = x_2^3 - ax_2$, $a > 0$, the (x_1, x_2) subsystem with $x_3 = \psi(x_1, x_2) + \bar{v}$ is input-to-state convergent (because it becomes a linear system with a Hurwitz system matrix). By Theorem 5.3, for any $b > 0$ the controller

$$u = -b(x_3 - x_2^3 + ax_2) + (3x_2^2 - a)(-x_1 - x_2^3 + x_3) + v \tag{5.26}$$

makes the closed-loop system (5.25), (5.26) with v as input input-to-state convergent.$\lhd$

5.3.2 Quadratic stability design

In this section we consider controller design procedures based on the notions of quadratic stability, stabilizability and detectability, which are defined below.

Definition 5.5. *A matrix function $\mathcal{A}(\zeta) \in \mathbb{R}^{d\times d}$ is called quadratically stable over a set Ξ if for some $\mathcal{P} = \mathcal{P}^T > 0$ and $\mathcal{Q} = \mathcal{Q}^T > 0$*

$$\mathcal{P}\mathcal{A}(\zeta) + \mathcal{A}(\zeta)^T\mathcal{P} \leq -\mathcal{Q} \quad \forall \zeta \in \Xi. \tag{5.27}$$

Definition 5.6. *A pair of matrix functions $\mathcal{A}(\zeta) \in \mathbb{R}^{d\times d}$ and $\mathcal{B}(\zeta) \in \mathbb{R}^{d\times k}$ is said to be quadratically stabilizable over Ξ if there exists a matrix $K \in \mathbb{R}^{k\times d}$ such that $\mathcal{A}(\zeta) + \mathcal{B}(\zeta)K$ is quadratically stable over Ξ.*

Definition 5.7. *A pair of matrix functions $\mathcal{A}(\zeta) \in \mathbb{R}^{d\times d}$ and $\mathcal{C}(\zeta) \in \mathbb{R}^{l\times d}$ is said to be quadratically detectable over Ξ if there exists a matrix $L \in \mathbb{R}^{d\times l}$ such that $\mathcal{A}(\zeta) + L\mathcal{C}(\zeta)$ is quadratically stable over Ξ.*

Notice that if $\mathcal{A}(\zeta) \equiv \mathcal{A}$ is constant, quadratic stability of $\mathcal{A}$ is equivalent to the matrix $\mathcal{A}$ being Hurwitz; quadratic stabilizability of constant matrices $(\mathcal{A}, \mathcal{B})$ and quadratic detectability of constant matrices $(\mathcal{A}, \mathcal{C})$ are equivalent to conventional stabilizability and detectability of the pairs of matrices $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{C})$, respectively. Similar to the case of constant matrices, the pair $(\mathcal{A}(\zeta), \mathcal{B}(\zeta))$ is quadratically stabilizable over Ξ if and only if $(\mathcal{A}^T(\zeta), \mathcal{B}^T(\zeta))$

is quadratically detectable over Ξ. This fact follows from pre- and postmultiplication by $\mathcal{P}^{-1}$ of the inequality

$$\mathcal{P}(\mathcal{A}(\zeta) + \mathcal{B}(\zeta)K) + (\mathcal{A}(\zeta) + \mathcal{B}(\zeta)K)^T\mathcal{P} \leq -\mathcal{Q}.$$

The purpose of the notion of quadratic stability introduced above becomes clear if one recalls Theorem 2.29. As follows from this theorem, if the system

$$\dot{z} = F(z, w), \quad z \in \mathbb{R}^d, \quad w \in \mathbb{R}^m, \tag{5.28}$$

is such that $F(z, w)$ is C^1 with respect to z, continuous with respect to w, and the Jacobian $\frac{\partial F}{\partial z}(z, w)$ is quadratically stable over $(z, w) \in \mathbb{R}^d \times \mathbb{R}^m$, then system (5.28) is input-to-state convergent. The notions of quadratic stabilizability and detectability are useful for controller and observer design as will be shown below. Consider the system

$$\begin{aligned} \dot{z} &= F(z, u, w), \\ y &= h(z, w), \end{aligned} \tag{5.29}$$

with state $z \in \mathbb{R}^d$, control $u \in \mathbb{R}^k$, external signal $w \in \mathbb{R}^m$, and measured output $y \in \mathbb{R}^l$. The functions $F(z, u, w)$, $h(z, w)$ are assumed to be C^1 with respect to z and u and continuous with respect to w.

Lemma 5.8. *Consider the system (5.29). Suppose the pair of matrix functions $\frac{\partial F}{\partial z}(z, u, w)$ and $\frac{\partial F}{\partial u}(z, u, w)$ is quadratically stabilizable over $\mathbb{R}^{d+k+m}$ with some matrix $K \in \mathbb{R}^{k \times d}$. Then the system*

$$\dot{z} = F(z, Kz + v, w) \tag{5.30}$$

with inputs v and w is input-to-state convergent.

Proof: The Jacobian of the right-hand side of system (5.30) equals

$$J(z, v, w) := \frac{\partial F}{\partial z}(z, Kz + v, w) + \frac{\partial F}{\partial u}(z, Kz + v, w)K.$$

By the choice of the matrix K, $J(z, v, w)$ is quadratically stable over $\mathbb{R}^{d+k+m}$. Hence, by Theorem 2.29 system (5.30) is input-to-state convergent. □

As we can see from this lemma, quadratic stabilizability of the pair $(\frac{\partial F}{\partial z}, \frac{\partial F}{\partial u})$ implies the existence of a feedback $u = Kz + v$ that makes the closed-loop system input-to-state convergent. The additional feedforward term v can be used, for example, for shaping steady-state solutions of the closed-loop system (5.30). The next lemma shows how the notion of quadratic detectability can be used for designing an observer with exponentially convergent error dynamics.

Lemma 5.9. *Consider system (5.29). Suppose the pair of matrix functions* $\frac{\partial F}{\partial z}(z,u,w)$ *and* $\frac{\partial h}{\partial z}(z,w)$ *is quadratically detectable over* $\mathbb{R}^{d+k+m}$ *with some matrix* $L \in \mathbb{R}^{d\times l}$. *Then the system*

$$\dot{\hat{z}} = F(\hat{z},u,w) + L(h(\hat{z},w) - y) \tag{5.31}$$

is an observer for system (5.29) with the observer error satisfying

$$|\hat{z}(t) - z(t)| \leq C\mathrm{e}^{-a(t-t_0)}|\hat{z}(t_0) - z(t_0)| \tag{5.32}$$

for some numbers $C > 0$ *and* $a > 0$ *independent of the particular inputs* $u(t)$, $w(t)$, *and solution* $z(t)$. *Moreover, the observer error dynamics*

$$\Delta\dot{z} = G(z+\Delta z,u,w) - G(z,u,w), \tag{5.33}$$

where $G(z,u,w) := F(z,u,w) + Lh(z,w)$, *is such that for any input* $z(\cdot) \in \overline{\mathbb{PC}}_d$, $w(\cdot) \in \overline{\mathbb{PC}}_m$ *and any feedback* $u = U(\Delta z,t)$ *all solutions of system (5.33) satisfy*

$$|\Delta z(t)| \leq C\mathrm{e}^{-a(t-t_0)}|\Delta z(t_0)|, \tag{5.34}$$

where the numbers $C > 0$ *and* $a > 0$ *are independent of* $z(t)$, $w(t)$ *and* $u = U(\Delta z,t)$.

Proof: Let us first prove the second part of the lemma. The Jacobian $\frac{\partial G}{\partial z}(z,u,w)$ equals $\frac{\partial F}{\partial z}(z,u,w) + L\frac{\partial h}{\partial z}(z,w)$. By the choice of the matrix L, $\frac{\partial G}{\partial z}(z,u,w)$ is quadratically stable over $\mathbb{R}^{d+k+m}$, i.e., there exist positive definite matrices $P > 0$ and $Q > 0$ such that

$$P\frac{\partial G}{\partial z}(z,u,w) + \frac{\partial G}{\partial z}^T(z,u,w)P \leq -Q$$

for all $(z,u,w) \in \mathbb{R}^{d+k+m}$. Hence, by Lemma 2.30 the derivative of the function $V(\Delta z) := 1/2\Delta z^T P\Delta z$ along solutions of system (5.33) satisfies

$$\frac{dV}{dt} = \Delta z^T P(G(z+\Delta z,u,w) - G(z,u,w)) \leq -a|\Delta z|_P^2 = -2aV(\Delta z), \tag{5.35}$$

where $|\Delta z|_P$ denotes $|\Delta z|_P = (\Delta z^T P\Delta z)^{1/2}$. In inequality (5.35), the number $a > 0$ depends only on the matrices P and Q and does not depend on the particular values of z, u, and w. This inequality, in turn, implies that there exists $C > 0$ depending only on the matrix P such that if $z(t)$ and $w(t)$ are defined for all $t \geq t_0$ then any solution of system (5.33) $\Delta z(t)$ is also defined for all $t \geq t_0$ and satisfies

$$|\Delta z(t)| \leq C\mathrm{e}^{-a(t-t_0)}|\Delta z(t_0)|, \quad \forall t \geq t_0. \tag{5.36}$$

We must still show that system (5.31) is an observer for system (5.29). Denote $\Delta z := \hat{z} - z(t)$. Since $z(t)$ is a solution of system (5.29), Δz satisfies (5.33).

By the previous analysis, we obtain that $\Delta z(t)$ satisfies (5.36). This implies (5.32). □

Lemma 5.9 provides conditions under which system (5.31) is an observer for system (5.29). The observer itself is designed in such a way that it is input-to-state convergent for y, u, and w viewed as inputs.

Lemmas 5.8 and 5.9 show how to design a state feedback controller that makes the closed-loop system input-to-state convergent and an observer for this system with an exponentially stable error dynamics. In fact, for such controllers and observers one can use the separation principle to design an output feedback controller that makes the closed-loop system input-to-state convergent. This statement follows from the next theorem.

Theorem 5.10. *Consider the system (5.29). Suppose the pair of matrix functions $\frac{\partial F}{\partial z}(z,u,w)$ and $\frac{\partial F}{\partial u}(z,u,w)$ is quadratically stabilizable over $\mathbb{R}^{d+k+m}$ with some matrix $K \in \mathbb{R}^{k\times d}$ and the pair of matrix functions $\frac{\partial F}{\partial z}(z,u,w)$ and $\frac{\partial h}{\partial z}(z,w)$ is quadratically detectable over $\mathbb{R}^{d+k+m}$ with some matrix $L \in \mathbb{R}^{d\times l}$. Then system (5.29) in closed loop with the controller*

$$\dot{\hat{z}} = F(\hat{z},u,w) + L(h(\hat{z},w) - y), \tag{5.37}$$
$$u = K\hat{z} + v, \tag{5.38}$$

with (v,w) as inputs is input-to-state convergent.

Proof: Denote $\Delta z := \hat{z} - z$. Then in the new coordinates $(z, \Delta z)$ the equations of the closed-loop system are

$$\dot{z} = F(z, Kz + K\Delta z + v, w), \tag{5.39}$$
$$\Delta\dot{z} = G(z + \Delta z, u, w) - G(z,u,w), \tag{5.40}$$
$$u = K(z + \Delta z) + v, \tag{5.41}$$

where $G(z,u,w) = F(z,u,w) + Lh(z,w)$. By the choice of K, system (5.39) with $(\Delta z, v, w)$ as inputs is input-to-state convergent for the class of inputs $\overline{\mathbb{PC}}_{d+k+m}$ (see Lemma 5.8). By the choice of the observer gain L, for any bounded inputs $z(t)$, $w(t)$, $v(t)$ and for the feedback $u = K(z(t) + \Delta z) + v(t)$ any solution of system (5.40), (5.41) satisfies

$$|\Delta z(t)| \le C\mathrm{e}^{-a(t-t_0)}|\Delta z(t_0)|, \tag{5.42}$$

where the numbers $C > 0$ and $a > 0$ are independent of $z(t)$, $w(t)$, and $v(t)$ (see Lemma 5.9). Hence, applying Property 2.28, we obtain that the closed-loop system (5.39)–(5.41) is input-to-state convergent. □

Remark. The controller proposed in Theorem 5.10 consists of the observer (5.37) and the linear state-feedback controller (5.38), which uses the estimates of the system states for feedback. As follows from the proof of the theorem, linearity of the controller is not essential. What *is* essential is that system (5.39) with Δz, v, and w as inputs is input-to-state convergent. Therefore, instead of the linear controller (5.38) one can use any controller $u = \psi(z) + v$ that makes the system

$$\dot{z} = F(z, \psi(z + \Delta z) + v, w)$$

input-to-state convergent with respect to the inputs Δz, v and w. For example, such a controller can be found using the backstepping method described in the previous section.$\lhd$

As we can see from Theorem 5.10, the notions of quadratic stability, stabilizability, and detectability can be very helpful in designing an output feedback controller that makes the corresponding closed-loop system input-to-state convergent. The question is how to check quadratic stability, stabilizability, and detectability. In general, this is not an easy task. Yet, in some particular cases this can be done efficiently, as shown in the following lemma.

Lemma 5.11. *Consider the matrix functions $\mathcal{A}(\zeta) \in \mathbb{R}^{d\times d}$, $\mathcal{B}(\zeta) \in \mathbb{R}^{d\times k}$, and $\mathcal{C}(\zeta) \in \mathbb{R}^{l\times d}$, $\zeta \in \Xi$, where Ξ is some set.*

(i) *Suppose there exist matrices $\mathcal{A}_1, \dots, \mathcal{A}_p$ such that*

$$\mathcal{A}(\zeta) \in \mathrm{co}\{\mathcal{A}_1, \dots, \mathcal{A}_p\}, \quad \forall \zeta \in \Xi,$$

and the LMI

$$\begin{aligned} \mathcal{P}\mathcal{A}_i + \mathcal{A}_i^T\mathcal{P} < 0, \quad i = 1, \dots, p, \\ \mathcal{P} = \mathcal{P}^T > 0, \end{aligned} \tag{5.43}$$

is feasible. Then $\mathcal{A}(\zeta)$ is quadratically stable over Ξ.

(ii) *Suppose there exist matrices $\mathcal{A}_1, \dots, \mathcal{A}_p$ and $\mathcal{B}_1, \dots, \mathcal{B}_p$ such that*

$$[\mathcal{A}(\zeta)\ \mathcal{B}(\zeta)] \in \mathrm{co}\{[\mathcal{A}_1\ \mathcal{B}_1], \dots, [\mathcal{A}_p\ \mathcal{B}_p]\}, \quad \forall \zeta \in \Xi,$$

and the LMI

$$\begin{aligned} \mathcal{A}_i\mathcal{P} + \mathcal{P}\mathcal{A}_i^T + \mathcal{B}_i\mathcal{Y} + \mathcal{Y}^T\mathcal{B}_i^T < 0, \quad i = 1, \dots, p, \\ \mathcal{P} = \mathcal{P}^T > 0, \end{aligned} \tag{5.44}$$

is feasible. Then the pair $\mathcal{A}(\zeta), \mathcal{B}(\zeta)$ is quadratically stabilizable over Ξ with the matrix $K = \mathcal{Y}\mathcal{P}^{-1}$, where $\mathcal{Y}$ and $\mathcal{P}$ satisfy (5.44).

(iii) *Suppose there exist matrices $\mathcal{A}_1, \dots, \mathcal{A}_p$ and $\mathcal{C}_1, \dots, \mathcal{C}_p$ such that*

$$[\mathcal{A}(\zeta)\ \mathcal{C}(\zeta)] \in \mathrm{co}\{[\mathcal{A}_1\ \mathcal{C}_1], \dots, [\mathcal{A}_p\ \mathcal{C}_p]\}, \quad \forall \zeta \in \Xi,$$

and the LMI

$$\begin{aligned} \mathcal{P}\mathcal{A}_i + \mathcal{A}_i^T\mathcal{P} + \mathcal{X}\mathcal{C}_i + \mathcal{C}_i^T\mathcal{X}^T < 0, \quad i = 1,\ldots,p, \\ \mathcal{P} = \mathcal{P}^T > 0, \end{aligned} \tag{5.45}$$

is feasible. Then the pair $\mathcal{A}(\zeta), \mathcal{C}(\zeta)$ is quadratically detectable over Ξ with the matrix $L = \mathcal{P}^{-1}\mathcal{X}$, where $\mathcal{X}$ and $\mathcal{P}$ satisfy (5.45).

Lemma 5.11 is a compilation of standard results on LMI applications to control (see, e.g., [5]). In general, the LMI conditions presented above are only sufficient for quadratic stability, stabilizability, and detectability. Yet, for the case of systems with one scalar output dependent nonlinearity, these conditions are not only sufficient, but also necessary. We consider such systems in the next section.

5.3.3 Controller design for Lur'e systems

In this section we consider controller design based on the notions of quadratic stability, stabilizability and detectability for systems with one scalar output dependent nonlinearity. In literature, systems with output dependent nonlinearities are often referred to as Lur'e systems, named after the Russian mathematician A.I. Lur'e. For such systems the notions of quadratic stability, stabilizability, and detectability simplify significantly. This simplification is due to the equivalence of quadratic stability and feasibility of certain LMIs, which will be stated below. Consider the system

$$\begin{aligned} \dot{z} &= Az + D\varphi(\zeta) + Bu + Ew, \\ \zeta &= C_\zeta z + H_\zeta w, \\ y &= Cz + Hw, \end{aligned} \tag{5.46}$$

with state $z \in \mathbb{R}^d$, control $u \in \mathbb{R}^k$, external signal $w \in \mathbb{R}^m$, measured output $y \in \mathbb{R}^l$, output $\zeta \in \mathbb{R}$, and scalar nonlinearity $\varphi(\zeta)$. The nonlinearity is assumed to be C^1 and to satisfy the condition

$$\sup_{\zeta\in\mathbb{R}} \frac{\partial\varphi}{\partial\zeta}(\zeta) = \gamma, \quad \inf_{\zeta\in\mathbb{R}} \frac{\partial\varphi}{\partial\zeta}(\zeta) = -\gamma, \tag{5.47}$$

for some finite $\gamma > 0$. If the nonlinearity $\varphi(\zeta)$ does not satisfy (5.47), but satisfies the condition

$$\sup_{\zeta\in\mathbb{R}} \frac{\partial\varphi}{\partial\zeta}(\zeta) = \alpha, \quad \inf_{\zeta\in\mathbb{R}} \frac{\partial\varphi}{\partial\zeta}(\zeta) = \beta,$$

for some finite $\alpha > \beta$, then by introducing the transformation $\tilde{\varphi}(\zeta) := \varphi(\zeta) - \frac{\alpha+\beta}{2}\zeta$ and $\tilde{A} := A + \frac{\alpha+\beta}{2}DC_\zeta$, $\tilde{E} := E + \frac{\alpha+\beta}{2}DH_\zeta$, system (5.46) can be written in an equivalent form

$$\dot{z} = \tilde{A}z + D\tilde{\varphi}(\zeta) + Bu + \tilde{E}w,$$

with the nonlinearity $\tilde{\varphi}(\zeta)$ satisfying condition (5.47) for $\gamma := (\alpha - \beta)/2$. So, we assume that all such transformations have been made and that the nonlinearity $\varphi(\zeta)$ satisfies (5.47). Denote the Jacobian of the right-hand side of (5.46) with respect to z by $\mathcal{A}(\zeta) := A + \frac{\partial\varphi}{\partial\zeta}(\zeta) DC_\zeta$. Denote $A_\gamma^- := A - \gamma DC_\zeta$ and $A_\gamma^+ := A + \gamma DC_\zeta$. The next lemma shows that quadratic stability of $\mathcal{A}(\zeta)$, quadratic stabilizability of the pair $(\mathcal{A}(\zeta), B)$, and quadratic detectability of the pair $(\mathcal{A}(\zeta), C)$ are equivalent to feasibility of certain LMIs.

Lemma 5.12. *Consider system (5.46).*

- *The following statements are equivalent:*
  (i) *The matrix function $\mathcal{A}(\zeta)$ is quadratically stable over $\mathbb{R}$.*
  (ii) *There exists a matrix $P = P^T > 0$ satisfying the LMI*
$$PA_\gamma^- + (A_\gamma^-)^T P < 0, \quad PA_\gamma^+ + (A_\gamma^+)^T P < 0. \tag{5.48}$$
  (iii) *The matrix A is Hurwitz and*
$$\sup_{\omega\in\mathbb{R}} |C_\zeta (i\omega I - A)^{-1} D| < \frac{1}{\gamma}. \tag{5.49}$$
- *The pair of matrix functions $(\mathcal{A}(\zeta), B)$ is quadratically stabilizable over $\mathbb{R}$ if and only if the following LMI is feasible:*
$$\begin{aligned} A_\gamma^- \mathcal{P} + \mathcal{P}(A_\gamma^-)^T + B\mathcal{Y} + \mathcal{Y}^T B^T &< 0, \\ A_\gamma^+ \mathcal{P} + \mathcal{P}(A_\gamma^+)^T + B\mathcal{Y} + \mathcal{Y}^T B^T &< 0, \\ \mathcal{P} = \mathcal{P}^T &> 0. \end{aligned} \tag{5.50}$$
  Under this condition, $\mathcal{A}(\zeta) + BK$ with $K := \mathcal{Y}\mathcal{P}^{-1}$, where $\mathcal{Y}$ and $\mathcal{P}$ satisfy (5.50), is quadratically stable over $\mathbb{R}$.
- *The pair of matrix functions $(\mathcal{A}(\zeta), C)$ is quadratically detectable over $\mathbb{R}$ if and only if the following LMI is feasible:*
$$\begin{aligned} \mathcal{P} A_\gamma^- + (A_\gamma^-)^T \mathcal{P} + \mathcal{X} C + C^T \mathcal{X}^T &< 0, \\ \mathcal{P} A_\gamma^+ + (A_\gamma^+)^T \mathcal{P} + \mathcal{X} C + C^T \mathcal{X}^T &< 0, \\ \mathcal{P} = \mathcal{P}^T &> 0. \end{aligned} \tag{5.51}$$
  Under this condition, $\mathcal{A}(\zeta) + LC$ with $L := \mathcal{P}^{-1}\mathcal{X}$, where $\mathcal{X}$ and $\mathcal{P}$ satisfy (5.51), is quadratically stable over $\mathbb{R}$.

Proof: The equivalence of (ii) and (iii) follows from Lemma 2.37. The implication (ii)⇒(i) holds because $\mathcal{A}(\zeta) \in \mathrm{co}\{A_\gamma^-, A_\gamma^+\}$. We still must prove the implication (i)⇒(ii). Suppose $\mathcal{A}(\zeta)$ is quadratically stabilizable over $\mathbb{R}$. Thus, there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that
$$P\mathcal{A}(\zeta) + \mathcal{A}^T(\zeta) P \le -Q, \quad \forall \zeta \in \mathbb{R}. \tag{5.52}$$

Due to condition (5.47), there exist sequences $\{\zeta_k^-\}_{k=1}^{+\infty}$ and $\{\zeta_k^+\}_{k=1}^{+\infty}$ such that $\mathcal{A}(\zeta_k^-) \to A_\gamma^-$ and $\mathcal{A}(\zeta_k^+) \to A_\gamma^+$ as $k \to +\infty$. Substituting these $\mathcal{A}(\zeta_k^-)$ and $\mathcal{A}(\zeta_k^+)$ in inequality (5.52), in the limit for $k \to +\infty$ we obtain (ii). This proves the first part of the lemma.

Let us show the equivalence of quadratic stabilizability of the pair $(\mathcal{A}(\zeta), B)$ and the feasibility of the LMI (5.50). The "if" part follows from Lemma 5.11. So, we only need to show that quadratic stabilizability of the pair $(\mathcal{A}(\zeta), B)$ implies the feasibility of the LMI (5.50). Since the pair $(\mathcal{A}(\zeta), B)$ is quadratically stabilizable, there exists a matrix K such that $\mathcal{A}(\zeta)+BK$ is quadratically stable over $\mathbb{R}$. By the result of the first part of the theorem, this implies that

$$P(A_\gamma^- + BK) + (A_\gamma^- + BK)^T P < 0, \tag{5.53}$$

$$P(A_\gamma^+ + BK) + (A_\gamma^+ + BK)^T P < 0, \tag{5.54}$$

for some matrix $P = P^T > 0$. Denote $\mathcal{P} := P^{-1}$ and $\mathcal{Y} := KP^{-1}$. Pre- and postmultiplication of inequalities (5.53) and (5.54) by P^{-1} implies that $\mathcal{P}$ and $\mathcal{Y}$ satisfy (5.50). This proves the second part of the lemma.

The last part of the lemma on quadratic detectability of the pair $(\mathcal{A}(\zeta), C)$ is proved in the same way as the part on quadratic stabilizability of the pair $(\mathcal{A}(\zeta), B)$. □

Condition (5.47) means that $\frac{\partial \varphi}{\partial \zeta}(\zeta)$ exactly "fits" the range $[-\gamma, \gamma]$. This condition allowed us to prove the equivalence of quadratic stability, stabilizability, and detectability to certain LMIs. In practice, however, it is sufficient to know that the nonlinearity $\varphi(\zeta)$ satisfies the condition $\left|\frac{\partial \varphi}{\partial \zeta}(\zeta)\right| \le \gamma$ for all $\zeta \in \mathbb{R}$. For system (5.46) with such a nonlinearity $\varphi(\zeta)$, the LMIs (5.48), (5.50), and (5.51) still guarantee quadratic stability of $\mathcal{A}(\zeta)$, stabilizability of $(\mathcal{A}(\zeta), B)$, and detectability of $(\mathcal{A}(\zeta), C)$, respectively. In the following we will denote the class of such nonlinearities by $\mathcal{F}_\gamma$, i.e.,

$$\mathcal{F}_\gamma := \left\{ \varphi \in C^1 : \left| \frac{\partial \varphi}{\partial \zeta}(\zeta) \right| \le \gamma \quad \forall \zeta \in \mathbb{R} \right\}.$$

The result of Lemma 5.12 together with Theorem 5.10 gives us the following corollary.

Corollary 5.13. *Consider system (5.46) with a nonlinearity $\varphi \in \mathcal{F}_\gamma$. Suppose the LMIs (5.50) and (5.51) are feasible. Then there exist matrices K and L such that system (5.46) in closed loop with the controller*

$$\dot{\hat{z}} = A\hat{z} + D\varphi(\hat{\zeta}) + Bu + Ew + L(\hat{y} - y), \tag{5.55}$$

$$\hat{\zeta} = C_\zeta \hat{z} + H_\zeta w, \quad \hat{y} = C\hat{z} + Hw, \tag{5.56}$$

$$u = K\hat{z} + v, \tag{5.57}$$

is input-to-state convergent.

An example illustrating an application of this controller design will be presented in Section 5.4.

Corollary 5.13 enables us to design an output feedback controller that makes the closed-loop system input-to-state convergent. An important observation regarding this controller design is that it requires accurate knowledge of the system parameters and the nonlinearity $\varphi(\zeta)$. In some cases, however, we may not know exactly the system parameters and the only available information about $\varphi(\zeta)$ is that it belongs to the class $\mathcal{F}_\gamma$. In this case, it may still be possible to design an output feedback controller that makes the closed-loop system input-to-state convergent. Such a controller design can be performed based on Lemma 5.12. Denote A°, B°, D°, C°, and C°_ζ to be the nominal values of the matrices A, B, D, C, and C_ζ. We will seek a robust controller of the form

$$\begin{aligned} \dot{\xi} &= G\xi + My, \\ u &= N_\xi \xi + N_y y + v. \end{aligned} \tag{5.58}$$

The following lemma gives sufficient conditions under which system (5.46) in closed loop with controller (5.58) is input-to-state convergent for all matrices A, B, D, C, and C_ζ close enough to their nominal values, for all matrices E, H, H_ζ, and for all nonlinearities $\varphi(\zeta)$ from the class $\mathcal{F}_\gamma$.

Lemma 5.14. *Consider the closed-loop system (5.46), (5.58) for the nominal parameters with $w \equiv 0$, $v \equiv 0$ and with φ as input:*

$$\begin{aligned} \dot{z} &= A^\circ z + B^\circ(N_\xi \xi + N_y C^\circ z) + D^\circ \varphi, \\ \dot{\xi} &= G\xi + MC^\circ z, \\ \zeta &= C^\circ_\zeta z. \end{aligned} \tag{5.59}$$

Suppose all poles of system (5.59) have negative real part and the transfer function $W^\circ_{\varphi\zeta}(s)$ of system (5.59) from input φ to output ζ satisfies

$$\|W^\circ_{\varphi\zeta}\|_\infty < \frac{1}{\gamma}. \tag{5.60}$$

Then system (5.46) in closed loop with controller (5.58) is input-to-state convergent for all matrices A, B, D, C, C_ζ close enough to their nominal values, for all matrices E, H, H_ζ, and for all nonlinearities $\varphi \in \mathcal{F}_\gamma$.

