Total Scalar Curvatures of Geodesic Spheres and of Boundaries of Geodesic Disks*

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Dedicated to Professor Lieven Vanhecke

Summary. Total curvatures of boundaries of geodesic disks in Riemannian manifolds are investigated. The first terms in the corresponding power series expansions are obtained for the total scalar curvature and the L^2-norms of the scalar curvature, the Ricci tensor and the curvature tensor. As an application, it is shown that these functions characterize the local geometry of most of the two-point homogeneous spaces.

1 Introduction

In the study of geometric properties of a Riemannian manifold (M, g), it is often useful to consider geometric objects naturally associated to (M, g). These can be special hypersurfaces like small geodesic spheres and tubes around geodesics, bundles with (M, g) as base manifold, or families of transformations reflecting symmetry properties of (M, g) [V88]. The existence of a relationship between the curvature of a Riemannian manifold and the volume of its geodesic spheres and tubes led some authors to state the following question:

> To what extent is the curvature (or the geometry) of a given Riemannian manifold (M, g) influenced, or even determined, by the volume properties of certain naturally defined families of geometric objects (for example geodesic spheres and tubes) in M?

This problem seems very difficult to handle in such a generality. However, when one looks at manifolds with a high degree of symmetry (e.g., two-point homogeneous spaces), these geometric objects have nice properties and one may expect to obtain characterizations of those spaces by means of such properties. For instance, the two-point homogeneous spaces may be characterized by using the spectrum of their geodesic

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spheres [CV81] or in most cases by the L^2-norm of the curvature tensor of geodesic spheres [DGH]. (See also [DGV] for more information on total curvatures of geodesic spheres.)

This work fits into the general program above. The family of geometric objects to be considered are the geodesic disks, which were previously investigated by O. Kowalski and L. Vanhecke with special attention to their volume properties [KV82], [KV83], [KV85]. Here, we are interested in the intrinsic geometry of the boundaries of these disks and we devote our attention to the study of their total scalar curvatures obtained by integrating the scalar curvature and the quadratic curvature invariants on these boundaries. In doing that, we compute the first terms in their power series expansions. Several conclusions are obtained from those coefficients. In particular, we note that

> *two-point homogeneous spaces are characterized by some of the total curvatures of the boundaries of geodesic disks among Riemannian manifolds with adapted holonomy.*

The paper is organized as follows. In Section 2, we recall some notation and basic notions on scalar curvature invariants. The first terms in the power series expansions of the corresponding total invariants are derived in §2.2. These are used in Section 3 to obtain the first terms in the power series expansions of the total curvatures of the boundaries. Finally, Section 4 is devoted to point out some applications of those expressions.

2 Preliminaries

Let (M^n, g) be an n-dimensional smooth Riemannian manifold of class C^∞. We denote by ∇ the Levi–Civita connection and put $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ for the curvature tensor, where X, Y are vector fields on M. Also, $R_{XYZW} = g(R_{XY}Z, W)$ and the Ricci tensor and the scalar curvature are given by $\rho_{XY} = \sum_{i=1}^n R_{Xe_iYe_i}$ and $\tau = \sum_{i=1}^n \rho_{e_ie_i}$ respectively, and with respect to an orthonormal basis $\{e_1, \dots, e_n\}$. For simplicity, here and in what follows, we use the notation $\rho_{ij} = \rho_{e_ie_j}$, $R_{ijkl} = R_{e_ie_je_ke_l}$, $\nabla_{ijk\dots} = \nabla_{e_ie_je_k\dots}$ and so on.

Finally, note that to avoid problems with the domains of exponential maps, the geodesic spheres and disks considered here are sufficiently small, i.e., their radius is always smaller than the injectivity radius at their center.

2.1 Scalar curvature invariants

A *scalar curvature invariant* is a polynomial in the components of the curvature tensor that does not depend on the choice of the orthonormal basis used to build it. The *order* of a scalar curvature invariant is, by definition, the number of derivatives of the metric tensor involved in it. Let $I(k, n)$ be the vector space of curvature invariants of order $2k$, $m \in M$ and $\{e_1, \dots, e_n\}$ an orthonormal basis of the tangent space at m, T_mM.

