The Geography of Non-Formal Manifolds

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Dedicated to Professor Lieven Vanhecke

Summary. We show that there exist non-formal compact oriented manifolds of dimension n and with first Betti number $b_1 = b \geq 0$ if and only if $n \geq 3$ and $b \geq 2$, or $n \geq (7 - 2b)$ and $0 \leq b \leq 2$. Moreover, we present explicit examples for each one of these cases.

1 Introduction

Simply connected compact manifolds of dimension less than or equal to 6 are formal [11, 10, 5]. A method to construct non-formal simply connected compact manifolds of any dimension $n \geq 7$ was given by the authors in [6]. An alternative method is given in [3] (see also [12] for an example in dimension 7). A natural question is whether there are examples of non-formal compact manifolds of any dimension whose first Betti number $b_1 = b \geq 0$ is arbitrary. We consider the following problem on the *geography* of manifolds:

For which pairs (n, b) with $n \geq 1$ and $b \geq 0$ are there compact oriented manifolds of dimension n and with $b_1 = b$ which are non-formal? Note that we can restrict to just considering connected manifolds. In this paper, we solve this problem completely by proving the following theorem.

Theorem 1. *There are compact oriented n-dimensional manifolds with $b_1 = b$ which are non-formal if and only if $n \geq 3$ and $b \geq 2$, or $n \geq (7 - 2b)$ and $0 \leq b \leq 2$.*

In the case of a simply connected manifold M, formality for M is equivalent to saying that its real homotopy type is determined by its real cohomology algebra. In the non-simply connected case, things are a little bit more complicated. If M is nilpotent,

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i.e., $\pi_1(M)$ is nilpotent and it acts nilpotently on $\pi_i(M)$ for $i \geq 2$, then formality means again that the real homotopy type is determined by the real cohomology algebra. In general, we shall say that M is formal, if the minimal model of the manifold (which is, by definition, the minimal model of the algebra of differential forms $\Omega^*(M)$) is determined by the real cohomology algebra (see Section 2 for precise definitions). Note that there are alternative (and non-equivalent) definitions of formality in the non-nilpotent situation (see [8]). This punctualization is important because the non-formal manifolds that we construct in Section 3 are necessarily not nilpotent (see Section 5). In the following table, the big dots mark the pairs (n, b_1) for which all manifolds of dimension n and first Betti number b_1 are formal. For any of the small dots, there are examples of non-formal manifolds. To prove Theorem 1 we need to do two things. On one hand, we need to verify that manifolds of dimension $n \leq 6$ with $b_1 = 0$ and manifolds of dimension $n \leq 4$ with $b_1 = 1$ are *always* formal. For this we use the results of [5]. On the other hand, we need to present examples of *non-formal* manifolds of dimension $n \geq 7$ with $b_1 = 0$, of dimension $n \geq 5$ with $b_1 = 1$ and of dimension $n \geq 3$ for any other $b_1 \geq 2$. For this we use a similar method to that of [6]. Note that both questions for the case $b_1 = 0$ are already solved, so here we have to focus on the case $b_1 = 1$.

Table 1. Geography of non-formal manifolds

$n \geq 7$	·	·	·	…
$n = 6$	•	·	·	…
$n = 5$	•	·	·	…
$n = 4$	•	•	·	…
$n = 3$	•	•	·	…
$n = 2$	•	•	•	•
	$b_1 = 0$	$b_1 = 1$	$b_1 = 2$	$b_1 \geq 3$

2 Minimal models and formality

We recall some definitions and results about minimal models [2, 7, 13]. Let (A, d) be a *differential algebra*, that is, A is a graded commutative algebra over the real numbers, with a differential d which is a derivation, i.e., $\mathrm{d}(a \cdot b) = (\mathrm{d}a) \cdot b + (-1)^{\deg(a)} a \cdot (\mathrm{d}b)$, where $\deg(a)$ is the degree of a. Morphisms between differential algebras are required to be degree preserving algebra maps which commute with the differentials. A differential algebra (A, d) is said to be *minimal* if:

1. A is free as an algebra, that is, A is the free algebra $\bigwedge V$ over a graded vector space $V = \oplus V^i$, and
2. there exists a collection of generators $\{a_\tau, \tau \in I\}$, for some well-ordered index set I, such that $\deg(a_\mu) \leq \deg(a_\tau)$ if $\mu < \tau$ and each $\mathrm{d}a_\tau$ is expressed in terms of preceding a_μ $(\mu < \tau)$. This implies that $\mathrm{d}a_\tau$ does not have a linear part, i.e., it lives in $\bigwedge V^{>0} \cdot \bigwedge V^{>0} \subset \bigwedge V$.

We shall say that a minimal differential algebra $(\bigwedge V, \mathrm{d})$ is a *minimal model* for a connected differentiable manifold M, if there exists a morphism of differential graded algebras $\rho\colon (\bigwedge V, \mathrm{d}) \longrightarrow (\Omega M, \mathrm{d})$, where ΩM is the de Rham complex of differential forms on M, inducing an isomorphism

$$\rho^*\colon H^*(\bigwedge V) \longrightarrow H^*(\Omega M, \mathrm{d}) = H^*(M)$$

on cohomology. If M is a simply connected manifold (or, more generally, a nilpotent space), the dual of the real homotopy vector space $\pi_i(M)\otimes\mathbf{R}$ is isomorphic to V^i for any i. Halperin in [7] proved that any connected manifold M has a minimal model unique up to isomorphism, regardless of its fundamental group. A minimal model $(\bigwedge V, \mathrm{d})$ of a manifold M is said to be *formal*, and M is said to be *formal*, if there is a morphism of differential algebras $\psi\colon (\bigwedge V, \mathrm{d}) \longrightarrow (H^*(M), \mathrm{d} = 0)$ that induces the identity on cohomology. Alternatively, the above property means that $(\bigwedge V, \mathrm{d})$ is a minimal model of the differential algebra $(H^*(M), 0)$. Therefore, $(\Omega M, \mathrm{d})$ and $(H^*(M), 0)$ share their minimal model, i.e., one can obtain the minimal model of M out of its real cohomology algebra. When M is nilpotent, the minimal model encodes its real homotopy type. In order to detect non-formality, we have Massey products. Let us recall its definition. Let M be a (not necessarily simply connected) manifold and let $a_i \in H^{p_i}(M)$, $1 \leq i \leq 3$, be three cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. Take forms α_i in M with $a_i = [\alpha_i]$ and write $\alpha_1 \wedge \alpha_2 = \mathrm{d}\xi$, $\alpha_2 \wedge \alpha_3 = \mathrm{d}\eta$. The Massey product of the classes a_i is defined as

$$\begin{aligned}\langle a_1, a_2, a_3\rangle &= [\alpha_1 \wedge \eta + (-1)^{p_1+1}\xi \wedge \alpha_3] \\ &\in \frac{H^{p_1+p_2+p_3-1}(M)}{a_1 \cup H^{p_2+p_3-1}(M) + H^{p_1+p_2-1}(M) \cup a_3}.\end{aligned}$$

We have the following result, for whose proof we refer to [2, 13, 14].

Theorem 2. *If M has a non-trivial Massey product, then M is non-formal.*

Therefore, the existence of a non-zero Massey product is an obstruction to the formality.

In order to prove formality, we extract the following notion from [5].

Definition 1. Let $(\bigwedge V, \mathrm{d})$ be a minimal model of a differentiable manifold M. We say that $(\bigwedge V, \mathrm{d})$ is *s-formal*, or M is a *s-formal manifold* ($s \geq 0$) if for each $i \leq s$ one can get a space of generators V^i of elements of degree i that decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:

1. $\mathrm{d}(C^i) = 0$,
2. the differential map $\mathrm{d}\colon N^i \longrightarrow \bigwedge V$ is injective,
3. any closed element in the ideal $I_s = I(\bigoplus_{i\leq s} N^i)$, generated by $\bigoplus_{i\leq s} N^i$ in $\bigwedge(\bigoplus_{i\leq s} V^i)$, is exact in $\bigwedge V$.