Proof: System (5.46) in closed loop with controller (5.58) has the form

$$\dot{\chi} = \hat{A}\chi + \hat{D}\varphi(\zeta) + \hat{E}w + \hat{H}v, \tag{5.61}$$

$$\zeta = \hat{C}\chi + H_\zeta w, \tag{5.62}$$

where $\chi := (z^T, \xi^T)^T$ and

$$\hat{A} := \begin{pmatrix} A + BN_yC & BN_\xi \\ MC & G \end{pmatrix}, \quad \hat{D} := \begin{pmatrix} D \\ 0 \end{pmatrix}, \quad \hat{E} := \begin{pmatrix} BN_yH + E \\ MH \end{pmatrix},$$

$$\hat{H} := \begin{pmatrix} B \\ 0 \end{pmatrix}, \quad \hat{C} := (C_\zeta, \ \ 0).$$

Recall that the norm $\|W_{\varphi\zeta}\|_\infty$ of the transfer function $W_{\varphi\zeta}(s)$ is defined as $\|W_{\varphi\zeta}\|_\infty = \sup_{\omega\in\mathbb{R}} |\hat{C}(i\omega I - \hat{A})^{-1}\hat{D}|$. Notice that $\|W_{\varphi\zeta}\|_\infty$ depends on the matrices $\hat{C}$, $\hat{A}$, and $\hat{D}$ continuously (at least in the domain where all eigenvalues of $\hat{A}$ have negative real parts). The matrices $\hat{C}$, $\hat{A}$, and $\hat{D}$ depend on the system matrices A, B, D, C, and C_ζ continuously. The matrix $\hat{A}^\circ$—the matrix $\hat{A}$ corresponding to the nominal system parameters—is Hurwitz. Therefore, the conditions of the theorem imply that $\hat{A}$ is Hurwitz and $\|W_{\varphi\zeta}\|_\infty < 1/\gamma$ for all matrices A, B, D, C, and C_ζ close enough to their nominal values. By Lemma 5.12, this implies that the Jacobian of the right-hand side of the closed-loop system, which is equal to $\mathcal{A}(\zeta) := \hat{A} + \hat{D}\hat{C}\frac{\partial\varphi}{\partial\zeta}(\zeta)$, is quadratically stable over $\mathbb{R}$. Hence, according to Theorem 2.29 the closed-loop system is input-to-state convergent provided that the matrices A, B, D, C, and C_ζ are close enough to their nominal values. Since condition (5.60) does not depend on the nonlinearity $\varphi(\zeta)$ and on the matrices E, H, and H_ζ, the input-to-state convergence property holds for all nonlinearities $\varphi \in \mathcal{F}_\gamma$ and all matrices E, H, and H_ζ. □

Remark. The problem of finding a linear controller (5.58) such that the corresponding transfer function $W_{\varphi\zeta}(s)$ satisfies condition (5.60) is a standard H_∞ optimization problem. There are many software packages for solving this problem. For example, one may use a standard MATLAB routine `hinflmi`. An example of this robust controller design will be given in Section 5.4.◁

An important assumption in the results presented in this section is that the nonlinearity $\varphi(\zeta)$ belongs to the class $\mathcal{F}_\gamma$ for some $\gamma > 0$. Below we give a result on controller design for systems with an arbitrary C^1 nonlinearity $\varphi(\zeta)$. Consider the system

$$\begin{aligned} \dot{z} &= Az + \varphi(y) + Bu + Ew, \\ y &= Cz + Hw, \end{aligned} \tag{5.63}$$

with state $z \in \mathbb{R}^d$, measured output $y \in \mathbb{R}$, control $u \in \mathbb{R}$, and external input $w \in \mathbb{R}^m$. We assume that system (5.63) has relative degree one, i.e., $CB \neq 0$. Without loss of generality, we assume $CB > 0$. The variable on which the nonlinearity depends is assumed to be measured, i.e., $\zeta = y$. The only available information on the nonlinearity $\varphi(y)$ is that it is C^1 and there exists a continuous scalar function $\psi(y)$ such that

$$\left|\frac{\partial\varphi}{\partial y}(y)\right| \leq \psi(y), \quad y \in \mathbb{R}. \tag{5.64}$$

Theorem 5.15. *Consider system (5.63) with the nonlinearity $\varphi(y)$ satisfying condition (5.64). Suppose all zeros of the system*

$$\begin{aligned} \dot{z} &= Az + Bu, \\ y &= Cz, \end{aligned} \tag{5.65}$$

have negative real parts and $CB > 0$ (i.e., system (5.65) has relative degree equal to one). Then there exists a C^1 function $U(y)$ such that system (5.63) in closed loop with the controller $u = U(y) + v$, where v is an additional scalar input, is input-to-state convergent for the class of inputs $\overline{\mathbb{PC}}_1 \times \overline{\mathbb{PC}}_m$. The function $U(y)$ can be chosen, for example, equal to

$$U(y) = -\kappa y - \mu \int_0^y |\psi(\tau)|^2 d\tau, \quad \forall \kappa \geq \kappa_*, \quad \mu \geq \mu_*, \tag{5.66}$$

where the numbers κ_* and μ_* depend only on the matrices A, B, and C and can be determined from the matrix inequalities

$$PA + A^T P - 2\kappa_* C^T C < 0, \quad PB = C^T,$$

$$\mu_* I \geq -P(PA + A^T P - 2\kappa_* C^T C)^{-1} P,$$

which are feasible.

Proof: See Appendix 9.13.

Remark 1. Theorem 5.15 can be extended to the case of y and u being vectors of dimensions larger than one. The idea of the proof remains the same as in the scalar case.$\triangleleft$

Remark 2. A possible way to relax the requirement that system (5.65) must be of relative degree one is by using filtered output transformations presented in [59]. Such transformations allow one to reduce the relative degree of the system while preserving the property that all zeros of the system lie in the left-half complex plane (minimum-phaseness).$\triangleleft$

Remark 3. Theorem (5.15) shows that any strongly minimum-phase system of the form (5.63) (i.e., system with $CB \neq 0$ and stable zeros) with a nonlinearity $\varphi(y)$ satisfying condition (5.64) can be made input-to-state convergent with a static output feedback of the form (5.66) provided that the gains κ and μ are high enough. The only essential information is the sign of CB (if $CB < 0$ then the formula for $U(y)$ must be with pluses) and the bound function $\psi(y)$. Such a characterization of controllers can be useful, for example, in adaptive control.$\triangleleft$

Remark 4. The result of Theorem 5.15 is closely related to the problem of passification (making a control system passive by means of feedback). We will not go into details on this point. The interested readers are referred to the literature on this subject, e.g., [18, 19].$\triangleleft$

5.4 Controller design for the global uniform output regulation problem

With the controller design methods given in the previous section, we can present results on controller design for the global (forward time) uniform output regulation problem. The results in this section are formulated in terms of Jacobians of the functions $f(x,u,w)$, $h_m(x,w)$, and $s(w)$, which are assumed to be at least C^1. In the following, we will use the following notation: $\chi := (x,u,w) \in \mathbb{R}^{n+k+m}$,

$$
\begin{aligned}
A(\chi) &:= \frac{\partial f}{\partial x}(x,u,w),\ B(\chi) := \frac{\partial f}{\partial u}(x,u,w),\\
E(\chi) &:= \frac{\partial f}{\partial w}(x,u,w),\ C(\chi) := \frac{\partial h_m}{\partial x}(x,w),\\
H(\chi) &:= \frac{\partial h_m}{\partial w}(x,w),\ S(\chi) := \frac{\partial s}{\partial w}(w).
\end{aligned}
$$

5.4.1 State feedback controller design

Let us first consider the state feedback case when the states x and w are available for measurements, i.e., $y = (x,w)$.

Theorem 5.16. *Consider system (5.1)–(5.3) with $y = (x,w)$ and exosystem (5.4). Suppose the regulator equations (5.5) are solvable and the corresponding continuous solutions $\pi(w)$ and $c(w)$ are globally defined (see Section 5.1 for details). If the pair $(A(\chi), B(\chi))$ is quadratically stabilizable over $\chi \in \mathbb{R}^{n+k+m}$, then the global (forward time) uniform output regulation problem is solved by a controller of the form*

$$u = c(w) + K(x - \pi(w)), \tag{5.67}$$

where the matrix K is such that the matrix function $A(\chi)+B(\chi)K$ is quadratically stable over $\chi \in \mathbb{R}^{n+k+m}$.

Proof: The controller (5.67) is such that for $\bar{y}_w(t) := (\pi(w(t)), w(t))$ it generates control $\bar{u}_w(t) = c(w(t))$. Therefore it has the generalized internal model property (see Section 5.1). Moreover, by the choice of the matrix K, the closed-loop system

$$\dot{x} = f(x, Kx + c(w) - K\pi(w), w)$$

is input-to-state convergent. Therefore, this controller solves the global (forward time) uniform output regulation problem (see Section 5.1 for details). □

As described in Section 5.2, controller (5.67) consists of two parts: $u = u_1 + u_2$, where $u_1 := c(w)$ and $u_2 = K(x - \pi(w))$. The first component u_1 guarantees the generalized internal model property, i.e., that for the input $\bar{y}_w(t) = (\pi(w(t)), w(t))$ the controller has the output $\bar{u}_w(t) = c(w(t))$. The second component u_2 guarantees that the closed-loop system is input-to-state convergent.

5.4.2 Output feedback controller design

If the full state (x, w) is not available for measurement, we can design an observer to asymptotically reconstruct the unmeasured variables in controller (5.67). If $w(t)$ is measured, a controller takes the form

$$u = c(w) + K(\hat{x} - \pi(w)),$$

where the state estimates $\hat{x}$ are generated, for example, by an observer of the form

$$\dot{\hat{x}} = f(\hat{x}, u, w) + L_x(h_m(\hat{x}, w) - y). \tag{5.68}$$

For such an observer design and related convergence analysis of the total closed-loop system, we can directly use Theorem 5.10. The main problem is that the state w of the exosystem is, in many cases, not available for measurements. This happens, for example, if the exosystem generates disturbances. Therefore, $w(t)$ can, in general, not be used in the controller. To overcome this difficulty, we extend the observer (5.68) with an observer for the exosystem and generate the controller from the formula $u = c(\hat{w}) + K(\hat{x} - \pi(\hat{w}))$, where $\hat{w}$ are the estimates of w. The main result on such a controller design is formulated in the following theorem.

Theorem 5.17. *Consider system (5.1)–(5.3) and exosystem (5.4). Suppose the regulator equations (5.5) are solvable and the corresponding solutions $\pi(w)$ and $c(w)$ are globally defined locally Lipschitz mappings. If the pair $(A(\chi), B(\chi))$ is quadratically stabilizable over $\chi \in \mathbb{R}^{n+k+m}$ and the pair*

$$\begin{bmatrix} A(\chi) & E(\chi) \\ 0 & S(\chi) \end{bmatrix}, \quad [C(\chi) \ H(\chi)], \tag{5.69}$$

is quadratically detectable over $\chi \in \mathbb{R}^{n+k+m}$, then the global (forward time) uniform output regulation problem is solved by a controller of the form

$$\begin{aligned} u &= c(\hat{w}) + K(\hat{x} - \pi(\hat{w})), \\ \dot{\hat{x}} &= f(\hat{x}, u, \hat{w}) + L_x(\hat{y} - y), \\ \dot{\hat{w}} &= s(\hat{w}) + L_w(\hat{y} - y), \\ \hat{y} &= h_m(\hat{x}, \hat{w}), \end{aligned} \tag{5.70}$$

where the matrices K and $L = [L_x^T, L_w^T]^T$ are such that the matrix functions $A(\chi) + B(\chi)K$ and

$$\begin{bmatrix} A(\chi) & E(\chi) \\ 0 & S(\chi) \end{bmatrix} + L[C(\chi) \ H(\chi)]$$

are quadratically stable over $\chi \in \mathbb{R}^{n+k+m}$.

Proof: Notice that controller (5.70) has the generalized internal model property. Namely, for every solution of the exosystem $w(t)$ lying in $\Omega(\mathcal{W})$, for the input $\bar{y}_w(t) = h_m(\pi(w(t)), w(t))$ system (5.70) has the solution $(\hat{x}(t), \hat{w}(t)) = (\pi(w(t)), w(t))$. This solution is bounded for all $t \in \mathbb{R}$ and for this solution the output of the controller equals $u = c(w(t))$. So, controller (5.70) indeed has the generalized internal model property. Moreover, system (5.1) in closed loop with controller (5.70) is input-to-state convergent. The proof of this part is identical to the proof of Theorem 5.10 since the observer error dynamics are globally exponentially stable. Therefore, this controller solves the global (forward time) uniform output regulation problem (see Section 5.1 for details). $\square$

Remark. As follows from Theorems 4.16 and 4.20, continuity of the mappings $\pi(w)$ and $c(w)$ is a necessary condition for the solvability of the global (forward time) uniform output regulation problem. In Theorem 5.17, the functions $\pi(w)$ and $c(w)$ are required to be locally Lipschitz. This additional requirement guarantees uniqueness of solutions of the closed-loop system. The requirement that $\pi(w)$ and $c(w)$ are globally defined is not very restrictive, since in many cases $\pi(w)$ and $c(w)$ can be extended from a neighborhood of $\Omega(\mathcal{W})$ ($\Omega(\mathcal{W}_+)$ in the case of the forward time variant of the problem) to the whole space $\mathbb{R}^m$.$\lhd$

5.4.3 Controller design for Lur'e systems

The conditions for controller design presented in the previous section become easily checkable when system (5.1)–(5.3) is a system with a scalar nonlinearity depending on an output. So, in this section we consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu + D\varphi(\zeta) + Ew, \\ \zeta &= C_\zeta x + H_\zeta w, \\ e &= C_r x + H_r w, \\ y &= Cx + Hw, \end{aligned} \tag{5.71}$$

with state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^k$, auxiliary output $\zeta \in \mathbb{R}$, regulated output $e \in \mathbb{R}^{l_r}$, and measured output $y \in \mathbb{R}^{l_m}$. The nonlinearity $\varphi(\zeta)$ is scalar and assumed to belong to the class $\mathcal{F}_\gamma$. We assume that the exogenous signal $w(t) \in \mathbb{R}^m$ is generated by the linear exosystem

$$\dot{w} = Sw, \tag{5.72}$$

where S is such that all its eigenvalues are simple and lie on the imaginary axis. This exosystem generates constant signals and harmonic signals at a fixed finite set of frequencies. Without loss of generality, we assume that S is skew-symmetric. It can easily be checked that in this case the exosystem satisfies the boundedness assumption **A1** in any ball $\mathcal{W}_r = \{w \in \mathbb{R}^m : \ |w| < r\}$. Moreover, $\Omega(\mathcal{W}_r) = \mathcal{W}_r$. In the following we assume that solutions of the exosystem (5.72) start in a ball $\mathcal{W}_r$ for some $r \in (0, +\infty]$. Thus, we are dealing with the regular variant of the global uniform output regulation problem. The regulator equations, in this case, have the form

$$\begin{aligned} \frac{d}{dt}\pi(w(t)) &= A\pi(w) + Bc(w) + D\varphi(C_\zeta \pi(w) + H_\zeta w) + Ew, \\ 0 &= C_r \pi(w) + H_r w, \end{aligned} \tag{5.73}$$

for all solutions of exosystem (5.72) lying in $\mathcal{W}_r$. Denote $A^-_{c\gamma} := A - \gamma D C_\zeta$, $A^+_{c\gamma} := A + \gamma D C_\zeta$. First, let us consider the static state feedback case with the states x and w being available for measurement, i.e., $y = (x, w)$.

Theorem 5.18. *Consider system (5.71) and exosystem (5.72) with $y = (x, w)$ and the nonlinearity $\varphi \in \mathcal{F}_\gamma$. Suppose the regulator equations (5.73) are solvable and the mappings $\pi(w)$ and $c(w)$ are globally defined continuous mappings. If the LMI*

$$\begin{aligned} A^+_{c\gamma}\mathcal{P}_c + \mathcal{P}_c(A^+_{c\gamma})^T + B\mathcal{Y} + \mathcal{Y}^T B^T &< 0, \\ A^-_{c\gamma}\mathcal{P}_c + \mathcal{P}_c(A^-_{c\gamma})^T + B\mathcal{Y} + \mathcal{Y}^T B^T &< 0, \\ \mathcal{P}_c = \mathcal{P}_c^T &> 0, \end{aligned} \tag{5.74}$$

is feasible, then the global uniform output regulation problem is solved by a controller of the form

$$u = c(w) + K(x - \pi(w)), \quad K := \mathcal{Y}\mathcal{P}_c^{-1}, \tag{5.75}$$

where $\mathcal{P}_c$ and $\mathcal{Y}$ satisfy (5.74).

Proof: This theorem is a corollary of Theorem 5.16 and Lemma 5.12. □

Next, we consider the case when only the output y is available for feedback. At this point, we will need the following notation: $\mathcal{C} := [C \;\; H]$,

$$A^-_{o\gamma} := \begin{bmatrix} A - \gamma D C_\zeta & E - \gamma D H_\zeta \\ 0 & S \end{bmatrix},$$

$$A^+_{o\gamma} := \begin{bmatrix} A + \gamma D C_\zeta & E + \gamma D H_\zeta \\ 0 & S \end{bmatrix}.$$

The following theorem provides conditions for output feedback controller design for Lur'e systems.

Theorem 5.19. *Consider system (5.71) with the nonlinearity $\varphi \in \mathcal{F}_\gamma$ and exosystem (5.72). Suppose the regulator equations (5.73) are solvable with the solutions $\pi(w)$ and $c(w)$ being globally defined locally Lipschitz mappings. If the LMI (5.74) and the LMI*

$$\begin{aligned} \mathcal{P}_o A_{o\gamma}^{+} + (A_{o\gamma}^{+})^T \mathcal{P}_o + \mathcal{X}\mathcal{C} + \mathcal{C}^T \mathcal{X}^T &< 0, \\ \mathcal{P}_o A_{o\gamma}^{-} + (A_{o\gamma}^{-})^T \mathcal{P}_o + \mathcal{X}\mathcal{C} + \mathcal{C}^T \mathcal{X}^T &< 0, \\ \mathcal{P}_o = \mathcal{P}_o^T &> 0, \end{aligned} \tag{5.76}$$

are feasible, then the global uniform output regulation problem is solved by a controller of the form

$$u = c(\hat{w}) + K(\hat{x} - \pi(\hat{w})), \tag{5.77}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + D\varphi(\hat{\zeta}) + E\hat{w} + L_x(\hat{y} - y), \tag{5.78}$$

$$\dot{\hat{w}} = S\hat{w} + L_w(\hat{y} - y), \tag{5.79}$$

$$\hat{\zeta} = C_\zeta \hat{x} + H_\zeta \hat{w}, \quad \hat{y} = C\hat{x} + H\hat{w}, \tag{5.80}$$

with $K = \mathcal{Y}\mathcal{P}_c^{-1}$*, where* $\mathcal{P}_c$ *and* $\mathcal{Y}$ *satisfy (5.74), and* $L = [L_x^T, L_w^T]^T = \mathcal{P}_o^{-1}\mathcal{X}$*, where* $\mathcal{P}_o$ *and* $\mathcal{X}$ *satisfy (5.76).*

Proof: This theorem is a corollary of Theorem 5.17 and Lemma 5.12. □

Remark. If ζ is measured, then the condition on feasibility of the LMI (5.76) can be relaxed by demanding that the pair of matrices

$$\begin{bmatrix} A & E \\ 0 & S \end{bmatrix}, \quad [C \ H],$$

is detectable. Under this condition the observer (5.78)–(5.80) can be replaced by the observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + D\varphi(\zeta) + E\hat{w} + L_1(\hat{y} - y), \\ \dot{\hat{w}} &= S\hat{w} + L_2(\hat{y} - y), \\ \hat{y} &= C\hat{x} + H\hat{w}, \end{aligned} \tag{5.81}$$

where $L := [L_1^T \ L_2^T]^T$ is taken such that the matrix

$$\begin{bmatrix} A & E \\ 0 & S \end{bmatrix} + L[C \ H]$$

is Hurwitz. Observer (5.81) has linear exponentially stable estimation error dynamics.◁

Let us illustrate the controller design presented in the last theorem with an example.

Example 5.20. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3 - x_2 + \sin(x_2), \\ \dot{x}_3 &= u, \\ e &= y = x_1 - w_1, \end{aligned} \tag{5.82}$$

and the exosystem

$$\dot{w}_1 = w_2, \tag{5.83}$$

$$\dot{w}_2 = -w_1. \tag{5.84}$$

The corresponding regulator equations admit the solution $\pi_1(w) = w_1$, $\pi_2(w) = w_2$, $\pi_3(w) = w_2 - w_1 - \sin(w_2)$, $c(w) = -w_1 - w_2 + w_1\cos(w_2)$ (see Example 5.1). The mappings $\pi(w)$ and $c(w)$ are globally defined and continuously differentiable. Let us apply Theorem 5.19. In our case,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$B = [0\,0\,1]^T$, $E \equiv 0$, $C_r = C = [1\,0\,0]$, $H_r = H = [-1\,0]$, $\zeta = x_2$, $C_\zeta = [0, 1, 0]$, $H_\zeta = 0$, and $\varphi(\zeta) = \sin(\zeta) \in \mathcal{F}_1$. Denote

$$A_c^- := \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_c^+ := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}_o^- := \begin{bmatrix} A_c^- & E \\ 0 & S \end{bmatrix}, \quad \mathcal{A}_o^+ := \begin{bmatrix} A_c^+ & E \\ 0 & S \end{bmatrix},$$

$\mathcal{C} := [C\ H]$. Numerical computations show that both LMIs (5.74) and (5.76) are feasible and, for example, the matrices $K = [-6\ -11,\ -6]^T$ and $L = [-153,\ -78,\ -13,\ -132,\ 52]$ can be used in controller (5.77)–(5.80).

Thus, all conditions of Theorem 5.19 are satisfied. By this theorem, controller (5.77)–(5.80) with the system matrices, mappings $\pi(w)$, $c(w)$ and controller parameters K, L specified above solves the global uniform output regulation problem for $\mathcal{W} = \mathbb{R}^2$. First, we perform simulations for the following initial conditions: $x(0) = (1, 2, 0)^T$, $w(0) = (1, 0)^T$, $\hat{x}(0) = 0$, $\hat{w}(0) = 0$. The results of these simulations are presented in Figures 5.1–5.4. Figure 5.1 shows the variable $x_1(t)$ and the external signal $w_1(t)$. The regulated output $e(t)$ is presented in Figure 5.2. Figure 5.3 shows the control $u(t)$ and in Figure 5.4 the states of the closed-loop system are presented. Further, we present simulation results for various initial conditions of the closed-loop system and the exosystem. The regulated output corresponding to these simulations is presented in Figure 5.5.$\triangleleft$

As can be seen from Theorems 5.18 and 5.19, the proposed controllers require accurate knowledge of the system model and the mappings $\pi(w)$ and $c(w)$. In practice, both the system model and the mappings $\pi(w)$ and $c(w)$ may not be known exactly or they may change if certain system parameters are varied. This raises the problem of robust controller design to cope with the uncertainties. This problem is addressed in the next section.

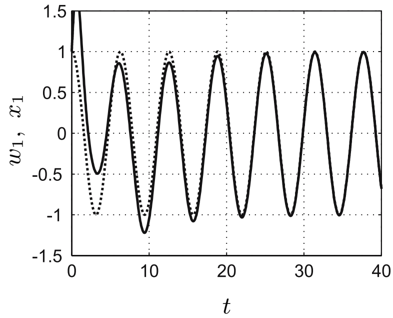

Fig. 5.1. Reference signal $w_1(t)$ (dotted) and $x_1(t)$ (solid).

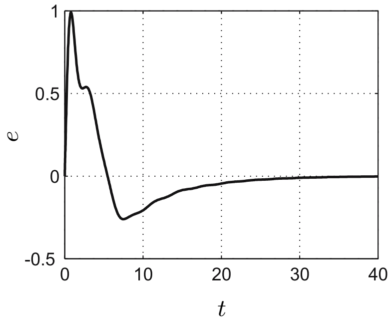

Fig. 5.2. Regulated output $e(t)$.

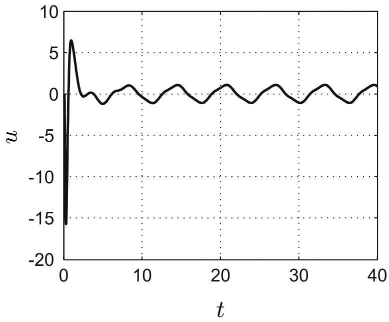

Fig. 5.3. Control $u(t)$.

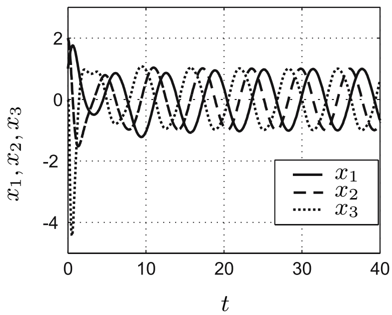

Fig. 5.4. System state $x(t)$.

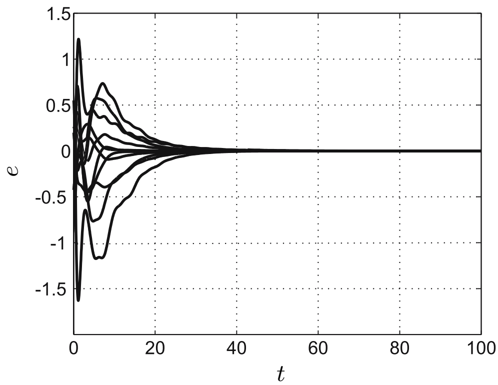

Fig. 5.5. The regulated output $e(t)$ corresponding to various initial conditions of the closed-loop system and the exosystem.

5.4.4 Robust controller design for Lur'e systems

In this section we design a controller that solves the global uniform output regulation problem not only for the nominal parameters of system (5.71), but also for the parameters from some neighborhood of the nominal ones and for all nonlinearities $\varphi \in \mathcal{F}_\gamma$ satisfying the additional condition $\varphi(0) = 0$. The exosystem is assumed to be a linear harmonic oscillator given by (5.72). To design a robust controller, we make the following assumptions.

R1 There exist matrices $\Lambda \in \mathbb{R}^{l_r \times l_m}$ and $\Psi \in \mathbb{R}^{1 \times l_m}$ such that $e = \Lambda y$ and $\zeta = \Psi y$.
R2 Both y and u are of the same dimension.

At this point, instead of solving the robust output regulation problem for system (5.71), we will solve it for the system

$$\begin{aligned} \dot{x} &= Ax + Bu + D\varphi(\Psi y) + Ew, \\ \bar{e} &= y = Cx + Hw, \end{aligned} \tag{5.85}$$

with the new regulated output $\bar{e}$. Obviously, since the original regulated output e is a linear function of $\bar{e}$, by solving the problem for system (5.85), we also solve it for the original system. The nominal parameters of system (5.85) are denoted by A°, B°, C°, D°.

Before presenting results on controller design for system (5.85), let us consider the case of system (5.85) without nonlinearity $\varphi(\zeta)$, i.e., the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew, \\ \bar{e} &= Cx + Hw. \end{aligned} \tag{5.86}$$

It is said that controller

$$\begin{aligned} \dot{\xi} &= G\xi + M\bar{e}, \\ u &= N_\xi \xi + N_y \bar{e}, \end{aligned} \tag{5.87}$$

solves the linear robust output regulation problem for system (5.86) and exosystem (5.72), if for all matrices A, B, and C close enough to their nominal values and for all matrices E and H, the closed-loop system (5.86), (5.87) with $w = 0$ is asymptotically stable, and for any solution of the closed-loop system (5.86), (5.87) and exosystem (5.72) it holds that $\bar{e}(t) \to 0$ as $t \to +\infty$.

It is known (see, e.g., [8]) that if controller (5.87) solves the linear robust output regulation problem, then for matrices A, B, and C close enough to their nominal values and for arbitrary matrices E and H there exist matrices $\Pi \in \mathbb{R}^{n \times m}$, $\Gamma \in \mathbb{R}^{k \times m}$, and $\Sigma \in \mathbb{R}^{q \times m}$ satisfying the following matrix equation:

$$\Pi S = A\Pi + B\Gamma + E, \tag{5.88}$$

$$0 = C\Pi + H, \tag{5.89}$$

$$\Sigma S = G\Sigma, \quad \Gamma = N_\xi \Sigma. \tag{5.90}$$

Equations (5.88) and (5.89) are the regulator equations for the linear case. This can be seen from the fact that if (5.88), (5.89) are postmultiplied by w, then they transform to the regulator equations with $\pi(w) = \Pi w$ and $c(w) = \Gamma w$. Equation (5.90) is a linear counterpart of the generalized internal model property of controller (5.87). Namely, if (5.90) is postmultiplied by w, then the resulting equation means that controller (5.87) with the input $\bar{y}_w(t) = \bar{e}_w(t) = C\Pi w(t) + Hw(t) \equiv 0$ (due to (5.89)) has a solution $\bar{\xi}_w(t) = \Sigma w(t)$, which is defined and bounded on $\mathbb{R}$ (because $w(t)$ is defined and bounded on $\mathbb{R}$), and along this solution $\bar{\xi}_w(t)$, the output of the controller equals $\bar{u}_w(t) = N_\xi \Sigma w(t) = \Gamma w(t) = c(w(t))$. This means that the input $\bar{e}_w(t) \equiv 0$ induces the output $\bar{u}_w(t) = \Gamma w(t)$ in controller (5.87).

After this intermezzo on the linear output regulation problem, we can formulate a technical result related to robust controller design for the global uniform output regulation problem for system (5.85) and exosystem (5.72).

Lemma 5.21. *Suppose controller (5.87) is such that*

(i) *it solves the linear robust output regulation problem for system (5.86) and exosystem (5.72);*
(ii) *for $w \equiv 0$, the transfer function $W^\circ_{\varphi\zeta}(s)$ of the closed-loop system with the nominal parameters*

$$\begin{aligned}
\dot{x} &= A^\circ x + B^\circ(N_\xi \xi + N_y C^\circ x) + D^\circ \varphi,\\
\dot{\xi} &= G\xi + MC^\circ x,\\
\zeta &= \Psi C^\circ x,
\end{aligned} \tag{5.91}$$

from input φ to output ζ satisfies $\|W^\circ_{\varphi\zeta}\|_\infty < 1/\gamma$.

Then controller (5.87) solves the global unform output regulation problem for system (5.85) and exosystem (5.72) for all matrices E and H, all nonlinearities $\varphi \in \mathcal{F}_\gamma$ satisfying $\varphi(0) = 0$, and for all matrices A, B, C, and D being close enough to their nominal values.

Proof: For all matrices A, B, C, and D being close enough to their nominal values, for all matrices E and H and for all nonlinearities $\varphi \in \mathcal{F}_\gamma$, system (5.85) in closed loop with (5.87) is input-to-state convergent. This fact follows from Lemma 5.14. Since controller (5.87) also solves the linear robust output regulation problem for system (5.86) and exosystem (5.72), for all matrices A, B, C, and D being close enough to their nominal values and for all matrices E and H there exist matrices Π, Γ and Σ satisfying (5.88)–(5.90). Since $y = \bar{e} = 0$ yields $\varphi(\Psi y) = 0$, (5.88)–(5.89) imply that the mappings $\pi(w) := \Pi w$ and $c(w) := \Gamma w$ are solutions to the regulator equations (5.73). Just like in the case of the linear output regulation problem for system (5.86) and exosystem (5.72), (5.90) implies that for any solution of the exosystem $w(t)$ the input $\bar{y}_w(t) = \bar{e}_w(t) \equiv 0$ induces the output $\bar{u}_w(t) = \Gamma w(t)$ in controller (5.87). By Theorem 4.16, controller (5.87) solves the global uniform output regulation problem for all A, B, C, and D close enough to their nominal values, for all E and H and for all $\varphi \in \mathcal{F}_\gamma$ satisfying $\varphi(0) = 0$. □

Remark. The problem of finding a controller that satisfies conditions (i) and (ii) in Lemma 5.21 has been solved in [1]. Yet, careful examination shows that the conditions under which the problem has been solved in [1] are not satisfied in our case. In particular, in [1] it is required that system (5.85) with input u and output $\bar{e}$ has relative degree zero, i.e., that u is directly present in the output $\bar{e}$. In our case, this condition is not satisfied. So, we proceed with our own controller design.◁

Necessary and sufficient conditions for solvability of the linear robust output regulation problem for linear system (5.86) and exosystem (5.72) are given by the following condition [8]:

R3 The pair (A°, B°) is stabilizable, the pair (A°, C°) is detectable, and for every λ being an eigenvalue of the matrix S the matrix

$$\begin{bmatrix} A^\circ - \lambda I & B^\circ \\ C^\circ & 0 \end{bmatrix}$$

has full row rank.