For $n \geq 2$, $I(1, n)$ has dimension 1 and is generated by τ. For $n \geq 4$, $I(2, n)$ has dimension 4 and a basis is given by,

$$\tau^2, \qquad \|\rho\|^2 = \sum \rho_{ij}^2, \qquad \|R\|^2 = \sum R_{ijkl}^2, \qquad \Delta\tau = \sum \nabla_{ii}^2 \tau. \tag{1}$$

A basis for $I(3, n)$ is given in [GV79]. For our purposes here, only invariants of order two and four are needed. Indeed, those allow to characterize important classes of Riemannian manifolds. We have for $n > 2$ [CV81]:

For any n-dimensional Riemannian manifold,

$$\|\rho\|^2 \geq \frac{1}{n}\tau^2, \tag{2}$$

with equality if and only if the manifold is an Einstein space.

For any n-dimensional Riemannian manifold,

$$\|R\|^2 \geq \frac{2}{n-1}\|\rho\|^2, \tag{3}$$

with equality if and only if the manifold has constant sectional curvature.

For a $2n$-dimensional Kähler manifold,

$$\|R\|^2 \geq \frac{4}{n+1}\|\rho\|^2, \tag{4}$$

with the equality holding if and only if M has constant holomorphic sectional curvature.

For a $4n$-dimensional quaternionic Kähler manifold,

$$\|R\|^2 \geq \frac{5n+1}{(n+2)^2}\|\rho\|^2, \tag{5}$$

with the equality holding precisely for the quaternionic space forms.

2.2 Total scalar curvatures of geodesic spheres

Our purpose here is to obtain the first two terms in the power series expansions of the integrals of the curvature invariants of order two and four on geodesic spheres. We denote by $G_m(r)$ the geodesic sphere with center $m \in M$ and radius r, that is, $G_m(r) = \{m' \in M/d(m, m') = r\}$. Since $r > 0$ is supposed to be smaller than the injectivity radius at m, the geodesic sphere $G_m(r)$ is a hypersurface of M and $G_m(r)$ $= \exp_m(S^{n-1}(r))$, where $S^{n-1}(r) = \{y \in T_m M/\|x\| = r\}$ is the sphere of radius r in the tangent space to M at the basepoint m. Moreover as a matter of notation, let $\widetilde{\tau}$, $\|\widetilde{\rho}\|^2, \ldots$ denote the scalar curvature, the square norm of the Ricci tensor, ... of the geodesic sphere $G_m(r)$, and set τ, $\|\rho\|^2, \ldots$ for the corresponding objects for the ambient manifold (M, g).

First of all, note that we will not consider the Laplacian of the scalar curvature since $\int_{G_m(r)} \widetilde{\Delta}\widetilde{\tau}du = 0$. Also, in what follows, $c_{n-1} = \frac{n\pi^{n/2}}{(n/2)!}$ where $(n/2)! = \Gamma((n/2)+1)$ stands for the volume of the unit sphere in the Euclidean n-space (cf. [G90]). In the lemma below, the first terms in the power series expansions of the total scalar curvature [CV81] and the L^2-norms of the scalar curvature, the Ricci tensor and the curvature tensor [DGH] of sufficiently small geodesic spheres are given.

Lemma 1 ([CV81], [DGH]). *Let (M, g) be an n-dimensional Riemannian manifold and $m \in M$. Then, we have:*

$$\int_{G_m(r)} \widetilde{\tau} = c_{n-1} r^{n-1} \left\{ \frac{(n-2)\,(n-1)}{r^2} - \frac{(n-3)\ (n-2)}{6n}\, \tau(m) \right.$$
$$+ \frac{1}{n\,(n+2)} \left[-\frac{(n+2)(n+3)}{120}\, \|R\|^2 + \frac{n^2+5\,n+21}{45}\, \|\rho\|^2 \right.$$
$$\left. \left. + \frac{n^2-7\,n-6}{72}\, \tau^2 - \frac{(n-3)\ (n-2)}{20}\, \Delta\tau \right](m)\, r^2 + O\left(r^3\right) \right\},$$

$$\int_{G_m(r)} \widetilde{\tau}^2 = c_{n-1} r^{n-1} \left\{ (n-2)^2\,(n-1)^2 r^{-4} - \frac{(n-5)\,(n-2)^2\,(n-1)}{6n}\, \tau(m) r^{-2} \right.$$
$$+ \frac{1}{n\,(n+2)} \left[-\frac{(n-2)\,(n-1)\,(n^2+13\,n+10)}{120}\, \|R\|^2 \right.$$
$$+ \frac{n^4+10\,n^3+43\,n^2-14\,n+120}{45}\, \|\rho\|^2$$
$$+ \frac{n^4-14\,n^3+29\,n^2-60\,n-188}{72}\, \tau^2$$
$$\left. \left. - \frac{(n-5)\,(n-2)^2\,(n-1)}{20}\, \Delta\tau \right](m) + O\left(r^2\right) \right\},$$