The condition of s-formality is weaker than that of formality. However, we have the following positive result proved in [5].

Theorem 3. *Let M be a connected and orientable compact differentiable manifold of dimension $2n$ or $(2n-1)$. Then M is formal if and only if is $(n-1)$-formal (that is, if and only if M is s-formal, for $s = n-1$, according to the previous definition).*

This result is very useful because it allows us to check that a manifold M is formal by looking at its s-stage minimal model, that is, $\bigwedge(\bigoplus_{i\leq s} V^i)$. In general, when computing the minimal model of M, after we pass the middle dimension, the number of generators starts to grow quite dramatically. This is due to the fact that Poincaré duality imposes that the Betti numbers do not grow and therefore there are a large number of cup products in cohomology vanishing, which must be killed in the minimal model by introducing elements in N^i, for i above the middle dimension. This makes Theorem 3 a very useful tool for checking formality in practice.

3 Non-formal manifolds with $b_1 = 1$ and dimensions 5 and 6

The 5-dimensional example

Let H be the Heisenberg group, that is, the connected nilpotent Lie group of dimension 3 consisting of matrices of the form

$$a = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbf{R}$. Then a global system of coordinates x, y, z for H is given by $x(a) = x$, $y(a) = y$, $z(a) = z$, and a standard calculation shows that a basis for the left invariant 1-forms on H consists of $\{dx, dy, dz - x\,dy\}$. Let Γ be the discrete subgroup of H consisting of matrices whose entries are integer numbers. So the quotient space $N = \Gamma\backslash H$ is a compact 3-dimensional nilmanifold. Hence the forms $dx, dy, dz - x\,dy$ descend to 1-forms α, β, γ on N and

$$d\alpha = d\beta = 0, \quad d\gamma = -\alpha \wedge \beta.$$

The non-formality of N is detected by a non-zero triple Massey product

$$\langle[\beta], [\alpha], [\alpha]\rangle = [-\alpha \wedge \gamma] \in \frac{H^2(N)}{[\beta] \cup H^1(N) + H^1(N) \cup [\alpha]} = H^2(N).$$

Now, let us consider the 5-dimensional manifold $X = N \times \mathbf{T}^2$, where $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. The coordinates of $\mathbf{R}^2$ will be denoted x_1, x_2. So $\{dx_1, dx_2\}$ defines a basis $\{\delta_1, \delta_2\}$ for the 1-forms on $\mathbf{T}^2$. We get a non-zero triple Massey product as follows:

$$\langle[\beta \wedge \delta_1], [\alpha], [\alpha]\rangle = [-\gamma \wedge \alpha \wedge \delta_1]. \tag{1}$$

Our aim now is to kill the fundamental group of X by performing a suitable surgery construction, in order to obtain a manifold with $b_1 = 1$. The projection $p(x, y, z) = (x, y)$ describes N as a fiber bundle $p : N \to \mathbf{T}^2$ with fiber $\mathbf{S}^1$. Actually, N is the total space of the unit circle bundle of the line bundle of degree 1 over the 2-torus. The fundamental group of N is therefore

$$\pi_1(N) \cong \Gamma = \langle \lambda_1, \lambda_2, \lambda_3 \mid [\lambda_1, \lambda_2] = \lambda_3, \lambda_3 \text{ central}\rangle, \tag{2}$$

where λ_3 corresponds to the fiber. The fundamental group of $X = N \times \mathbf{T}^2$ is

$$\pi_1(X) = \pi_1(N) \oplus \mathbf{Z}^2. \tag{3}$$

Consider the following submanifolds embedded in X:

$$T_1 = p^{-1}(\{0\} \times \mathbf{S}^1) \times \{0\} \times \{0\},$$
$$T_2 = \{\xi\} \times \mathbf{S}^1 \times \mathbf{S}^1,$$

with ξ a point in N. These are 2-dimensional tori with trivial normal bundle. Consider now another 5-manifold Y with an embedded 2-dimensional torus T with trivial normal bundle. Then, we may perform the *fiber connected sum* of X and Y identifying T_1 and T, denoted $X \#_{T_1=T} Y$, in the following way: take (open) tubular neighborhoods $\nu_1 \subset X$ and $\nu \subset Y$ of T_1 and T respectively; then $\partial \nu_1 \cong \mathbf{T}^2 \times \mathbf{S}^2$ and $\partial \nu \cong \mathbf{T}^2 \times \mathbf{S}^2$; take an orientation reversing diffeomorphism $\phi : \partial \nu_1 \xrightarrow{\cong} \partial \nu$; the fiber connected sum is defined to be the (oriented) manifold obtained by gluing $X - \nu_1$ and $Y - \nu$ along their boundaries by the diffeomorphism ϕ. In general, the resulting manifold depends on the identification ϕ, but this will not be relevant for our purposes.

Lemma 1. *Suppose Y is simply connected, then the fundamental group of $X \#_{T_1=T} Y$ is the quotient of $\pi_1(X)$ by the image of $\pi_1(T_1)$.*

Proof. Since the codimension of T_1 is bigger than or equal to 3, we have that $\pi_1(X - \nu_1) = \pi_1(X - T_1)$ is isomorphic to $\pi_1(X)$. The Seifert–Van Kampen theorem establishes that $\pi_1(X \#_{T_1=T} Y)$ is the amalgamated sum of $\pi_1(X - \nu_1) = \pi_1(X)$ and $\pi_1(Y - \nu) = \pi_1(Y) = 1$ over the image of $\pi_1(\partial \nu_1) = \pi_1(T_1 \times \mathbf{S}^2) = \pi_1(T_1)$, as required. □

We shall take for Y the 5-sphere $\mathbf{S}^5$. We embed a 2-dimensional torus $\mathbf{T}^2$ in $\mathbf{R}^5$. This torus has a trivial normal bundle since its tangent bundle is trivial (being parallelizable) and the tangent bundle of $\mathbf{R}^5$ is also trivial. After compactifying $\mathbf{R}^5$ by one point, we get a 2-dimensional torus $T \subset \mathbf{S}^5$ with trivial normal bundle. In the same way, we may consider another copy of the 2-dimensional torus $T \subset \mathbf{S}^5$ and perform the fiber connected sum of X and $\mathbf{S}^5$ identifying T_2 and T. We may do both fiber connected sums along T_1 and T_2 simultaneously, since T_1 and T_2 are disjoint. Call

$$M = X \#_{T_1=T} \mathbf{S}^5 \#_{T_2=T} \mathbf{S}^5 \tag{4}$$

the resulting manifold. By Lemma 1, $\pi_1(M)$ is the quotient of $\pi_1(X)$ by the images of $\pi_1(T_1)$ and $\pi_1(T_2)$. This kills the $\mathbf{Z}^2$ summand in (3) and it also kills λ_2 and λ_3 in (2). Therefore $\pi_1(M) = \langle \lambda_1 \rangle \cong \mathbf{Z}$, i.e., $b_1(M) = 1$.

Our goal now is to prove that M is non-formal. We shall do this by proving the non-vanishing of a suitable triple Massey product. More specifically, let us prove that the Massey product (1) survives to M. For this, let us describe geometrically the cohomology classes $[\alpha \wedge \delta_1]$ and $[\beta]$. Consider the following submanifolds of X:

$$B_1 = p^{-1}(\mathbf{S}^1 \times \{a_2\}) \times \{b_1\} \times \mathbf{S}^1,$$
$$B_2 = p^{-1}(\{a_1\} \times \mathbf{S}^1) \times \mathbf{S}^1 \times \mathbf{S}^1,$$

