$\mathbb{Z}_2$ and $\mathbb{Z}$-Deformation Theory for Holomorphic and Symplectic Manifolds*

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Dedicated to Professor Lieven Vanhecke

Summary. We present and investigate, within the general frame of deformation theory, new $\mathbb{Z}_2$-constructions for generalized moduli spaces of holomorphic and symplectic structures.

1 Introduction

Deformation theories are one of the keystone settings of contemporary geometry, appearing in very different areas and providing, through moduli space constructions, a highly powerful tool to produce new invariants (cf. [6] and [7] for recent accounts).

This paper, in the first part, describes a tuning up of a general machine for deformation theory, enhancing the relationships between $\mathbb{Z}$ and $\mathbb{Z}_2$-theories. Then, after presenting equivalence classes of A^∞-algebras as an example of deformation space, to show how vast the range covered by deformation theories is, it deals with complex/holomorphic deformations and symplectic deformation. In the latter case, a totally new non-naïf theory is constructed.

By means of the results established in the first part, both in the complex/holomorphic case and the symplectic case, we define and discuss the corresponding $\mathbb{Z}_2$-theories (complex/holomorphic and supersymplectic structures).

2 $\mathbb{Z}_2$-theory and $\mathbb{Z}$-theory of deformations of DLA

2.1 $\mathbb{Z}_2$-theory: superstructures

We start with a quick overview of superstructures (or $\mathbb{Z}_2$-structures).

Definition 1. 1. A *super vector space* is a vector space V together with a decomposition

$$V = V^{(0)} \oplus V^{(1)}.$$

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Vectors in $V^{(0)}$ and in $V^{(1)}$ are called *homogeneous* of *degree* (or *parity*) 0 and 1, respectively. The degree of the homogeneous vector v is denoted by $|v|$.
A vector supersubspace of V is a subspace $W \subset V$ of the form

$$W = W^{(0)} \oplus W^{(1)},$$

where $W^{(j)}$ is a vector subspace of $V^{(j)}$, $j = 0, 1$.

2. A *super algebra* is an algebra A together with a vector space decomposition

$$A = A^{(0)} \oplus A^{(1)},$$

in such a way that

$$A^{(j)} A^{(k)} \subset A^{(j+k)}, \quad j, k \in \mathbb{Z}_2.$$

The bracket [,], defined on homogeneous elements as

$$[a,\, b] := ab - (-1)^{|a||b|} ba,$$

is called the *super commutator* of A and A is said to be *supercommutative* if its super commutator vanishes identically.

3. A *super Lie algebra* is a super vector space $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ together with a bilinear map,

$$[\, ,\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$

such that
   a) $[\mathfrak{g}^{(j)},\, \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(j+k)}$, $j, k \in \mathbb{Z}_2$
   b) for homogeneous elements a, b, c, we have:
      i. $[a,\, b] = -(-1)^{|a||b|}[b,\, a]$,
      ii. $[a,\, [b,\, c]] = [[a,\, b],\, c] + (-1)^{|a||b|}[b,\, [a,\, c]]$.

Note that:

- given a vector space V, the exterior algebra $\wedge^* V$ has a natural structure of supercommutative super algebra, just setting

$$V^{(0)} = \wedge^{\text{even}} V \ , \ V^{(1)} = \wedge^{\text{odd}} V;$$

- given a super vector space $V = V^{(0)} \oplus V^{(1)}$ with projections p_1 and p_2, then $End(V)$ has a natural structure of super algebra, just setting

$$End(V)^{(0)} := \{f \in End(V) \mid f(V^{(j)}) \subset V^{(j)} \ , \ j \in \mathbb{Z}_2\},$$

$$End(V)^{(1)} := \{f \in End(V) \mid f(V^{(j)}) \subset V^{(j+1)} \ , \ j \in \mathbb{Z}_2\},$$

the relation,

$$f = (p_1 \circ f \circ p_1 + p_2 \circ f \circ p_2) + (p_1 \circ f \circ p_2 + p_2 \circ f \circ p_1),$$

proves that

$$End(V) = End(V)^{(0)} \oplus End(V)^{(1)};$$

- given a super algebra A, the super commutator [,] defines on A the structure of super Lie algebra.

Definition 2. Let

$$A = A^{(0)} \oplus A^{(1)}$$

be a super algebra.

1. A *super derivation* D of degree $|D|$ on A is an element of $End(A)^{(|D|)}$ such that

$$D(ab) = (Da)b + (-1)^{|D||a|} a(Db).$$

   This amounts to say that, for every $a \in A$, we have

$$[D, L_a] - L_{Da} = 0,$$

   L_a being the left multiplication by a.
2. A *differential* d on A is a super derivation of degree 1 such that $d^2 = 0$. The couple (A, d) is called a *differential super algebra* (DSA).

Definition 3. Let

$$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$$

be a super Lie algebra.

1. A *super derivation* D of degree $|D|$ on $\mathfrak{g}$ is an element of $End(\mathfrak{g})^{(|D|)}$ such that:

$$D[a, b] = [Da, b] + (-1)^{|D||a|}[a, Db].$$

2. A *differential* d on $\mathfrak{g}$ is a super derivation of degree 1 such that $d^2 = 0$. The couple $(\mathfrak{g}, d)$ is called a *differential super Lie algebra* (DSLA).

Note that, in both cases, the super commutator of the super derivations D and F is a super derivation of degree $|D| + |F|$.

Let $(\mathfrak{g}, [\,,\,], d)$ be a DSLA. We set

$$Z^{(p)} = Z^{(p)}(\mathfrak{g}, d) := \{a \in \mathfrak{g}^{(p)} \mid da = 0\}\ ,\ Z = Z^{(0)} \oplus Z^{(1)},$$

$$B^{(p)} = B^{(p)}(\mathfrak{g}, d) := d(\mathfrak{g}^{(p-1)})\ ,\ B = B^{(0)} \oplus B^{(1)},$$

$$H^{(p)} = H^{(p)}(\mathfrak{g}, d) = Z^{(p)}(\mathfrak{g}, d)/B^{(p)}(\mathfrak{g}, d)\ ,\ H = H^{(0)} \oplus H^{(1)}.$$

Let $(\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}, [\,,\,], d)$ be a DSLA. For $\gamma \in \mathfrak{g}^{(1)}$, set

$$d_\gamma a := da + [\gamma, a].$$

Clearly,

$$d_\gamma[a, b] = [d_\gamma a, b] + (-1)^{|a|}[a, d_\gamma b]$$

and

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0 \Longrightarrow d_\gamma^2 = 0,$$

i.e.,

$$\gamma \quad \text{satisfies the Maurer-Cartan (MC) equation} \Longrightarrow d_\gamma^2 = 0.$$

Set

$$\mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g}) = \{\gamma \in \mathfrak{g}^{(1)} \mid \gamma \text{ satisfies } MC\}.$$

Given $\alpha \in \mathfrak{g}$, set

$$ad(\exp(\alpha)) := \sum_{h=0}^{\infty} \frac{1}{h!}(\alpha\delta(\alpha))^h$$

(where, of course, $\alpha\delta(\alpha)(\epsilon) := [\alpha, \epsilon]$). Therefore $\exp(\mathfrak{g})$ acts on the left on $\mathfrak{g}$. $\exp(\mathfrak{g}^{(0)})$ acts on the d_γ's on the left as

$$d_\gamma \mapsto ad(\exp(\alpha))d_\gamma ad(\exp(-\alpha))$$

and this induces a left action of $\exp(\mathfrak{g}^{(0)})$ on $\mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g})$ given by

$$(\alpha, \gamma) \mapsto \chi(\alpha)\gamma := \gamma - \sum_{h=0}^{\infty} \frac{1}{(h+1)!}(\alpha\delta(\alpha))^h(d_\gamma\alpha).$$

As usual, the results of the present section hold in the framework of formal power series, i.e., modulo convergence. Convergence can be rigorously established in the class of Artin rings and their projective limits.