$$\int_{G_m(r)} \|\widetilde{\rho}\|^2 = c_{n-1} r^{n-1} \left\{ (n-2)^2\,(n-1)\, r^{-4} - \frac{(n-5)\,(n-2)^2}{6n}\, \tau(m) r^{-2} \right.$$
$$+ \frac{1}{n\,(n+2)} \left[-\frac{n^3-9n^2-16n-20}{120}\, \|R\|^2 \right.$$
$$+ \frac{n^3+31n^2-16n-120}{45}\, \|\rho\|^2 + \frac{n^3-13n^2-16n+44}{72}\, \tau^2$$
$$\left. \left. - \frac{(n-5)(n-2)^2}{20}\, \Delta\tau \right](m) + O\left(r^2\right) \right\},$$

$$\int_{G_m(r)} \|\widetilde{R}\|^2 = c_{n-1} r^{n-1} \left\{ 2\,(n-2)\,(n-1)\, r^{-4} - \frac{(n-5)\,(n-2)}{3n}\, \tau(m) r^{-2} \right.$$
$$+ \frac{1}{n\,(n+2)} \left[\frac{59n^2-93n-10}{60}\, \|R\|^2 + \frac{2\left(n^2-37n+60\right)}{45}\, \|\rho\|^2 \right.$$
$$\left. \left. + \frac{n^2-11n+2}{36}\, \tau^2 - \frac{(n-5)\,(n-2)}{10}\, \Delta\tau \right](m) + O\left(r^2\right) \right\}.$$

3 Total scalar curvatures of boundaries of geodesic disks

Geodesic disks were introduced by O. Kowalski and L. Vanhecke as a generalization of the notion of a two-dimensional disk in the Euclidean space $\mathbb{E}^3$. In a series of papers ([KV82], [KV83], [KV85]) they investigated their volume properties in relation to local homogeneity and obtained a characterization of the two-point homogeneous spaces by means of the volumes of their small geodesic disks. Since the boundaries of geodesic disks are compact submanifolds, we are interested in their total scalar curvatures obtained by integrating the corresponding scalar curvature invariants of order two and four.

Recall that the *geodesic disk* $\overline{D}_m^{\xi}(r)$ of radius r, centered at $m \in M$ and orthogonal to $\xi \in T_m M$, is defined by

$$\begin{aligned}\overline{D}_m^{\xi}(r) &= \{\exp_m(su)/u \in T_m M,\ \|u\| = 1,\ g(u,\xi) = 0,\ 0 \le s \le r\} \\ &= \{m' \in M/d(m,m') \le r\} \cap \exp_m(\{\xi\}^{\perp})\end{aligned}$$

where $\exp_m : T_m M \to M$ is the exponential map at m. For the purpose of this paper and the investigation of total scalar curvatures, we consider the boundaries

$$D_m^{\xi}(r) = \{m' \in M/d(m,m') = r\} \cap \exp_m(\{\xi\}^{\perp}).$$

In order to obtain the first terms in the power series expansions of the total curvatures of these boundaries, the following result will be extensively used. It relates scalar curvature invariants of order two and four of $\exp_m(\{\xi\}^{\perp})$ with the corresponding objects in the ambient space.

Lemma 2. *Let (M,g) be an n-dimensional Riemannian manifold and $\xi \in T_m M$ a unit vector. If $\widetilde{R}$, $\widetilde{\rho}$, $\widetilde{\tau}$, ... denote the objects in $\exp_m(\{\xi\}^{\perp})$ and R, ρ, τ, ... denote the corresponding objects on (M,g), then the following hold at m:*