where the a_i and b_i are generic points of $\mathbf{S}^1$. It is easy to check that $B_i \cap T_j = \emptyset$ for all i and j. So B_i may be also considered as submanifolds of M. Let η_i be the 2-forms representing the Poincaré dual to B_i in X. By [1], η_i can be taken supported in a small tubular neighborhood of B_i. Therefore the support of η_i lies inside $X - T_1 - T_2$, so we also have naturally $\eta_i \in \Omega^2(M)$. Note that in X we have clearly that $[\eta_1] = [\beta \wedge e_1]$ and $[\eta_2] = [\alpha]$, where e_1 is (the pull-back of) a differential 1-form on $\mathbf{S}^1$ (considered as the first of the two circle factors in $X = N \times \mathbf{S}^1 \times \mathbf{S}^1$) cohomologous to δ_1 and supported in a neighborhood of $b_1 \in \mathbf{S}^1$. Thus $[\eta_1] = [\beta \wedge \delta_1]$ in X.

Lemma 2. *The triple Massey product* $\langle[\eta_1], [\eta_2], [\eta_2]\rangle$ *is well-defined on M and equals to* $[-\gamma \wedge \alpha \wedge e_1]$.

Proof. Let α' be the pull-back to N of the 1-form supported in a neighborhood of a_1 in the first factor of $\mathbf{S}^1 \times \mathbf{S}^1$ under the projection $p : N \to \mathbf{T}^2$. Analogously, let β' be the pull-back to N of the 1-form supported in a neighborhood of a_2 in the second factor of $\mathbf{S}^1 \times \mathbf{S}^1$. Therefore $[\alpha'] = [\alpha]$ and $[\beta'] = [\beta]$. Clearly,

$$(\alpha' \wedge e_1) \wedge \beta' = \mathrm{d}\gamma' \wedge e_1,$$

where $\mathrm{d}\gamma' = \alpha' \wedge \beta'$. It can be supposed easily that γ' is zero in a neighborhood of $\xi \in N$. Therefore the support of $\gamma' \wedge e_1$ is disjoint from T_1 and T_2. Hence $\gamma' \wedge e_1$ is well-defined as a form in M. So the triple Massey product

$$\langle[\eta_1], [\eta_2], [\eta_2]\rangle = [-\gamma' \wedge \alpha \wedge e_1]$$

is well-defined in M. □

Finally, let us see that this Massey product is non-zero in

$$\frac{H^3(M)}{[\beta' \wedge e_1] \cup H^1(M) + H^2(M) \cup [\alpha']}.$$

Consider $B_3 = p^{-1}(\mathbf{S}^1 \times \{a_3\}) \times \mathbf{S}^1 \times \{b_2\}$, for generic points a_3, b_2 of $\mathbf{S}^1$. Then the Poincaré dual of B_3 is defined by a 2-form $\beta'' \wedge e_2$ supported near B_3, where β'' is Poincaré dual to $p^{-1}(\mathbf{S}^1 \times \{a_3\})$, $[\beta''] = [\beta]$, and e_2 is (the pull-back of) a differential 1-form on $\mathbf{S}^1$ (considered as the second of the two circle factors in $X = N \times \mathbf{S}^1 \times \mathbf{S}^1$) cohomologous to δ_2 and supported in a neighborhood of $b_2 \in \mathbf{S}^1$. Again this 2-form can be considered as a form in M. Now, for any $[\varphi] \in H^1(M)$, $[\varphi'] \in H^2(M)$ we have,

$$([\gamma' \wedge \alpha \wedge e_1] + [\beta' \wedge e_1 \wedge \varphi] + [\alpha' \wedge \varphi']) \cdot [\beta'' \wedge e_2] = 1,$$

since the first product gives 1, the second is zero and the third is zero because $\alpha' \wedge \beta''$ is exact in N and hence in M. This result and Theorem 2 prove the following:

Theorem 4. *The manifold M, defined by (4), is a compact oriented non-formal* 5-*manifold with* $b_1 = 1$.