We assume that condition **R3** is satisfied and proceed with a design of a robust regulator. The design closely follows the design of a robust controller for the linear robust output regulation problem (see, e.g., [8]). Let S_{min} be a $p \times p$ matrix whose characteristic polynomial coincides with the minimal polynomial of S. Construct a block-diagonal $kp \times kp$ matrix Φ that has k blocks S_{min} on its diagonal, where k is the number of inputs (see Assumption **R2**). Choose a $kp \times k$ matrix N and a $k \times kq$ matrix Γ such that (Φ, Γ) is controllable and (Φ, N) is observable. Consider the augmented system

$$\begin{aligned} \dot{x} &= A^\circ x + B^\circ \Gamma \xi_1 + B^\circ v + D^\circ \varphi, \\ \dot{\xi}_1 &= \Phi \xi_1 + N C^\circ x, \\ \zeta &= \Psi C^\circ x. \end{aligned} \tag{5.92}$$

Next, we need to find a controller

$$\begin{aligned} \dot{\xi}_2 &= K\xi_2 + LC^\circ x, \\ v &= M\xi_2 + RC^\circ x, \end{aligned} \tag{5.93}$$

such that system (5.92) in closed loop with this controller is asymptotically stable for $\varphi = 0$ and the transfer function $W^\circ_{\varphi\zeta}(s)$ from input φ to output ζ satisfies $\|W^\circ_{\varphi\zeta}\|_\infty < 1/\gamma$. As follows from the linear regulator theory [8], the total controller

$$\begin{aligned} \dot{\xi}_1 &= \Phi\xi_1 + Ny, \\ \dot{\xi}_2 &= K\xi_2 + Ly, \\ u &= \Gamma\xi_1 + M\xi_2 + Ry, \end{aligned} \tag{5.94}$$

solves the linear robust output regulation problem for system (5.86) and exosystem (5.72). At the same time, the transfer function $W^{\circ}_{\varphi\zeta}(s)$ satisfies $\|W^{\circ}_{\varphi\zeta}\|_{\infty} < 1/\gamma$. Therefore, by Lemma 5.21 controller (5.94) solves the global uniform output regulation problem for system (5.85) and exosystem (5.72) for all matrices A, B, C, and D close enough to their nominal values, for all E and H, and for all $\varphi \in \mathcal{F}_{\gamma}$ satisfying $\varphi(0) = 0$.

The problem of finding a controller (5.93) that guarantees $\|W^{\circ}_{\varphi\zeta}\|_{\infty} < 1/\gamma$ is a standard problem in H_{∞} optimization, for which efficient solvers are available, for example, in MATLAB. Notice that the proposed robust controller design follows the decomposition strategy from Section 5.2. First, we design a ξ_1-subsystem with the generalized internal model property and then we find a ξ_2-subsystem that makes the overall closed-loop system input-to-state convergent. Let us illustrate the proposed robust controller design with an example.

Example 5.22. Consider system (5.71) with the nominal system matrices

$$A^{\circ} = \begin{bmatrix} 1 & -2 & 0 \\ 40 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}, \quad B^{\circ} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad D^{\circ} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C^{\circ} = [1, 0, 0].$$

The exosignal w is generated by the exosystem

$$\begin{aligned} \dot{w}_1 &= w_2, \\ \dot{w}_2 &= -w_1. \end{aligned} \tag{5.95}$$

The outputs of the system are equal: $\zeta = e = y = Cx + Hw$. The matrices E and H can be chosen arbitrarily. The value γ for the class of nonlinearities $\mathcal{F}_{\gamma}$ is chosen $\gamma = 0.1$. Notice that with such a choice of system matrices assumptions **R1**–**R3** hold. Following the design procedure given above, we set

$$\Phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Gamma = [1 \ 0].$$

Next, we search for a controller (5.93) that would satisfy the inequality $\|W^{\circ}_{\varphi\zeta}\| < 1/\gamma$. Such controller is found using the MATLAB routine `hinflmi`. The obtained controller is validated by means of simulations. In the simulations the matrices A, B, C, and D are taken equal to their nominal values and the nonlinearity is chosen $\varphi(\zeta) = \gamma \sin(\zeta)$.

For the initial conditions $x(0) = [1, 2, 3]^T$, $\xi_1(0) = 0$, $\xi_2(0) = 0$, $w_1(0) = 1$, $w_2(0) = 0$, and for the matrices E and H

$$E := \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad H := [0, \ 1],$$

the simulation results are shown in Figures 5.6 and 5.7. Results of several simulations with randomly chosen matrices E and H and random initial conditions for the closed-loop system and exosystem are given in Figure 5.8.$\triangleleft$

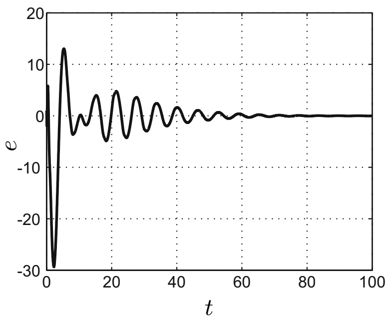

Fig. 5.6. Regulated output e.

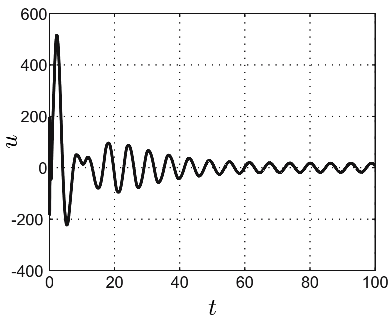

Fig. 5.7. Control u.

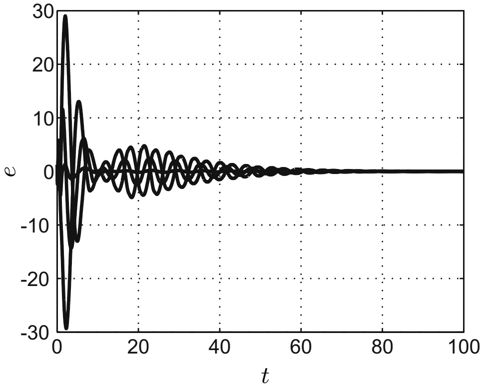

Fig. 5.8. Simulation results for various initial conditions and for various E and H.

5.5 Summary

In this chapter we have presented several controller design methods for solving the global (forward time) uniform output regulation problem. All designs are based on the assumption that the regulator equations are solvable and that the corresponding solutions of these equations are known. Under this assumption and under the assumption that the global uniform output regulation problem is solvable, a controller that solves the problem can be decomposed into two parts:

- the first part guarantees the generalized internal model property of the controller, which is necessary for the output regulation to occur;
- the second part guarantees that the overall closed-loop system is input-to-state convergent.

In Section 5.2 we have discussed certain ways how to design the first part of the controller. The problem of making a system input-to-state convergent by means of feedback has been discussed in more detail in Section 5.3. We have presented controller design methods based on backstepping, quadratic stability, and H_∞ optimization methods. When being applied to Lur'e systems, these methods are formulated in a simple and easily verifiable format. Although these controller design methods have been developed in the scope of the output regulation problem, they can be used independently for different nonlinear control problems, e.g., for the problem of tracking arbitrary time-varying reference signals and for the nonlinear observer design problem.

With the design tools on making a closed-loop system input-to-state convergent at our disposal, we have presented controller design methods for the global uniform (forward time) output regulation problem. Under the assumption that the states of the system and the exosystem are measured, a state feedback controller design has been presented. If only some output is available for measurements, we have shown how to design an observer-based output feedback controller. These two controller designs are based on the quadratic stability approach. For Lur'e systems, the conditions that need to be satisfied for such controller designs to be feasible reduce to checking the solvability of certain LMIs. For the case of a Lur'e system with uncertain parameters and an unknown nonlinearity from the class of nonlinearities with a bounded derivative, we have presented a robust controller design that copes with such uncertainties.

6

The local output regulation problem: convergence region estimates

In the previous chapter we presented several controller design methods providing solutions for the *global* uniform output regulation problem. These methods allow one to solve this problem for certain classes of nonlinear systems. If a system does not belong to one of these classes, it may happen that either the *global* uniform output regulation problem is not solvable, or it is solvable, but we do not know how to find a solution. At the same time, it may still be possible to find a controller that solves the *local* exponential output regulation problem. There are many results on controller design for the local exponential output regulation problem for different classes of systems (see, e.g., [8, 34, 38]). Despite the fact that the local output regulation problem is well studied, one question remained open: given a controller solving the exponential output regulation problem locally, in *some* neighborhood of the origin, how do we determine (or estimate) this neighborhood of admissible initial conditions? Without answering this question, solutions to the local exponential output regulation problem may not be satisfactory from an engineering point of view. In this chapter we address this estimation problem. In the first part of the chapter we consider this problem for the so-called *exact* variant of the local output regulation problem. In this variant, the regulated output tends to zero for all solutions of the closed-loop system and the exosystem starting close enough to the origin.

For certain systems it can be very difficult to find a controller that guarantees that the regulated output tends *exactly* to zero. At the same time, there are relatively simple design procedures for finding controllers guaranteeing that output regulation occurs *approximately*: for small initial conditions of the closed-loop system and the exosystem, the regulated output tends to small values with the order of magnitude determined by the controller design. The problem of estimating the sets of admissible initial conditions is also relevant for such an approximate local output regulation problem. This problem will be considered in the second part of the chapter.

We will consider only controllers solving the (approximate) local *exponential* output regulation problem, which in Chapter 3 was also referred to as the conventional local output regulation problem. For the sake of brevity, the word exponential in the name of the problem will be omitted, and by the local output regulation problem we will indicate the local exponential output regulation problem. This abbreviated name of the problem is also consistent with the name used in the literature.

6.1 Estimates for the local output regulation problem

6.1.1 Estimation problem statement

First, we recall the local output regulation problem (see also Section 3.3). Consider systems modeled by equations of the form

$$\dot{x} = f(x, u, w), \tag{6.1}$$

$$e = h_r(x, w), \tag{6.2}$$

$$y = h_m(x, w), \tag{6.3}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^k$, regulated output $e \in \mathbb{R}^{l_r}$, measured output $y \in \mathbb{R}^{l_m}$, and exogenous input $w \in R^m$ generated by the linear exosystem

$$\dot{w} = Sw. \tag{6.4}$$

The functions $f(x, u, w)$, $h_r(x, w)$, and $h_m(x, w)$ have continuous partial derivatives of some high order and satisfy $f(0, 0, 0) = 0$, $h_r(0, 0) = 0$, $h_m(0, 0) = 0$. It is assumed that exosystem (6.4) is *neutrally stable* (see Definition 3.1). The assumption of linearity of the exosystem is introduced to avoid unnecessary technical complications. All results presented below can be extended to the case of general neutrally stable exosystems. Due to the neutral stability assumption, the spectrum of S consists of eigenvalues on the imaginary axis with their geometric and algebraic multiplicity being equal. Without loss of generality, we assume that S is skew-symmetric, and therefore any solution of system (6.4) has the property $|w(t)| \equiv \text{Const}$. Notice that if the right-hand side of (6.1) depends on a vector p of unknown constant parameters, w and p can be united and treated together as an external signal (w, p) generated by an extended exosystem given by (6.4) and $\dot{p} = 0$. This extended exosystem also satisfies the neutral stability assumption. We assume that such an extension has already been made and that (6.4) corresponds to an extended exosystem.

The local output regulation problem is to find, if possible, a feedback of the form

$$\begin{aligned} \dot{\xi} &= \eta(\xi, y), \\ u &= \theta(\xi, y), \end{aligned} \tag{6.5}$$

with sufficiently smooth functions $\eta(\xi, y)$ and $\theta(\xi, y)$ satisfying $\eta(0, 0) = 0$, $\theta(0, 0) = 0$ such that

a) for $w(t) \equiv 0$ the closed-loop system

$$\dot{x} = f(x, \theta(\xi, h_m(x, w)), w), \tag{6.6}$$

$$\dot{\xi} = \eta(\xi, h_m(x, w)), \tag{6.7}$$

has an asymptotically stable linearization at the origin;

b) for every solution of the closed-loop system and exosystem (6.4) starting close enough to the origin $(x, \xi, w) = (0, 0, 0)$ it holds that $e(t) = h_r(x(t), w(t)) \to 0$ as $t \to +\infty$.

A controller solving the local output regulation problem makes the output $e(t)$ tend to zero at least for small initial conditions $(x(0), \xi(0), w(0))$. Without specifying the region of admissible initial conditions for which output regulation occurs, such a solution may not be satisfactory from an engineering point of view [70]. Thus, we come to the following **estimation problem:** *given the closed-loop system (6.6), (6.7) and the neutrally stable exosystem (6.4), estimate the region of admissible initial conditions for which the regulated output* $e(t) = h_r(x(t), w(t))$ *tends to zero.*

Denote $z := (x^T, \xi^T)^T \in \mathbb{R}^d$. Then the closed-loop system (6.6), (6.7) can be written as

$$\begin{aligned} \dot{z} &= F(z, w), \\ e &= \bar{h}_r(z, w) := h_r(x, w), \end{aligned} \tag{6.8}$$

where $F(z, w)$ is the right-hand side of (6.6), (6.7). As shown in Section 4.1 (see also [8, 38, 39]), the controller (6.5) solves the local output regulation problem if and only if the corresponding closed-loop system (6.8) satisfies the following conditions:

A) *the Jacobian matrix* $\frac{\partial F}{\partial z}(0, 0)$ *is Hurwitz;*

B) *there exists a* C^1 *mapping* $\alpha(w)$ *defined in a neighborhood* W *of the origin, with* $\alpha(0) = 0$*, such that*

$$\begin{aligned} \frac{\partial \alpha}{\partial w}(w) S w &= F(\alpha(w), w), \\ 0 &= \bar{h}_r(\alpha(w), w), \qquad \textit{for all} \;\; w \in W. \end{aligned} \tag{6.9}$$

We will give a solution to the estimation problem formulated above based on the functions $F(z, w)$ and $\alpha(w)$, which are found at the stage of controller design [8, 38]. To simplify the subsequent analysis, it is assumed that the closed-loop system (6.8) and the mapping $\alpha(w)$ are defined globally for all $z \in \mathbb{R}^d$ and $w \in \mathbb{R}^m$ (i.e., $W = \mathbb{R}^m$). If this assumption does not hold, one should restrict the subsequent results to the sets $Z \subset \mathbb{R}^d$ and $W \subset \mathbb{R}^m$ for which $F(z, w)$ and $\alpha(w)$ are well defined.

Before proceeding with solving the estimation problem, we discuss the main idea of the solution. First, we find two sets $\mathcal{C} \subseteq \mathbb{R}^d$ and $\mathcal{W}_c \subseteq \mathbb{R}^m$ having the following property: if $w(t) \in \mathcal{W}_c$ for $t \geq 0$, then any two solutions $z_1(t)$ and $z_2(t)$ of system (6.8) lying in $\mathcal{C}$ for all $t \geq 0$ converge to each other:

$|z_1(t) - z_2(t)| \to 0$ as $t \to \infty$. We call the set $\mathcal{C}$ a *convergence set* and the set $\mathcal{W}_c$ a *companion* of the set $\mathcal{C}$. Such sets exist, due to condition A). This condition implies that near the origin, for small $w(t)$, the closed-loop system (6.8) behaves like a linear asymptotically stable system and, in particular, all its solutions are exponentially stable (this statement will be made precise later on). Second, we find a set $\mathcal{Y} \subset \mathcal{C} \times \mathcal{W}_c$ of initial conditions $(z(0), w(0))$ such that any trajectory $(z(t), w(t))$ starting in this set satisfies the following conditions: $w(t) \in \mathcal{W}_c$, $\alpha(w(t)) \in \mathcal{C}$ and $z(t) \in \mathcal{C}$ for all $t \geq 0$. As follows from condition B), $\bar{z}_w(t) := \alpha(w(t))$ is a solution of system (6.8) along which $e(t) \equiv 0$. Thus, by the properties of $\mathcal{C}$ and $\mathcal{W}_c$, it holds that $z(t) \to \bar{z}_w(t) := \alpha(w(t))$ as $t \to +\infty$ and hence $e(t) = \bar{h}_r(z(t), w(t)) \to \bar{h}_r(\alpha(w(t)), w(t)) \equiv 0$. So, $\mathcal{Y}$ is an estimate of the set of admissible initial conditions $(z(0), w(0))$ for which output regulation occurs.

6.1.2 Convergence sets and the Demidovich condition

In this section we discuss how to find a convergence set $\mathcal{C}$ and its companion set $\mathcal{W}_c$ for the closed-loop system (6.8). As follows from Lemma 2.30, if a convex set $\mathcal{C} \subseteq \mathbb{R}^d$ and a set $\mathcal{W}_c \subseteq \mathbb{R}^m$ are such that the Demidovich condition

$$P\frac{\partial F}{\partial z}(z, w) + \frac{\partial F}{\partial z}^T(z, w)P \leq -Q, \quad \forall z \in \mathcal{C}, \quad w \in \mathcal{W}_c, \tag{6.10}$$

holds for some positive definite matrices $P = P^T$ and $Q = Q^T$, then there exists $\beta > 0$ such that

$$(z_1 - z_2)^T P(F(z_1, w) - F(z_2, w)) \leq -\beta(z_1 - z_2)^T P(z_1 - z_2) \tag{6.11}$$

for all $z_1, z_2 \in \mathcal{C}$ and for any $w \in \mathcal{W}_c$. The number $\beta > 0$ depends only on the matrices P and Q. Consider the function $V(z_1, z_2) = 1/2|z_1 - z_2|_P^2$, where $|z|_P$ denotes $|z|_P := (z^T P z)^{1/2}$. Inequality (6.11) implies that for any piecewise-continuous input $w(t)$ satisfying $w(t) \in \mathcal{W}_c$ for $t \geq t_0$ and for any two solutions $z_1(t)$ and $z_2(t)$ of system (6.8) corresponding to this input and satisfying $z_1(t) \in \mathcal{C}$ and $z_2(t) \in \mathcal{C}$ for all $t \geq t_0$, it holds that

$$\frac{d}{dt}V(z_1(t), z_2(t)) \leq -2\beta V(z_1(t), z_2(t)). \tag{6.12}$$

This, in turn, implies that $z_1(t)$ exponentially tends to $z_2(t)$:

$$|z_1(t) - z_2(t)|_P \leq \mathrm{e}^{-\beta(t-t_0)}|z_1(t_0) - z_2(t_0)|_P.$$

Consequently, we see that a convex set $\mathcal{C}$ and a set $\mathcal{W}_c$ for which the Demidovich condition (6.10) is satisfied represent a convergence set and its companion, respectively.

Another consequence of inequality (6.12) is that if $\bar{z}(t)$ is a solution of system (6.8) and the ellipsoid $\mathcal{E}_P(\bar{z}(t), r) := \{z : |z - \bar{z}(t)|_P < r\}$ is contained in $\mathcal{C}$ for all $t \geq 0$, then $\mathcal{E}_P(\bar{z}(t), r)$ is invariant. This observation results in the following corollary.

Corollary 6.1. *Suppose $\mathcal{C}$ and $\mathcal{W}_c$ satisfy the Demidovich condition (6.10). Let $w(t) \in \mathcal{W}_c$ for all $t \geq 0$ and $\bar{z}(t)$ be a solution of (6.8) such that $\bar{z}(t) \in \mathcal{C}$ for all $t \geq 0$. If the ellipsoid $\mathcal{E}_P(\bar{z}(t), r)$ is contained in $\mathcal{C}$ for all $t \geq 0$, then any solution of (6.8) starting in $z(0) \in \mathcal{E}_P(\bar{z}(0), r)$ exponentially tends to $\bar{z}(t)$.*

To solve the estimation problem stated in Section 6.1.1, we need to find sets $\mathcal{C}$ and $\mathcal{W}_c$ satisfying the Demidovich condition (6.10) for some $P = P^T > 0$ and $Q = Q^T > 0$. To reduce the number of arbitrary parameters in the Demidovich condition, we rewrite it in the form

$$\sup_{z \in \mathcal{C},\ w \in \mathcal{W}_c} \Lambda\left(P\frac{\partial F}{\partial z}(z,w) + \frac{\partial F}{\partial z}^T(z,w)P\right) =: -a < 0, \tag{6.13}$$

where $\Lambda(\cdot)$ denotes the largest eigenvalue of a symmetric matrix. Condition (6.13) is equivalent to condition (6.10) for $Q := aI$ and it is more convenient for finding the sets $\mathcal{C}$ and $\mathcal{W}_c$. Since $\frac{\partial F}{\partial z}(0,0)$ is Hurwitz (this is the case due to condition A)), one can choose a matrix $P = P^T > 0$ satisfying the matrix inequality $P\frac{\partial F}{\partial z}(0,0) + \frac{\partial F}{\partial z}^T(0,0)P < 0$. By continuity, $P\frac{\partial F}{\partial z}(z,w) + \frac{\partial F}{\partial z}^T(z,w)P$ is negative definite at least for small z and w. Hence, the Demidovich condition (6.13) is satisfied for $\mathcal{C}(\mathcal{R}) := \{z : |z| < \mathcal{R}\}$ and $\mathcal{W}(\rho) := \{w : |w| < \rho\}$ for some small $\mathcal{R}$ and ρ. If $P\frac{\partial F}{\partial z}(z,w) + \frac{\partial F}{\partial z}^T(z,w)P$ depends only on part of the coordinates z, then the Demidovich condition is satisfied for $\mathcal{C}_N(\mathcal{R}) := \{z : |Nz| < \mathcal{R}\}$ and $\mathcal{W}_c(\rho) := \{w : |w| < \rho\}$, where the matrix N is such that Nz consists of the coordinates that are present in $P\frac{\partial F}{\partial z}(z,w) + \frac{\partial F}{\partial z}^T(z,w)P$. Having chosen the matrix N, the numbers ρ and $\mathcal{R}$ can be found numerically solving inequality (6.13) with $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(\rho)$, where $\mathcal{R}$ and ρ are the parameters to be found. In some simple cases such $\mathcal{R}$ and ρ can be found analytically.

6.1.3 Estimation results

Having found a convergence set $\mathcal{C}_N(\mathcal{R})$ and its companion $\mathcal{W}_c(\rho)$, we can solve the estimation problem stated in Section 6.1.1. Prior to formulating the solution, let us introduce the following function:

$$m_N(w_0) := \sup_{t \geq 0} |N\alpha(w(t, w_0))|, \tag{6.14}$$

where $w(t, w_0)$ is a solution of the exosystem (6.4) satisfying $w(0, w_0) = w_0$. The function $m_N(w_0)$ indicates whether $\alpha(w(t, w_0))$ lies in the set $\mathcal{C}_N(\mathcal{R})$: if $m_N(w_0) < \mathcal{R}$, then $\alpha(w(t, w_0)) \in \mathcal{C}_N(\mathcal{R})$ for all $t \geq 0$. Denote δ to be the smallest number such that the inequality $|Nz| \leq \delta |z|_P$ is satisfied for all $z \in \mathbb{R}^d$. The number δ can be found from the formula $\delta = \|NP^{-1/2}\|$, where $\|\cdot\|$ is the matrix norm induced by the vector norm $|\cdot|$. Indeed,

$$\delta = \sup_{|z|_P=1} |Nz| = \sup_{|P^{1/2}z|=1} |Nz| = \sup_{|\tilde{z}|=1} |NP^{-1/2}\tilde{z}| = \|NP^{-1/2}\|.$$

The following theorem gives an estimate of the set of admissible initial conditions in the form of a neighborhood of the output-zeroing manifold $z = \alpha(w)$.

Theorem 6.2. *Let the local output regulation problem be solved. Suppose the closed-loop system (6.8) satisfies the Demidovich condition (6.13) with* $\mathcal{C}_N(\mathcal{R}) := \{z : |Nz| < \mathcal{R}\}$ *and* $\mathcal{W}_c(\rho) := \{w : |w| < \rho\}$ *for some* $\mathcal{R} > 0$, $\rho > 0$ *and some matrix* N*. Then any trajectory* $(z(t), w(t))$ *of the closed-loop system (6.8) and the exosystem (6.4) starting in the set*

$$\mathcal{Y} := \left\{ (z_0, w_0) : |w_0| < \rho,\ m_N(w_0) < \mathcal{R},\ |z_0 - \alpha(w_0)|_P < \frac{1}{\delta}(\mathcal{R} - m_N(w_0)) \right\} \tag{6.15}$$

satisfies

$$|z(t) - \alpha(w(t))| \leq C\mathrm{e}^{-\beta t}|z(0) - \alpha(w(0))| \tag{6.16}$$

for some $\beta > 0$ *and* $C > 0$ *independent of* $z(0), w(0)$*, and*

$$e(t) = \bar{h}_r(z(t), w(t)) \to 0, \quad \text{as} \ \ t \to \infty.$$

Proof: We need to show that (6.16) holds for any solution $(z(t), w(t))$ that starts in $(z(0), w(0))$ satisfying the relations: $|w(0)| < \rho$, $m_N(w(0)) < \mathcal{R}$ and $z(0) \in \mathcal{E}_P(\alpha(w(0)), r)$, where $\mathcal{E}_P(\bar{z}, r) := \{z : |z - \bar{z}|_P < r\}$ and $r := (\mathcal{R} - m_N(w(0)))/\delta$. Due to the conditions on $(z(0), w(0))$ and the properties of the exosystem, $|w(t)| \equiv |w(0)| < \rho$ and the solution $\bar{z}_w(t) := \alpha(w(t))$ satisfies

$$|N\bar{z}_w(t)| \leq \sup_{t\geq 0} |N\alpha(w(t))| = m_N(w(0)) < \mathcal{R}.$$

Hence, $\bar{z}_w(t) \in C_N(\mathcal{R})$ and $w(t) \in \mathcal{W}_c(\rho)$ for all $t \geq 0$. Let us show that $\mathcal{E}_P(\bar{z}_w(t), r) \subset \mathcal{C}_N(\mathcal{R})$ for all $t \geq 0$. Suppose $z \in \mathcal{E}_P(\bar{z}_w(t), r)$ for some $t \geq 0$. Then

$$\begin{aligned} |Nz| \leq |N\bar{z}_w(t)| + |N(z - \bar{z}_w(t))| &\leq m_N(w(0)) + \delta|z - \bar{z}_w(t)|_P \\ &< m_N(w(0)) + \delta r = \mathcal{R}. \end{aligned}$$

Consequently, $\mathcal{E}_P(\bar{z}_w(t), r) \subset \mathcal{C}_N(\mathcal{R})$ for all $t \geq 0$. The sets $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(\rho)$ satisfy the Demidovich condition (6.10). By Corollary 6.1, we obtain (6.16). Finally, $e(t) = \bar{h}_r(z(t), w(t)) \to \bar{h}_r(\alpha(w(t)), w(t)) \equiv 0$ as $t \to +\infty$. □

The relation between the sets $\mathcal{Y}$, $\mathcal{C}_N(\mathcal{R})$, and $\mathcal{W}_c(\rho)$ is schematically shown in Figure 6.1. If we want the closed-loop system (6.8) and the exosystem (6.4) to start in the set $\mathcal{Y}$, we need to guarantee that, first, the exosystem starts in a point w_0 in the set $\mathcal{L} := \{w_0 : |w_0| < \rho, m_N(w_0) < \mathcal{R}\}$ and, second, that the closed-loop system (6.8) starts in the set $\mathcal{D}(w_0) := \{z_0 : (z_0, w_0) \in \mathcal{Y}\}$. As can be seen from Figure 6.2, the sets $\mathcal{D}(w_0)$ may be different for different

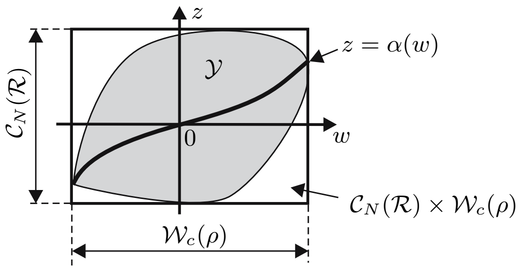

Fig. 6.1. Relation between the sets $\mathcal{Y}$, $\mathcal{C}_N(\mathcal{R})$, and $\mathcal{W}_c(\rho)$: $\mathcal{Y}$ is an invariant set inside $\mathcal{C}_N(\mathcal{R}) \times \mathcal{W}_c(\rho)$.

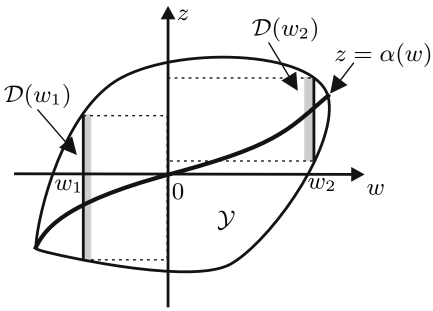

Fig. 6.2. The sets $\mathcal{Y}$ and $\mathcal{D}(w)$: for different w_1 and w_2, the sets $\mathcal{D}(w_1)$ and $\mathcal{D}(w_2)$ may be different.

values of w_0. Thus, the knowledge of w_0 is important. In practice, however, we may not know the exact value of w_0. For example, if the exosystem generates disturbances, then, knowing the level of disturbances, we can establish that $w_0 \in \mathcal{L}$, but still the exact value of w_0 is unknown. To cope with this difficulty, in the next result we find sets Z_0 and W_0 such that in whatever point $w_0 \in W_0$ the exosystem is initialized, output regulation will occur if the closed-loop system starts in $z_0 \in Z_0$. Prior to formulating the result, we define the functions

$$\sigma(r) := \sup_{|w| \le r} (|N\alpha(w)| + \delta|\alpha(w)|_P), \quad R(r) := (\mathcal{R} - \sigma(r))/\delta. \tag{6.17}$$

The function $\sigma(r)$ is nondecreasing and $\sigma(0) = 0$. Let $r_* > 0$ be the largest number such that $r_* \le \rho$ and $\sigma(r) < \mathcal{R}$ for all $r \in [0, r_*)$. Now, we can formulate the result.

Theorem 6.3. *The conclusion of Theorem 6.2 holds for any trajectory of the closed-loop system (6.8) and the exosystem (6.4) starting in*

$$z(0) \in E_P(R(r)) := \{z : |z|_P < R(r)\}, \quad w(0) \in B_w(r) := \{w : |w| < r\},$$

for some $r \in [0, r_*)$.

Proof: The proof of this theorem is based on the fact that for every $r \in [0, r_*)$ the set $E_P(R(r)) \times B_w(r)$ is a subset of $\mathcal{Y}$, as shown in Figure 6.3. If $E_P(R(r)) \times B_w(r) \subset \mathcal{Y}$ for any $r \in [0, r_*)$, then the statement of Theorem 6.3 follows from Theorem 6.2. Let us show $E_P(R(r)) \times B_w(r) \subset \mathcal{Y}$ for any $r \in [0, r_*)$. Suppose $z_0 \in E_P(R(r))$ and $w_0 \in B_w(r)$ for some fixed $r \in [0, r_*)$. According to the definition of $\mathcal{Y}$, we first need to show that $|w_0| < \rho$. This is true because $|w_0| < r < r_* \leq \rho$. Next, we show that $m_N(w_0) < \mathcal{R}$. By the definition of $\sigma(r)$, it holds that $|N\alpha(w)| \leq \sigma(r)$ for all $|w| \leq r$. The choice of $|w_0| < r$ implies $|w(t, w_0)| \equiv |w_0| < r$. Hence, by the definition of $m_N(w_0)$ we obtain

$$m_N(w_0) = \sup_{t \geq 0} |N\alpha(w(t, w_0))| \leq \sup_{|w|<r} |N\alpha(w)| \leq \sigma(r).$$

The choice of $r < r_*$ implies $\sigma(r) < \mathcal{R}$ and, consequently, $m_N(w_0) < \mathcal{R}$.