$$\begin{aligned}\|\widetilde{R}\|^2 &= \|R\|^2 + 4\sum_{i,j=1}^{n} R_{\xi i \xi j}^2 - 4\sum_{i,j,k=1}^{n} R_{\xi ijk}^2, \\ \|\widetilde{\rho}\|^2 &= \|\rho\|^2 + \rho_{\xi\xi}^2 - 2\sum_{i=1}^{n}\rho_{\xi i}^2 + \sum_{i,j=1}^{n} R_{\xi i\xi j}^2 - 2\sum_{i,j=1}^{n}\rho_{ij}R_{\xi i\xi j}, \\ \widetilde{\tau} &= \tau - 2\rho_{\xi\xi}, \\ \widetilde{\Delta}\widetilde{\tau} &= \Delta\tau - 2\Delta\rho_{\xi\xi} + 2\nabla_{\xi\xi}^2\rho_{\xi\xi} - \nabla_{\xi\xi}^2\tau + \frac{4}{9}\rho_{\xi\xi}^2 \\ &\quad - \frac{4}{9}\sum_{i=1}^{n}\rho_{\xi i}^2 + \frac{4}{3}\sum_{i,j=1}^{n} R_{\xi i\xi j}^2 - \frac{2}{3}\sum_{i,j,k=1}^{n} R_{\xi ijk}^2.\end{aligned}$$

Proof. It follows from the work in [KV82], after some calculations. □

Now, the first terms in the power series expansions of the total curvatures of the boundaries of geodesic disks are obtained from the corresponding ones for the geodesic spheres in Lemma 1 after using the identities in Lemma 2. As a matter of notation $\hat{\tau}$, $\|\hat{\rho}\|^2$, ... denote the curvature objects on the boundaries $D_m^\xi(r)$, while τ, $\|\rho\|^2$, ... stand for the corresponding objects on (M, g). We omit the calculations which are straightforward and immediately state the different expansions separately in Theorem 1–Theorem 4.

Theorem 1. *Let (M, g) be an n-dimensional Riemannian manifold, $m \in M$ and $\xi \in T_m M$ a unit vector. Then, for sufficiently small radius r, one has the following expansion for the total scalar curvature of the boundaries $D_m^\xi(r)$:*

$$\int_{D_m^\xi(r)} \hat{\tau} = c_{n-2} r^{n-2} \left\{ (n-2)(n-3)r^{-2} + A_{(0)}(m) + A_{(2)}(m)r^2 + O(r) \right\}$$

where

$$A_{(0)} = -\frac{(n-3)(n-4)}{6(n-1)}[\tau - 2\rho_{\xi\xi}],$$

$$\begin{aligned} A_{(2)} = \frac{1}{(n-1)(n+1)} \Bigg\{ &\frac{n^2-9n+2}{72}\tau^2 - \frac{(n+2)(n+1)}{120}\|R\|^2 + \frac{n^2+3n+17}{45}\|\rho\|^2 \\ &- \frac{(n-3)(n-4)}{20}\Delta\tau + \frac{(n-3)(n-4)}{20}[\nabla^2_{\xi\xi}\tau - 2\nabla^2_{\xi\xi}\rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}] \\ &- \frac{(n+2)(n+11)}{45}\sum_{i=1}^n \rho_{\xi i}^2 - \frac{(n-4)(7n-11)}{90} \\ &\times \sum_{i,j=1}^n R_{\xi i\xi j}^2 - \frac{2(n^2+3n+17)}{45}\sum_{i,j=1}^n R_{\xi i\xi j}\rho_{ij} + \frac{n^2-2n+7}{15} \\ &\times \sum_{i,j,k=1}^n R_{\xi ijk}^2 - \frac{n^2-9n+2}{18}\tau\rho_{\xi\xi} + \frac{(n-1)(n-4)}{18}\rho_{\xi\xi}^2 \Bigg\}. \end{aligned}$$

Theorem 2. *Let (M, g) be an n-dimensional Riemannian manifold, $m \in M$ and $\xi \in T_m M$ a unit vector. Then, for sufficiently small radius r, one has the following expansion for the L^2-norm of the scalar curvature of the boundaries $D_m^\xi(r)$:*

$$\int_{D_m^\xi(r)} \hat{\tau}^2 = c_{n-2} r^{n-2} \left\{ (n-2)^2(n-3)^2 r^{-4} + B_{(-2)}(m)r^{-2} + B_{(0)}(m) + O(r) \right\}$$

where

$$B_{(-2)} = -\frac{(n-3)^2(n-6)(n-2)}{6(n-1)}[\tau - 2\rho_{\xi\xi}],$$

$$B_{(0)} = \frac{1}{(n-1)(n+1)} \left\{ \frac{n^4 - 18n^3 + 77n^2 - 164n - 84}{72} \tau^2 \right.$$

$$- \frac{(n-2)(n-3)(n^2+11n-2)}{120} \|R\|^2 + \frac{n^4+6n^3+19n^2-74n+168}{45} \|\rho\|^2$$