The 6-dimensional example

A compact oriented non-simply connected and non-formal manifold M' of dimension 6 is obtained in an analogous fashion to the construction of the 5-dimensional manifold M. We start with $X' = N \times \mathbf{T}^3$ and consider the 3-dimensional tori with trivial normal bundle

$$T_1' = p^{-1}(\{0\} \times \mathbf{S}^1) \times \{0\} \times \{0\} \times \mathbf{S}^1,$$

$$T_2' = \{\xi\} \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1.$$

Define

$$M' = X' \#_{T_1'=T'} \mathbf{S}^6 \#_{T_2'=T'} \mathbf{S}^6, \tag{5}$$

where T' is an embedded 3-torus in $\mathbf{S}^6$ with trivial normal bundle. Then M' is a non-formal 6-manifold with $b_1 = 1$, which can be proved in a similar way to Theorem 4.

4 Proof of theorem 1

Let us first prove the affirmative results in Theorem 1.

Proposition 1. *Let M be a connected, compact and orientable manifold of dimension n and first Betti number $b_1 = b$.*

- *If $n \leq 2$, then M is formal.*
- *If $n \leq 6$ and $b = 0$, then M is formal.*
- *If $n \leq 4$ and $b = 1$, then M is formal.*

Proof. The first item is well-known: The circle and any oriented surface are formal. However, it follows from Theorem 3 very easily. Since M is connected, M is 0-formal. Hence M is formal as $n \leq 2$. Second item follows from [5, 10, 11]. Let us recall briefly the proof. Since M has $b_1 = 0$, it follows that in the minimal model $V^1 = 0$. This implies that $N^2 = 0$ since there are no decomposable elements of degree 3 and hence no element of V^2 can kill any element of degree 3 in the minimal model. Thus M is 2-formal and hence formal, by Theorem 3, since $n \leq 6$. The third item is proved similarly. Since M has $b_1 = 1$, in the minimal model $(\bigwedge V, d)$ we have that $V^1 = C^1$ is generated by one element ξ. There cannot be any element in N^1 since there are no decomposable elements of degree 2 (the only such element is $\xi \cdot \xi = 0$). Thus M is 1-formal and hence formal, by Theorem 3, since $n \leq 4$. □

With this result, we only need to find non-formal (connected, compact, orientable) manifolds under the conditions $n \geq \max\{3, 7 - 2b_1\}$ to complete the proof of Theorem 1.

- Non-formal manifolds with $n \geq 7$ and $b_1 = 0$ are constructed by the authors in [6]. Actually, those examples are simply connected. An alternative method is given in [3]. Oprea [12] also constructed examples of dimension 7 for other purposes.
- Non-formal manifolds of dimensions $n = 5$ or 6 and first Betti number $b_1 = 1$. These are the manifolds M and M' given by (4) and (5) in Section 3.

- Non-formal manifolds of dimension $n \geq 7$ and $b_1 = 1$. Take the non-formal 5-dimensional manifold M of Section 3 and consider $M \times \mathbf{S}^{n-5}$. This is again non-formal (by [5, Lemma 2.11]) and has $b_1(M \times \mathbf{S}^{n-5}) = b_1(M) = 1$.
- Case $n = 3$ and $b_1 = 2$. The manifold N considered as the beginning of Section 3 is non-formal.
- Case $n = 3$ and $b_1 \geq 3$. Consider $N\#(b_1-2)(\mathbf{S}^1 \times \mathbf{S}^2)$, which is non-formal because the Massey product $\langle[\beta], [\alpha], [\alpha]\rangle = [\alpha \wedge \gamma]$ is again defined and non-zero (as it happened for N).
- Case $n = 4$ and $b_1 \geq 3$. Consider $\left(N\#(b_1 - 3)(\mathbf{S}^1 \times \mathbf{S}^2)\right) \times \mathbf{S}^1$, which is non-formal being a product of a non-formal manifold with other manifold.
- Case $n \geq 5$ and $b_1 \geq 2$. We just consider $\left(N\#(b_1 - 2)(\mathbf{S}^1 \times \mathbf{S}^2)\right) \times \mathbf{S}^{n-3}$.
- Case $n = 4$ and $b_1 = 2$. A non-formal example can be constructed by a nilmanifold which is non-formal. For example (see [4]), let E be the total space of the $\mathbf{S}^1$-bundle over N with Chern class $c_1 = [\beta \wedge \gamma] \in H^2(N)$. The nilmanifold E is defined by the equations,