Next, we need to show that $|z_0 - \alpha(w_0)|_P < (\mathcal{R} - m_N(w_0))/\delta$. The triangle inequality implies

$$|z_0 - \alpha(w_0)|_P \leq |z_0|_P + |\alpha(w_0)|_P. \tag{6.18}$$

By the choice of z_0 and by the definition of $R(r)$, we obtain

$$\begin{aligned} |z_0|_P < R(r) = (\mathcal{R} - \sigma(r))/\delta &= (\mathcal{R} - \sup_{|w| \leq r} (|N\alpha(w)| + \delta|\alpha(w)|_P))/\delta \\ &\leq (\mathcal{R} - m_N(w_0))/\delta - |\alpha(w_0)|_P. \end{aligned}$$

Substituting this inequality in (6.18), we obtain

$$|z_0 - \alpha(w_0)|_P < \frac{1}{\delta}(\mathcal{R} - m_N(w_0)).$$

This completes the proof. □

The estimates presented in Theorem 6.3 are based on a pair of numbers $(\mathcal{R}, \rho)$ such that the Demidovich condition (6.13) is satisfied for the sets $\mathcal{C}_N(\mathcal{R})$

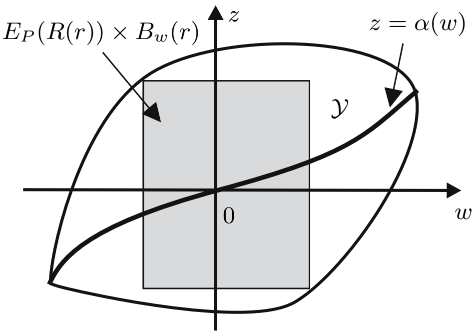

Fig. 6.3. Relation between the sets $\mathcal{Y}$, $E_P(R(r))$, and $B_w(r)$.

and $\mathcal{W}_c(\rho)$. In fact, there is a whole set of pairs $(\mathcal{R}, \rho)$ and the corresponding family of sets of $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(\rho)$ for which condition (6.13) is satisfied. For estimation purposes, among all such sets $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(\rho)$ we want to find the ones that are maximal in a certain sense. To this end, denote $\mathcal{R}_*(\rho)$ to be the largest number such that the Demidovich condition (6.13) is satisfied for all sets $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(\rho) = \{w : |w| < \rho\}$ with $\mathcal{R} \in [0, \mathcal{R}_*(\rho))$. One can easily check that $\mathcal{R}_*(\rho)$ is a nonincreasing function of ρ. This function can be found numerically. Having found $\mathcal{R}_*(\rho)$, we can enlarge the estimates presented in Theorem 6.3 by redefining $R(r)$ (see (6.17)) in the following way:

$$R(r) := (\mathcal{R}_*(r) - \sigma(r))/\delta. \tag{6.19}$$

In this case, the convergence of solutions to the output-zeroing manifold will be exponential, but the numbers $C > 0$ and $\beta > 0$ in (6.16) may depend on the initial conditions $(z(0), w(0))$.

6.1.4 Example: the TORA system

Let us illustrate the application of Theorem 6.3. Consider the so-called TORA system (transitional oscillator with a rotational actuator), which is shown in Figure 6.4. This system consists of a cart of mass M which is attached to a wall with a spring of stiffness k. The cart is affected by a disturbance force F_d. An arm of mass m rotates around the axis in the center of the cart. The center of mass of the arm CM is located at distance l from the rotational axis. J is the inertia of the arm with respect to the rotational axis. The arm is actuated by a control torque T_u. The cart and the arm move in the horizontal plane and, therefore, there is no effect of gravity. The horizontal displacement of the cart is denoted by e and the angular displacement of the arm is denoted by θ. This is a nonlinear mechanical benchmark system that was introduced in [85] (see also [45, 47]). This is an idealized system in which no friction, flexibility, motor dynamics, and so on are taken into account. A real TORA system will be used for experiments in Chapter 7.

The control problem is to find a control law for the torque T_u such that the horizontal displacement e tends to zero in the presence of the harmonic disturbance force F_d of known frequency, but unknown amplitude and phase. This is a particular case of the output regulation problem. We will find a controller solving this problem locally, i.e., for small initial conditions $e(0)$, $\dot{e}(0)$, $\theta(0)$, and $\dot{\theta}(0)$ and for disturbances with small amplitudes. After finding such a local controller, we will estimate the set of admissible initial conditions of the closed-loop system and admissible amplitudes of the disturbance force. These estimates will be found based on Theorem 6.3.

The equations of motion for the TORA system are given by [85]:

$$\begin{aligned} (M+m)\ddot{e} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + ke &= F_d, \\ J\ddot{\theta} + ml\ddot{e}\cos\theta &= T_u. \end{aligned} \tag{6.20}$$

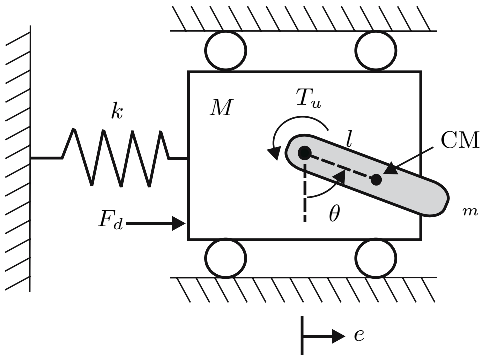

Fig. 6.4. The TORA system.

The disturbance force F_d can be considered as an output of the linear harmonic oscillator

$$\dot{w}_1 = \omega w_2, \quad \dot{w}_2 = -\omega w_1, \quad F_d = w_1. \tag{6.21}$$

For simplicity, we assume that e, $\dot{e}$, θ, $\dot{\theta}$, w_1, and w_2 are measured. All parameters of system (6.20) and exosystem (6.21) are known. To solve the local output regulation problem, we transform system (6.20) into a simpler form. After the time transformation $\tau := \sqrt{k/(m+M)}\, t$ and coordinate transformation

$$\begin{aligned}
x_1 &:= \sqrt{\frac{M+m}{J}} e + \phi \sin\theta, \quad \phi := \frac{ml}{\sqrt{J(M+m)}}, \\
x_2 &:= \sqrt{\frac{M+m}{J}} \frac{de}{d\tau} + \phi \frac{d\theta}{d\tau} \cos\theta, \\
x_3 &:= \theta, \quad x_4 := \frac{d\theta}{d\tau},
\end{aligned}$$

and feedback transformation

$$T_u = \frac{Jk}{M+m}((1-\phi^2\cos^2 x_3)v - \phi\cos x_3(x_1 - (1+x_4^2)\phi\sin x_3 - \mu F_d)),$$

where $\mu = \frac{1}{k}\sqrt{\frac{M+m}{J}}$ and v is a new control, system (6.20) takes the form (see [85] for details):

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + \phi \sin x_3 + \mu F_d, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= v, \\
e &= (x_1 - \phi \sin x_3)\sqrt{\tfrac{J}{M+m}}.
\end{aligned} \tag{6.22}$$

The exosystem (6.21) transforms into

$$\dot{w}_1 = \hat{\omega} w_2, \quad \dot{w}_2 = -\hat{\omega} w_1, \quad F_d = w_1, \tag{6.23}$$

where $\hat{\omega} := \sqrt{(m+M)/k}\ \omega$ is the new excitation frequency.

Let us solve the local output regulation problem for system (6.22) and exosystem (6.23). We will use the controller design method from [8]. According to this method, first we need to solve the regulator equations

$$\begin{aligned} \frac{d}{dt}\pi(w(t)) &= f(\pi(w), c(w), w), \\ 0 &= h_r(\pi(w), w), \end{aligned} \tag{6.24}$$

where the functions $f(x, v, w)$ and $h_r(x, w)$ correspond to (6.22). The mappings $\pi(w)$ and $c(w)$ satisfying (6.24) in some neighborhood of the origin are given by the formulas

$$\pi_1(w) := -\frac{\mu w_1}{\hat{\omega}^2}, \quad \pi_2(w) := -\frac{\mu w_2}{\hat{\omega}}, \quad \pi_3(w) := -\arcsin\left(\frac{\mu w_1}{\hat{\omega}^2 \phi}\right), \tag{6.25}$$

$$\pi_4(w) := -\frac{\mu \hat{\omega} w_2}{\sqrt{\hat{\omega}^4 \phi^2 - \mu^2 w_1^2}}, \quad c(w) := \frac{\mu \hat{\omega}^2 w_1 (\hat{\omega}^4 \phi^2 - \mu^2 (w_1^2 + w_2^2))}{\left(\sqrt{\hat{\omega}^4 \phi^2 - \mu^2 w_1^2}\right)^3}.$$

At the next step, choose a matrix K such that

$$\frac{\partial f}{\partial x}(0,0,0) + K \frac{\partial f}{\partial u}(0,0,0)$$

is a Hurwitz matrix. This is possible, because at the origin the pair of matrices $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u})$ is stabilizable. Then the controller $v = c(w) + K(x - \pi(w))$ solves the local output regulation problem. Indeed, it is easy to check that for this controller the closed-loop system satisfies conditions A) and B) given earlier with the mapping $\alpha(w) := \pi(w)$ and the function

$$F(x, w) := \begin{pmatrix} x_2 \\ -x_1 + \phi \sin x_3 + \mu w_1 \\ x_4 \\ c(w) + K(x - \pi(w)) \end{pmatrix}.$$

Since the controller is static, the state of the closed-loop system z coincides with the system state x. Having found a controller solving the output regulation problem in *some* neighborhood of the origin, let us estimate this set of admissible initial conditions $(x(0), w(0))$ for the following values of the parameters: $\phi = 0.5$, $\mu = 0.04$, $\hat{\omega} = 1$, $K = (12, -4, -8, -5)$. To this end, we apply Theorem 6.3.

First, we must choose a matrix $P = P^T > 0$ such that

$$P \frac{\partial F}{\partial x}(0,0) + \frac{\partial F}{\partial x}^T (0,0) P < 0.$$

We find such P from the Lyapunov equation

$$P\frac{\partial F}{\partial x}(0,0) + \frac{\partial F}{\partial x}^T(0,0)P = -Q,$$

where Q is the diagonal matrix $\mathrm{diag}(2, 8, 1, 1)$. The estimation procedure depends on the choice of the matrix P. Therefore, different choices of the matrix Q (and, consequently, of the matrix P) may result in different estimates. The particular value of the matrix Q presented above has been found by trial and error to obtain better (larger in some sense) estimates. For convenience, the matrix P corresponding to the chosen matrix Q is normalized such that $\|P\| = 1$. Since $\frac{\partial F}{\partial x}(x, w)$ depends only on x_3, the matrix N for the set $\mathcal{C}_N(\mathcal{R})$ is chosen equal to $N = (0, 0, 1, 0)$, i.e., such that $Nx = x_3$. So, the convergence set $\mathcal{C}$ is sought in the form $\mathcal{C}_N(\mathcal{R}) := \{x : |x_3| < \mathcal{R}\}$ (see Section 6.1.2 for details). Since $\frac{\partial F}{\partial x}(x, w)$ does not depend on w, the companion set $\mathcal{W}_c$ can be taken equal to $\mathbb{R}^m$ and $\mathcal{R}_*(\rho) \equiv \mathcal{R}_* = \mathrm{Const}$. In our example, the matrix

$$J(x, w) = P\frac{\partial F}{\partial x}(x, w) + \frac{\partial F}{\partial x}^T(x, w)P$$

depends only on x_3. The number $\mathcal{R}_*$ is the largest number such that $J(x_3)$ is negative definite for all $|x_3| < \mathcal{R}_*$. This $\mathcal{R}_*$ can be found numerically in the following way. We check whether $J(\mathcal{R})$ and $J(-\mathcal{R})$ are negative definite for increasing $\mathcal{R}$ starting with $\mathcal{R} = 0$ until one of the matrices $J(\mathcal{R})$ or $J(-\mathcal{R})$ ceases being negative definite. The step for the increment of $\mathcal{R}$ is taken equal to 0.01. The value of $\mathcal{R}$ at which one of the matrices $J(\mathcal{R})$ or $J(-\mathcal{R})$ ceases being negative definite is an approximate value of $\mathcal{R}_*$. This procedure gives $\mathcal{R}_* \approx 1.03$. Finally, computation of $R(r)$ using formula (6.19) gives us estimates of the admissible initial conditions set: $E_P(R(r)) \times B_w(r)$. The function $R(r)$ is shown in Figure 6.5. The apparent linearity of the graph of $R(r)$ can be explained by the fact that it corresponds to small values of the state variables, for which the nonlinearity present in the system does not play a significant role.

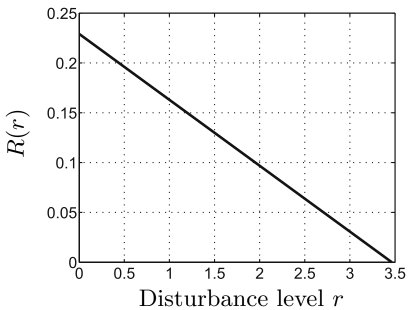

Fig. 6.5. $R(r)$ and r for the estimates $E_P(R(r)) \times B_w(r)$.

To compare the estimates with the actual set for which output regulation occurs, we perform simulations. First, we fix a disturbance level $r = 1$. The estimate of the set of admissible $x(0)$ corresponding to this level of disturbances is equal to $E_P(R_*(r))$. This is a four-dimensional set. To visualize it, we take a cross-section of this set for fixed $x_3 = 0$ and $x_4 = 0$. This cross-section is shown in Figure 6.6 by a solid ellipsoid. Then we compute the cross-section of the actual region of convergence corresponding to $x_3 = 0$, $x_4 = 0$, and the level of disturbances $r = 1$. In Figure 6.6, it is shown by the dotted ellipsoid. This cross-section is found by numerical integration of system (6.22) and exosystem (6.23) in the following way. A point (x_1, x_2) belongs to this cross-section if for the initial condition $x(0) = (x_1, x_2, 0, 0)^T$ output regulation occurs for all initial conditions of the exosystem satisfying $|w(0)| \leq r$. As can be seen from Figure 6.6, the estimates are fairly conservative. One possible reason for such conservativeness is a bad choice of the matrix P. A different choice of P may result in better estimates. At the moment, it is an open question how to choose P to obtain the best (in some sense) estimates. Another explanation for such conservative estimates is that the estimation procedure is based on quadratic Lyapunov functions analysis, which is conservative by itself when applied to nonlinear systems.

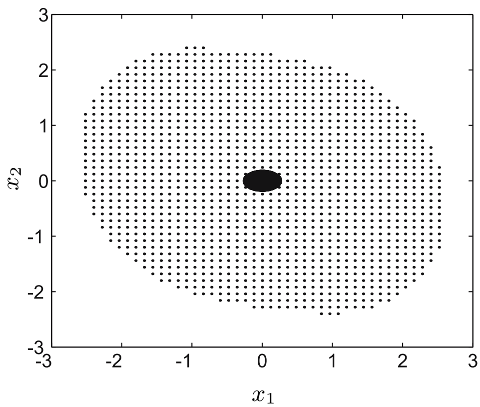

Fig. 6.6. The actual (dotted) and the estimated (solid) sets of admissible initial conditions: cross-section for $x_3 = 0$, $x_4 = 0$, and $|w_0| = 1$.

6.2 Estimates for the approximate local output regulation problem

Even though the local output regulation can be solvable, it can be extremely difficult to find a controller that solves it. Condition B) is the one that is especially difficult to satisfy. At the same time, in many cases it is easy to find a controller satisfying (6.9) in condition B) approximately (see [8, 33, 35]), i.e., satisfying the condition

B*) *there exists a* C^1 *mapping* $\tilde{\alpha}(w)$ *defined in a neighborhood* W *of the origin, with* $\tilde{\alpha}(0) = 0$, *such that*

$$\begin{aligned} \frac{\partial \tilde{\alpha}}{\partial w}(w) S w &= F(\tilde{\alpha}(w), w) + \varepsilon_1(w), \\ 0 &= \bar{h}_r(\tilde{\alpha}(w), w) + \varepsilon_2(w), \end{aligned} \tag{6.27}$$

for all $w \in W$, *where* $\varepsilon_1(w)$ *and* $\varepsilon_2(w)$ *are small (in some sense) continuous functions satisfying* $\varepsilon_1(0) = 0$ *and* $\varepsilon_2(0) = 0$.

It is known (see [35]) that if the closed-loop system satisfies conditions A) and B*), then for all sufficiently small initial conditions $z(0)$ and $w(0)$ the regulated output $e(t)$ converges to a function $\tilde{e}(w(t))$, where $\tilde{e}(w)$ is of the same order of magnitude as $\varepsilon_1(w)$ and $\varepsilon_2(w)$. It means that if for some $\nu \geq 1$, $C_1 > 0$ and $C_2 > 0$ it holds that $|\varepsilon_i(w)| \leq C_i |w|^\nu$, $i = 1, 2$, for all w in some neighborhood of the origin W, then there exists $C_e > 0$ such that $|\tilde{e}(w)| \leq C_e |w|^\nu$ for all $w \in W$. This is called approximate local output regulation. Since it is required that the initial conditions be sufficiently small, the problem of estimating this set of admissible initial conditions is also relevant in the case of such approximate output regulation [71]. This estimation problem can be solved using the same techniques as in the case of exact output regulation.

The main idea is to find a set of initial conditions $\tilde{\mathcal{Y}} \subset \mathcal{C} \times \mathcal{W}_c$ (where $\mathcal{C}$ and $\mathcal{W}_c$ satisfy the Demidovich condition) such that if $(z(0), w(0)) \in \tilde{\mathcal{Y}}$, then $z(t) \in \mathcal{C}$, $\tilde{\alpha}(w(t)) \in \mathcal{C}$ and $w(t) \in \mathcal{W}_c$ for all $t \geq 0$. As follows from (6.27), $\tilde{z}(t) := \tilde{\alpha}(w(t))$ can be considered as a solution of the perturbed system

$$\dot{z} = F(z, w(t)) + \varepsilon_1(w(t)) \tag{6.28}$$

and along this solution the regulated output equals $-\varepsilon_2(w(t))$. Since $\tilde{z}(t)$ is exponentially stable (because of the Demidovich condition), a small perturbation $\varepsilon_1(w(t))$ implies, in the limit, a small difference between $z(t)$ and $\tilde{\alpha}(w(t))$. Hence, $\tilde{\mathcal{Y}}$ is an estimate of the set of admissible initial conditions. Estimates in the form of direct product $\tilde{Z}_0 \times \tilde{W}_0$ can be found in a similar way as in Theorem 6.3. In the derivation of estimation results, we will need the following technical lemma.

Lemma 6.4. *Consider the closed-loop system*

$$\dot{z} = F(z, w(t)) \tag{6.29}$$

and the perturbed system

$$\dot{z} = F(z, w(t)) + \epsilon(t), \tag{6.30}$$

where $\epsilon(t)$ *is a continuous function of time. Suppose* $\mathcal{C}$ *and* $\mathcal{W}_c$ *are such that system (6.29) satisfies the Demidovich condition (6.13). Let* $w(t) \in \mathcal{W}_c$ *for* *all* $t \geq 0$*, and* $\bar{z}(t)$ *be a solution of (6.30) such that the ellipsoid* $\mathcal{E}_P(\bar{z}(t), r) := \{z : \ |z - \bar{z}(t)|_P \leq r\}$ *is contained in* $\mathcal{C}$ *for all* $t \geq 0$*. If the perturbation term satisfies* $|\epsilon(t)|_P \leq ar/(2\|P\|)$ *for* $t \geq 0$*, then any solution of the unperturbed system (6.29) starting in* $z(0) \in \mathcal{E}_P(\bar{z}(0), r)$ *satisfies*

$$\limsup_{t\to+\infty} |z(t) - \bar{z}(t)|_P \leq \frac{2\|P\|}{a} \limsup_{t\to+\infty} |\epsilon(t)|_P. \tag{6.31}$$

Proof: See Appendix 9.14.

6.2.1 Estimation results

Having found the sets $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(\mathcal{R})$ for which the closed-loop system (6.8) satisfies the Demidovich condition, we can solve the estimation problem for the approximate local output regulation problem. Prior to formulating the solution, let us introduce the following functions:

$$\tilde{m}_N(w_0) := \sup_{t\geq 0} |N\tilde{\alpha}(w(t, w_0))|, \quad q(w_0) := \sup_{t\geq 0} |\varepsilon_1(w(t, w_0))|_P. \tag{6.32}$$

The following theorem gives an estimate of the set of admissible initial conditions in the form of a neighborhood of the approximate output-zeroing manifold $z = \tilde{\alpha}(w)$.

Theorem 6.5. *Consider the closed-loop system (6.8) and the exosystem (6.4) satisfying conditions* A) *and* B*)*. Suppose the closed-loop system (6.8) satisfies the Demidovich condition (6.13) with* $\mathcal{C}_N(\mathcal{R}) := \{z : |Nz| < \mathcal{R}\}$ *and* $\mathcal{W}_c(\rho) := \{w : |w| < \rho\}$ *for some* $\mathcal{R} > 0$, $\rho > 0$ *and some matrix* N*. Then any trajectory* $(z(t), w(t))$ *of the closed-loop system (6.8) and the exosystem (6.4) starting in the set*

$$\begin{aligned}\tilde{\mathcal{Y}} := \Big\{(z_0, w_0) : |w_0| < \rho, \tilde{m}_N(w_0) + \frac{2\delta\|P\|}{a} q(w_0) < \mathcal{R}, \\ |z_0 - \tilde{\alpha}(w_0)|_P < \frac{1}{\delta}(\mathcal{R} - \tilde{m}_N(w_0))\Big\}\end{aligned} \tag{6.33}$$

satisfies

$$\limsup_{t\to+\infty} |z(t) - \tilde{\alpha}(w(t))|_P \leq \frac{2\|P\|}{a} \limsup_{t\to+\infty} |\varepsilon_1(w(t))|_P \tag{6.34}$$

and, consequently,

$$\limsup_{t\to+\infty} |e(t)| \leq \bar{C} \limsup_{t\to+\infty} |\varepsilon_1(w(t))|_P + \limsup_{t\to+\infty} |\varepsilon_2(w(t))|, \tag{6.35}$$

for some number $\bar{C} > 0$ *independent of the particular solution* $(z(t), w(t))$*.*

The proof of this theorem is very similar to the proof of Theorem 6.2. It is provided in Appendix 9.15.

The next theorem is a counterpart of Theorem 6.3. It provides estimates in the form of a direct product of two sets $Z_0 \times W_0$ such that in whatever point $w_0 \in W_0$ the exosystem is initialized, approximate output regulation will occur if the closed-loop system starts in $z_0 \in Z_0$. Prior to formulating the result, we define the functions

$$\tilde{\sigma}(r) := \sup_{|w_0|<r} (|N\tilde{\alpha}(w_0)| + \delta|\tilde{\alpha}(w_0)|_P), \qquad \tilde{R}(r) := (\mathcal{R} - \tilde{\sigma}(r))/\delta,$$
$$\eta(r) := \sup_{|w_0|<r} \left(|N\tilde{\alpha}(w_0)| + \frac{2\delta\|P\|}{a}|\varepsilon_1(w_0)|_P \right). \tag{6.36}$$

Let $r_* > 0$ be the largest number such that $r_* \leq \rho$, $\tilde{\sigma}(r) < \mathcal{R}$ and $\eta(r) < \mathcal{R}$ for all $r \in [0, r_*)$. The estimates for the sets of admissible $z(0)$ and $w(0)$ are given by the next theorem.

Theorem 6.6. *The conclusion of Theorem 6.5 holds for any trajectory of the closed-loop system (6.8) and the exosystem (6.4) starting in*

$$z(0) \in E_P(\tilde{R}(r)) := \{z : |z|_P < \tilde{R}(r)\}, \quad w(0) \in B_w(r) := \{w : |w| < r\},$$

for some $r \in [0, r_*)$.

Similar to the proof of Theorem 6.3, the proof of this theorem is based on the fact that for every $r \in [0, r_*)$ the set $E_P(\tilde{R}(r)) \times B_w(r)$ is a subset of $\tilde{\mathcal{Y}}$. The proof is given in Appendix 9.16.

6.2.2 Example

Let us illustrate the application of Theorem 6.6. Consider the local output regulation problem for the TORA system (6.22) (see Section 6.1.4). This time we assume that the disturbance force F_d equals

$$F_d = \lambda \arctan(w_1/\lambda)$$

for $\lambda = 3$, as shown in Figure 6.7. As in the previous example, w_1 is generated by the linear harmonic oscillator (6.23).

Although this problem looks similar to the problem considered in Section 6.1.4, it is difficult to solve. The difficulty is in solving the regulator equations (6.24). At the same time, for the controller

$$v = c(w) + K(x - \pi(w)),$$

with the mappings $\pi(w)$ and $c(w)$ defined in (6.25), the closed-loop system satisfies the conditions A) and B*) with

$$\tilde{\alpha}(w) := \pi(w), \quad \varepsilon_1(w) := (0, \mu(\lambda \arctan(w_1/\lambda) - w_1), 0, 0)^T, \quad \varepsilon_2(w) \equiv 0.$$

Therefore, this controller solves the approximate local output regulation problem, i.e., for small initial conditions of the closed-loop system and the exosystem the regulated output tends to small values, as shown in Figure 6.8 (see [8, 33, 35] for details on controller design for the approximate local output regulation problem). The values of the parameters ϕ, μ, ω, and K are taken the same as in Section 6.1.4 and $\lambda = 3$.

Let us apply Theorem 6.6 to estimate the set of admissible $(x(0), w(0))$. First, we need to find a pair of sets $\mathcal{C}_N(\mathcal{R})$ and $\mathcal{W}_c(r)$ satisfying the Demidovich condition (6.13). Since $\frac{\partial F}{\partial x}(x, w)$ does not depend on w, the companion set $\mathcal{W}_c$ can be taken equal to $\mathbb{R}^m$. For the convergence set $\mathcal{C}_N(\mathcal{R})$ we choose $\mathcal{R} = 0.88$. The corresponding a equals $a = 0.083$. We have chosen arbitrary $\mathcal{R}$ from the range of $\mathcal{R}$s for which the corresponding a is positive. Such a range has been determined numerically in the previous example and it equals $[0,\ 1.03)$. Finally, after computing $\tilde{R}(r)$, $\eta(r)$, and r_*, we obtain estimates of the admissible initial conditions set $E_P(\tilde{R}(r)) \times B_w(r)$, where $\tilde{R}(r)$ is given in Figure 6.9.

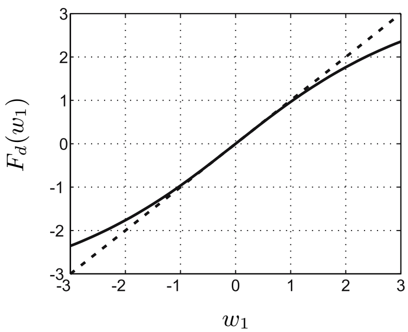

Fig. 6.7. Nonlinear disturbance force: $F_d(w_1) = \lambda \arctan(w_1/\lambda)$, for $\lambda = 3$ – solid, $F_d(w_1) = w_1$ – dashed.

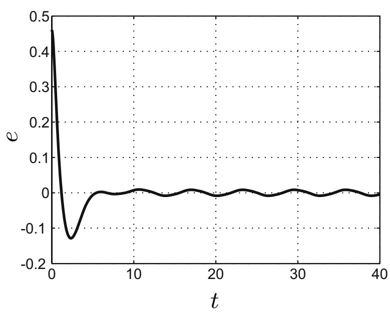

Fig. 6.8. Approximate output regulation: the regulated output converges to small (in some sense) values.

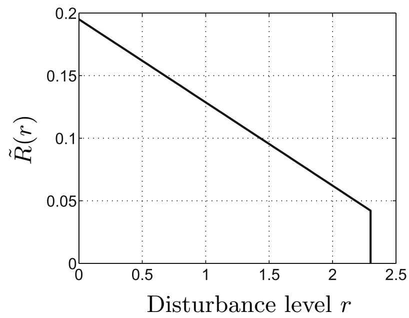

Fig. 6.9. Approximate output regulation: the function $\tilde{R}(r)$ for the estimates $E_P(\tilde{R}(r)) \times B_w(r)$.

Theorem 6.6 provides the estimates for $r \in [0, r_*)$. In our case, $r_* \approx 2.3$ (for $r = r_*$, the function $\eta(r)$ reaches $\mathcal{R}$). For $r > r_*$, Theorem 6.6 does not guarantee that *both* $x(t)$ starting in $E_P(\tilde{R}(r))$ and $\tilde{\alpha}(w(t))$ with $w(t)$ starting in $B_w(r)$ will lie in the convergence set $\mathcal{C}_N(\mathcal{R})$. Thus, Lemma 6.4 cannot be applied and the inequalities (6.34) and (6.35) may not hold.

Note that the mappings $\tilde{\alpha}(w)$ and $c(w)$ and, thus, the closed-loop system are defined only for $|w_1| < \omega^2\phi/\mu$. For the given values of the system parameters this constraint is given by $|w_1| < 12.5$. The obtained estimates satisfy this condition. Just like in the case of exact output regulation, the estimates are rather conservative, which can be explained either by an inappropriate choice of the matrix P or by the approach itself, since it is based on quadratic Lyapunov functions.

6.3 Summary

In this chapter we have considered the problem of estimating the sets of admissible initial conditions for a solution to the (approximate) local output regulation problem. The presented solutions to this estimation problem are based on the Demidovich condition. If a controller solves the local output regulation problem, then the closed-loop system satisfies the Demidovich condition at least in some neighborhood of the origin. This neighborhood is estimated, and, based on these estimates, we provide a way to compute estimates of the sets of admissible initial conditions for the (approximate) local output regulation problem. The obtained estimates consist of initial conditions for which the trajectories of the forced closed-loop system exponentially converge to the (approximate) output-zeroing manifold. The results are illustrated by application to a disturbance rejection problem in the TORA system. Since the exosystem is allowed to generate constant signals, the obtained results are also suitable for systems with parametric uncertainties. Although the analysis in this chapter has been performed under the assumption of linearity of the exosystem, the results can be extended to the case of general neutrally stable exosystems.

The obtained estimates are, in general, fairly conservative since they are based on a quadratic stability analysis and strongly depend on the choice of the matrices N and P. Despite this conservatism, the results can be useful in the following situations. First, one can directly use the estimates in practice (for certain simple systems they may be quite satisfactory). Second, if the estimates are too conservative, one can use them as a starting point for obtaining larger estimates by means of, for example, backward integration. The third way is to use the estimates as a criterion for choosing/tuning certain controller parameters. Since controller design admits some freedom in choosing certain controller parameters (like the matrix K in the TORA example), one can pick such parameters that guarantee larger estimates.