$$- \frac{(n-3)^2(n-6)(n-2)}{20} \Delta\tau + \frac{(n-3)^2(n-6)(n-2)}{20}$$

$$\times [\nabla^2_{\xi\xi}\tau - 2\nabla^2_{\xi\xi}\rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}] - \frac{n^4 + 26n^3 - 31n^2 - 4n + 228}{45}$$

$$\times \sum_{i=1}^{n} \rho^2_{\xi i} - \frac{7n^4 - 78n^3 + 223n^2 - 488n + 276}{90}$$

$$\times \sum_{i,j=1}^{n} R^2_{\xi i \xi j} - \frac{2(n^4 + 6n^3 + 19n^2 - 74n + 168)}{45}$$

$$\times \sum_{i,j=1}^{n} R_{\xi i \xi j}\rho_{ij} + \frac{(n-2)(n-3)(n^2+n+8)}{15}$$

$$\times \sum_{i,j,k=1}^{n} R^2_{\xi ijk} - \frac{n^4 - 18n^3 + 77n^2 - 164n - 84}{18} \tau\rho_{\xi\xi}$$

$$\left. + \frac{n^4 - 10n^3 + 57n^2 - 136n - 60}{18} \rho^2_{\xi\xi} \right\}.$$

Theorem 3. *Let (M, g) be an n-dimensional Riemannian manifold, $m \in M$ and $\xi \in T_m M$ a unit vector. Then, for sufficiently small radius r, one has the following expansion for the L^2-norm of the Ricci tensor of the boundaries $D^\xi_m(r)$:*

$$\int_{D^\xi_m(r)} \|\hat{\rho}\|^2 = c_{n-2} r^{n-2} \left\{ (n-2)(n-3)^2 r^{-4} + C_{(-2)}(m) r^{-2} + C_{(0)}(m) + O(r) \right\}$$

where

$$C_{(-2)} = -\frac{(n-3)^2(n-6)}{6(n-1)} [\tau - 2\rho_{\xi\xi}],$$

$$C_{(0)} = \frac{1}{(n-1)(n+1)} \left\{ \frac{n^3 - 16n^2 + 13n + 46}{72} \tau^2 - \frac{n^3 - 12n^2 + 5n - 14}{120} \|R\|^2 \right.$$

$$+ \frac{n^3 + 28n^2 - 75n - 74}{45} \|\rho\|^2 - \frac{(n-3)^2(n-6)}{20} \Delta\tau + \frac{(n-3)^2(n-6)}{20}$$

$$\times [\nabla^2_{\xi\xi}\tau - 2\nabla^2_{\xi\xi}\rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}] - \frac{n^3 + 68n^2 - 195n - 94}{45}$$

$$\times \sum_{i=1}^{n} \rho_{\xi i}^2 - \frac{7n^3 - 164n^2 + 435n - 218}{90}$$

$$\times \sum_{i,j=1}^{n} R_{\xi i \xi j}^2 - \frac{2(n^3 + 28n^2 - 75n - 74)}{45}$$

$$\times \sum_{i,j=1}^{n} R_{\xi i \xi j} \rho_{ij} + \frac{n^3 - 12n^2 + 25n - 34}{15}$$

$$\times \sum_{i,j,k=1}^{n} R_{\xi ijk}^2 - \frac{n^3 - 16n^2 + 13n + 46}{18} \tau \rho_{\xi\xi} + \frac{n^3 - 35n + 38}{18} \rho_{\xi\xi}^2 \Bigg\} .$$

Theorem 4. *Let (M, g) be an n-dimensional Riemannian manifold, $m \in M$ and $\xi \in T_m M$ a unit vector. Then, for sufficiently small radius r, one has the following expansion for the L^2-norm of the curvature tensor of the boundaries $D_m^\xi(r)$:*

$$\int_{D_m^\xi(r)} \|\hat{R}\|^2 = c_{n-2} r^{n-2} \left\{ 2(n-2)(n-3) r^{-4} + D_{(-2)}(m) r^{-2} + D_{(0)}(m) + O(r) \right\}$$

where

$$D_{(-2)} = -\frac{(n-3)(n-6)}{3(n-1)} [\tau - 2\rho_{\xi\xi}],$$

$$D_{(0)} = \frac{1}{(n-1)(n+1)} \Bigg\{ \frac{n^2 - 13n + 14}{36} \tau^2 + \frac{59n^2 - 211n + 142}{60} \|R\|^2$$