$$d\alpha = d\beta = 0, \quad d\gamma = -\alpha \wedge \beta, \quad d\eta = \beta \wedge \gamma,$$

  where $\{\alpha, \beta, \gamma, \eta\}$ is a basis for the differential 1-forms on E. Then $[\beta] \cup [\alpha] = [\alpha] \cup [\alpha] = 0$, so that the Massey product $\langle[\beta], [\alpha], [\alpha]\rangle$ is well-defined, and it is non-zero because it is represented by the cohomology class of $\gamma \wedge \alpha$ which is non-zero in cohomology.

5 Final remarks

Note that the examples of non-formal manifolds with $b_1 = 1$ that we have constructed have Abelian fundamental group, since it is isomorphic to $\mathbf{Z}$. However, these manifolds are not nilpotent. Actually, if a manifold M with $b_1 = 1$ is nilpotent, then M is 2-formal. Furthermore, if the dimension is $n \leq 6$ and M is compact oriented, then it is formal. To prove that for a nilpotent manifold M with $b_1 = 1$ we have that M is 2-formal, it is enough to check that $N^2 = 0$. This would follow from the fact that no decomposable element of degree 3 (i.e., elements in $V^1 \cdot V^2$) is exact. Let ξ be the generator of V^1 and let $a \in V^2$ be a non-zero closed element. Suppose that $[\xi] \cup [a] = 0$ and let us reach to a contradiction. We use the following lemma of Lalonde–McDuff–Polterovich [9], which has been communicated to us by J. Oprea.

Lemma 3. *Suppose that $\gamma \in \pi_1(M)$, $A \in \pi_2(M)$, $h \in H^1(M; \mathbf{Z})$ and $\alpha \in H^2(M; \mathbf{Z})$, satisfy that $h(\gamma) \neq 0$ and $\alpha(A) \neq 0$. Then if $\alpha \cup h = 0$, the action of γ on A is non-trivial.*

In our case, take $h = [\xi] \in H^1(M)$ (after suitable rescaling if necessary to make it an integral class). Let $\gamma \in \pi_1(M)$ be any element with $h(\gamma) \neq 0$. Then, $h(\gamma^n) \neq 0$ for any $n > 0$. Now take $\alpha = [a]$ and consider any element $A \in \pi_2(M)$ with $\alpha(A) \neq 0$ (this exists since we are assuming that M is nilpotent and in this case $V^2 = (\pi_2(M) \otimes \mathbf{R})^*$). Then Lemma 3 implies that γ^n acts on A non-trivially. Hence γ acts non-nilpotently on $\pi_2(M)$, which is a contradiction.

We end up with some questions that arise naturally once Theorem 1 is answered.

1. Are there any restrictions on the Betti numbers for the existence of non-formal manifolds? Alternatively, solve the following *geography problem*:
   For which tuples $(n, b_1, \ldots, b_s)$ with $n \geq 1$, $s = [n/2]$ and $b_i \geq 0$ is there a compact oriented manifold M of dimension n, with Betti numbers $b_i(M) = b_i$, $i = 1, \ldots, s$, and which is *non-formal*?
2. Another alternative question is the following: Given a finitely presented group Γ and an integer n with $n \geq \max\{3, 2b_1(\Gamma) - 7\}$, are there always non-formal n-manifolds M with fundamental group $\pi_1(M) \cong \Gamma$?

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