7

Experimental case study

The output regulation problem for nonlinear systems has been studied from a theoretical point of view in a series of publications. For a number of nonlinear mechanical systems, the output regulation problem has been studied in the papers [32, 41, 43, 83, 84] and in the monograph [42]. Despite the significant interest in this problem, most of the known results are theoretical with only a few papers aiming at experimental validation of the proposed solutions [4, 55].

In this chapter we address the nonlinear output regulation problem from an experimental point of view. We study the disturbance rejection problem for the TORA system considered in Section 6.1.4. This problem is a particular case of the local output regulation problem. First, in Section 7.1 we design a simple state feedback controller solving this problem. Second, this controller is implemented in an experimental setup described in Section 7.2. Third, in Section 7.3 we present experimental results. Moreover, we identify the causes that limit the performance of this controller in practice and show that output regulation is attained experimentally.

The reason for this experimental study is twofold. The first reason is to check whether the controllers from the nonlinear output regulation theory are applicable in an experimental setting in the presence of disturbances and modeling uncertainties, which are inevitable in practice. The second reason is to identify factors that can deteriorate the controller performance and therefore require specific attention already at the stage of controller design. The results presented in this chapter should be considered as the first steps in the problem of experimental output regulation. As any other first step into terra incognita, these results will probably raise more questions and new challenges than provide ultimate answers.

7.1 Controller design for the TORA system

In this section we design a simple controller for the disturbance rejection problem considered in Section 6.1.4. This problem is a particular case of the local output regulation problem. A controller solving this problem has already been presented in Section 6.1.4. In that controller design, we first transformed the system model into a simple form by means of certain coordinate and feedback transformations and then designed a controller for the simplified model. The controller designed in such a way is convenient for illustrating the estimation results presented in the previous chapter because it is easier to compute estimates of the set of admissible initial conditions for closed-loop systems having a relatively simple form. Due to the nonlinear feedback transformations employed in the simplification of the system model, the resulting controller becomes a rather sophisticated nonlinear controller. In practice, however, it is convenient, important, and in some cases even critical that the controller is simple. Computational limitations of digital signal processors used in the controller implementation are one of the reasons for simplicity of the controller. Another reason, which is more philosophical, is that the more sophisticated a controller is the more difficult it is to analyze its performance and to identify possible problems, which are inevitable in practice. Since in this chapter we aim at the experimental implementation of an output regulation controller, we need to design a simple controller for the TORA system.

We start with the equations of motion for the ideal TORA system (see Section 6.1.4):

$$\begin{aligned} \bar{M}\ddot{e} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + ke &= F_d, \\ J\ddot{\theta} + ml\ddot{e}\cos\theta &= T_u. \end{aligned} \tag{7.1}$$

Recall (see Figure 6.4) that in this model e is the horizontal displacement of the cart, θ is the angular position of the arm, F_d is the disturbance force, and T_u is the control torque applied to the arm. The parameters of the system are $\bar{M} := M + m$, where M is the mass of the cart and m is the mass of the rotating arm, l is the distance between the rotational axis of the arm and its center of mass, k is the stiffness of the spring, and J is the total inertia of the arm with respect to the rotational axis. Notice that $J > ml^2$. The disturbance force F_d is generated by the linear exosystem

$$\dot{w}_1 = \omega w_2, \quad \dot{w}_2 = -\omega w_1, \quad F_d = w_1, \tag{7.2}$$

where ω is the oscillation frequency. The initial conditions of the exosystem (7.2) determine the amplitude and phase of the excitation. Recall that the control problem is to asymptotically regulate $e(t)$ to zero for all sufficiently small initial conditions of the closed-loop system and for all sufficiently small initial conditions of the exosystem. This is a particular case of the local output regulation problem. For simplicity it is assumed that e, $\dot{e}$, θ, $\dot{\theta}$, w_1, and w_2 are measured.

A controller solving this problem is sought in the form

$$T_u = c(w) + K(x - \pi(w)), \tag{7.3}$$

where $x := [e, \dot{e}, \theta, \dot{\theta}]^T$ is the state of the system (7.1), $w := [w_1, w_2]^T$ is the state of the exosystem (7.2), $c(w)$ and $\pi(w)$ are the solutions of the regulator equations, and the matrix K is such that for $w = 0$ the closed-loop system (7.1), (7.3) has an asymptotically stable linearization at the origin (see, e.g., Section 4.1 or [8] for more information on controller design for the local output regulation problem).

The requirement on the matrix K is equivalent to the requirement that $A + BK$ is a Hurwitz matrix, where the matrices

$$A := \begin{bmatrix} 0 & 1\,0\,0 \\ -\frac{kJ}{\bar{M}J - m^2l^2} & 0\,0\,0 \\ 0 & 0\,0\,1 \\ \frac{kml}{\bar{M}J - m^2l^2} & 0\,0\,0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ -\frac{ml}{\bar{M}J - m^2l^2} \\ 0 \\ \frac{\bar{M}}{\bar{M}J - m^2l^2} \end{bmatrix},$$

follow from the linearization of the system (7.1) at the origin with $F_d = 0$ and T_u viewed as input. Notice that in the model (7.1), $J > ml^2$ and $\bar{M} > m$. Therefore, $\bar{M}J - m^2l^2 > 0$. One can easily check that this inequality implies controllability of the pair (A, B). Hence, we can always choose a matrix K such that $A + BK$ is Hurwitz. Next we need to solve the regulator equations. In other words, we need to find continuous mappings $\pi(w) := [\pi_1(w), \pi_2(w), \pi_3(w), \pi_4(w)]^T$ and $c(w)$ defined in a neighborhood of the origin $w = 0$ and satisfying $\pi(0) = 0$ and $c(0) = 0$ such that, for any sufficiently small solution of the exosystem $w(t)$, for the disturbance force $F_d(t) = w_1(t)$ and controller action $T_u(t) = c(w(t))$, the function $\bar{x}_w(t) = \pi(w(t))$ is a solution of system (7.1) and along this solution the displacement $e(t)$ equals zero. By substitution one can easily check that the mappings

$$\pi_1(w) = 0, \quad \pi_2(w) = 0, \quad \pi_3(w) = -\arcsin\left(\frac{w_1}{ml\omega^2}\right), \tag{7.4}$$

$$\pi_4(w) = -\frac{\omega w_2}{(m^2l^2\omega^4 - w_1^2)^{1/2}}, \quad c(w) = \frac{\omega^2 w_1 (m^2l^2\omega^4 - w_1^2 - w_2^2)J}{(m^2l^2\omega^4 - w_1^2)^{3/2}}, \tag{7.5}$$

satisfy the regulator equations. Consequently, we have found a controller solving the local output regulation problem. Controller (7.3) admits some freedom in the choice of the matrix K. This freedom can be used, for example, in tuning the controller to obtain desirable performance of the closed-loop system. Controller (7.3) is implemented in the experimental setup described in the next section.

7.2 Experimental setup

To obtain experimental validation of the proposed controller, an experimental setup for the TORA system has been built. This setup has been constructed by adapting an existing X-Y positioning system (the H-bridge setup available in the Dynamics and Control Technology Laboratory at Eindhoven University of Technology) shown in Figure 7.1.

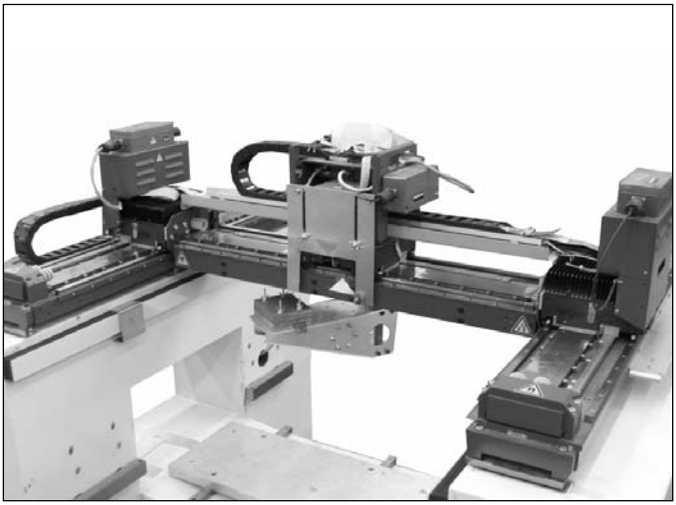

Fig. 7.1. The adapted H-bridge setup.

The adapted H-bridge setup is schematically shown in Figure 7.2. It consists of the following components. The two parallel axes Y1 and Y2 are equipped with Linear Magnetic Motor Systems LiMMS Y1 and LiMMS Y2

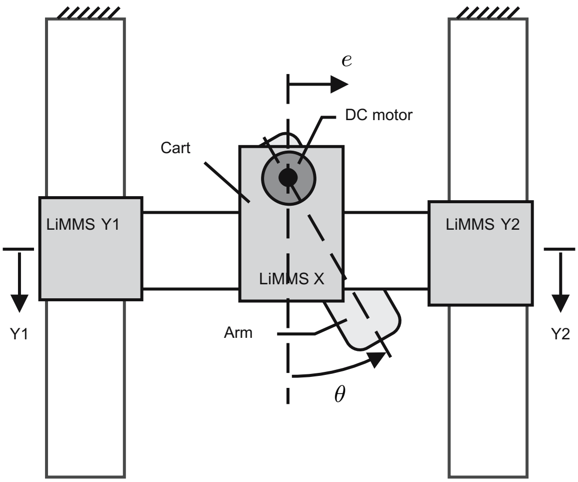

Fig. 7.2. The adapted H-bridge setup scheme, top view.

that can move along their axes. The Y1 and Y2 carriages support the X axis. In all experiments that are performed on this setup the Y1 and Y2 carriages are controlled with a low-level PID controller to maintain a fixed position. Therefore, in the following we will assume that these two carriages stand still and that the X axis is fixed.

In the following we will refer to the X-LiMMS carriage moving along the X axis as the cart. The mass of the cart is M [kg]. The displacement of the cart e [m] is measured using a linear incremental encoder (Heidenhain LIDA 201) with a 1 μm resolution. The force applied to the cart by the linear motor is proportional to the (voltage) control signal u_F, which is fed to the linear motor through a proportional current amplifier, i.e., $F = \kappa_F u_F$. The constant κ_F has the value of 74.4 N/V ([27]). In addition to the actuating force, a friction force F_f is present in the bearings of the cart. Moreover, a cogging force F_c, which is caused by the interaction of the permanent magnets in the X rail and the iron-core coils of the electromagnets in the cart, acts on the cart (see [27] for details on the cogging force). We assume that the friction force depends only on the cart velocity, i.e., $F_f = F_f(\dot{e})$, and the cogging force depends only on the position of the cart, i.e., $F_c = F_c(e)$. This assumption, although being a simplification of reality, helps with dealing with these two forces.

To implement the TORA system at the H-bridge setup, additional hardware has been added to the cart (see Figure 7.3). A vertical shaft supported by a set of (angular contact) ball bearings is attached to the back of the cart, thus forming a rotational joint. An arm of mass m [kg] is attached to the lower end of the shaft. The center of mass of the arm is located at the distance of l [m] from the shaft center line. The angular position of the shaft (and consequently of the arm) θ is measured by a rotational incremental encoder (Maxon, HEDL55) with a (quadrature decoded) resolution of 0.18° at the motor shaft. A 48V, 150W DC motor (Maxon RE40) fitted with a ceramic planetary gearhead (Maxon GP42C) drives the shaft via an adapted flexible coupling (ROBA-DX, type 931.333). The gear ratio equals $g_r = 113$. The backlash in the gearhead is approximately 0.5° at the output shaft. The total inertia of all rotating parts (the arm, shaft, coupling, gearhead, and motor) with respect to the shaft is J [kg m^2]. Due to the friction in the motor, gearhead, and ball bearings of the shaft, an additional friction torque $T_f = T_f(\dot{\theta})$ acts on the arm. The torque T generated by the DC motor is proportional to the current i [A] fed to the motor, i.e., $T = \kappa_T i$, where $\kappa_T = 60.3\ \mathrm{mN \cdot m/A}$ is the motor constant. The current i is generated by an analog current amplifier. It is proportional to the (voltage) control signal u_T fed to the amplifier, i.e., $i = \kappa_A u_T$, where $\kappa_A = 1.2\, A/V$ is the amplifier constant. More details on the adapted H-bridge setup can be found in [46, 72].

Taking into account all the active forces and torques, we obtain the following model of the setup consisting of the cart moving along the fixed X axis and the (horizontally) rotating arm attached to the cart

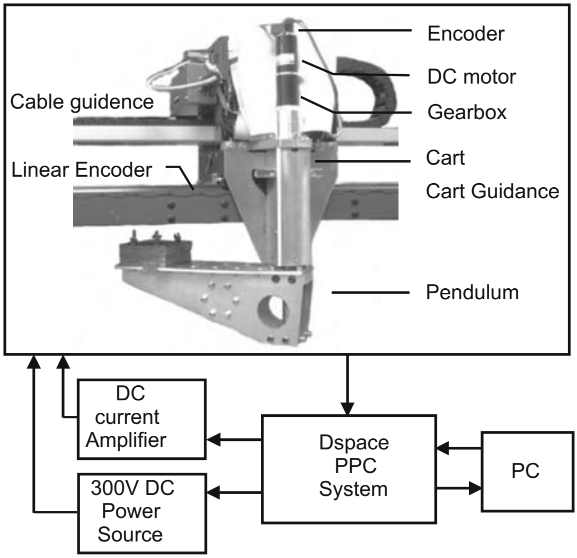

Fig. 7.3. The adapted H-bridge setup: rear view and connection scheme.

$$\begin{aligned}\bar{M}\ddot{e} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) &= F - F_f(\dot{e}) + F_c(e), \\ J\ddot{\theta} + ml\ddot{e}\cos\theta &= T - T_f(\dot{\theta}),\end{aligned} \tag{7.6}$$

where $\bar{M} := M + m$, the actuator force acting on the cart equals $F = \kappa_F u_F$ and the actuator torque acting on the arm equals $T = g_r \kappa_T \kappa_A u_T$, where u_F and u_T are the control signals for the cart and for the arm, respectively.

The cogging force $F_c(e)$ and the friction force $F_f(\dot{e})$ are identified using the methods presented in [6]. The corresponding graphs are presented in Figures 7.4 and 7.5, respectively. The friction torque $T_f(\dot{e})$ has been identified using constant angular velocity tests. The resulting graph is given in Figure 7.6.

Initial estimates of the mass $\bar{M}$, the product ml, and the inertia J are computed from the CAD drawings, material data, and specifications of the motor and gearhead. These estimates are $\bar{M} = 20.965$ kg, $ml = 1.2514$ kg m and $J = 0.5405$ kg m^2. They will be used as a starting point to obtain more accurate estimates based on closed-loop experiments.

To implement the TORA system in the resulting setup, we need to compensate for the friction in the cart and the arm and for the cogging force in the X axis. Moreover, we need to implement the virtual spring action $-ke$ and the disturbance force F_d along the X axis. For the cart, this is achieved by the controller

$$u_F = \frac{1}{\kappa_F}(\hat{F}_f(\dot{e}) - \hat{F}_c(e) - ke + F_d), \tag{7.7}$$

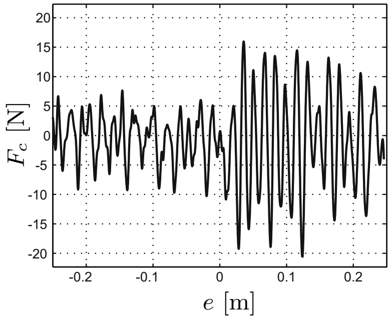

Fig. 7.4. The identified cogging force $F_c(e)$.

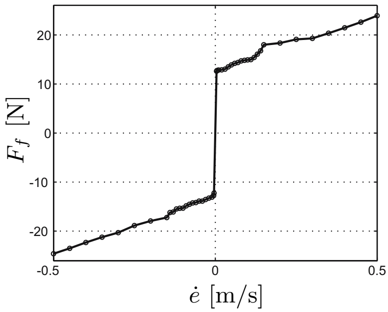

Fig. 7.5. The identified friction force $F_f(\dot{e})$.

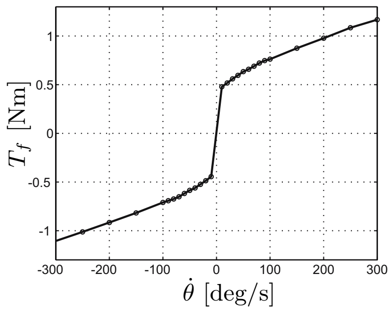

Fig. 7.6. The identified friction torque $T_f(\dot{\theta})$.

where $\hat{F}_f(\dot{e})$ and $\hat{F}_c(e)$ are the friction compensation and cogging compensation forces (based on the identified values of these forces), k [N/m] is the stiffness of the virtual spring (which we can set arbitrarily), and $F_d(t) = w_1(t)$ is the disturbance force acting on the cart. In the experiments performed on the setup, parameter k is set equal to $k = 500$ N/m. The exosystem (7.2), with $w(t) = [w_1(t), w_2(t)]^T$, is integrated in the PC/dSpace-system and the disturbance force $F_d(t) = w_1(t)$ is computed from the obtained solutions.

Next, we need to implement friction compensation in the rotating arm. This is achieved by the controller

$$u_T = \frac{1}{g_r \kappa_T \kappa_A}(T_u + \hat{T}_f(\dot{\theta})), \tag{7.8}$$

where $\hat{T}_f(\dot{\theta})$ is the friction compensation torque based on the identified friction torque in the arm (see Figure 7.6), and T_u is a new control input.

After implementing the low-level controllers (7.7), (7.8) and the exosystem (7.2), the resulting system takes the form

$$\begin{aligned} \bar{M}\ddot{e} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + ke &= F_d + \varepsilon_F, \\ J\ddot{\theta} + ml\ddot{e}\cos\theta &= T_u + \varepsilon_T, \end{aligned} \tag{7.9}$$

where $F_d(t) = w_1(t)$ is the disturbance force, T_u is the control torque (new input), and ε_F and ε_T are the residual terms due to nonexact friction and cogging compensation and uncertainties in the system parameters. System (7.9) is now in the form of system (7.1) (if the residual terms are not taken into account) for which the controller (7.3) solves the local output regulation problem. This controller requires the values of e and θ, which are measured by the encoders, the derivatives $\dot{e}$ and $\dot{\theta}$, which are obtained by numerical differentiation and filtering of the measured signals e and θ, and the values of $w_1(t)$ and $w_2(t)$, which are computed in the dSpace-system.

7.3 Experiments

In this section we present results of experiments performed on the adapted H-bridge setup in closed loop with the controller (7.3).

7.3.1 Parameter settings

The gain matrix K in the controller (7.3) is set to $K := [29,\ -1.5,\ -11,\ -1.9]$. The eigenvalues of the linearized closed-loop system corresponding to this K and to the initial estimates of the system parameters given in the previous section equal $-1.0313 \pm 5.8493i$ and $-0.9121 \pm 3.8901i$. The choice of the matrix K is determined by several requirements. The first and the third entries in the matrix K, which correspond to the displacement of the cart e and angular position of the arm θ, must be large enough to compensate for the residual friction and backlash present in the system. At the same time, the real part of the eigenvalues of the linearized closed-loop system must be less than a certain threshold to guarantee fast convergence rates and sufficient robustness properties of the closed-loop system. In theory, for any matrix K such that $A + BK$ is Hurwitz, controller (7.3) solves the output regulation problem in *some* neighborhood of the origin, i.e., for initial conditions of the closed-loop system and the exosystem being small enough. This neighborhood of admissible initial conditions essentially depends on the choice of K. Thus, our choice of the matrix K must be such that the resulting set of admissible initial conditions is relatively large in order to test this controller in experiments (the problem of estimating this neighborhood of admissible initial conditions for a system in closed loop with a controller solving the local output regulation problem has been considered in Chapter 6). Finally, the control signal resulting from the controller with this matrix K must not exceed, in most operating conditions, the bounds imposed by the amplifier and DC motor specifications. Taking these requirements into account, a combination of some optimization procedures with trial and error resulted in the matrix K presented above.

The estimates for the parameters J and ml are tuned based on closed-loop experiments using the output regulation controller (to obtain better performance). The new estimates are $\hat{J} = 0.4270\ \mathrm{N \cdot m^2}$ (21% smaller than the initial estimate) and $\widehat{ml} = 1.3389$ kg · m (7% larger than the initial estimate). These estimates are used in the feedforward part of the output regulation controller in the experiments presented below.

The friction compensation torque in the rotating arm $\hat{T}_f(\dot{\theta})$ is set 1.5 times larger than the identified friction torque $T_f(\dot{\theta})$ given in Figure 7.6. It has been noticed that for this friction compensation in the rotating arm the controller has a better performance. Such a large deviation from the identified values may be explained by the fact that the friction in the gearhead, which is the main contributor to the friction in the arm motion, depends not only on the angular velocity $\dot{\theta}$, but also on the torque applied to the shaft. The identification of the friction torque has been performed for very low torques (constant velocity experiments), while in the experiments with the TORA controller the torques are much higher. The cogging compensation force $\hat{F}_c(e)$ is set equal to the identified cogging force presented in Figure 7.4. The friction compensation force $\hat{F}_f(\dot{e})$ in the cart motion is set to 90% of the identified friction force presented in Figure 7.5 to avoid overcompensation. Moreover, for a cart velocity $\dot{e}$ of magnitude less than 0.035 m/s, it is set to

$$\hat{F}_f(\dot{e}) := \frac{|\dot{e}|0.90}{0.035} F_f(\dot{e}).$$

The resulting friction compensation force is shown in Figure 7.7. This undercompensation of the friction in the cart motion reduces the friction-induced limit-cycling, which is observed in experiments if the friction compensation force is set equal the real friction force, see, e.g., [75]. At the same time, friction undercompensation makes the equilibrium set, in terms of the position of the cart, larger. In the experiments, this equilibrium set can be easily observed when the cart sticks in a point e_*, which is close but not equal to zero.

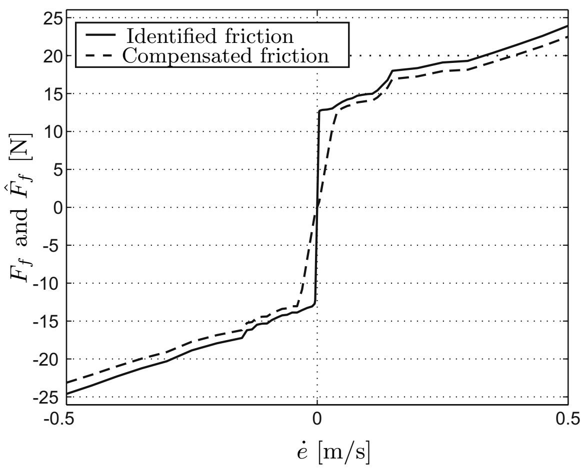

Fig. 7.7. The identified friction force $F_f(\dot{e})$ and the friction compensation force $\hat{F}_f(\dot{e})$.

In the experiments, the frequency of the disturbance force $F_d(t)$ (the frequency of the exosystem) is set to 1 Hz, which corresponds to ω in the exosystem (7.2) equal to $\omega = 2\pi$ rad/s.

7.3.2 Experimental results

All experiments are performed for the initial conditions of the exosystem equal to $w_1(0) = 0$, $w_2(0) = \mathcal{A}$. These initial conditions correspond to the disturbance force $F_d(t) := \mathcal{A}\sin(\omega t)$. We perform the experiments for two values of the amplitude $\mathcal{A}$: $\mathcal{A} = 15$ N and $\mathcal{A} = 25$ N.

Two types of experiments are performed. In the experiments of the first type, the system starts in a given initial condition $e(0) = e_0$ [m], $\dot{e}(0) = 0$ [m/s], $\theta(0) = \theta_0$ [deg], $\dot{\theta}(0) = 0$ [deg/s]. For each value of the amplitude $\mathcal{A}$ we perform three experiments corresponding to different initial conditions e_0 and θ_0. These initial conditions are given in Table 7.1.

	e_0 [m]	θ_0 [deg]
Experiment # 1	-0.2	20
Experiment # 2	0.2	20
Experiment # 3	0.1	90

Table 7.1. Initial conditions e_0 and θ_0 used in the experiments.

The results of the experiments corresponding to the disturbance amplitudes $\mathcal{A} = 15$ N and $\mathcal{A} = 25$ N are presented in Figures 7.8 and 7.9, respectively. In these figures the controller effort is represented by the current $i = \kappa_A u_T$ [A] fed by the amplifier to the DC motor.

In the experiments of the second type, the system is affected by the disturbance force $F_d(t)$ of amplitude $\mathcal{A}$. Initially, only the feedback part in the controller (7.3) is active, i.e., $T_u = Kx$, and there is no compensation for the disturbance force $F_d(t)$. Since there is no disturbance compensation, the cart starts oscillating. At an arbitrary time instant t_* the feedforward part of the controller is activated, i.e., $T_u = c(w) + K(x - \pi(w))$. This results in disturbance rejection in the position of the cart e. The results of the experiments corresponding to the disturbance amplitudes $\mathcal{A} = 15$ N and 25 N are presented in Figures 7.10 and 7.11, respectively.

From these experimental results we can immediately draw the following conclusion. The output regulation controller (7.3) does compensate a significant part of the harmonic disturbance force acting on the cart. This results in stabilization of the cart at an equilibrium position close to zero. The remaining regulation error is due to the residual friction in the cart motion.

In Figure 7.12 the cart displacement signal related to an experiment, performed at a different time, is depicted. Clearly, exact output regulation is not attained and a limit cycle of small amplitude remains. In this respect, it

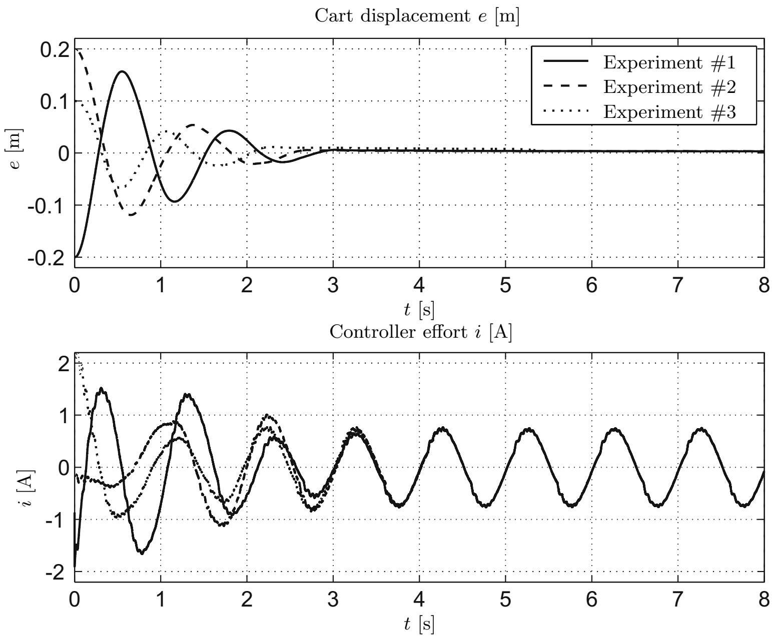

Fig. 7.8. Experiments for a disturbance force of amplitude $\mathcal{A} = 15$ N and predefined initial conditions.

should be noted that the friction characteristics in the setup are subject to change due to temperature and humidity change in the laboratory. However, exactly the same friction compensation as in the previous experiments was used. Consequently, the limit cycling can be caused by an interaction of several factors: friction and friction compensation in the cart motion, friction and friction compensation in the rotating arm, feedback controller and backlash in the gearhead. As can be seen from Figure 7.13, the backlash, for example, manifests itself through the peaks in the angular velocity of the arm. When the shaft reaches the border of the backlash zone, an impact occurs that causes the peaks in $\dot{\theta}$. These peaks, in turn, are amplified by the controller (7.3) and fed back into the DC motor actuating the arm. This provides the system with extra energy to compensate for the energy dissipation due to friction. These problems require an additional investigation, which is outside the scope of our research.

7.4 Summary

In this chapter, experimental results on the local output regulation problem for the TORA system have been presented. We have constructed a simple state-feedback controller that solves a disturbance rejection problem for the

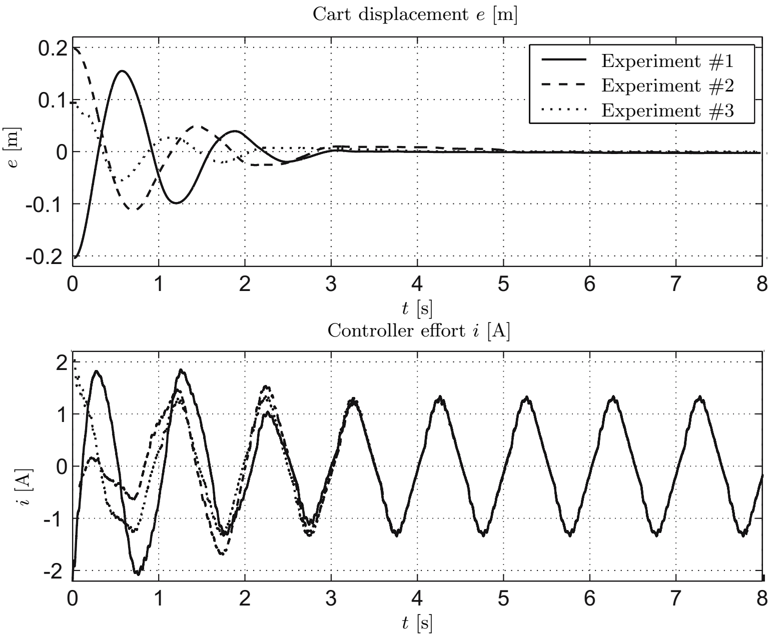

Fig. 7.9. Experiments for a disturbance force of amplitude $\mathcal{A} = 25$ N and predefined initial conditions.

TORA system. This problem is a particular case of the local output regulation problem. To validate this controller in experiments, an experimental setup for the TORA system has been built from an existing H-bridge setup. The proposed state-feedback controller has been implemented in this setup and tested in a series of experiments.

As follows from the results of these experiments, for the setup in closed loop with the proposed controller output regulation occurs, though only approximately. This means that the regulated output $e(t)$ does not tend to zero exactly, but either sticks in an equilibrium position close to zero or keeps on oscillating with a small amplitude. These phenomena are due to nonexact compensation of the friction and the backlash problem in the gearhead of the rotating arm. At the stage of controller design for the output regulation problem, these factors have not been taken into account.