$$+ \frac{2(n^2 - 39n + 98)}{45} \|\rho\|^2 - \frac{(n-3)(n-6)}{10} \Delta\tau$$

$$+ \frac{(n-3)(n-6)}{10} [\nabla_{\xi\xi}^2 \tau - 2\nabla_{\xi\xi}^2 \rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}]$$

$$- \frac{2(n^2 - 69n + 178)}{45} \sum_{i=1}^{n} \rho_{\xi i}^2 + \frac{173n^2 - 657n + 514}{45}$$

$$\times \sum_{i,j=1}^{n} R_{\xi i \xi j}^2 - \frac{4(n^2 - 39n + 98)}{45} \sum_{i,j=1}^{n} R_{\xi i \xi j} \rho_{ij} - \frac{2(29n^2 - 101n + 62)}{15}$$

$$\times \sum_{i,j,k=1}^{n} R_{\xi ijk}^2 - \frac{n^2 - 13n + 14}{9} \tau \rho_{\xi\xi} + \frac{(n-2)(n-23)}{9} \rho_{\xi\xi}^2 \Bigg\} .$$

4 Characterizations of the model spaces

The purpose of this section is to obtain characterizations of the two-point homogeneous spaces by means of the total curvatures of the boundaries of geodesic disks as an application of the expansions in Theorems 1–4. First of all, we recall that by a two-point homogeneous space we mean one of the following spaces: Euclidean n-space, the n-dimensional spheres and the hyperbolic spaces, the projective and hyperbolic n-spaces over the complex numbers or over the quaternions, and the Cayley projective or hyperbolic plane. Furthermore, we say that the holonomy of a Riemannian manifold (M, g) is *adapted* to one of these models, if the holonomy group of (M, g) is a subgroup of the holonomy group of the given model space, that is, the holonomy of (M, g) is contained in $O(n)$, $U(n)$, $Sp(1) \cdot Sp(n)$ or Spin(9) respectively. Moreover, note that in what follows, we will omit the Cayley plane since its holonomy group completely characterizes its local geometry. In fact, if a manifold has holonomy group contained in Spin(9), then it is flat or locally isometric to the Cayley plane or its non-compact dual [A67].

We begin with the following:

Lemma 3. *Let (M, g) be an n-dimensional Riemannian manifold. Suppose that one of the following holds:*

(i) $4 < n$ *and the total scalar curvature of the boundaries of geodesic disks coincides with the corresponding one in an Einstein manifold;*
(ii) $3 < n \neq 6$ *and any of the L^2-norms of the scalar curvature, the Ricci tensor or the curvature tensor of the boundaries of geodesic disks coincides with the corresponding one in an Einstein manifold.*

Then, (M, g) is an Einstein manifold with the same scalar curvature as the model space.

Proof. (i) is obtained from the coefficient $A_{(0)}$ in Theorem 1 and (ii) follows immediately from the corresponding coefficients of r^{-2} in the expansions in Theorems 2–4. □

Recall that a Riemannian manifold is said to be 2-*stein* if (M, g) is Einsteinian and satisfies

$$\sum_{i,j=1}^{n} R_{xixj}^2 = \lambda g(x,x)^2$$

for all x. Also, (M, g) is said to be *super-Einstein* if it is Einstein and

$$\sum_{i,j,k=1}^{n} R_{xijk}^2 = \mu g(x,x)$$

for all x. It was shown in [CV81] that 2-stein manifolds are super-Einstein, but the converse is not true. (For instance, irreducible symmetric spaces are super-Einstein, but they are not necessarily 2-stein.)

Lemma 4. *Let (M, g) be an n-dimensional Einstein manifold. If*

$$a\|R\|^2 + b\sum_{i,j,k=1}^{n} R_{\xi ijk}^2 + c\sum_{i,j=1}^{n} R_{\xi i\xi j}^2 = k \tag{6}$$

for some real constants a, b, c, k with $(n+4)b + 3c \neq 0$, $c \neq 0$ and for all unit vectors ξ, then (M, g) is 2*-stein.*

Proof. Put

$$\omega_{xyvw} = \sum_{i,j=1}^{n} R_{xiyj} R_{viwj}, \qquad \eta_{xy} = \sum_{i,j,k=1}^{n} R_{xijk} R_{yijk}.$$

Then, for all vectors $x, y \in T_m M$ and all $\alpha, \beta \in \mathbb{R}$, it follows from (6) that

$$a\|R\|^2 g(\alpha x + \beta y, \alpha x + \beta y)^2 + b\eta_{\alpha x+\beta y, \alpha x+\beta y} g(\alpha x + \beta y, \alpha x + \beta y) \\ + c\omega_{\alpha x+\beta y,\dots,\alpha x+\beta y} = kg(\alpha x + \beta y, \alpha x + \beta y)^2.$$