Set

$$\mathrm{Def}_{\mathbb{Z}_2}(\mathfrak{g}) := \mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g})/\exp(\mathfrak{g}^{(0)}).$$

Definition 4. $\mathrm{Def}_{\mathbb{Z}_2}(\mathfrak{g})$ is called the $\mathbb{Z}_2$-deformation space of the DSLA $\mathfrak{g}$.

Note that:

- if $t \mapsto \gamma(t)$ is a smooth curve in $\mathfrak{MC}(\mathfrak{g})$, with $\gamma(0) = \gamma$, then,

$$d\gamma(t) + \frac{1}{2}[\gamma(t), \gamma(t)] = 0$$

and so

$$0 = d\gamma\prime(0) + [\gamma, \gamma\prime(0)] = d_\gamma\gamma\prime(0).$$

Consequently,

$$T_\gamma\mathfrak{MC}(\mathfrak{g}) \subset Z^{(1)}(\mathfrak{g}, d_\gamma).$$

- $$\frac{d}{dt}\chi(t\alpha)\gamma_{|t=0} = -d_\gamma\alpha$$

and so,

$$\gamma \mapsto -d_\gamma\alpha,$$

represents the fundamental vector field of α associated to the given action.

- Setting

  $$\hat{\gamma}(t) := \chi(t\alpha)\gamma(t),$$

  we get

  $$\hat{\gamma}'(0) = \gamma'(0) - d_\gamma \alpha.$$

  Consequently, if $\epsilon \in Z^{(1)}(\mathfrak{g}, d_\gamma)$ is tangent to $\mathfrak{MC}(\mathfrak{g})$ at γ, then any element of $[\epsilon] \in H^{(1)}(\mathfrak{g}, d_\gamma)$ is tangent to $\mathfrak{MC}(\mathfrak{g})$ at γ and is related to ϵ by the action induced by χ. Therefore, if $\langle\gamma\rangle \in \mathrm{Def}_{\mathbb{Z}_2}(\mathfrak{g})$ then

  $$T_{\langle\gamma\rangle}\mathrm{Def}_{\mathbb{Z}_2}(\mathfrak{g}) \subset H^{(1)}(\mathfrak{g}, d_\gamma).$$

We set the following

Definition 5. If

$$T_{\langle\gamma\rangle}\mathrm{Def}_{\mathbb{Z}_2}(\mathfrak{g}) = H^{(1)}(\mathfrak{g}, d_\gamma),$$

we say that the deformation theory of the DSLA $\mathfrak{g}$ is totally unobstructed at $\langle\gamma\rangle$.

We are mainly interested, as we shall see, in infinitesimal deformations at 0 or, more precisely, formal developments of deformations.

2.2 $\mathbb{Z}$-theory

As a special case of superstructures we have $\mathbb{Z}$-graded structures.

Definition 6. A *graded vector space* is a vector space V together with a decomposition,

$$V = \bigoplus_{p\in\mathbb{Z}} V_p,$$

with the agreement that $V_p = \{0\}$, if $p < 0$; again vectors in the V_p's are called *homogeneous* and they are assigned to have *degree* p. In the same way, we can consider graded algebras, graded Lie algebras, differential graded algebras (DGA), differential graded Lie algebras (DGLA) etc., with the same definitions as before (indices in $\mathbb{Z}$).

In particular, if

$$\left(\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p,\ [\ ,\ ], d\right)$$

is a DGLA, we set

$$\mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g}) = \{\gamma \in \mathfrak{g}_1 \mid \gamma \text{ satisfies } MC\}.$$

Then, exactly as the $\mathbb{Z}_2$-case, we have a left action of $\exp(\mathfrak{g}_0)$ on $\mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g})$ and we set

$$\mathrm{Def}_{\mathbb{Z}}(\mathfrak{g}) := \mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g})/\exp(\mathfrak{g}_0).$$

Note that any graded structure has a natural underlying superstructure.

2.3 Formal deformations

We want to describe the basic setting of formal deformations. Let $(\mathfrak{g}, [\ ,\ ], d)$ be a DSLA and let H be its cohomology. Let H^* be the super vector space dual of H and let $K := \Bbbk[[H^*]]$ be the completed supersymmetric algebra of H^*.

In particular, if $\{v_1, \ldots, v_N\}$ is a super basis of H, the dual superbasis $\{x_1, \ldots, x_N\}$ satisfies $|x_j| = |v_j| - 1\ ,\ 1 \le j \le N$.

Set

$$\mathfrak{g}_K := \mathfrak{g} \otimes K\ ,\ d_K := d \otimes 1 \text{ etc.,}$$

extend the structure of DSLA to $\mathfrak{g}_K$ in the standard way, i.e.,

- $[a \otimes \alpha,\ b \otimes \beta] = (-1)^{|\alpha||b|}[a,\ b] \otimes \alpha\beta,$
- $|a \otimes \alpha| = |a||\alpha|.$

Finally, let $\mathfrak{m}_K$ be the maximal ideal of K.
Note that

- $\mathfrak{g} \otimes \mathfrak{m}_K$ an ideal (and hence a subalgebra) of $\mathfrak{g}_K$;
- $\omega \in \mathfrak{g}_K$ can be written as

$$\omega = \sum_{j=0}^{\infty} \omega_j,$$

  where the ω_j's are homogeneous polynomials of degree j in the H^*-variables;
- $v_h \mapsto v_h x_h\ ,\ 1 \le h \le N\ ,$ identifies H with a degree-one homogeneous polynomial in $(H \otimes \mathfrak{m}_K)^{(1)}$.

Set

$$\mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]] := \mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g} \otimes \mathfrak{m}_K) = \left\{\gamma \in (\mathfrak{g} \otimes \mathfrak{m}_K)^{(1)} \mid d_K\gamma + \frac{1}{2}[\gamma,\ \gamma] = 0\right\},$$

$$\mathrm{Def}_{\mathbb{Z}_2}[[\mathfrak{g}]] := \mathrm{Def}(\mathfrak{g} \otimes \mathfrak{m}_K).$$

Definition 7. We say that the deformation theory of the DSLA $\mathfrak{g}$ is formally totally unobstructed (at $\langle\gamma\rangle$), if the deformation theory of $\mathfrak{g} \otimes \mathfrak{m}_K$ is totally unobstructed at $\langle\gamma\rangle$.

2.4 $\mathbb{Z}$-theory versus $\mathbb{Z}_2$-theory

It is a very interesting fact that $\mathbb{Z}_2$-deformation theory fibers in a natural manner over $\mathbb{Z}$-deformation theory.