In practice there is always some type of (non)parametric uncertainty present in the system. It can be either due to inaccurately identified parameters of the system or friction, backlash, or other parasitic phenomena, which are not taken into account in the system model. These uncertainties may significantly reduce the performance of a controller. This performance deterioration may manifest itself, for example, in a steady-state regulation error, as illustrated by the experimental results on the TORA system presented above. As follows from the experiments on the TORA system performed for different

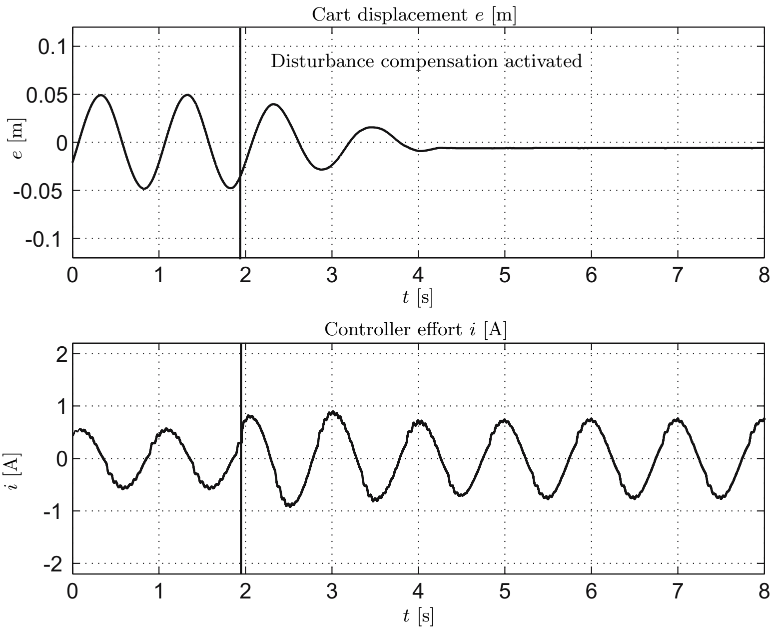

Fig. 7.10. Experiments for a disturbance force of amplitude $\mathcal{A} = 15$ N. Disturbance compensation is activated during the experiment.

values of the controller gain K while tuning the controller, this steady-state regulation error can be reduced by a proper choice of the gain K. Also, this gain matrix K essentially determines the region of admissible initial conditions for which this local controller works. Moreover, it determines the rate of convergence for the closed-loop system. In this experimental case study, the choice of the matrix K, which takes into account these practically important design issues, has been done using ad hoc optimization and trial-and-error methods. It should be noted that the problem of a systematic choice or tuning controller parameters for nonlinear output regulation that takes into account the above-mentioned design considerations has not been studied in the literature so far. This fact urges the need for further work in this direction.

Even with the ad hoc tuning of the controller and with many uncertainties present in the system, the experimental results show relatively good performance of the closed-loop system. These successful experiments indicate that the output regulation theory can be successfully applied in experiments.

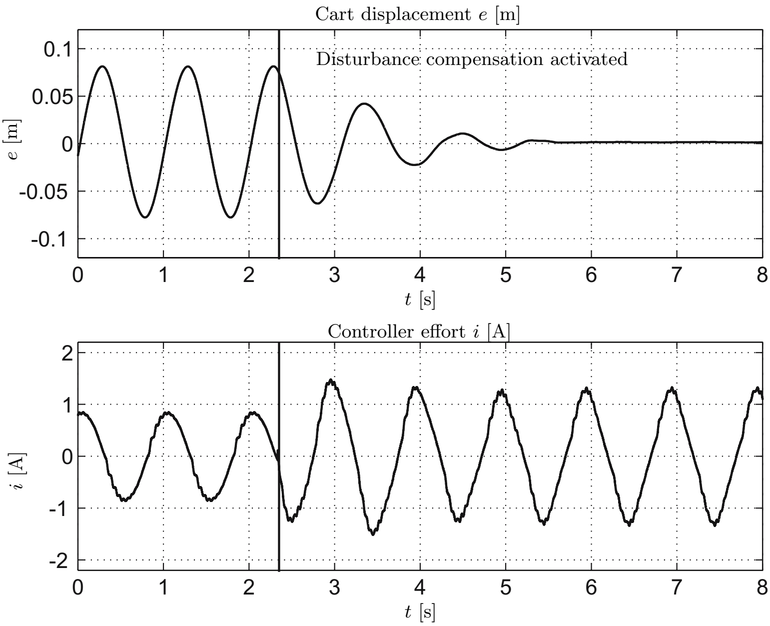

Fig. 7.11. Experiments for a disturbance force of amplitude $\mathcal{A} = 25$ N. Disturbance compensation is activated during the experiment.

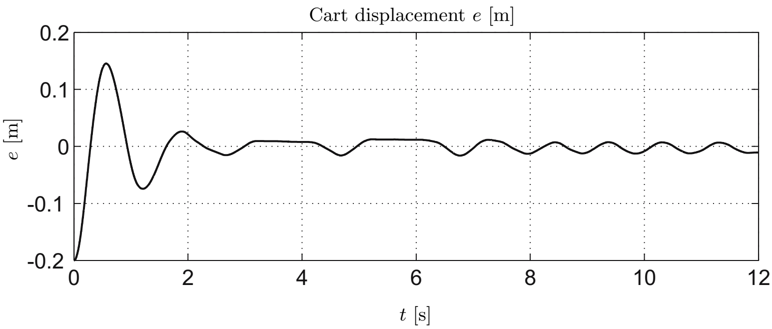

Fig. 7.12. Limit cycling in the cart motion. The disturbance force amplitude is $\mathcal{A} = 15$ N.

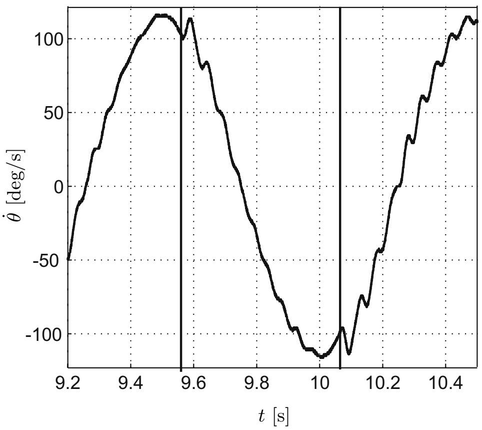

Fig. 7.13. Angular velocity of the arm. Impact instants due to backlash are indicated with the vertical lines.

8

Concluding remarks

The problem of asymptotic regulation of the output of a dynamical system, which includes both tracking and disturbance rejection problems, plays an important role in control theory. A particular case, when the reference signals and/or disturbances are generated by an autonomous system of differential equations, is the output regulation problem. This problem has been systematically studied in this book. Our treatment of the output regulation problem is based on the notion of convergent systems. In the development of this approach we have passed several stages.

Convergent systems. We have extended and elaborated the notions of convergent systems originally developed by B.P. Demidovich. We have introduced the notions of the uniform and exponential convergence, the UBSS property, and the input-to-state convergence property. Then we studied various properties of convergent systems. It appears that convergent systems, although nonlinear, have rather simple dynamics and enjoy many stability properties comparable to those of asymptotically stable linear systems. This makes them convenient to deal with. Finally, we have proposed sufficient conditions for various convergence properties for systems with smooth and nonsmooth right-hand sides. These sufficient conditions and properties of convergent systems serve as tools in the subsequent treatment of the output regulation problem. Moreover, they can be used for other control problems as well.

The uniform output regulation problem. Having developed a mathematical apparatus for convergent systems, we have formulated the uniform output regulation problem. This is a new problem formulation for the output regulation problem based on the notion of convergent systems. In this problem formulation one needs to find a controller such that for any input generated by the exosystem, the corresponding closed-loop system has a unique (globally) uniformly asymptotically stable steady-state solution and the regulated output tends to zero along all solutions of the closed-loop system. We have formulated the global, local, as well as robust variants of the uniform output regulation problem. This new problem setting includes as particular cases the output regulation problem for linear systems and the conventional local output regulation problem for nonlinear systems. Moreover, it has several advantages over other existing problem formulations. It allows one to deal with exosystems having complex dynamics, e.g., exosystems with a (chaotic) attractor with an unbounded domain of attraction. Up to now most of the results on the output regulation problem dealt only with exosystems having relatively simple dynamics, for example, with linear harmonic oscillators. Another advantage of this new problem formulation is that it allows one to treat the local and global variants of the uniform output regulation problem in a unified way. As becomes clear from the solvability analysis of the global uniform output regulation problem, many of the known controllers solving the global output regulation problem in some other problem settings in fact solve the global uniform output regulation problem.

Solvability analysis. One of the main advantages of the chosen problem setting for the uniform output regulation problem is that it allows one to obtain relatively simple results on the solvability of the problem. For the global, global robust, and local variants of the uniform output regulation problem, we have provided necessary and sufficient conditions for the solvability of these problems as well as results on the characterization of all controllers solving these problems. These results extend the solvability results for the conventional local output regulation problem, which are based on the center manifold theorem. The solvability analysis of the uniform output regulation problem is based on certain invariant manifold theorems. They serve as nonlocal counterparts of the center manifold theorem. These invariant manifold theorems, although obtained in the scope of the output regulation problem, can be applied in other fields of systems and control theory as well. For example, they can be used for the analysis of synchronization phenomena, the computation of periodic solutions of nonlinear systems excited by harmonic inputs, and for the performance analysis of nonlinear convergent systems.

Controller design. The analysis of the global uniform output regulation problem provides necessary and sufficient conditions under which a controller solves the problem. How to design a controller satisfying these conditions is a separate problem. We have addressed this problem for several classes of nonlinear systems and provided several results on controller design for the global uniform output regulation problem. One of these controller designs is based on the notions of quadratic stabilizability and detectability, which extend the conventional notions of stabilizability and detectability from linear systems theory to the case of nonlinear systems. The controller design based on these notions extends known controllers solving the linear and the local nonlinear output regulation problems to the case of the global uniform output regulation problem for nonlinear systems. For the case of a Lur'e system with a nonlinearity having a bounded derivative and an exosystem being a linear harmonic oscillator, feasibility conditions for such controller design can be easily verified by checking the feasibility of certain LMIs. Moreover, for this class of systems and exosystems we provide a robust controller design, which copes not only with uncertainties in the system parameters, but also with an uncertain nonlinearity from a class of nonlinearities with a given bound on their derivatives. The controller design results obtained at this stage allow us to solve the global uniform output regulation problem for new classes of nonlinear systems.

Convergence region estimates. If a solution to the global uniform output regulation problem cannot be found, it can still be possible to find a controller that solves the corresponding local output regulation problem. The resulting controller solves the output regulation problem for initial conditions of the closed-loop system and the exosystem lying in *some* neighborhood of the origin. In this book we have presented estimation results that, given a controller solving the local output regulation problem, provide estimates of this neighborhood of admissible initial conditions. These results are obtained for both the exact and the approximate local output regulation problem. The proposed estimation results enhance the applicability of the controller design procedures for the nonlinear local output regulation problem. As in the rest of the book, the notion of convergence plays a central role in these estimation results.

Experimental case study. To check the applicability of controllers from the nonlinear output regulation theory in practice, we have performed an experimental case study for the TORA system. For this system we have considered a disturbance rejection problem, which is a particular case of the local output regulation problem. We have designed a controller solving this problem and checked its performance in experiments. To this end, an experimental setup for the TORA system has been built and the proposed controller has been implemented in this setup. Despite the uncertainties and several parasitic effects, such as residual uncompensated friction, backlash, and a residual cogging force present in the system, the proposed controller has demonstrated good performance in the experiments by achieving approximate output regulation (the regulated output converges to small values). The residual regulation error is due to modeling uncertainties, which are inevitable in practice. This is one of the first experimental works in the field of nonlinear output regulation.

The results presented in this book can be extended in several ways. Below we briefly review some possible extensions.

In this book the uniform output regulation problem has been considered for systems modeled by ordinary differential equations (ODEs). At the same time, many practical systems cannot be modeled by ODEs and may require a model in the form of integral equations or partial differential equations (PDEs). The analysis and controller design methods for the output regulation problem for systems given by integral equations or PDEs is a possible step for further research. Another direction of research can be related to the output regulation problem for systems given in the form of differential equations with nonsmooth and discontinuous right-hand sides. Some preliminary results in this direction have been presented in [67].

The problem of controller design for the uniform output regulation problem requires a lot of further research, since the class of nonlinear systems for which the available controller design methods apply is rather limited. One of the main problems is designing controllers that make the corresponding closed-loop system globally uniformly convergent. This interesting problem has connections to different areas of nonlinear control theory. Controller design methods for convergent systems can be beneficial for both tracking and disturbance rejection problems.

In the problem of estimating the set of admissible initial conditions for the local output regulation problem there are several unanswered questions. The estimation procedures presented in this book depend on several parameters, which can be chosen in many ways. How to choose these parameters to obtain the largest (in some sense) estimates is still an open question. Another question is how to choose controller parameters to increase the set of admissible initial conditions.

For a given system and exosystem, the output regulation problem can be solved (if it is solvable) by many controllers. All these controllers achieve the control goal of regulating the output of the system, but the performance of the closed-loop system depends on the particular controller. In this respect, we face the problem of performance analysis for nonlinear systems. This problem has been thoroughly investigated for linear systems, but for nonlinear systems there are more questions than answers, starting with the question of how to quantify the performance of a nonlinear system. A possible approach to tackling the performance analysis problem for nonlinear convergent systems can be based on the invariant manifold theorems presented in this book. These theorems allow to extend the Bode plot defined for linear systems to the case of nonlinear convergent systems. The extended Bode plot can be applied for performance analysis of nonlinear convergent systems. This fact opens an interesting research direction in nonlinear (control) systems theory.

The notion of convergent systems seems to be very useful in many areas of systems and control theory. The research on properties of convergent systems, analysis, and design tools for convergent systems has started relatively recently. At the moment, there is a need for new design and analysis tools for convergent systems.

At the end of the book we can conclude that the approach to the output regulation problem based on the notion of convergent systems appears to be very effective. Moreover, the results presented in this book can be applied not only to the output regulation problem, but to other problems in systems and control theory as well.

9

Appendix

9.1 Proof of Lemma 2.11

The proof of this lemma is based on ideas from [15, 89]. Since system (2.6) is locally ISS, there exist $k_z > 0$ and $k_w > 0$, a class $\mathcal{KL}$ function $\beta(r,s)$, and a class $\mathcal{K}$ function $\gamma(r)$ such that for any initial state $z(t_0)$ with $|z(t_0)| \le k_z$ and any input $w(t)$ satisfying $\sup_{t \ge t_0} |w(t)| \le k_w$, the solution $z(t)$ exists for all $t \ge t_0$ and satisfies

$$|z(t)| \le \beta(|z(t_0)|, t - t_0) + \gamma\left(\sup_{t_0 \le \tau \le t} |w(\tau)|\right). \tag{9.1}$$

Choose $\tilde{k}_w > 0$ such that $\tilde{k}_w \le k_w$ and $\beta(\gamma(\tilde{k}_w), 0) + \gamma(\tilde{k}_w) < k_z$. Such $\tilde{k}_w$ exists because $\beta(r,0)$ and $\gamma(r)$ are continuous and satisfy $\beta(0,0) = 0$ and $\gamma(0) = 0$. We will show that for any input $w(t)$ satisfying $\sup_{t \ge t_0} |w(t)| \le \tilde{k}_w$ there exists a solution $z_w(t)$ that is defined for all $t \in \mathbb{R}$ and satisfies $\sup_{t \in \mathbb{R}} |z_w(t)| \le k_z$. If such a solution exists, then it satisfies the inequality

$$|z_w(t)| \le \beta(|z_w(t_0)|, t - t_0) + \gamma\left(\sup_{t_0 \le \tau \le t} |w(\tau)|\right)$$

for all $t_0 \in \mathbb{R}$ and $t \ge t_0$. In the limit for $t_0 \to -\infty$ the term $\beta(|z_w(t_0)|, t - t_0)$ tends to zero because $|z_w(t_0)|$ is bounded on $\mathbb{R}$ and $\beta(r,s)$ is a $\mathcal{KL}$ function. Hence, we obtain $|z_w(t)| \le \gamma(\sup_{\tau \in \mathbb{R}} |w(\tau)|)$ for all $t \in \mathbb{R}$. This implies the statement of the lemma.

Let us show that a solution $z_w(t)$ described above does exist. Choose $\varepsilon > 0$ such that for $\rho := \gamma(\tilde{k}_w) + \varepsilon$ it holds that $\beta(\rho, 0) + \gamma(\tilde{k}_w) \le k_z$ and $\rho < k_z$. Such ε exists due to continuity of $\beta(r,0)$ and the choice of $\tilde{k}_w$. Denote $\bar{B}_\rho$ to be the closed ball $\bar{B}_\rho := \{z : |z| \le \rho\}$. By the formula (9.1), any solution $z(t)$ satisfying $z(t_0) \in \bar{B}_\rho$ and corresponding to some input $w(t)$ such that $\sup_{t \ge t_0} |w(t)| \le \tilde{k}_w$ is defined for all $t \ge t_0$ and satisfies the inequality

$\sup_{t \geq t_0} |z(t)| \leq \beta(\rho, 0) + \gamma(\tilde{k}_w)$. By the choice of $\tilde{k}_w$ and ρ this implies that $\sup_{t \geq t_0} |z(t)| \leq k_z$.

Choose $T > 0$ such that $\beta(\rho, T) < \varepsilon$. Such T exists because $\beta(r, s)$ is a decreasing function of s. For every $n \geq 0$, define the set

$$V_n := \{\hat{z} \in \mathbb{R}^d : \ \hat{z} := z(0, z_0, -nT), z_0 \in \bar{B}_\rho\},$$

where $z(t, z_0, t_0)$ is a solution of system (2.6) starting in $z(t_0) = z_0$. Since $\bar{B}_\rho$ is a compact set, by continuity of the solution $z(t, z_0, t_0)$ on z_0 we obtain that V_n is also a compact set. By the definition of the set V_n, if $\hat{z} \in V_n$, then the solution of system (2.6) satisfying $z(0) = \hat{z}$ is defined at least for $t \in [-nT, 0]$ and $|z(-nT)| \leq \rho$. Hence, by the reasoning presented above, such $z(t)$ is defined for all $t \geq -nT$ and satisfies $\sup_{t \geq -nT} |z(t)| \leq k_z$.

Let us show that $V_{n+1} \subset V_n$ for all $n \geq 0$. As follows from the definition of V_n, it is enough to show that any solution starting in $z(-(n+1)T) \in \bar{B}_\rho$ satisfies $z(-nT) \in \bar{B}_\rho$. Suppose $z(t)$ satisfies $|z(-(n+1)T)| \leq \rho$. Then it is defined for all $t \geq -(n+1)T$ and satisfies (9.1). Then, by the properties of functions $\beta(r, s)$ and $\gamma(r)$, this solution satisfies $|z(-nT)| \leq \beta(\rho, T) + \gamma(\tilde{k}_w)$. By the choice of T we obtain that $|z(-nT)| \leq \varepsilon + \gamma(\tilde{k}_w) = \rho$. Indeed, it holds that $V_{n+1} \subset V_n$ for all $n \geq 0$.

Since the sets V_n are compact and $V_{n+1} \subset V_n$ for all $n \geq 0$, their intersection is nonempty, i.e., there exists $z_* \in \bigcap_{n=0}^{+\infty} V_n$. Consider the solution $z_w(t)$ starting in $z_w(0) = z_*$. Since $z_* \in V_n$ for all $n \geq 0$, the solution $z_w(t)$ is defined for all $t \in \mathbb{R}$ and it satisfies $\sup_{t \in \mathbb{R}} |z_w(t)| \leq k_z$.

The proof of the case of system (2.6) being ISS is identical to the local case. □

9.2 Proof of Property 2.24

Notice that since $F(z, w)$ is C^1, the Jacobians $\frac{\partial F}{\partial z}(z, w)$ and $\frac{\partial F}{\partial w}(z, w)$ are bounded in some neighborhood of the origin $(z, w) = (0, 0)$. Thus, all conditions of Lemma 2.10 are satisfied. By Lemma 2.10, system (2.19) is locally ISS. Hence, by Lemma 2.11, there exists $\tilde{k}_w > 0$ and a class $\mathcal{K}$ function $\gamma(r)$ such that for any input $w(t)$ satisfying $|w(t)| \leq \tilde{k}_w$, for all $t \in \mathbb{R}$, system (2.19) has a solution $z_w(t)$ that is defined for all $t \in \mathbb{R}$ and satisfies $\sup_{t \in \mathbb{R}} |z_w(t)| \leq \gamma(\sup_{t \in \mathbb{R}} |w(t)|)$. Since system (2.19) is locally uniformly convergent for the class of inputs $\mathcal{N}$, there exist $r > 0$ and a neighborhood of the origin $\mathcal{Z} \subset \mathbb{R}^d$ such that system (2.19) is uniformly convergent in $\mathcal{Z}$ for any input $w(\cdot) \in \mathcal{N}$ satisfying $|w(t)| < r$ for all $t \in \mathbb{R}$. Choose $k_w > 0$ such that $k_w < \tilde{k}_w$, $k_w < r$ and such that the closed ball $\bar{B}_{\gamma(k_w)} := \{z : |z| \leq \gamma(k_w)\}$ lies in $\mathcal{Z}$. Such k_w exists, because $\gamma(0) = 0$, the function $\gamma(r)$ is continuous and $\mathcal{Z}$ is a neighborhood of the origin. Consider some input $w(\cdot) \in \mathcal{N}$ satisfying $|w(t)| < k_w$ for all $t \in \mathbb{R}$. Due to uniform convergence, for this input there exists a steady-state solution $\bar{z}_w(t)$ that is defined and bounded for all

$t \in \mathbb{R}$ and uniformly asymptotically stable in $\mathcal{Z}$. Moreover, there exists a solution $z_w(t)$ defined for all $t \in \mathbb{R}$ and lying in the compact set $\bar{B}_{\gamma(k_w)} \subset \mathcal{Z}$. Hence, by Property 2.4 $\bar{z}_w(t) \equiv z_w(t)$ for $t \in \mathbb{R}$. Thus, for any input $w(\cdot) \in \mathcal{N}$ satisfying $|w(t)| < k_w$ for all $t \in \mathbb{R}$ the corresponding steady-state solution satisfies (2.25). □

9.3 Proof of Property 2.25

Consider two inputs $w_1(\cdot)$ and $w_2(\cdot) \in \overline{\mathbb{PC}}_m$ satisfying $w_1(t) - w_2(t) \to 0$ as $t \to +\infty$ and the corresponding steady-state solutions $\bar{z}_{w_1}(t)$ and $\bar{z}_{w_2}(t)$. By the definition of convergence, both $\bar{z}_{w_1}(t)$ and $\bar{z}_{w_2}(t)$ are defined and bounded for all $t \in \mathbb{R}$. Consider the system

$$\Delta\dot{z} = F(\bar{z}_{w_2}(t) + \Delta z, w_2(t) + \Delta w) - F(\bar{z}_{w_2}(t), w_2(t)). \tag{9.2}$$

This system describes the dynamics of $\Delta z = z(t) - \bar{z}_{w_2}(t)$, where $z(t)$ is some solution of system (2.19) with the input $w_2(t) + \Delta w(t)$. Since $F(z, w) \in C^1$ and both $\bar{z}_{w_2}(t)$ and $w_2(t)$ are bounded uniformly with respect to $t \in \mathbb{R}$, the partial derivatives

$$\frac{\partial F}{\partial z}(\bar{z}_{w_2}(t) + \Delta z, w_2(t) + \Delta w)$$

and

$$\frac{\partial F}{\partial w}(\bar{z}_{w_2}(t) + \Delta z, w_2(t) + \Delta w)$$

are bounded in some neighborhood of the origin $(\Delta z, \Delta w) = (0, 0)$, uniformly in $t \in \mathbb{R}$. Also, for $\Delta w \equiv 0$ system (9.2) has a uniformly globally asymptotically stable equilibrium $\Delta z = 0$. By Lemma 2.10, system (9.2) is locally ISS with respect to the input Δw. This implies (see Remark to Definition 2.8) that there exist $k_z > 0$ and $k_w > 0$ such that for any input $\Delta w(t)$ satisfying $|\Delta w(t)| \le k_w$ for all $t \ge t_0$ and $\Delta w(t) \to 0$ as $t \to +\infty$, it holds that any solution $\Delta z(t)$ starting in $|\Delta z(t_0)| \le k_z$ tends to zero, i.e., $\Delta z(t) \to 0$ as $t \to +\infty$.

Choose $t_0 \in \mathbb{R}$ such that $|w_1(t) - w_2(t)| \le k_w$ for all $t \ge t_0$. Consider a solution of the system

$$\dot{z} = F(z, w_1(t)) \tag{9.3}$$

starting in a point $z(t_0)$ satisfying $|z(t_0) - \bar{z}_{w_2}(t_0)| \le k_z$. By the reasoning presented above, $\Delta z(t) := z(t) - \bar{z}_{w_2}(t) \to 0$ as $t \to +\infty$. At the same time, $\bar{z}_{w_1}(t)$ is a uniformly globally asymptotically stable solution of system (9.3). Hence, $z(t) - \bar{z}_{w_1}(t) \to 0$ as $t \to +\infty$. Therefore, $\bar{z}_{w_2}(t) - \bar{z}_{w_1}(t) \to 0$ as $t \to +\infty$. □

9.4 Proof of Property 2.27

Consider some input $w(\cdot) \in \overline{\mathbb{PC}}_m$. Since the y-subsystem is input-to-state convergent, there exists a solution $\bar{y}_w(t)$ that is defined and bounded on $\mathbb{R}$. Since the z-subsystem with (y, w) as inputs is input-to-state convergent, there exists a steady-state solution $\bar{z}_w(t)$ corresponding to the input $(\bar{y}_w(t), w(t))$. This $\bar{z}_w(t)$ is defined and bounded on $\mathbb{R}$.

Let $(z(t), y(t))$ be some solution of system (2.27) with some input $\tilde{w}(t)$. Denote $\Delta z := z - \bar{z}_w(t)$, $\Delta y := y - \bar{y}_w(t)$ and $\Delta w = \tilde{w} - w(t)$. Then Δz and Δy satisfy the equations

$$\Delta\dot{z} = F(\bar{z}_w(t) + \Delta z, \bar{y}_w(t) + \Delta y, w(t) + \Delta w) - F(\bar{z}_w(t), \bar{y}_w(t), w(t)), \quad (9.4)$$

$$\Delta\dot{y} = G(\bar{y}_w(t) + \Delta y, w(t) + \Delta w) - G(\bar{y}_w(t), w(t)). \quad (9.5)$$

Since both the z-subsystem and the y-subsystem of system (2.27) are input-to-state convergent, system (9.4) with $(\Delta y, \Delta w)$ as input is ISS and system (9.5) with Δw as input is ISS. Hence, by Theorem 2.12 the interconnected system (9.4), (9.5) is ISS. In the original coordinates (z, y) this means that system (2.27) is ISS with respect to the solution $(\bar{z}_w(t), \bar{y}_w(t))$. This implies that system (2.27) is input-to-state convergent. □

9.5 Proof of Theorem 2.29

As follows from Lemma 2.30, the Demidovich condition (2.29) guarantees that every solution of system (2.19) corresponding to an input $w(\cdot) \in \overline{\mathbb{PC}}(\mathcal{W})$ is globally exponentially stable. Namely, condition (2.29) implies that inequality (2.31) is satisfied for any two points $z_1, z_2 \in \mathbb{R}^d$ and for any $w \in \mathcal{W}$. This, in turn, implies that for a given input $w(t)$ taking its values in $\mathcal{W}$ and for any two solutions $z_1(t)$ and $z_2(t)$ corresponding to this input, the derivative of the function $V(z_1, z_2) = \frac{1}{2}(z_1 - z_2)^T P(z_1 - z_2)$ satisfies

$$\frac{d}{dt}V(z_1(t), z_2(t)) = (z_1 - z_2)^T P(F(z_1, w) - F(z_2, w)) \leq -2\beta V(z_1, z_2). \quad (9.6)$$

The last inequality guarantees that the difference between any two solutions $z_1(t)$ and $z_2(t)$ corresponding to some input $w(\cdot) \in \overline{\mathbb{PC}}(\mathcal{W})$ exponentially tends to zero. In particular,

$$|z_1(t) - z_2(t)|_P \leq \mathrm{e}^{-\beta(t-t_0)}|z_1(t_0) - z_2(t_0)|_P,$$

where $|z|_P$ denotes $|z|_P := (z^T P z)^{1/2}$.

It remains to show that for any $w(\cdot) \in \overline{\mathbb{PC}}_m$ there exists a solution $\bar{z}_w(t)$ that is defined and bounded on $\mathbb{R}$. To show this, consider the function $W(z) := \frac{1}{2}z^T P z$. The derivative of this function along a solution $z(t)$ corresponding to some bounded input $w(t)$ equals

$$\frac{d}{dt}W(z(t)) = z^T PF(z,w) = (z-0)^T P(F(z,w) - F(0,w)) + z^T PF(0,w).$$

Applying inequality (2.31), we obtain

$$\frac{d}{dt}W(z(t)) \leq -\beta|z|_P^2 + |z^T PF(0,w)|.$$

Notice that the term $|z^T PF(0,w)|$ can be estimated using the Cauchy inequality in the following way. Let the matrix Π be such that $P = \Pi^T \Pi$. Then $z^T PF(0,w) = (\Pi z)^T (\Pi F(0,w))$, i.e., $z^T PF(0,w)$ equals the scalar product of Πz and $\Pi F(0,w)$. By the Cauchy inequality, we obtain

$$|(\Pi z)^T (\Pi F(0,w))| \leq |\Pi z||\Pi F(0,w)| = |z|_P |F(0,w)|_P.$$

Here we have used the equality $|z|_P = |\Pi z|$. With this estimate of the term $|z^T PF(0,w)|$, we obtain

$$\frac{d}{dt}W(z(t)) \leq -\beta|z|_P^2 + |z|_P|F(0,w(t))|_P \leq -|z|_P(\beta|z|_P - \sup_{t\in\mathbb{R}}|F(0,w(t))|_P).$$

In this inequality, $\sup_{t\in\mathbb{R}}|F(0,w(t))|_P$ is finite, since $F(0,w)$ is continuous in $w \in \mathcal{W}$ and, since $w(\cdot) \in \overline{\mathbb{PC}}(\mathcal{W})$, $w(t)$ belongs to a compact subset of $\mathcal{W}$ for all $t \in \mathbb{R}$. So, we obtain that for a given input $w(\cdot) \in \overline{\mathbb{PC}}(\mathcal{W})$, the inequality $dW/dt(z) \leq 0$ holds for $|z|_P \geq r$, where $r := \sup_{t\in\mathbb{R}}|F(0,w(t))|_P/\beta$. Consequently, the set $\{z : |z|_P \leq r\}$ is a compact positively invariant set. The existence of a solution $\bar{z}_w(t)$ that is defined and bounded for all $t \in \mathbb{R}$ follows from Lemma 2.31. Thus, for every input $w(\cdot) \in \overline{\mathbb{PC}}(\mathcal{W})$ there exists a solution $\bar{z}_w(t)$ that is defined and bounded for all $t \in \mathbb{R}$ and is globally exponentially stable. Hence, system (2.19) is globally exponentially convergent for the class of inputs $\overline{\mathbb{PC}}(\mathcal{W})$. The fact that the steady-state solution $\bar{z}_w(t)$ lies in the set $\{z : |z|_P \leq r\}$, where $r := \sup_{t\in\mathbb{R}}|F(0,w(t))|_P/\beta$, determines that system (2.19) has the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\mathcal{W})$. Namely, for any compact set $\mathcal{K}_w \subset \mathcal{W}$ the compact set $\mathcal{K}_z$ in the definition of the UBSS property can be chosen equal to

$$\mathcal{K}_z := \left\{ z : |z|_P \leq \sup_{w\in\mathcal{K}_w} |F(0,w)|_P/\beta \right\}.$$

Let us show that for the case of $\mathcal{W} = \mathbb{R}^d$ system (2.19) is input-to-state convergent for the class of inputs $\overline{\mathbb{PC}}_m$. Consider some input $w(\cdot) \in \overline{\mathbb{PC}}_m$ and the corresponding steady-state solution $\bar{z}_w(t)$. Let $z(t)$ be a solution of system (2.19) corresponding to some input $\hat{w}(\cdot) \in \overline{\mathbb{PC}}_m$. Denote $\Delta z := z - \bar{z}_w(t)$ and $\Delta w := \hat{w} - w(t)$. Then Δz satisfies the equation

$$\Delta\dot{z} = F(\bar{z}_w(t) + \Delta z, w(t) + \Delta w) - F(\bar{z}_w(t), w(t)). \tag{9.7}$$

We will use Theorem 2.9 to show that this system is ISS. Then input-to-state stability of system (9.7) implies that system (2.19) is input-to-state convergent for the class of inputs $\overline{\mathbb{PC}}_m$.