Expand the previous expression and take the coefficients of $\alpha^2\beta^2$. Then, put $y = e_i$ and take the trace to obtain

$$2a\|R\|^2(n+2)g(x,x) + b(\|R\|^2 g(x,x) + (n+4)\eta_{xx}) \\ + 2c\left(\sum_{i,j=1}^{n} \rho_{ij} R_{xixj} + \frac{3}{2}\eta_{xx}\right) = 2(n+2)kg(x,x). \quad (7)$$

Since (M, g) is assumed to be Einsteinian, (7) becomes

$$[b(n+4) + 3c]\eta_{xx} = -\left[2(n+2)a\|R\|^2 + b\|R\|^2 + \frac{2c\tau^2}{n^2} - 2(n+2)k\right] g_{xx}, \quad (8)$$

and contracting this gives

$$[b(n+4) + 3c]\|R\|^2 = -n\left[2(n+2)a\|R\|^2 + b\|R\|^2 + \frac{2c\tau^2}{n^2} - 2(n+2)k\right]. \quad (9)$$

Now, from (8) and (9), one has

$$[b(n+4) + 3c]\eta_{xx} = \frac{b(n+4) + 3c}{n}\|R\|^2 g_{xx},$$

and thus $\eta = \frac{\|R\|^2}{n} g$. Hence, it follows from (6) that

$$\omega_{xxxx} = -\frac{1}{c}\left(\frac{na + b}{n}\|R\|^2 - k\right) g_{xx}^2$$

which shows that (M, g) is 2-stein. □

Lemma 5. *Let (M, g) be an n-dimensional Riemannian manifold. Suppose that one of the following holds:*

(i) $4 < n$ *and the total scalar curvature of the boundaries* $D_m^\xi(r)$ *does not depend on the normal direction* ξ*, or*

(ii) $3 < n \neq 6$ *and any of the* L^2*-norms of the scalar curvature, the Ricci tensor or the curvature tensor of the boundaries* $D_m^\xi(r)$ *does not depend on the normal direction* ξ*.*

Then, (M, g) *is 2-stein.*

Proof. We first show (i). Since the total scalar curvature of the boundaries of geodesic disks does not depend on the normal direction, the coefficients $A_{(0)}$ and $A_{(2)}$ are independent of the unit ξ. Therefore, it follows from $A_{(0)} = -[(n-3)(n-4)/6(n-1)][\tau - 2\rho_{\xi\xi}]$ that (M, g) is an Einstein space. Moreover, for an Einstein manifold, $\rho = (\tau/n)g$ holds and so, the coefficient $A_{(2)}$ becomes

$$A_{(2)} = \frac{1}{(n-1)(n+1)}\left\{\frac{(n-4)(5n^3-37n^2+62n+92)}{360n^2}\tau^2 - \frac{(n+2)(n+1)}{120}\|R\|^2 - \frac{(n-4)(7n-11)}{90}\sum_{i,j=1}^{n} R^2_{\xi i\xi j} + \frac{n^2-2n+7}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk}\right\}.$$

So, (M, g) is 2-stein as an application of Lemma 4. The case (ii) is obtained in an analogous way. Indeed, if one assumes either $B_{(-2)}$ or $C_{(-2)}$ or $D_{(-2)}$ to be independent of ξ and $\dim M \neq 3, 6$, then (M, g) is an Einstein space. The fact that it is also 2-stein follows from Lemma 4 after consideration of the coefficients $B_{(0)}$, $C_{(0)}$ and $D_{(0)}$, which now become

$$B_{(0)} = \frac{1}{(n-1)(n+1)}\left\{\frac{5n^6-102n^5+789n^4-2712n^3+3352n^2+1520n-5712}{360n^2}\tau^2 - \frac{(n-2)(n-3)(n^2+11n-2)}{120}\|R\|^2 - \frac{7n^4-78n^3+223n^2-488n+276}{90} \times \sum_{i,j=1}^{n} R^2_{\xi i\xi j} + \frac{(n-2)(n-3)(n^2+n+8)}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk}\right\}, \tag{10}$$