In fact, let $(\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p,\ [\ ,\ ], d)$ be a DGLA, let $\pi_j\ :\ \mathfrak{g} \longrightarrow \mathfrak{g}_j\ ,\ j \in \mathbb{Z}$ be the natural projections, and let $\tilde{\mathfrak{g}} := \bigoplus_{j>1} \mathfrak{g}_j$; consider on $\mathfrak{g}$ the underlying structure of DSLA. Then we have:

Lemma 1. $\pi_1\ :\ \mathfrak{g} \longrightarrow \mathfrak{g}_1$ *induces a surjective map*

$$\pi\ :\ \mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g}) \longrightarrow \mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g})$$

such that:

1. *for every* $\gamma_1 \in \mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g})$,

$$\pi^{-1}(\gamma_1) = \gamma_1 + \mathfrak{MC}_{\mathbb{Z}_2}(\tilde{\mathfrak{g}},\ d_{\gamma_1});$$

2. *for every* $\alpha \in \mathfrak{g}^{(0)}$,

$$\pi_1 \circ \chi(\alpha) = \chi(\pi_0(\alpha)) \circ \pi_1,$$

and thus we obtain a surjective map

$$\pi\ :\ Def_{\mathbb{Z}_2}(\mathfrak{g}) \longrightarrow Def_{\mathbb{Z}}(\mathfrak{g})$$

and

$$\pi^{-1}(\langle\gamma_1\rangle) \approx Def_{\mathbb{Z}_2}(\tilde{\mathfrak{g}},\ d_{\gamma_1}).$$

Proof. Let $\gamma \in \mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g})$. Write $\gamma = \gamma_1 + \sigma$ with $\gamma_1 = \pi_1(\gamma)$ and $\sigma \in \tilde{\mathfrak{g}} \cap \mathfrak{g}^{(1)}$. Then,

$$\begin{aligned} d\gamma + \frac{1}{2}[\gamma,\ \gamma] = 0 \\ &= d\gamma_1 + d\sigma + \frac{1}{2}[\gamma_1,\ \gamma_1] + \frac{1}{2}[\sigma,\ ,\ \sigma] + [\gamma_1\ ,\ \sigma] \\ &= d\gamma_1 + \frac{1}{2}[\gamma_1,\ \gamma_1] + d_{\gamma_1}\sigma + \frac{1}{2}[\sigma,\ ,\ \sigma], \end{aligned}$$

and thus,

$$\pi_1(\gamma) \in \mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g}), \qquad \sigma \in \mathfrak{MC}_{\mathbb{Z}_2}(\tilde{\mathfrak{g}}).$$

This gives the surjectivity and 1. at once. 2. is now obvious.

At formal level, we have:

$$\mathfrak{g} \otimes \mathfrak{m}_K = \bigoplus_{p \in \{\mathbb{Z}} (\mathfrak{g} \otimes \mathfrak{m}_K)_p,$$

where

$$(\mathfrak{g} \otimes (\mathfrak{m}_K))_p = \bigoplus_{r+s=p} (\mathfrak{g})_r \otimes (\mathfrak{m}_K)_s.$$

In particular,

$$(\mathfrak{g} \otimes (\mathfrak{m}_K)_1 = \mathfrak{g}_0 \otimes (\mathfrak{m}_K)_1 \mathfrak{g}_1 \otimes \mathfrak{m}_K)_0,$$

and

$$(\mathfrak{m}_K)_0 = \mathfrak{m}_{\hat{K}},$$

where

$$\hat{K} = \mathbb{k}[[x_1, \ldots, x_n]] \qquad \text{with} \qquad n = dim_{\mathbb{k}}\, H^1.$$

Therefore, we have a further reduction; the results are summarized in the following lemma, which can be proved exactly as the previous one.

Lemma 2. *Let*

$$p\ :\ (\mathfrak{g}\otimes\mathfrak{m}_K)_1\ \longrightarrow\ \mathfrak{g}_1(\otimes\mathfrak{m}_K)_0$$

be the natural projection. Then

1. *p induces a surjective map:*

$$p\ :\ \mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g}\otimes\mathfrak{m}_K)\ \longrightarrow\ \mathfrak{MC}_{\mathbb{Z}}[[\mathfrak{g}]]:=\left\{\alpha\in\mathfrak{g}_1\otimes\mathfrak{m}_K)_0\,|\,d\alpha+\frac{1}{2}[\alpha,\ \alpha]=0\right\};$$

2. *for every $\gamma_1\in\mathfrak{MC}_{\mathbb{Z}}[[\mathfrak{g}]]$,*

$$p^{-1}(\gamma_1)=\gamma_1+\mathfrak{E}_{\mathbb{Z}}[[\mathfrak{g},d_{\gamma_1}]]:=\left\{\alpha\in\mathfrak{g}_0\otimes\mathfrak{m}_K)_1\,|\,d_{\gamma_1}\alpha+\frac{1}{2}[\alpha,\ \alpha]=0\right\};$$

3. *p is* $\exp(\mathfrak{g}_0\otimes(\mathfrak{m}_K)_0$*-invariant;*
4. *setting $\tilde{\pi}:=p\circ\pi_1$ we have that, for every $\gamma_1\in\mathfrak{MC}_{\mathbb{Z}}[[\mathfrak{g}]]$,*

$$\tilde{\pi}_1(\gamma_1)=\gamma_1+\mathfrak{F}(\gamma_1),$$

where

$$\mathfrak{F}(\gamma_1)=\{(\beta,\ \sigma)\,|\,\beta\in\mathfrak{E}_{\mathbb{Z}}[[\mathfrak{g},d_{\gamma_1}]],\ \sigma\in\mathfrak{MC}_{\mathbb{Z}_2}(\tilde{\mathfrak{g}},\ d_{\gamma_1+\beta})\};$$

5. *we obtain a surjective map,*

$$\tilde{\pi}\ :\ Def_{\mathbb{Z}_2}[[\mathfrak{g}]]\ \longrightarrow\ Def_{\mathbb{Z}}[[\mathfrak{g}]]:=\mathfrak{MC}_{\mathbb{Z}}[[\mathfrak{g}]]/\exp(\mathfrak{g}_0\otimes(\mathfrak{m}_K)_0),$$

and $\tilde{\pi}^{-1}(\langle\gamma_1\rangle)\approx\{(\langle\beta\rangle,\ \langle\sigma\rangle)\}$, where

$$\langle\beta\rangle\in\mathfrak{E}_{\mathbb{Z}}[[\mathfrak{g},d_{\gamma_1}]]/\exp(\mathfrak{g}_0\otimes(\mathfrak{m}_K)_0),\quad\langle\sigma\rangle\in Def_{\mathbb{Z}_2}(\tilde{\mathfrak{g}},\ d_{\gamma_1+\beta}),$$

2.5 A special case

Let us begin with some general facts.

Definition 8. A differential $\mathbb{k}$-vector space $(V,\ d)$ is a $\mathbb{k}$-vector space V equipped with $d\in Hom_{\mathbb{k}}(V,\ V)$ satisfying $d^2=0$; set $Z:=\operatorname{Ker} d,\ B:=Im\,d,\ H:=Z/B$.