Consider the function $V(\Delta z) = \frac{1}{2}(\Delta z)^T P \Delta z$. Its derivative along solutions of system (9.7) satisfies

$$\begin{aligned}\frac{dV}{dt} &= \Delta z^T P\{F(\bar{z}_w(t) + \Delta z, w(t) + \Delta w(t)) - F(\bar{z}_w(t), w(t))\} \\ &= \Delta z^T P\{F(\bar{z}_w(t) + \Delta z, w(t) + \Delta w(t)) - F(\bar{z}_w(t), w(t) + \Delta w(t))\} \\ &\quad + \Delta z^T P\{F(\bar{z}_w(t), w(t) + \Delta w(t)) - F(\bar{z}_w(t), w(t))\}.\end{aligned} \tag{9.8}$$

Applying Lemma 2.30 to the first component in formula (9.8), we obtain

$$\Delta z^T P\{F(\bar{z}_w(t) + \Delta z, w(t) + \Delta w(t)) - F(\bar{z}_w(t), w(t) + \Delta w(t))\} \leq -\beta|\Delta z|_P^2. \tag{9.9}$$

Applying the Cauchy inequality to the second component in formula (9.8), we obtain the following estimate:

$$|\Delta z^T P\{F(\bar{z}_w(t), w(t)+\Delta w(t)) - F(\bar{z}_w(t), w(t))\}| \leq |\Delta z|_P |\delta(t, \Delta w)|_P, \tag{9.10}$$

where

$$\delta(t, \Delta w) := F(\bar{z}_w(t), w(t) + \Delta w) - F(\bar{z}_w(t), w(t)).$$

Since $F(z, w)$ is continuous and $\bar{z}_w(t)$ and $w(t)$ lie in compact sets for all $t \in \mathbb{R}$, the function $\delta(t, \Delta w)$ is continuous in Δw uniformly in $t \in \mathbb{R}$. This, in turn, implies that $\tilde{\rho}(r) := \sup_{t\in\mathbb{R}} \sup_{|\Delta w|\leq r} |\delta(t, \Delta w)|_P$ is a continuous nondecreasing function. Define the function $\rho(r) := \tilde{\rho}(r) + r$. This function is continuous, strictly increasing, and $\rho(0) = 0$. Thus, it is a class $\mathcal{K}$ function. Also, due to the definition of $\rho(r)$, we obtain the following estimate:

$$|\delta(t, \Delta w)|_P \leq \rho(|\Delta w|).$$

After substituting this estimate together with estimates (9.10) and (9.9) in formula (9.8), we obtain

$$\frac{dV}{dt} \leq -\beta|\Delta z|_P^2 + |\Delta z|_P \rho(|\Delta w|). \tag{9.11}$$

From this formula we obtain that

$$\frac{dV}{dt} \leq -\frac{\beta}{2}|\Delta z|_P^2, \quad \forall\ |\Delta z|_P \geq \frac{2}{\beta}\rho(|\Delta w|). \tag{9.12}$$

Thus, applying Theorem 2.9, we obtain that system (9.7) is input-to-state stable. This completes the proof of the theorem. □

9.6 Proof of Theorem 2.33

We will show that the conditions of the theorem imply the existence of $\beta > 0$ such that for any $w \in \mathcal{W}$ and for any two points $z_1, z_2 \in \mathbb{R}^d$ it holds that

$$(z_1 - z_2)^T P(F(z_1, w) - F(z_2, w)) \leq -\beta |z_1 - z_2|_P^2. \tag{9.13}$$

Once this relation is proved, the rest of the proof repeats the proof of Theorem 2.29. Basically, the proof of inequality (9.13) replaces Lemma 2.30 in the proof of Theorem 2.29.

Notice that inequality (9.13) holds automatically for $z_1 = z_2$. Hence, we need to prove it only for the case $z_1 \neq z_2$. Inequality (9.13) will be proved in several steps. First, consider two points $z_1, z_2 \in \mathbb{R}^d$, $z_1 \neq z_2$, such that the open line segment (z_1, z_2) connecting these points, thus not containing these points, lies entirely in the set $\mathcal{C} := \mathbb{R}^d \setminus \Gamma$, as shown in Figure 9.1. Then, by Lemma 2.30, inequality (9.13) holds for these z_1 and z_2 and for any $w \in \mathcal{W}$.

Second, consider two points $z_1, z_2 \in \mathbb{R}^d$ such that the open line segment (z_1, z_2) connecting these points intersects the set Γ in a finite number of points, as shown in Figure 9.2. Denote $y_1 := z_1$, $y_p := z_2$, and y_i, $i = 2, \ldots, p-1$—the points of intersection of the line segment (z_1, z_2) with the set Γ. The points $y_1, \ldots, y_p$ are ordered in such a way that any open line segment (y_i, y_{i+1}), $i = 1, \ldots, p-1$, does not intersect Γ and $y_i \neq y_{i+1}$ for $i = 1, \ldots, p-1$. Denote $e := (z_1 - z_2)/|z_1 - z_2|_P$. Since all points y_i, $i = 1, \ldots, p$, lie on the same closed line segment $[z_1, z_2]$, and they are ordered, we obtain

$$e = \frac{y_i - y_{i+1}}{|y_i - y_{i+1}|_P}, \quad i = 1, \ldots, p-1. \tag{9.14}$$

Taking this fact into account, we obtain

$$(z_1 - z_2)^T P(F(z_1, w) - F(z_2, w)) = |z_1 - z_2|_P \sum_{i=1}^{p-1} e^T P(F(y_i, w) - F(y_{i+1}, w))$$

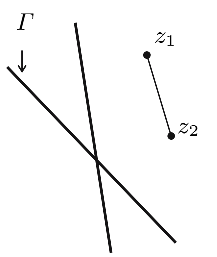

Fig. 9.1. The line segment (z_1, z_2) does not intersect Γ.

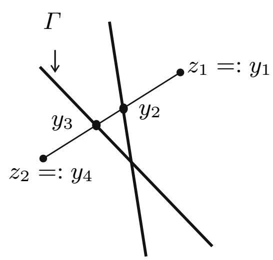

Fig. 9.2. The line segment (z_1, z_2) intersects the set Γ in a finite number of points. The points $y_1, \ldots, y_4$ are ordered.

$$= |z_1 - z_2|_P \sum_{i=1}^{p-1} \frac{(y_i - y_{i+1})^T P(F(y_i, w) - F(y_{i+1}, w))}{|y_i - y_{i+1}|_P}. \tag{9.15}$$

Notice that for each pair of points y_i and y_{i+1}, $i = 1, \dots, p-1$, the open line segment (y_i, y_{i+1}) connecting these points, but not including them, lies entirely in the set $\mathcal{C} := \mathbb{R}^d \setminus \Gamma$. Thus, as follows from the first step of the proof,

$$(y_i - y_{i+1})^T P(F(y_i, w) - F(y_{i+1}, w)) \leq -\beta |y_i - y_{i+1}|_P^2.$$

Substituting this inequality in (9.15), we obtain

$$(z_1 - z_2)^T P(F(z_1, w) - F(z_2, w)) \leq -\beta |z_1 - z_2|_P \sum_{i=1}^{p-1} |y_i - y_{i+1}|_P.$$

Since all points y_i, $i = 1, \dots, p$, lie on the same line segment $[z_1, z_2]$ and they are ordered,

$$\sum_{i=1}^{p-1} |y_i - y_{i+1}|_P = |y_1 - y_p|_P = |z_1 - z_2|_P. \tag{9.16}$$

This fact implies

$$(z_1 - z_2)^T P(F(z_1, w) - F(z_2, w)) \leq -\beta |z_1 - z_2|_P^2.$$

In the third step, consider two points $z_1, z_2 \in \mathbb{R}^d$ such that the open line segment (z_1, z_2) connecting these points intersects the set Γ in infinite number of points. In our case, this means that the line segment (z_1, z_2) belongs to one or more hyperplanes constituting the set Γ. These hyperplanes are given by the equations $H_j^T z + h_j = 0$, $j = 1, \dots, k$. Consider the sequences $\{z_{1i}\}_{i=1}^{+\infty}$ and $\{z_{2i}\}_{i=1}^{+\infty}$ such that $z_{1i} \to z_1$ and $z_{2i} \to z_2$ as $i \to +\infty$ and such that each line segment (z_{1i}, z_{2i}), $i = 1, 2, \dots$, intersects the set Γ in a finite number of points. Such sequences exist because for small variations of either z_1 or z_2 in the direction transversal to the hyperplane $H_j^T z + h_j = 0$ to which the line segment (z_1, z_2) belongs, the varied line segment $(\tilde{z}_1, \tilde{z}_2)$ does not lie in the hyperplane $H_j^T z + h_j = 0$. Since each line segment (z_{1i}, z_{2i}), $i = 1, 2, \dots$ intersects the set Γ in a finite number of points, from the second step we obtain

$$(z_{1i} - z_{2i})^T P(F(z_{1i}, w) - F(z_{2i}, w)) \leq -\beta |z_{1i} - z_{2i}|_P^2, \quad i = 1, 2, \dots.$$

By continuity of the vectorfield $F(z, w)$, in the limit for $i \to +\infty$ we obtain

$$(z_1 - z_2)^T P(F(z_1, w) - F(z_2, w)) \leq -\beta |z_1 - z_2|_P^2.$$

This completes the proof. □

9.7 Proof of Lemma 2.35

Let us first prove sufficiency. Suppose conditions (2.36) are satisfied. To check continuity, we need to check whether $A_i z + b_i = A_j z + b_j$ for all z lying in the set $\Pi := \{z : H^T z + h = 0\}$. Conditions (2.36) imply that $A_i z + b_i - A_j z - b_j = G_H(H^T z + h)$. Hence, if z belongs to the set Π, then $A_i z + b_i - A_j z - b_j = 0$.

At the next step, we prove necessity. Suppose

$$A_i z + b_i - A_j z - b_j = 0 \tag{9.17}$$

for all z lying in the set Π. Every point in Π can be expressed as $z = z_* + \tilde{z}$, where z_* is some point such that $H^T z_* + h = 0$ and $\tilde{z}$ is an arbitrary point satisfying $H^T \tilde{z} = 0$. Substituting this expression in (9.17), we obtain

$$A_i(z_* + \tilde{z}) + b_i - A_j(z_* + \tilde{z}) + b_j = 0.$$

Since (9.17) holds, in particular, for $z = z_*$, from the last expression we obtain $(A_i - A_j)\tilde{z} = 0$ for all $\tilde{z}$ satisfying $H^T \tilde{z} = 0$. Hence, $(A_i - A_j)\tilde{z} = 0$ for all $\tilde{z} \in \mathrm{Ker} H^T$. Let A_{ij}^r be the rth row of the matrix $A_i - A_j$. Then

$$A_{ij}^r \tilde{z} = 0 \quad \forall \tilde{z} \in \mathrm{Ker} H^T. \tag{9.18}$$

Since H is a nonzero vector, $\mathrm{Ker} H^T$ is a $(d-1)$-dimensional subspace. The relation $A_{ij}^r \tilde{z} = 0$ for all $\tilde{z} \in \mathrm{Ker} H^T$ implies that A_{ij}^r lies in the orthogonal compliment to $\mathrm{Ker} H^T$, which is spanned by H^T. Hence, $A_{ij}^r = \alpha_r H^T$ for some scalar α_r. Repeating this analysis for all rows of the matrix $A_i - A_j$, we obtain $A_i - A_j = G_H H^T$, where the vector G_H equals $G_H = (\alpha_1, \ldots, \alpha_d)^T$. After substituting this equality in (9.17), we obtain $G_H H^T z + b_i - b_j = 0$ for all z satisfying $H^T z + h = 0$. This, in turn, implies $b_i - b_j = G_H h$. □

9.8 Proof of Theorem 2.40

Conditions (2.44) and (2.45) imply that the set $N(\rho) := \{z : V_2(z) \leq \alpha_5 \circ \gamma(\rho)\}$ is positively invariant for every input $w(\cdot) \in \overline{\mathbb{PC}}_m$ satisfying

$$|w(t)| \leq \rho \quad \forall \;\; t \in \mathbb{R}. \tag{9.19}$$

Condition (2.44) implies that for every $\rho \geq 0$ the set $N(\rho)$ is bounded. Hence for any input $w(\cdot) \in \overline{\mathbb{PC}}_m$ every solution of system (2.19) remains in a bounded set $N(\rho)$ for some $\rho \geq \sup_{t \in \mathbb{R}} |w(t)|$.

Consider an input $w(t)$ satisfying (9.19). Then the set $N(\rho)$ is a compact positively invariant set. By Lemma 2.31 there exists a solution $\bar{z}_w(t)$ lying in $N(\rho)$ for all $t \in \mathbb{R}$. By condition (2.44), this solution satisfies $|\bar{z}_w(t)| \leq \alpha_4^{-1} \circ \alpha_5 \circ \gamma(\rho)$ for all $t \in \mathbb{R}$. This is a uniform bound on $|\bar{z}_w(t)|$ for all $w(t)$ satisfying (9.19). Substituting $\bar{z}_w(t)$ for z_2 in (2.42) and (2.43) gives conditions for uniform global asymptotic stability of the solution $\bar{z}_w(t)$ (see [51]). As a consequence, system (2.19) is globally uniformly convergent for the class of inputs $\overline{\mathbb{PC}}_m$. Moreover, since the steady-state solutions $\bar{z}_w(t)$ are bounded on $\mathbb{R}$ uniformly with respect to inputs satisfying (9.19), system (2.19) has the UBSS property. $\square$

9.9 Proof of Theorem 2.41

We will prove this theorem in the following sequence: (i)⇒(iii)⇒(ii)⇒(i).

(i)⇒(iii). Since (2.19) is locally exponentially convergent, for the input $w(t) \equiv 0$ the steady-state solution $\bar{z}_w(t) \equiv 0$ is locally exponentially stable. This implies that the linearization of system $\dot{z} = F(z, 0)$ at the origin is exponentially stable (see [51], Theorem 3.13). This is equivalent to the matrix $\partial F/\partial z(0, 0)$ being Hurwitz.

(iii)⇒(ii). Since $\frac{\partial F}{\partial z}(0, 0)$ is Hurwitz, there exists a positive definite matrix $P = P^T > 0$ such that

$$P\frac{\partial F}{\partial z}(0,0) + \frac{\partial F}{\partial z}^T(0,0)P < 0. \tag{9.20}$$

Denote $J(z, w) = P\frac{\partial F}{\partial z}(z, w) + \frac{\partial F}{\partial z}^T(z, w)P$. Since $F(z, w) \in C^1$, the function $J(z, w)$ is continuous. This implies, since $J(0, 0)$ is negative definite, that $J(z, w)$ is negative definite for all small z and w. Consider a neighborhood of the origin $z = 0$ in the form of the ellipsoid $E_P(R) := \{z : |z|_P < R\}$, where $|z|_P := (z^T P z)^{1/2}$. Choose $R > 0$ and $r > 0$ small enough to guarantee that $J(z, w) < -Q$ for some $Q > 0$, all $z \in E_P(R)$, and all $|w| < r$. Note that the set $E_P(R)$ is convex. Therefore, by Lemma 2.30, this implies the existence of $\beta > 0$ such that

$$(z_1 - z_2)^T P(F(z_1, w) - F(z_2, w)) \le -\beta|z_1 - z_2|_P^2 \tag{9.21}$$

holds for any $z_1, z_2 \in E_P(R)$ and any $|w| < r$. Relation (9.21), in turn, implies that for any input $w(t)$ satisfying $|w(t)| < r$ for all $t \in \mathbb{R}$, any two solutions $z_1(t)$ and $z_2(t)$ lying in $E_P(R)$ for all $t \ge t_0$ exponentially converge to each other, i.e.,

$$|z_1(t) - z_2(t)|_P \le \mathrm{e}^{-\beta(t-t_0)}|z_1(t_0) - z_2(t_0)|_P. \tag{9.22}$$

Choose a positive number $\bar{r} \le r$ such that $\sup_{|w|<\bar{r}} |F(0, w)|_P < \beta R$. This is possible because $F(0, w)$ is continuous and $F(0, 0) = 0$. With this choice of $\bar{r}$ the set $E_P(R)$ will be invariant for any input $w(t)$ satisfying $|w(t)| < \bar{r}$. Namely, consider the function $W(z) := \frac{1}{2}|z|_P^2$. Its derivative along a solution of (2.19) equals

$$\frac{dW}{dt} = z^T PF(z, w) = (z - 0)^T P(F(z, w) - F(0, w)) + z^T PF(0, w).$$

Applying inequality (9.21) and Cauchy inequality, we obtain

$$\frac{dW}{dt} \leq -|z|_P \left(\beta |z|_P - \sup_{t\in\mathbb{R}} |F(0, w(t))|_P \right).$$

Thus, if $|w(t)| \leq \bar{r}$ for all $t \in \mathbb{R}$, then for $|z|_P = R$ it holds that

$$\frac{dW}{dt} \leq -R \left(\beta R - \sup_{|w|\leq\bar{r}} |F(0, w)|_P \right).$$

By the choice of $\bar{r}$ we obtain $dW/dt \leq 0$ for all $|z|_P = R$. Thus, the set $E_P(R)$ is compact and positively invariant with respect to system (2.19) with any input $w(t)$ satisfying the inequality $|w(t)| \leq \bar{r}$ for all $t \in \mathbb{R}$. This implies that, first, by Lemma 2.31, for every $w(t)$ satisfying $|w(t)| \leq \bar{r}$ for all $t \in \mathbb{R}$ there exists a solution $\bar{z}_w(t)$ that is defined for all $t \in \mathbb{R}$ and lies in $E_P(R)$ for all $t \in \mathbb{R}$. Second, since $E_P(R)$ is invariant, any two solutions starting in $E_P(R)$ satisfy (9.22). Hence, for any input $w(t)$ satisfying $|w(t)| \leq \bar{r}$, for all $t \in \mathbb{R}$, the corresponding steady-state solution $\bar{z}_w(t)$ is defined and bounded on $\mathbb{R}$ and it is exponentially stable in $E_P(R)$, i.e., the system (2.19) is convergent in $\mathcal{Z} := E_P(R)$. Thus, system (2.19) is locally exponentially convergent for the class of inputs $\overline{\mathbb{PC}}_m$.

(ii)⇒(i). This implication follows from the following inclusion:

$$\mathcal{N} \bigcap \{w(\cdot) : |w(t)| < \bar{r}, t \in \mathbb{R}\} \subset \overline{\mathbb{PC}}_m \bigcap \{w(\cdot) : |w(t)| < \bar{r}, t \in \mathbb{R}\}.$$

Thus, if system (2.19) is exponentially convergent in some neighborhood of the origin for the class of inputs $\overline{\mathbb{PC}}_m$ subjected to the constraint $|w(t)| < \bar{r}$, for some $\bar{r} > 0$ and all $t \in \mathbb{R}$, then it is also exponentially convergent in the same neighborhood for the class of inputs $\mathcal{N}$ subjected to the constraint $|w(t)| < \bar{r}$. □

9.10 Proof of Lemma 4.3

(i)⇒(ii). We prove the existence of $\alpha(w)$ by constructing this mapping. Since system (4.8) is uniformly convergent in $\mathcal{Z}$ for the class of inputs $\mathcal{I}_s(\mathcal{W})$, for any solution of the exosystem $w(t, w_0)$ starting in $w(0, w_0) = w_0 \in \mathcal{W}$ there exists a unique steady-state solution $\bar{z}_w(t)$ that is bounded on $\mathbb{R}$ and uniformly asymptotically stable in $\mathcal{Z}$. To emphasize the dependency of this steady-state solution on w_0, we denote it by $\bar{z}(t, w_0)$. Construct the mapping $\alpha(w)$ in the following way: for every $w_0 \in \mathcal{W}$ and every $t \in \mathbb{R}$ set $\alpha(w(t, w_0)) := \bar{z}(t, w_0)$ or, equivalently, $\alpha(w_0) = \bar{z}(0, w_0)$. By the definition of the mapping $\alpha(w)$, the graph $z = \alpha(w)$ is invariant with respect to (4.8) and (4.9) and any solution $z(t) = \alpha(w(t))$ lying on this manifold is uniformly asymptotically stable in $\mathcal{Z}$.

It remains to show that the mapping $z = \alpha(w)$, constructed above, is continuous, i.e., that for any $w_1 \in \mathcal{W}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|w_1 - w_2| < \delta$ implies $|\alpha(w_1) - \alpha(w_2)| < \varepsilon$. In the following we assume that $\varepsilon > 0$ and $w_1 \in \mathcal{W}$ are chosen arbitrarily and are fixed, while w_2 varies in the closed ball $\mathcal{K}_0 := \{w : |w_1 - w| \leq r\}$, where $r > 0$ is chosen such that $\mathcal{K}_0 \subset \mathcal{W}$. Such r exists because $w_1 \in \mathcal{W}$ and $\mathcal{W}$ is an open set.

As a preliminary observation, notice that by the conditions given in (i), for the compact set $\mathcal{K}_0$ there exists a compact set $\mathcal{K}_z \subset \mathcal{Z}$ such that $\bar{z}(t, w_0) \in \mathcal{K}_z$ for all $w_0 \in \mathcal{K}_0$ and all $t \in \mathbb{R}$. Hence, since $\alpha(w(t, w_0)) \equiv \bar{z}(t, w_0)$, for any w_1 and w_2 from the set $\mathcal{K}_0$ it holds that $\alpha(w(t, w_i)) \in \mathcal{K}_z$ for $i = 1, 2$ and for all $t \in \mathbb{R}$.

To prove continuity of $\alpha(w)$, we introduce the function

$$\varphi_T(w_1, w_2) := \hat{z}(0, -T, \alpha(w(-T, w_2)), w_1),$$

where $T > 0$ will be specified later and $\hat{z}(t, t_0, z_0, w_*)$ is the solution of the time-varying system

$$\dot{\hat{z}} = F(\hat{z}, w(t, w_*)) \tag{9.23}$$

satisfying the initial conditions $\hat{z}(t_0, t_0, z_0, w_*) = z_0$.

The function $\varphi_T(w_1, w_2)$ has the following meaning, which is illustrated in Figure 9.3. First, consider the steady-state solution $\alpha(w(t, w_2))$, which is a solution of system (9.23) with the input $w(t, w_2)$. At time $t = 0$, $\alpha(w(0, w_2)) = \alpha(w_2)$. Next, we shift along $\alpha(w(t, w_2))$ to time $t = -T$ and appear in $\alpha(w(-T, w_2))$. Then we switch the input in system (9.23) to $w(t, w_1)$, shift forward to the time instant $t = 0$ along the solution $\hat{z}(t)$ starting in $\hat{z}(-T) = \alpha(w(-T, w_2))$, and appear in $\hat{z}(0) = \varphi_T(w_1, w_2)$. Notice that $\varphi_T(w_0, w_0) = \alpha(w_0)$ (there is no switch of inputs and we just shift back and forth along the same solution $\alpha(w(t, w_0))$). Thus,

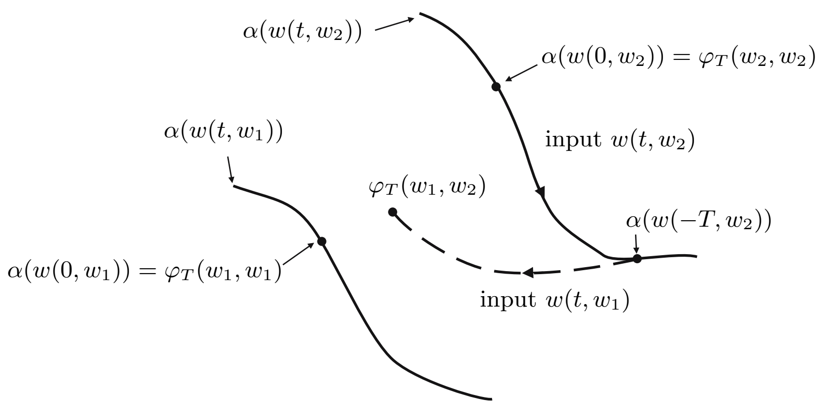

Fig. 9.3. The construction of the function $\varphi_T(w_1, w_2)$.

$$\begin{aligned} \alpha(w_1) - \alpha(w_2) &= \varphi_T(w_1, w_1) - \varphi_T(w_2, w_2) \\ &= \varphi_T(w_1, w_1) - \varphi_T(w_1, w_2) \\ &\quad + \varphi_T(w_1, w_2) - \varphi_T(w_2, w_2). \end{aligned} \tag{9.24}$$

By the triangle inequality, this implies

$$\begin{aligned} |\alpha(w_1) - \alpha(w_2)| \leq& \ |\varphi_T(w_1, w_1) - \varphi_T(w_1, w_2)| \\ &+ |\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)|. \end{aligned} \tag{9.25}$$

First, we will show that there exists $T > 0$ such that

$$|\varphi_T(w_1, w_1) - \varphi_T(w_1, w_2)| < \varepsilon/2 \quad \forall \ \ w_2 \in \mathcal{K}_0. \tag{9.26}$$

Second, we will show that given $T > 0$ satisfying (9.26), there exists $\delta > 0$ such that

$$|\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)| < \varepsilon/2 \quad \forall \ \ w_2 : |w_1 - w_2| < \delta. \tag{9.27}$$

Combining inequalities (9.26) and (9.27), we obtain $|\alpha(w_1) - \alpha(w_2)| < \varepsilon$ for all w_2 satisfying $|w_1 - w_2| < \delta$. Due to the arbitrary choice of $\varepsilon > 0$ and $w_1 \in \mathcal{W}$, this proves continuity of $\alpha(w)$ in the set $\mathcal{W}$.

To show (9.26), notice that $\varphi_T(w_1, w_1) = \hat{z}_1(0)$ and $\varphi_T(w_1, w_2) = \hat{z}_2(0)$, where $\hat{z}_1(t)$ and $\hat{z}_2(t)$ are solutions of the system

$$\dot{\hat{z}} = F(\hat{z}, w(t, w_1)), \tag{9.28}$$

with the initial conditions

$$\hat{z}_1(-T) = \alpha(w(-T, w_1)), \quad \hat{z}_2(-T) = \alpha(w(-T, w_2)).$$

By the conditions given in (i), $\hat{z}_1(t) = \alpha(w(t, w_1))$ is a solution of system (9.28); it is uniformly asymptotically stable in $\mathcal{Z}$ and lies in the compact set $\mathcal{K}_z$. By the definition of uniform asymptotic stability in the set $\mathcal{Z}$, this implies that $\hat{z}_1(t)$ attracts all other solutions $\hat{z}(t)$ of system (9.28) uniformly over the initial conditions $t_0 \in \mathbb{R}$ and $\hat{z}(t_0)$ from any compact subset of $\mathcal{Z}$. In particular, for the compact set $\mathcal{K}_z$ and for $\varepsilon > 0$ there exists $T = T(\varepsilon, \mathcal{K}_z) > 0$ such that $\hat{z}(t_0) \in \mathcal{K}_z$ implies

$$|\hat{z}_1(t) - \hat{z}(t)| < \varepsilon/2, \quad \forall \ t \geq t_0 + T(\varepsilon, \mathcal{K}_z), \ t_0 \in \mathbb{R}. \tag{9.29}$$

By the definition of $\hat{z}_2(t)$, $\hat{z}_2(-T) = \alpha(w(-T, w_2))$. Since $\alpha(w(t, w_2)) \in \mathcal{K}_z$ for all $t \in \mathbb{R}$ (see above), $\hat{z}_2(-T) \in \mathcal{K}_z$. Thus, for $t_0 = -T$ and $t = 0$ formula (9.29) implies

$$|\hat{z}_1(0) - \hat{z}_2(0)| < \varepsilon/2, \tag{9.30}$$

which is equivalent to (9.26).

To show (9.27), notice that for a fixed $T > 0$, the function $\hat{z}(0, -T, z_0, w_0)$ is continuous with respect to z_0 and w_0. Thus, it is uniformly continuous over the compact set $G := \{(z_0, w_0) : \ z_0 \in \mathcal{K}_z, w_0 \in \mathcal{K}_0\}$. Hence, there exists $\delta > 0$, such that if $z_0 \in \mathcal{K}_z$, $w_1 \in \mathcal{K}_0$, $w_2 \in \mathcal{K}_0$, and $|w_1 - w_2| < \delta$, then

$$|\hat{z}(0, -T, z_0, w_1) - \hat{z}(0, -T, z_0, w_2)| < \varepsilon/2. \tag{9.31}$$

Recall that by the definition of $\varphi_T(w_1, w_2)$

$$\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2) = \hat{z}(0, -T, z_0, w_1) - \hat{z}(0, -T, z_0, w_2), \tag{9.32}$$

where $z_0 := \alpha(w(-T, w_2))$. Since $w_1 \in \mathcal{K}_0$, $w_2 \in \mathcal{K}_0$, and $\alpha(w(-T, w_2)) \in \mathcal{K}_z$, it follows from (9.31) and (9.32) that

$$|w_1 - w_2| < \delta \quad \Rightarrow \quad |\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)| < \varepsilon/2.$$

Thus, we have shown (9.27). This completes the proof of continuity of $\alpha(w)$ and completes the proof of implication (i)$\Rightarrow$(ii).