$$C_{(0)} = \frac{1}{(n-1)(n+1)}\left\{\frac{5n^5-92n^4+605n^3-1622n^2+548n+2696}{360n^2}\tau^2 - \frac{n^3-12n^2+5n-14}{120}\|R\|^2 - \frac{7n^3-164n^2+435n-218}{90} \times \sum_{i,j=1}^{n} R^2_{\xi i\xi j} + \frac{n^3-12n^2+25n-34}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk}\right\}, \tag{11}$$

$$D_{(0)} = \frac{1}{(n-1)(n+1)}\left\{\frac{5n^4-77n^3+14n^2+1180n-2072}{180n^2}\tau^2 + \frac{59n^2-211n+142}{60}\|R\|^2 + \frac{173n^2-657n+514}{45} \times \sum_{i,j=1}^{n} R^2_{\xi i\xi j} - \frac{2(29n^2-101n+62)}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk}\right\}. \tag{12}$$

□

Now we are ready to derive the desired characterizations of the two-point homogeneous spaces for $n > 4$.

Theorem 5. *Let (M, g) be an n-dimensional Riemannian manifold with holonomy adapted to a two-point homogeneous space. If $4 < n$ and the total scalar curvature of sufficiently small boundaries $D_m^\xi(r)$ coincides with that of a two-point homogeneous space, then (M, g) is locally isometric to that model space.*

Proof. It follows from Lemma 5-(i) that (M, g) is 2-stein and thus super-Einstein [CV81], from where it follows that

$$\sum_{i,j=1}^{n} R_{\xi i \xi j}^2 = \frac{1}{n(n+2)}\left(\frac{3}{2}\|R\|^2 + \frac{1}{n}\tau^2\right), \qquad \sum_{i,j,k=1}^{n} R_{\xi ijk}^2 = \frac{1}{n}\|R\|^2. \tag{13}$$

Then, the coefficient $A_{(0)}$ in the power series expansion of the total scalar curvature of the boundaries of geodesic disks becomes

$$A_{(2)} = \frac{1}{n(n^2-1)(n+2)}\left\{\frac{(n-4)(5n^4-27n^3-12n^2+188n+228)}{360n}\tau^2 - \frac{(n-4)(n^3+n^2+26n+6)}{120}\|R\|^2\right\}.$$

Now the result is obtained by just comparing this with the corresponding coefficient $A_{(2)}$ in the model spaces and using the equations (2)–(5). □

Here it is worthwhile to emphasize that dimension four is excluded in previous theorem. Since the boundaries of the geodesic disks in a 4-dimensional manifold are compact surfaces, the total curvature $\int_{D_m^\xi(r)} \hat{\tau}$ is the Gauss Bonnet integral, and thus a topological invariant.

Theorem 6. *Let (M, g) be an n-dimensional Riemannian manifold with holonomy adapted to a two-point homogeneous space. If $3 < n \neq 6$ and the L^2-norms of the scalar curvature or the Ricci tensor or the curvature tensor of sufficiently small boundaries of geodesic disks coincides with that of a two-point homogeneous space, then (M, g) is locally isometric to that model space.*

Proof. Proceeding as in the previous theorem and using (13), the equations (10), (11) and (12) of the corresponding coefficients become

$$B_{(0)} = \frac{1}{n(n^2-1)(n+2)} \times \left\{\frac{5n^7-92n^6+585n^5-1162n^4-1760n^3+7332n^2-720n-12528}{360n}\tau^2 - \frac{n^6-9n^4-190n^3+714n^2-840n-216}{120}\|R\|^2\right\},$$

$$C_{(0)} = \frac{n-3}{n(n^2-1)(n+2)} \left\{ \frac{5n^5 - 67n^4 + 220n^3 + 220n^2 - 1380n - 2088}{360n} \tau^2 - \frac{(n^2 - 14n - 2)(n^2 - n + 18)}{120} \|R\|^2 \right\},$$

$$D_{(0)} = \frac{1}{n(n^2-1)(n+2)} \left\{ \frac{(n-3)(5n^4 - 52n^3 - 296n^2 + 1012n + 696}{180n} \tau^2 + \frac{(n-3)(59n^3 - 148n^2 - 34n - 12)}{60} \|R\|^2 \right\}.$$

Now the result follows by comparing these with the corresponding coefficients in the model spaces and using the characterizations (2)–(5). □

Explicit formulas for the total scalar curvatures of the boundaries of geodesic disks in the two-point homogeneous spaces are not yet available. However, by making use of the expansions in Theorems 1–4, the first terms in their power series expansions can be explicitly computed.

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