Lemma 3. *Let $(V,\ d)$ be a differential $\mathbb{k}$-vector space; then there exist vector subspaces $\mathcal{H}$ and S with*

1 *$\mathcal{H}\oplus B=Z$ (and so $\mathcal{H}\approx H$),*
2 *$S\cap Z=\{0\}$,*

in such a way that

$$V=\mathcal{H}\oplus dS\oplus S. \tag{1}$$

(1) *is called a* Hodge decomposition *for $(V,\ d)$.*

Moreover, given (1), $Q \in Hom_{\mathbb{k}}(V, V)$ *is defined in such a way that*

$$\alpha = \pi_{\mathcal{H}}(\alpha) + dQ(\alpha) + Q(d\alpha),$$

i.e., Q is a cohomological homotopy between I and $\pi_{\mathcal{H}}$ and $\alpha \in \mathfrak{g}$ is d-exact if and only if $d\alpha = 0$ and $\pi_{\mathcal{H}}(\alpha) = 0$ and in this case $\alpha = dQ(\alpha)$. Finally, if V is a super vector space (resp. a graded vector space) and d is compatible with the grading, then it is possible to choose $\mathcal{H}$ and S to be supersubspaces (resp. graded subspaces) obtaining a super (resp., graded) Hodge decomposition.

Proof. Let $\mathcal{H} \subset Z$ be a vector subspace such that

$$\mathcal{H} \oplus B = Z.$$

Let $R \subset \mathfrak{g}$ be a vector subspace such that:

- $\mathfrak{g} = \mathcal{H} \oplus R$,
- $B \subset R$.

Clearly, $R \cap Z = B$.

Let $S \subset R$ be a supersubspace such that $R = B \oplus S$. Then

$$S \cap Z = 0 \quad \text{and} \quad B = dS.$$

Finally, if

$$\alpha = \pi_{\mathcal{H}}(\alpha) + d\beta + \gamma,$$

just set $Q(\alpha) = \beta$. Then $dQ(d\alpha) = d\alpha = d\gamma$ and thus $\gamma = Q(d\alpha)$; note also that $Q^2 = 0$. Concerning the last statement, just observe that we can perform the whole construction preserving the grading.

We have now the following

Lemma 4. *Let* $(\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}, d)$ *be a DSLA. Then the following facts are equivalent:*

1. *there exists a quasi isomorphism,*

$$\phi : (\mathfrak{g}, [\ ,\ ], d) \longrightarrow (H, 0, 0);$$

2. *we have:*

$$[\mathfrak{g}, \mathfrak{g}] \cap Z \subset B; \tag{2}$$

3. *there exists super Hodge decomposition* $\mathfrak{g} = \mathcal{H} \oplus dS \oplus S$, *such that*

$$[\mathfrak{g}, \mathfrak{g}] \subset dS \oplus S. \tag{3}$$

Proof. 1. $\Longrightarrow$ 2. Since Φ is a quasi-isomorphism, we have, in particular

$$[\mathfrak{g}, \mathfrak{g}] \cap Z \subset \operatorname{Ker}\Phi \cap Z = B.$$

2. $\Longrightarrow$ 3. Let $\mathcal{H} \subset Z$ be a supersubspace such that

$$\mathcal{H} \oplus B = Z.$$

Clearly $\mathcal{H} \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$. Then as in the general construction of Hodge decomposition, just choose $R \subset \mathfrak{g}$ to be a supersubspace such that:

- $\mathfrak{g} = \mathcal{H} \oplus R$,
- $[\mathfrak{g}, \mathfrak{g}] + B \subset R$.

3. $\Longrightarrow$ 1. Just set $\Phi(\alpha) := [\pi_{\mathcal{H}}(\alpha)]$.

We recall that a dGBV algebra (A, Δ, d) satisfying the Δd-lemma is an example of DSLA meeting the condition of lemma (4) (cf.[3] and [5] [1]).

We have the following

Lemma 5. *Assume the DSLA* $(\mathfrak{g}, [\,,\,], d)$ *satisfies the conditions of lemma* (4)*; fix* $\mathcal{H}$, S *and hence* Φ *and* Q*. Let*

$$a \,:\, \mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]] \longrightarrow (Z \otimes \mathfrak{m}_K)^{(1)},$$

be defined by

$$a(\gamma) := \gamma + \frac{1}{2} Q_K([\gamma, \gamma]).$$

Then:

1. a *is one-to-one with inverse map,*

$$b := \alpha = \sum_{j=1}^{\infty} \alpha_j \mapsto \gamma = \sum_{j=1}^{\infty} \gamma_j,$$

where:

$$\gamma_1 = \alpha_1$$
$$\vdots$$
$$\gamma_j = -\frac{1}{2} \sum_{r+s=j} Q_K([\gamma_r, \gamma_s]) + \alpha_j.$$

2.

$$a(\chi(\beta)\gamma) = a(\gamma) \, mod((B \otimes \mathfrak{m}_K)_1)$$
$$a^{-1}(\alpha + d\epsilon) = a^{-1}(\epsilon) \, \mathrm{mod}(\exp((\mathfrak{g} \otimes \mathfrak{m}_K)_0))$$

and so

$$a^* \,:\, \langle\gamma\rangle \mapsto [a(\gamma)]$$

establishes a bijection

$$Def_{\mathbb{Z}_2}[[\mathfrak{g}]] \longrightarrow (H \otimes \mathfrak{m}_K)^{(1)}.$$

3. *Let*

$$\widetilde{\mathfrak{MC}_{\mathbb{Z}_2}}[[\mathfrak{g}]] := \{\gamma \in (\mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]] \mid \gamma_j \in \operatorname{Ker}\Phi \otimes \mathfrak{m}_K \;,\; j \geq 2\}.$$

Then

a) $\widetilde{\mathfrak{MC}_{\mathbb{Z}_2}}[[\mathfrak{g}]]$ *is* $\exp((\mathfrak{g} \otimes \mathfrak{m}_K)^{(0)}) - invariant$

b) $a^* \;:\; \widetilde{Def}_{\mathbb{Z}_2}[[\mathfrak{g}]] \;:=\; \widetilde{\mathfrak{MC}_{\mathbb{Z}_2}}[[\mathfrak{g}]]/\exp((\mathfrak{g} \otimes \mathfrak{m}_K)^{(0)}) \longrightarrow H$

Proof. 1. First note that, given $\gamma \in \mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]]$, we have that $[\gamma,\, \gamma]$ is d_K-exact and, because of (4),

$$[\gamma,\, \gamma] = d_K Q_K([\gamma,\, \gamma]).$$

Therefore

$$da(\gamma) = d_K\gamma + \frac{1}{2} d_K Q_K([\gamma,\, \gamma]) = 0.$$

Now we can first check that, given $\alpha \in (Z \otimes \mathfrak{m}_K)^{(1)}$, we have

$$db(\alpha) + \frac{1}{2}[b(\alpha),\, b(\alpha)] = 0. \tag{4}$$

Now (4) amounts to

$$d\gamma_j = -\frac{1}{2} \sum_{r+s=j} [\gamma_r,\, \gamma_s],$$

and this can be shown recursively. It is certainly true for $j = 1$. Assume it is true for $l < j$, then:

$$\begin{aligned} d \sum_{r+s=j} [\gamma_r,\, \gamma_s] &= \sum_{r+s=j} ([d\gamma_r,\, \gamma_s] - [\gamma_r,\, d\gamma_s]) \\ &= -\frac{1}{2} \sum_{r+s=j} \sum_{p+q=r} [[\gamma_p,\, \gamma_q],\, \gamma_s] + \frac{1}{2} \sum_{r+s=j} \sum_{t+u=s} [\gamma_r,\, [\gamma_t],\, \gamma_u] \\ &= - \sum_{r+s+t=j} [[\gamma_r,\, \gamma_s],\, \gamma_t] \\ &= 0, \quad \text{by Jacobi identity.} \end{aligned}$$