Let us prove relation (4.10). Suppose $\alpha_1 : \mathcal{W} \to \mathcal{Z}$ and $\alpha_2 : \mathcal{W} \to \mathcal{Z}$ are continuous mappings such that the sets

$$\mathcal{M}_1(\mathcal{W}) := \{(z, w) : z = \alpha_1(w), \ w \in \mathcal{W}\},$$
$$\mathcal{M}_2(\mathcal{W}) := \{(z, w) : z = \alpha_2(w), \ w \in \mathcal{W}\},$$

are invariant with respect to systems (4.8) and (4.9). Consider some solution of the exosystem $w(t)$ lying in $\mathcal{W}$ (i.e., $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$). For this solution, the functions $z_1(t) := \alpha_1(w(t))$ and $z_2(t) := \alpha_2(w(t))$ are solutions of system (4.8) lying in the set $\mathcal{Z}$ for all $t \in \mathbb{R}$. Since system (4.8) is uniformly convergent in $\mathcal{Z}$ for the input $w(t)$, there exists a steady-state solution $\bar{z}_w(t)$ attracting all solutions of system (4.8) starting in $\mathcal{Z}$. This implies $|\bar{z}_w(t) - z_i(t)| \to 0$ as $t \to +\infty$ for $i = 1, 2$. By the triangle inequality, the last expression implies $|z_1(t) - z_2(t)| \to 0$ as $t \to +\infty$. Hence, we have proved relation (4.10).

Let us show that if a solution $w(t)$ of the exosystem (4.9) lies in a compact set $\mathcal{K}_w \subset \mathcal{W}$ for all $t \in \mathbb{R}$, then $\alpha_1(w(t)) \equiv \alpha_2(w(t))$ for $t \in \mathbb{R}$. Since $\alpha_1(w)$ and $\alpha_2(w)$ are continuous, $z_1(t) := \alpha_1(w(t))$ and $z_2(t) := \alpha_2(w(t))$ are two solutions of system (4.8) corresponding to the same input $w(t)$ and lying in the compact set $\mathcal{K}_z := \alpha_1(\mathcal{K}_w) \bigcup \alpha_2(\mathcal{K}_w) \subset \mathcal{Z}$ for all $t \in \mathbb{R}$. Since system (4.8) is uniformly convergent in the set $\mathcal{Z}$ for the class of inputs $\mathcal{I}_s(\mathcal{W})$, the steady-state solution is uniformly asymptotically stable in $\mathcal{Z}$. By Property 2.5, the steady-state solution $\bar{z}_w(t)$ is unique and, by Property 2.4, we obtain $\bar{z}_w(t) \equiv z_i(t)$ for $i = 1, 2$ and $t \in \mathbb{R}$. Hence, $z_1(t) \equiv z_2(t)$ or, equivalently, $\alpha_1(w(t)) \equiv \alpha_2(w(t))$ for $t \in \mathbb{R}$.

If system (4.9) satisfies the boundedness assumption **A1** in the set $\mathcal{W}$, then any solution $w(t)$ starting in $w(0) \in \mathcal{W}$ lies in a compact subset of $\mathcal{W}$. Therefore, by the reasoning presented above, the mapping $\alpha(w)$ defined in (ii) is unique.

Let us show that under the boundedness assumption **A1** on system (4.9), (ii) implies (i). We show that system (4.8) is uniformly convergent in $\mathcal{Z}$ for the class of inputs $\mathcal{I}_s(\mathcal{W})$. Recall that the class $\mathcal{I}_s(\mathcal{W})$ contains all solutions of system (4.9) starting in $\mathcal{W}$. Due to assumption **A1**, any solution $w(t)$ of system (4.9) starting in $w(0) \in \mathcal{W}$ lies in some compact set $\mathcal{K}_w \subset \mathcal{W}$ for all $t \in \mathbb{R}$. For any $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$, define $\bar{z}_w(t) := \alpha(w(t))$. Because the graph $\mathcal{M}(\mathcal{W})$ is invariant with respect to systems (4.8) and (4.9), $\bar{z}_w(t)$ is a solution of system (4.8) that is defined and bounded on $\mathbb{R}$. By the conditions given in (ii), $\bar{z}_w(t)$ is uniformly asymptotically stable in $\mathcal{Z}$. Hence, system (4.8) is uniformly convergent in $\mathcal{Z}$ for the class of inputs $\mathcal{I}_s(\mathcal{W})$. Due to the boundedness assumption **A1**, for any compact set of initial conditions $\mathcal{K}_0 \subset \mathcal{W}$ there exists a compact set $\mathcal{K}_w \subset \mathcal{W}$ such that if $w(0) \in \mathcal{K}_0$, then $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$. Since $\alpha(w)$ is a continuous mapping from $\mathcal{W}$ to $\mathcal{Z}$, for the compact set $\mathcal{K}_w \subset \mathcal{W}$ the set $\mathcal{K}_z := \alpha(\mathcal{K}_w) \subset \mathcal{Z}$ is also compact. Therefore, if a solution of the exosystem $w(t)$ starts in $w(0) \in \mathcal{K}_0$, then the corresponding steady-state solution $\bar{z}_w(t) = \alpha(w(t))$ lies in the compact set $\mathcal{K}_z$ for all $t \in \mathbb{R}$. Hence, we have proved (i). This completes the proof of the lemma. $\square$

9.11 Proof of Theorem 4.6

The proof of this theorem is based on the fact that any system (4.9) with a compact positively invariant set $\mathcal{W}_+$ can be extended to some neighborhood of $\mathcal{W}_+$ in such a way that the extended system satisfies the boundedness assumption **A1** in this neighborhood and, at the same time, it has the same dynamics on $\mathcal{W}_+$ as the original system (4.9). This statement is formulated in the following lemma.

Lemma 9.1. *Consider system (4.9). Suppose $\mathcal{W}_+ \subset \mathbb{R}^m$ is a compact set that is positively invariant with respect to system (4.9). Then for any open set $\widetilde{\mathcal{W}} \supset \mathcal{W}_+$ there exists a system*

$$\dot{w} = \tilde{s}(w) \tag{9.33}$$

such that $\tilde{s}(w)$ is a locally Lipschitz function, $\tilde{s}(w) = s(w)$ for all $w \in \mathcal{W}_+$ and system (9.33) satisfies the boundedness assumption **A1** *in the set $\widetilde{\mathcal{W}}$.*

Proof: Since $\mathcal{W}_+$ is a compact set and $\widetilde{\mathcal{W}}$ is a neighborhood of $\mathcal{W}_+$, we can choose $R > 0$ such that the set $\mathcal{L} := \{w : \mathrm{dist}(w, \mathcal{W}_+) \leq R\}$ lies in $\widetilde{\mathcal{W}}$. Consider system (4.8) and system (9.33) with $\tilde{s}(w)$ being a locally Lipschitz function such that $\tilde{s}(w) = s(w)$ for all $w \in \mathcal{W}_+$, and $\tilde{s}(w) = 0$ for all w satisfying $\mathrm{dist}(w, \mathcal{W}_+) \geq R$. For example, $\tilde{s}(w)$ can be chosen equal to $\tilde{s}(w) := \psi(\mathrm{dist}(w, \mathcal{W}_+))s(w)$, where $\psi(x)$ is a smooth scalar function satisfying $\psi(x) = 1$ for $x = 0$ and $\psi(x) = 0$ for $x \geq R$. In particular, one can choose $\psi(x)$ to be equal to (see [51]):

$$\psi(x) := 1 - \frac{1}{b}\int_0^x \mathrm{e}^{\left(\frac{-1}{y}\right)}\mathrm{e}^{\left(\frac{-1}{R-y}\right)}dy \quad \text{for} \quad 0 \leq x < R,$$

and $\psi(x) := 0$ for $x \geq R$, where b is chosen such that $\psi(R) = 0$. Since $\mathrm{dist}(w, \mathcal{W}_+)$ is a globally Lipschitz function of w, $\psi(x)$ is smooth, and $s(w)$ is locally Lipschitz, the function $\tilde{s}(w)$ defined above is locally Lipschitz.

Notice that system (9.33) satisfies the boundedness assumption **A1** in the open set $\widetilde{\mathcal{W}}$. Namely, the right-hand side of (9.33) is constructed in such a way that it is not equal to zero only inside the compact set $\mathcal{L}$, which lies strictly inside $\widetilde{\mathcal{W}}$. Moreover, the set $\mathcal{L}$ is invariant because $\tilde{s}(w) = 0$ on the boundary of $\mathcal{L}$. Thus, if a solution of system (9.33) starts in a point $w(0) \in \widetilde{\mathcal{W}}$, then it either lies in the compact set $\mathcal{L}$ or remains constant. Hence, if a solution of system (9.33) starts in a compact set $\mathcal{K}_0 \subset \widetilde{\mathcal{W}}$, then it remains in the compact set $\mathcal{K}_w := \mathcal{K}_0 \bigcup \mathcal{L}$ for all $t \in \mathbb{R}$. □

Now we can prove Theorem 4.6. Since system (9.33) satisfies the boundedness assumption **A1** in the set $\widetilde{\mathcal{W}}$, $\mathcal{I}_{\tilde{s}}(\widetilde{\mathcal{W}})$—the class of solutions of system (9.33) starting in the open invariant set $\widetilde{\mathcal{W}}$—satisfies $\mathcal{I}_{\tilde{s}}(\widetilde{\mathcal{W}}) \subset \overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$. Therefore, the fact that system (4.8) is globally uniformly convergent and has the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$ implies that it is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_{\tilde{s}}(\widetilde{\mathcal{W}})$. Applying Theorem 4.4, we conclude that there exists a continuous mapping $\alpha : \widetilde{\mathcal{W}} \to \mathcal{Z}$ such that the set $\mathcal{M} := \{(z, w) : z = \alpha(w),\ w \in \widetilde{\mathcal{W}}\}$ is invariant with respect to systems (4.8) and (9.33) and every solution $z(t) = \alpha(w(t))$ on this manifold is globally uniformly asymptotically stable. Since the dynamics of systems (4.9) and (9.33) coincide in the positively invariant set $\mathcal{W}_+$, the set $\mathcal{M}(\mathcal{W}_+) = \{(z, w) : z = \alpha(w),\ w \in \mathcal{W}_+\}$ is positively invariant with respect to (4.8) and (4.9) and for every solution $w(t)$ of system (4.9) starting in $w(0) \in \mathcal{W}_+$, the solution of system (4.8) $\bar{z}_w(t) := \alpha(w(t))$ considered for $t \geq 0$ is globally uniformly asymptotically stable.

The mapping $\alpha(w)$ depends on the choice of the auxiliary system (9.33), which can be made in many ways. So, in general, such a mapping $\alpha(w)$ is not unique. If $\alpha_1(w)$ and $\alpha_2(w)$ are two such mappings, then for any solution of system (4.9) starting in $w(0) \in \mathcal{W}_+$, the functions $z_1(t) := \alpha_1(w(t))$ and $z_2(t) := \alpha_2(w(t))$ are two solutions of system (4.8) corresponding to the input $w(t)$, which is well defined for all $t \geq 0$. By Property 2.22, there exists a solution $\tilde{z}_w(t)$ defined for $t \geq 0$ that is globally asymptotically stable. Hence, it attracts both $\alpha_1(w(t))$ and $\alpha_2(w(t))$ as $t \to +\infty$. This implies (4.11).

If $w(0) \in \mathcal{W}_\pm$, then the corresponding solution $w(t)$ of system (4.9) lies in the compact set $\mathcal{W}_+$ for all $t \in \mathbb{R}$. Since both $\alpha_1(w)$ and $\alpha_2(w)$ are continuous in $\mathcal{W}$, $\alpha_1(w(t))$ and $\alpha_2(w(t))$ are two solutions of system (4.8) corresponding to the same input $w(t)$ and lying in the compact set $\mathcal{K}_z := \alpha_1(\mathcal{W}_+) \bigcup \alpha_2(\mathcal{W}_+)$ for all $t \in \mathbb{R}$. But due to the global uniform convergence of system (4.8), by Property 2.4 these solutions must coincide with the steady-state solution. Hence, $\alpha_1(w(t)) \equiv \alpha_2(w(t))$ for $t \in \mathbb{R}$. Since this relation holds for any solution $w(t) \in \mathcal{W}_\pm$, we obtain $\alpha_1(w) = \alpha_2(w)$ for all $w \in \mathcal{W}_\pm$. □

9.12 Proof of Theorem 4.8

(il)⇒(iil). This implication is proved using Lemma 4.3. To apply this lemma, we will show that there exists a neighborhood of the origin $\mathcal{Z} \subset \mathbb{R}^d$, and an invariant neighborhood of the origin $\widehat{\mathcal{W}} \subset \mathbb{R}^m$ such that **a)** system (4.8) is uniformly convergent in $\mathcal{Z}$ for the class of inputs $\mathcal{I}_s(\widehat{\mathcal{W}})$ and **b)** there exists a compact set $\mathcal{K}_z \subset \mathcal{Z}$ such that for any $w(\cdot) \in \mathcal{I}_s(\widehat{\mathcal{W}})$ the corresponding steady-state solution $\bar{z}_w(t)$ lies in $\mathcal{K}_z$ for all $t \in \mathbb{R}$. Once **a)** and **b)** are proved, then condition (i) in Lemma 4.3 is satisfied for the sets $\mathcal{Z}$ and $\widehat{\mathcal{W}}$ defined above. By this lemma, there exists a continuous mapping $\alpha : \widehat{\mathcal{W}} \to \mathcal{Z}$ such that the graph $\mathcal{M}(\widehat{\mathcal{W}}) := \{(z, w) = \alpha(w), w \in \widehat{\mathcal{W}}\}$ is invariant with respect to systems (4.8) and (4.9). As follows from Lemma 4.3, the mapping $\alpha(w)$ is uniquely defined for all solutions of the exosystem lying in some compact subset of $\widehat{\mathcal{W}}$ for all $t \in \mathbb{R}$. Consider a closed ball $\bar{B}_\rho := \{w : |w| \le \rho\}$, where $\rho > 0$ is such that $\bar{B}_\rho \subset \widehat{\mathcal{W}}$. Such a ball exists, since $\widehat{\mathcal{W}}$ is a neighborhood of the origin. Choose $\mathcal{W}$ to be an invariant neighborhood of the origin such that $\mathcal{W} \subset \bar{B}_\rho$. Such a neighborhood exists, since $w(t) \equiv 0$ is Lyapunov stable in forward and backward time. Therefore, all solutions of system (4.9) starting in $\mathcal{W}$ lie in a compact subset of $\widehat{\mathcal{W}}$ for all $t \in \mathbb{R}$. Thus, the mapping $\alpha(w)$ satisfies the conditions in (iil), and, by Lemma 4.3, any other function $\tilde{\alpha}(w)$ satisfying these conditions coincides with $\alpha(w)$ for all $w \in \mathcal{W}$. The fact that $\bar{z}_w(t) \equiv 0$ is the steady-state solution corresponding to the input $w(t) \equiv 0$ implies $\alpha(0) = 0$. This proves (iil).

Let us show that the neighborhoods of the origin $\mathcal{Z}$ and $\widehat{\mathcal{W}}$ described above do exist. Notice that in our case system (4.8) satisfies the conditions of Property 2.24. Let $\mathcal{Z}$ be a neighborhood of the origin provided by Property 2.24. Then we can choose $\varepsilon > 0$ such that the closed ball $\bar{B}_\varepsilon := \{z : |z| \le \varepsilon\}$ lies in $\mathcal{Z}$. As follows from Property 2.24, there exists $\delta > 0$ such that system (4.8) is uniformly convergent in $\mathcal{Z}$ for all solutions of the exosystem satisfying $|w(t)| \le \delta$ for all $t \in \mathbb{R}$ and the corresponding steady-state solutions $\bar{z}_w(t)$ lie in the compact set $\bar{B}_\varepsilon \subset \mathcal{Z}$ (such δ can be chosen equal to $\delta := \gamma^{-1}(\varepsilon)$, where $\gamma(r)$ is the class $\mathcal{K}$ function from Property 2.24). Because $w(t) \equiv 0$ is stable in forward and backward time, there exists an invariant neighborhood of the origin $\widehat{\mathcal{W}}$ such that if $w(0) \in \widehat{\mathcal{W}}$ then $|w(t)| \le \delta$ for all $t \in \mathbb{R}$. By the reasoning presented above, system (4.8) is uniformly convergent in $\mathcal{Z}$ for the class of inputs $\mathcal{I}_s(\widehat{\mathcal{W}})$ and for any $w(\cdot) \in \mathcal{I}_s(\widehat{\mathcal{W}})$ the corresponding steady-state solution $\bar{z}_w(t)$ lies in the compact set $\bar{B}_\varepsilon \subset \mathcal{Z}$. Hence, $\mathcal{Z}$ and $\widehat{\mathcal{W}}$ are the required neighborhoods of the origin. This completes the proof of the implication (il)⇒(iil).

(iil)⇒(il). We will show that there exist a neighborhood of the origin $\mathcal{Z} \subset \mathbb{R}^d$ and an invariant neighborhood of the origin $\mathcal{W}_* \subset \mathbb{R}^m$ such that system (4.8) is uniformly convergent in $\mathcal{Z}$ for the class of inputs $\mathcal{I}_s(\mathcal{W}_*)$. Since $\mathcal{W}$ defined in (iil) is a neighborhood of the origin, there exists a closed ball $\bar{B}_\delta := \{w : \ |w| \le \delta\}$ such that $\bar{B}_\delta \subset \mathcal{W}$. Since $w(t) \equiv 0$ is stable in forward and backward time, there exists an invariant neighborhood of the origin $\mathcal{W}_*$ satisfying $\mathcal{W}_* \subset \bar{B}_\delta$. Hence, any solution of the system (4.9) starting in $\mathcal{W}_*$ lies in a compact subset of $\mathcal{W}$. Since $\alpha(w)$ is continuous for all $w \in \mathcal{W}$, for any solution of the exosystem starting in $\mathcal{W}_*$ the function $\bar{z}_w(t) := \alpha(w(t))$ is a solution of system (4.8) that is defined and bounded on $\mathbb{R}$ and that is uniformly asymptotically stable in $\mathcal{Z}$. Hence, by the definition of uniform convergence, system (4.8) is uniformly convergent in $\mathcal{Z}$ for the class of inputs $\mathcal{I}_s(\mathcal{W}_*)$. This completes the proof of the implication (iil)⇒(il) and the proof of the theorem. □

9.13 Proof of Theorem 5.15

Consider system (5.63) in closed loop with control $u = U(y) + v$, where the function $U(y)$ equals

$$U(y) = -\kappa y - \mu \int_0^y |\psi(\tau)|^2 d\tau$$

for some $\kappa \in \mathbb{R}$ and $\mu \geq 0$. The derivative of $U(y)$ equals $\frac{\partial U}{\partial y}(y) = -\kappa - \mu|\psi(y)|^2$. Denote $\xi(y) := \frac{\partial \varphi}{\partial y}(y)$. Then the Jacobian of the right-hand side of the closed-loop system equals

$$\mathcal{A}(y) = A + \xi(y)C + B(-\kappa - \mu|\psi(y)|^2)C.$$

Due to condition (5.64), $|\xi(y)| \leq |\psi(y)|$ for all $y \in \mathbb{R}$. Let us show that there exist $\kappa_* \in \mathbb{R}$, $\mu_* \geq 0$ and matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that

$$J := P(A + \xi C + B(-\kappa - \mu|\psi|^2)C) + (A + \xi C + B(-\kappa - \mu|\psi|^2)C)^T P \leq -Q, \tag{9.34}$$

for all $\kappa \geq \kappa_*$, $\mu \geq \mu_*$, and all $\xi \in \mathbb{R}^d$, $\psi \in \mathbb{R}$ satisfying the condition $|\xi| \leq |\psi|$. Inequality (9.34) implies that the Jacobian $\mathcal{A}(y)$ is quadratically stable over $\mathbb{R}$ and, therefore, the closed-loop system is input-to-state convergent (see Definition 5.5 and Theorem 2.29). Rewrite J in the following form:

$$J := P(A - \kappa BC) + (A - \kappa BC)^T P + P\xi C + C^T \xi^T P - \mu|\psi|^2(PBC + C^T B^T P). \tag{9.35}$$

Since $CB > 0$ and system (5.65) has all its zeros with negative real part, it follows from [18, 19] that there exist $\kappa_* \in \mathbb{R}$ and a matrix $P = P^T > 0$ such that

$$P(A - \kappa_* BC) + (A - \kappa_* BC)^T P =: -2Q < 0, \quad PB = C^T. \tag{9.36}$$

Notice that this implies

$$P(A - \kappa BC) + (A - \kappa BC)^T P \leq -2Q, \quad \forall \kappa \geq \kappa_*. \tag{9.37}$$

Namely,

$$\begin{aligned}
&P(A - \kappa BC) + (A - \kappa BC)^T P \\
&= P(A - \kappa_* BC) + (A - \kappa_* BC)^T P - (\kappa - \kappa_*)(PBC + C^T B^T P) \\
&= -2Q - 2(\kappa - \kappa_*)C^T C \le -2Q.
\end{aligned}$$

In this reasoning we have used the fact that $PB = C^T$. From (9.35) and (9.37) we obtain

$$J \le -2Q + P\xi C + C^T \xi^T P - 2\mu|\psi|^2 C^T C$$

for all $\kappa \ge \kappa_*$. The inequalities $\mu \ge 0$ and $|\psi| \ge |\xi|$ imply

$$\begin{aligned}
J &\le -2Q + P\xi C + C^T \xi^T P - 2\mu|\xi|^2 C^T C \\
&= -Q - (Q - P\xi C - C^T \xi^T P + C^T \xi^T (2\mu I)\xi C).
\end{aligned}$$

By completion of squares, we obtain

$$Q - P\xi C - C^T \xi^T P + C^T \xi^T P Q^{-1} P \xi C = (\Pi - \Pi^{-1} P\xi C)^T (\Pi - \Pi^{-1} P\xi C) \ge 0,$$

where $\Pi = \Pi^T > 0$ is such that $Q = \Pi^T \Pi$. Hence,

$$J \le -Q + C^T \xi^T (PQ^{-1}P - 2\mu I)\xi C$$

for all $\kappa \ge \kappa_*$, and all $|\psi| \ge |\xi|$. If μ_* is such that $2\mu_* I \ge PQ^{-1}P$, then for all $\mu \ge \mu_*$ and all $\xi \in \mathbb{R}^d$ it holds that $J \le -Q$. Matrix inequalities for finding κ_* and μ_* directly follow from inequality (9.36), which is feasible, and condition $2\mu_* I \ge PQ^{-1}P$. This completes the proof. □

9.14 Proof of Lemma 6.4

Denote $\zeta := z - \bar{z}$. It satisfies the equation

$$\dot{\zeta} := F(z, w(t)) - F(\bar{z}, w(t)) - \epsilon(t). \tag{9.38}$$

Consider the Lyapunov function $V(\zeta) := 1/2\zeta^T P\zeta$. Its derivative satisfies

$$\frac{dV}{dt} = \zeta^T P(F(z, w(t)) - F(\bar{z}, w(t)) - \epsilon(t)).$$

Notice that in the region $|\zeta|_P \le r$ both $\bar{z}(t)$ and $z = \zeta + \bar{z}(t)$ belong to $\mathcal{C}$. Since $w(t)$ belongs to $\mathcal{W}_c$ for all $t \ge 0$, and the sets $\mathcal{C}$ and $\mathcal{W}_c$ satisfy the Demidovich condition (6.13), we can apply Lemma 2.30. By formula (2.31),

$$\frac{dV}{dt} \le -\beta|\zeta|_P^2 - \zeta^T P\epsilon(t) \tag{9.39}$$

for some $\beta > 0$. Due to the remark to Lemma 2.30, the number β equals $\beta = a/\|P\|$, where a is from the Demidovich condition (6.13). Taking into account this fact and applying the Cauchy inequality to the second term in formula (9.39), we obtain

$$\frac{dV}{dt} \le -\frac{a}{\|P\|}|\zeta|_P^2 + |\zeta|_P \lambda(t_0), \quad \text{for} \;\; t \ge t_0 \ge 0, \tag{9.40}$$

where $\lambda(t_0) := \sup_{t \ge t_0} |\epsilon(t)|_P$. By the conditions of the lemma, $\lambda(t_0) < ar/(2\|P\|)$ for any $t_0 \ge 0$. Thus, from (9.40) we can conclude that the ellipsoid $\bar{E}_P(r) := \{\zeta : |\zeta|_P < r\}$ is invariant with respect to (9.38). Application of Theorem 5.1 from [51] implies that for any solution starting in $\bar{E}_P(r)$ and any η satisfying $\frac{2\|P\|}{a}\lambda(t_0) < \eta < r$ there exists $T > 0$ such that $|\zeta(t)|_P \le \eta$ for all $t \ge t_0 + T$. Due to the arbitrary choice of $\eta > 2\|P\|/a\lambda(t_0)$, any solution of (9.38) starting in $\bar{E}_P(r)$ satisfies

$$\limsup_{t \to +\infty} |\zeta(t)|_P \le \frac{2\|P\|}{a}\lambda(t_0).$$

Since the left-hand side does not depend on t_0, we can conclude that

$$\limsup_{t \to +\infty} |\zeta(t)|_P \le \frac{2\|P\|}{a} \lim_{t_0 \to +\infty} \lambda(t_0) = \frac{2\|P\|}{a} \limsup_{t \to +\infty} |\epsilon(t)|_P.$$

This completes the proof. □

9.15 Proof of Theorem 6.5

We need to show that (6.34) holds for any solution $(z(t), w(t))$ that starts in $(z(0), w(0))$ satisfying the relations $|w(0)| < \rho$, $\tilde{m}_N(w_0) + \frac{2\delta\|P\|}{a}q(w_0) < \mathcal{R}$ and $z(0) \in E_P(\tilde{\alpha}(w(0)), r)$, where $E_P(\bar{z}, r) := \{z : |z - \bar{z}|_P < r\}$ and $r := (\mathcal{R} - \tilde{m}_N(w(0)))/\delta$. Due to the properties of the exosystem, we obtain $|w(t)| \equiv |w(0)| < \rho$ and the solution $\bar{z}_w(t) := \tilde{\alpha}(w(t))$ of the system

$$\dot{z} = F(z, w(t)) + \varepsilon_1(w(t)) \tag{9.41}$$

satisfies

$$|N\bar{z}_w(t)| \le \sup_{t \ge 0} |N\tilde{\alpha}(w(t))| = \tilde{m}_N(w(0)) < \mathcal{R}.$$

Hence, $\bar{z}_w(t) \in C_N(\mathcal{R})$ and $w(t) \in \mathcal{W}_c(\rho)$ for all $t \ge 0$. Let us show that $E_P(\bar{z}_w(t), r) \subset \mathcal{C}_N(\mathcal{R})$ for all $t \ge 0$. Suppose $z \in E_P(\bar{z}_w(t), r)$ for some $t \ge 0$. Then

$$\begin{aligned} |Nz| \le |N\bar{z}_w(t)| + |N(z - \bar{z}_w(t))| &\le \tilde{m}_N(w(0)) + \delta|z - \bar{z}_w(t)|_P \\ &< \tilde{m}_N(w(0)) + \delta r = \mathcal{R}. \end{aligned}$$

Consequently, $E_P(\bar{z}_w(t), r) \subset \mathcal{C}_N(\mathcal{R})$ for all $t \ge 0$. As follows from the second inequality in the definition of $\tilde{\mathcal{Y}}$, the term $\varepsilon_1(w(t))$ satisfies

$$|\varepsilon_1(w(t))|_P \leq \sup_{t\geq 0} |\varepsilon_1(w(t))|_P = q(w(0)) < \frac{a}{2\|P\|} r.$$

Thus, by Lemma 6.4 we obtain that any solution of system (6.8) starting in $z(0) \in E_P(\bar{z}_w(0), r)$ satisfies (6.34). Since the set $\tilde{\mathcal{Y}}$ is bounded and the function $\bar{h}_r(z, w)$ is C^1, there exists a constant $L > 0$ such that

$$|\bar{h}_r(z_1, w) - \bar{h}_r(z_2, w)| \leq L|z_1 - z_2|_P$$

for all $(z_i, w) \in \tilde{\mathcal{Y}}$, $i = 1, 2$. With this inequality, inequality (6.34) implies (6.35) with the constant $\bar{C} := 2\|P\|L/a$. □

9.16 Proof of Theorem 6.6

It is sufficient to show that $E_P(R(r)) \times B_w(r) \subset \tilde{\mathcal{Y}}$ for any $r \in [0, r_*)$. Then the statement of Theorem 6.6 follows from Theorem 6.5. Suppose $z_0 \in E_P(R(r))$ and $w_0 \in B_w(r)$ for some fixed $r \in [0, r_*)$. According to the definition of $\tilde{\mathcal{Y}}$, we first need to show that $|w_0| < \rho$. This is true because $|w_0| < r < r_* \leq \rho$. Next, we show that $\tilde{m}_N(w_0) + \frac{2\delta\|P\|}{a}|q(w_0)|_P < \mathcal{R}$. By the definition of $\eta(r)$, it holds that $|N\tilde{\alpha}(w)| + \frac{2\delta\|P\|}{a}|\varepsilon_1(w)|_P \leq \eta(r)$ for all $|w| < r$. The choice of $|w_0| < r$ implies $|w(t, w_0)| \equiv |w_0| < r$. Hence, by the definition of $\tilde{m}_N(w_0)$ and $q(w_0)$ we obtain

$$\tilde{m}_N(w_0) = \sup_{t\geq 0} |N\tilde{\alpha}(w(t, w_0))| \leq \sup_{|w|<r} |N\tilde{\alpha}(w)|,$$

$$q(w_0) = \sup_{t\geq 0} |\varepsilon_1(w(t, w_0))|_P \leq \sup_{|w|<r} |\varepsilon_1(w)|_P.$$

Thus, we obtain

$$\tilde{m}_N(w_0) + \frac{2\delta\|P\|}{a}|q(w_0)|_P \leq \sup_{|w|<r} \left(|N\tilde{\alpha}(w)| + \frac{2\delta\|P\|}{a}|\varepsilon_1(w)|_P \right) = \eta(r).$$

The choice of $r < r_*$ implies that $\eta(r) < \mathcal{R}$ and consequently

$$\tilde{m}_N(w_0) + \frac{2\delta\|P\|}{a}|q(w_0)|_P < \mathcal{R}.$$

Next, we need to show that $|z_0 - \tilde{\alpha}(w_0)|_P < (\mathcal{R} - \tilde{m}_N(w_0))/\delta$. The triangle inequality implies

$$|z_0 - \tilde{\alpha}(w_0)|_P \leq |z_0|_P + |\tilde{\alpha}(w_0)|_P. \tag{9.42}$$

By the choice of z_0 and by the definition of $\tilde{R}(r)$,

$$|z_0|_P < \tilde{R}(r) = (\mathcal{R} - \tilde{\sigma}(r))/\delta = \Big(\mathcal{R} - \sup_{|w_0|<r} (|N\tilde{\alpha}(w_0)| + \delta|\tilde{\alpha}(w_0)|_P)\Big)/\delta \leq$$
$$(\mathcal{R} - \tilde{m}_N(w_0))/\delta - |\tilde{\alpha}(w_0)|_P.$$

Substituting this inequality in (9.42), we obtain $|z_0 - \tilde{\alpha}(w_0)|_P < (\mathcal{R} - \tilde{m}_N(w_0))/\delta$. This completes the proof. □

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