Therefore,

$$b \;:\; (Z \otimes \mathfrak{m}_K)^{(1)} \longrightarrow \mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]].$$

Then,

$$ab(\alpha) = \sum_{j=1}^{\infty} \beta_j = b(\alpha) + \frac{1}{2} Q([\underline{(}\alpha),\, b(\alpha)]) = \alpha.$$

In fact,

$$\beta_j = \gamma_j + \frac{1}{2}\sum_{r+s=j} Q([\gamma_r, \gamma_s])$$
$$= -\frac{1}{2}\sum_{r+s=j} Q([\gamma_r, \gamma_s]) + \alpha_j + \frac{1}{2}\sum_{r+s=j} Q([\gamma_r, \gamma_s]) = \alpha_j,$$

$$ba(\gamma) = \sum_{j=0}^{\infty} \epsilon_j = \gamma,$$

can be shown recursively. Definitely true for $j = 1$, assume it holds true for $l < j$. Then,

$$\epsilon_j = -\frac{1}{2}\sum_{r+s=j} Q([\epsilon_r, \epsilon_s]) + \gamma_j + \frac{1}{2}\sum_{r+s=j} Q([\gamma_r, \gamma_s]) = \gamma_j.$$

2. We can easily show by direct computation that:

$$a(\chi(\eta)\gamma) = a(\gamma) + dQ(\chi(\eta)\gamma - \gamma).$$

Vice versa, given $\epsilon \in (\mathfrak{g} \otimes \mathfrak{m}_K)^{(0)}$, we can construct recursively $\eta \in (\mathfrak{g} \otimes \mathfrak{m}_K)^{(0)}$ such that,

$$a^{-1}(\alpha + d\epsilon) = \chi(\eta)a^{-1}(\alpha),$$

i.e.,

$$\alpha + d\epsilon = \alpha + dQ(\chi(\eta)a^{-1}(\alpha) - a^{-1}(\alpha)).$$

Set $\eta_1 = \epsilon_1$ and assume η_l has been constructed for $l < j$. Note that, in general,

$$(\chi(\eta)\gamma - \gamma)_j = A_j - d\eta_j,$$

where A_j depends on γ_r, η_s for $0 < r, s < j$.
Therefore:

$$(dQ(\chi(\eta)\gamma - \gamma))_j = dQ(A_j) - d\eta_j.$$

Thus choose $\eta_j = Q(A_j) - \epsilon_j$.
3. is clear.

Finally, if $(\mathfrak{g}, [\ ,\ ], d)$ is a DGLA, we have the following, easy to prove lemma:

Lemma 6. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]] & \xrightarrow{a} & (Z \otimes \mathfrak{m}_K)^{(1)} \\ \big\downarrow \tilde{\pi} & & \big\downarrow \tilde{\pi} \\ \mathfrak{MC}_{\mathbb{Z}}[[\mathfrak{g}]] & \xrightarrow{a} & (Z \otimes \mathfrak{m}_K)^{(1)} \end{array}$$

Moreover,

$$a^* \circ \tilde{\pi}^* = \tilde{\pi}^* \circ a^*$$

and analogous results hold true for $\widetilde{\mathfrak{MC}}$, *provided* $|\Phi| = 0$ *in* $\mathbb{Z}$.

Finally, note that, if we want to efficiently define a at the $\mathbb{Z}$-level only, we just need to replace (2) with

$$[\mathfrak{g}_1,\ \mathfrak{g}_1] \cap Z_2 \subset B_2.$$

3 An example: A^∞-algebras and deformation theory

As a first example of deformation space, we consider the following. Let $(V = V^{(0)} \oplus V^{(1)},\ d)$ be a differentiable $\mathbb{k}$-super vector space. We can extend d and the superstructure to the tensor algebra $\mathbb{T}(V)$. In particular,

- $d(R \otimes S) = dR \otimes S + (-1)^{|R|} R \otimes dS$;
- if $L \in \mathrm{Hom}_{\mathbb{k}}(V^{\otimes r},\ V^{\otimes s})$, then

  $$dL = d \circ L - (-1)^{|L|} L \circ d$$

  and

  $$d(L \circ M) = dL \circ M + (-1)^{|L|} L \circ dM.$$

Set

$$C^p(V) := \mathrm{Hom}_{\mathbb{k}}(V^{\otimes(p+1)},\ V),$$

and given $R \in C^p(V)$, set

$$\|R\| = (|R| + p) \mod 2.$$

Given $R \in C^p(V),\ S \in C^q(V)$, let $[R,\ S] \in C^{p+q}(V)$ be defined as

$$[R,\ S] := \sum_{k=1}^{p+1} (-1)^{p(k-1)} R \circ (I^{\otimes(k-1)} \otimes S \otimes I^{\otimes((p+1-k)}) + \\ - (-1)^{\|R\|\|S\|} \sum_{k=1}^{q+1} (-1)^{q(k-1)} S \circ (I^{\otimes(k-1)} \otimes R \otimes I^{\otimes((q+1-k)}).$$

Then,

$$d[R,\ S] = [dR,\ S] + (-1)^{\|R\|}[R,\ dS]$$

and

$$\left(C(V) := \bigoplus_{p \in \mathbb{Z}} C^p(V),\ [\ ,\ ],\ d \right),$$

is a DSLA.

Let $\mathcal{A}(V)$ be the completion of $C(V)$ and extend in an obvious way the DSLA structure to $\mathcal{A}(V)$. Let $\mathcal{A}^*(V)$ be the sub DSLA of $\mathcal{A}(V)$ of elements with no components in $C^0(V)$. Then,

a structure of A^∞-algebra on V is a solution of the MC equation in $\mathcal{A}^*(V)$.

See [11] and [12] for examples of A^∞-algebras related to complex and symplectic geometry.

4 Complex and holomorphic deformation theory

4.1 Preliminaries

Let

$$J_n = \begin{pmatrix} O & -I_n \\ I_n & 0 \end{pmatrix}.$$

We consider the faithful representation,

$$\rho : \mathfrak{gl}(n, \mathbb{C}) \longrightarrow \mathfrak{gl}(2n, \mathbb{R}),$$

$$\rho : A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

In the sequel, we shall identify

$$\mathfrak{gl}(n, \mathbb{C}) \quad \text{with} \quad \rho(\mathfrak{gl}(n, \mathbb{C})) = \{X \in \mathfrak{gl}(2n, \mathbb{R}) \mid XJ_n - J_nX = 0\}.$$

Moreover,

$$\mathfrak{gl}(2n, \mathbb{R}) = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{s}(n),$$

where

$$s(n) := \{X \in \mathfrak{gl}(2n, \mathbb{R}) \mid XJ_n + J_nX = 0\},$$

with projections

$$R : \mathfrak{gl}(2n, \mathbb{R}) \longrightarrow \mathfrak{gl}(n, \mathbb{C}), \qquad X \mapsto \frac{1}{2}(X - J_nXJ_n),$$

$$S : \mathfrak{gl}(2n, \mathbb{R}) \longrightarrow \mathfrak{s}(n), \quad X \mapsto \frac{1}{2}(X + J_nXJ_n).$$

Let

$$\mathfrak{W}(n) := \{P \in GL(2n, \mathbb{R}) \mid P^2 = -I\}.$$

Clearly,

- $P \in \mathfrak{W}(n) \iff P = AJ_nA^{-1}$,
- $P = AJ_nA^{-1} = BJ_nB^{-1} \iff B^{-1}A \in GL(n, \mathbb{C})$.

Consequently,

$$\mathfrak{W}(n) = GL(2n, \mathbb{R})/GL(n, \mathbb{C})$$

and

$$GL(2n, \mathbb{R}) \mapsto \mathfrak{W}(n)$$

is a $GL(n, \mathbb{C})$-principal bundle with projection $\pi(A) = AJ_nA^{-1}$. In particular, there exists a neighborhood U of J_n and a section σ over U, i.e., a map $\sigma : U \longrightarrow GL(2n, \mathbb{R})$ such that:

a. $\sigma(J_n) = I$,
b. for every $P \in U$, $\sigma(P)J_n\sigma(P)^{-1} = P$.

Moreover, since $R(\sigma(J_n)) = I$, if U is sufficiently small, then, for every $P \in U$, $R(\sigma(P)) \in GL(n, \mathbb{C})$ and so $\tilde{\sigma}(P) := \sigma(P)(R(\sigma(P)))^{-1}$ is a section over U with $R(\tilde{\sigma}(P)) \equiv I$. It is obvious that $\tilde{\sigma}$ is uniquely characterized by these conditions, namely,

- $\tilde{\sigma}(J_n) = I$,
- $R(\tilde{\sigma})(P) \equiv I$.

In other words, every $P \in U$ can be expressed in a unique way as

$$P = (I + L)J_n(I + L)^{-1} \quad \text{with} \quad LJ_n = -J_nL. \tag{5}$$

We can give a complete description of those elements in $\mathfrak{W}(n)$ which are expressible as (5). Let

$$\mathcal{A}(n) := \{X \in \mathfrak{s}(n) \mid det(I + X) \neq 0\},$$
$$\mathfrak{P}(n) := \{P \in \mathfrak{W}(n) \mid det(I - J_nP) \neq 0\}.$$

Then, we have the following:

Lemma 7. *Set*

$$r(P) := (I - J_nP)^{-1}(I + J_nP).$$

Then r diffeomorphically sends $\mathfrak{P}(n)$ into $\mathcal{A}(n)$

Proof. Just note that

$$r(P) = 2(I - J_nP)^{-1} - I = -(I - PJ_n)^{-1}(I + PJ_n),$$

and that, clearly,

$$r^{-1}(L) = (I + L)J_n(I + L)^{-1}.$$

Note also that the elements $P \in \mathfrak{W}(n)$ are in one-to-one correspondence with complex subspaces W of $\mathbb{C}^{2n} = (\mathbb{R}^{2n})^{\mathbb{C}}$, satisfying

$$\mathbb{C}^{2n} = W \oplus \bar{W}. \tag{6}$$

In fact, given $P \in \mathfrak{W}(n)$, just set $W = V_P^{0,1}$; vice versa, given W satisfying (6), set $P = -it_2 \circ t_1^{-1}$, where

$$t_1 := p_{1|\bar{W}} : \bar{W} \longrightarrow \mathbb{R}^{2n},$$
$$t_2 := p_{2|\bar{W}} : \bar{W} \longrightarrow i\mathbb{R}^{2n}.$$

Given $\bar{W}$ sufficiently close to $V_{J_n}^{0,1}$, $\bar{W}$ can be described as the graph of a $\mathbb{C}$-linear map $L : V_{J_n}^{0,1} \longrightarrow V_{J_n}^{1,0}$ (and so $LJ_n = -J_nL$). Consequently,

$$\bar{W} = \{(I + L)X + i(I + L)JX \mid X \in \mathbb{R}^{2n}\},$$

and the corresponding element of $\mathfrak{W}(n)$ is $P = (I + L)J_n(I + L)^{-1}$.

4.2 Starting deformation theory

Let $(M,\ J)$ be a complex manifold and let $\mathfrak{H}(M)$ be the Lie algebra of smooth vector fields on M. Given $X,\ Y \in \mathfrak{H}(M)$, set

$$\begin{aligned}(\bar{\partial}_J X)(Y) &:= \frac{1}{4}([Y,\ X] + J[JY,\ X] + [JY, JX] - J[Y, JX]) \\ &= \frac{1}{2}([Y,\ X] + J[JY,\ X]) - \frac{1}{4}N_J(X,\ Y),\end{aligned} \tag{7}$$

where, as usual,

$$N_J \in \wedge_J^{0,2}(M) \otimes TM,$$

defined as

$$N_J(X,\ Y) := [JX,\ JY] - [X,\ Y] - J[JX,\ Y] - J[X,\ JY],$$

is the Nijenhuis tensor of J and

$$N_J = 0 \quad \Longleftrightarrow \quad J \text{ is integrable.}$$

Then we have:

- $\bar{\partial}_J X \in \wedge_J^{0,1} \otimes TM$,
- $\bar{\partial}_J JX = J\bar{\partial}_J X$, i.e., $\bar{\partial}_J J = 0$.

Note also that, given $f \in C^\infty(M,\ \mathbb{C})$, then

$$(\bar{\partial}_J)^2 f(X,\ Y) = -\frac{1}{8}(N_J(X,\ Y) - iJN_J(X,\ Y))f.$$

Let $(M,\ J)$ be a holomorphic manifold and set

- $\mathfrak{g} = \mathcal{A} := \wedge_J^{0,*}(M) \otimes TM$,
- $[X,\ Y] = [X * Y] := \frac{1}{2}([X,\ Y] - [JX,\ JY])$, for $X,\ Y \in \mathfrak{H}(M)$.
  A straightforward computation shows that $[\ *\ ]$ is a Lie algebra bracket (note that for a general complex structure J, we have:

  $$\mathfrak{S}[X * [Y * Z]] = \frac{1}{4}\mathfrak{S}[JN,\ N_J(JY,\ Z)]),$$

- $d = \bar{\partial}_J$ where, now, for $X,\ Y \in \mathfrak{H}$,

  $$(\bar{\partial}_J X)(Y) := \frac{1}{2}([Y,\ X] + J[JY,\ X]).$$

Then:

1. Define $|\alpha \otimes X| := |\alpha|$ and so,

$$\mathcal{A} = \bigoplus_{p\in\mathbb{Z}} \mathcal{A}_p,$$

where

$$\mathcal{A}_p = \begin{cases} \wedge_J^{0,p}(M) \otimes TM, & \text{if } 0 \le p \le n, \\ 0\ , & \text{otherwise.} \end{cases}$$

2. Extend $[\,*\,]$ to $\mathcal{A}$ in the following way:
   a) if $L \in \wedge_J^{0,1}(M) \otimes TM$, define $[L * L]$ by means of the formula

$$[L * L](X,\ Y) = [L(X) * L(Y)] - L([L(X) * Y] + [X * L(Y)] - L([X * Y]));$$

   b) given $R,\ S \in \wedge_J^{0,1}(M) \otimes TM$, define $[R * S]$ by polarization, i.e.,

$$[R * S] := \frac{1}{2}([R + S * R + S] - [R * R] - [S * S]);$$

   c) given $\alpha \in \wedge_J^p(M),\ \beta \in \wedge_J^q(M)$, define

$$[\alpha \wedge R * \beta \wedge S] := (-1)^q \alpha \wedge \beta \wedge [R * S];$$

   d) extend to the general case by bilinearity.

   Note that, in terms of local complex coordinates $z_1, \dots, z_n$, under the identification

$$TM \longleftrightarrow T^{1,0}M, \quad X \longleftrightarrow \frac{1}{2}(X - iJX),$$

   we have that, given $R \in \mathcal{A}_p\ ,\ S \in \mathcal{A}_q$,

$$R = \sum_{j=1}^{n} \sum_{|I|=p} r_{jI} d\bar{z}_I \otimes \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} r_j \otimes \frac{\partial}{\partial z_j},$$

$$S = \sum_{j=1}^{n} \sum_{|K|=q} s_{jK} d\bar{z}_I \otimes \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} s_j \otimes \frac{\partial}{\partial z_j}.$$

   Then,

$$[R * S] = \sum_{j,k=1}^{n} \left(r_j \wedge \frac{\partial}{\partial z_j} s_k - (-1)^{pq} s_j \wedge \frac{\partial}{\partial z_j} r_k \right) \otimes \frac{\partial}{\partial z_k},$$

   where, of course,

$$\frac{\partial}{\partial z_j} s_k = \sum_{|K|=q} \frac{\partial}{\partial z_j} s_{kK} d\bar{z}_K,$$

   (see e.g. [8]).
3. Extend $\bar{\partial}$ to $\mathcal{A}$ by setting

$$\bar{\partial}_J(\alpha \otimes X) = \bar{\partial}_J \alpha \otimes X + (-1)^{|\alpha|} \alpha \wedge \bar{\partial}_J X.$$

Then $(\mathcal{A},\ [\,*\,],\ \bar\partial_J)$ is a DGLA; note that $\mathcal{A}_0 = \mathfrak{H}(M)$ and, consequently, $\exp(\mathcal{A}_0)$, is the connected component of id_M in $\mathcal{D}\mathrm{iff}(M)$. Let $\tilde J$ be another complex structure on M with $det(I - J\tilde J) \neq 0$. Then we can write in a unique way,

$$\tilde J = AJA^{-1},$$

with

$$A = I + L \quad \text{and} \quad LJ + JL = 0,$$

i.e.,

$$L \in \wedge^{0,1}_J(M) \otimes TM.$$

A tedious but straightforward computation yields the following

Lemma 8. *Let $L,\ A,\ \tilde J$ be as before and let*

$$\rho(A) \ := \ (A^*)^{-1} \otimes A \in Aut(T^*M \otimes TM).$$

Then:

- $\rho^{-1}(A)N_{\tilde J} = -4(\bar\partial_J L + \frac{1}{2}[L * L]);$
- $\rho^{-1}(A) \circ \bar\partial_{\tilde J} \circ \rho(A) = \bar\partial_J + [L * \cdot],$

i.e., on TM :

- $A^{-1}N_{\tilde J}(AX,\ AY) = -4(\bar\partial_J L + \frac{1}{2}[L * L](X,\ Y)),$
- $A^{-1}(\bar\partial_{\tilde J} AX)(AY) = (\bar\partial_J X)(Y) + [L * X](Y).$

Proof. It is enough to consider the case

$$J = J_n, \quad A(0) = I \ \ (\text{i.e., } \mathrm{L}(0) = 0),$$

and perform the computations at 0.

Consequently,

- $(\bar\partial_J)_L = \bar\partial_J + [L * \cdot]$ corresponds to $\bar\partial_{\tilde J}$;
- $L \in \mathfrak{MC}_{\mathbb{Z}}(\mathcal{A})$, $det(I + L) \neq 0$, $\Longleftrightarrow$, $\tilde J = (I + L)J(I + L)^{-1}$ is a holomorphic structure and so $L \ \mapsto \ (I + L)J(I + L)^{-1}$ establishes a bijection:

$$\mathfrak{MC}^*_{\mathbb{Z}}(\mathcal{A}) := \{L \in \mathfrak{MC}_{\mathbb{Z}}(\mathcal{A}) \mid det(I + L) \neq 0\}$$
$$\downarrow$$
$$\{\text{holomorphic strucures } \tilde J \text{ s.t. } det(I - \tilde J J) \neq 0\};$$

- two $\exp(\mathcal{A}_0)$-equivalent elements of $\mathfrak{MC}_{\mathbb{Z}}(\mathcal{A})$ correspond to diffeomorphic holomorphic structures.

We have also the following, easy to prove

Lemma 9. *Let L, A, $\tilde{J}$ be as before. Then, on $\wedge_J^{0,*}(M)$ we have*

$$\rho^{-1}(A) \circ \bar{\partial}_{\tilde{J}} \circ \rho(A) \;=\; \bar{\partial}_J + L \wedge \partial_J,$$

where, more generally,

$$\wedge \;:\; (\wedge_J^{0,p}(M) \otimes T_J^{1,0}M) \times (\wedge_J^{0,q}(M) \otimes \wedge_J^{1,0}(M)) \;\longrightarrow\; \wedge_J^{0,p+q}(M)$$

is defined by means of the duality pairing.

Lemma 9 suggests the possibility of considering operators on $\wedge_J^{0,*}(M)$ of the form:

$$\alpha \mapsto \bar{\partial}_J \alpha + L \wedge \partial_J \alpha,$$

with

$$L \in \wedge_J^{\mathrm{odd}}(M) \otimes TM\,, \quad L = \sum_{p=1}^{[\frac{n+1}{2}]} L_p \quad \text{with} \quad L_p \in \wedge_J^{0,2p-1} M,$$

possibly with $L_1 = 0$, i.e., including L_1 into a new $\tilde{J}$ on the basis of Lemma 8. Therefore, we can set the following:

Definition 9. A supercomplex (resp. superholomorphic) structure on M is the datum $\mathcal{J} = (J,\, L)$ of a complex (resp. holomorphic) structure J on M and $L \in \mathcal{A}^{(1)} = \wedge_J^{\mathrm{odd}}(M) \otimes TM$.

Given a superholomorphic structure $\mathcal{J} = (J,\, L)$, set, on $\wedge_J^{0,*}(M)$:

$$\bar{\daleth} = \bar{\daleth}_{\mathcal{J}} = \bar{\partial}_J + L \wedge \partial_J.$$

Clearly $\bar{\daleth}$ is a parity one derivation and

$$\bar{\daleth}^2 = (\bar{\partial}_J L + \frac{1}{2}[L * L]) \wedge (\partial + L \wedge \bar{\partial}_J).$$

Moreover, $\bar{\daleth}$ extends to $\mathcal{A}$ as

$$\bar{\daleth} = \bar{\partial}_J + [L * \cdot\,]\,,$$

and it satisfies

$$\bar{\daleth}(\alpha \otimes X) = \bar{\daleth}\alpha \otimes X + (-1)^{|\alpha|}\alpha \wedge \bar{\daleth}X,$$

for $X \in \mathcal{A}_0$, $\alpha \in \wedge_J^{0,*}(M)$. Clearly $\bar{\daleth}$ reflects the $\mathbb{Z}_2$-deformation theory of $\mathcal{A}$. Thus, in particular, we have

$$\bar{\daleth}^2 = 0 \quad \Longleftrightarrow \quad \bar{\partial}_J L + \frac{1}{2}[L * L] = 0,$$

which gives by lemma 1,

$$\bar{\partial}_J L_1 + \frac{1}{2}[L_1 * L_1] = 0,$$

i.e.,

$$\tilde{J} := (I + L)J(I + L)^{-1}$$

is holomorphic. This leads to a superholomorphic structure $\tilde{\mathcal{J}} = (\tilde{J}, \tilde{L})$ with $\tilde{L}_1 = 0$. Note that:

- if $n = 2$, then superholomorphic structures coincide with complex structures (because $L = L_1$!),
- if $n = 3$, then $L = L_1 + L_2$ and

$$\bar{\partial}_J L + \frac{1}{2}[L * L] = 0 \quad \Longleftrightarrow \quad \bar{\partial}_J L_1 + \frac{1}{2}[L_1 * L_1] = 0,$$

  and so, assuming $L_1 = 0$, we obtain for

$$\alpha \in \wedge_J^{0,*}(M), \quad \alpha = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3, \quad \text{with } \alpha_p \in \wedge_J^{0,p}(M) \;\; 0 \leq p \leq 3 :$$

$$\bar{\daleth}\alpha = 0 \quad \Longleftrightarrow \quad \begin{cases} \bar{\partial}_J \alpha_0 = 0, \\ \bar{\partial}_J \alpha_1 = 0, \\ \bar{\partial}_J \alpha_2 + L_2 \wedge \partial_J \alpha_0 = 0. \end{cases}$$

4.3 A very simple example

Let $M = \mathbb{T}^{2n} = \mathbb{C}^n/\mathbb{Z}^{2n}$ and let $\mathcal{J} = (J, L)$, where

- $J = J_{sdt}$,
- $L = \sum_{p=2}^n L_p \;,\; L_p = \sum_{j=1}^n \sum_{|I|=2p-1} a_{\bar{I}j} dz_{\bar{I}} \otimes \frac{\partial}{\partial z_j} \;,\; a_{\bar{I}j} \in \mathbb{C}.$

Clearly,

$$\bar{\daleth}^2_{\mathcal{J}} = 0.$$

5 Symplectic deformation theory

5.1 Preliminaries

Let (V, κ) be a $2n$-dimensional symplectic vector space. Define the *symplectic Hodge operator*

$$\bigstar : \wedge^r V^* \longrightarrow \wedge^{2n-r} V^*,$$

by means of the relation,

$$\alpha \wedge \bigstar \beta = \kappa(\alpha, \beta)\frac{\kappa^n}{n!},$$

$\alpha, \beta \in \wedge^r V^*$. It is easy to check that $\bigstar^2 = I$.

Consider the following endomorphisms of $\wedge^* V^*$:

- $L : \alpha \mapsto \kappa \wedge \alpha$,
- $\Lambda := -\bigstar L \bigstar$,
- $H = \sum_{r=0}^{2n} (n-r) p_r$, where

$$p_r : \wedge^* V^* \longrightarrow \wedge^r V^*,$$

  is the natural projection.

It is easy to check that

$$[L,\ \Lambda] = H\ ,\ [L,\ H] = -2L\ ,\ [\Lambda,\ H] = 2\Lambda,$$

and so $\wedge^* V^*$ has the natural structure of the $\mathfrak{sl}(2,\ \mathbb{C})$-module.

We have

Lemma 10. *For* $0 \leq p \leq n$*,*

$$L^p : \wedge^{n-p} V^* \longrightarrow \wedge^{n+p} V^*,$$

is an isomorphism and so, in particular, for $0 \leq p < n$*,*

$$L : \wedge^p V^* \longrightarrow \wedge^{p+2} V^*,$$

is injective.

We have also

Lemma 11. *Let* $0 \leq p \leq n$*.*

- *If* $\alpha \in \wedge^p V^*$*, then*

$$\bigstar(\alpha \wedge \kappa^{n-p}) = (-1)^{\frac{1}{2}p(p-1)} (n-p)!(\alpha + \Lambda\alpha \wedge \kappa). \tag{8}$$

- $$\bigstar\, \kappa^p = \frac{p!}{(n-p)!} \kappa^{n-p}. \tag{9}$$

For every $A \in End(V)$, we define ${}^T A \in End(V)$ by means of the relation

$$\kappa(Av,\ w) = \kappa(v,\ {}^T Aw).$$

Let,

$$\mathcal{S}_\kappa(V) := \{A \in End(V) \mid A = {}^T A\},$$
$$\mathcal{S}^*_\kappa(V) = \mathcal{S}_\kappa(V) \cap Aut(V).$$

We can immediately check that

$$A \in \mathcal{S}^*_\kappa(V) \quad \Longleftrightarrow \quad A^{-1} \in \mathcal{S}^*_\kappa(V).$$

Clearly, given $A \in \mathcal{S}_\kappa(V)$,

$$\kappa_A(v,\ w) := \kappa(Av,\ w)$$

defines an element of $\wedge^2 V^*$ and

$$\natural_\kappa\ :\ \mathcal{S}_\kappa(V)\ \longrightarrow\ \wedge^2 V^*\ ,\ A\ \mapsto\ \kappa_A,$$

is a bijection sending $\mathcal{S}^*_\kappa(V)$ into symplectic forms.

Let $\tilde{\kappa}$ now be a symplectic form on V. Then there exists a uniquely defined $A \in \mathcal{S}_\kappa(V)$ §[a] such that:

$$\tilde{\kappa} = \kappa_A.$$

Consequently, if $\alpha,\ \beta \in \wedge^r V^*$, then,

$$\tilde{\kappa}(\alpha,\ \beta) = \kappa(\rho(A)\alpha,\ \beta) = \kappa(\alpha,\ \rho(A)\beta),$$

where, as before,

$$\rho(A)(\zeta_1 \wedge \cdots \wedge \zeta_r) = (A^*)^{-1}\zeta_1 \wedge \cdots \wedge (A^*)^{-1}\zeta_r.$$

Moreover,

$$\tilde{\kappa}^n = e^{n\lambda}\kappa^n,$$

where $\lambda = \lambda(A) = \frac{1}{2n}\log|det\ A|$.

Therefore, if $\tilde{\bigstar}$ is the symplectic Hodge operator with respect to $\tilde{\kappa}$, we have

$$\alpha \wedge \tilde{\bigstar}\beta = \tilde{\kappa}(\alpha,\ \beta)\frac{\tilde{\kappa}^n}{n!} = \kappa(\alpha,\ e^{n\lambda}\rho(A)\beta)\frac{\kappa^n}{n!} = \alpha \wedge \bigstar e^{n\lambda}\rho(A)\beta,$$

and so, setting $C = C(A) := e^{n\lambda}\rho(A)$, we have

$$\tilde{\bigstar} = \bigstar C = C^{-1}\bigstar.$$

Let $(M,\ \kappa)$ be an almost symplectic manifold. Set

$$d^\bigstar := (-1)^{r+1}\bigstar d\bigstar,$$

on r-forms. Clearly, $(d^\bigstar)^2 = 0$ and if $\tilde{\kappa}$ is another almost symplectic structure, then

$$d^{\tilde{\bigstar}} = C^{-1}d^\bigstar C.$$

We have the following

Lemma 12. *Let $(M,\ \kappa)$ be an almost symplectic manifold. Set*

$$\eth_\kappa := [L,\ d^\bigstar].$$

Then the following facts are equivalent:

1. $d\kappa = 0$*, i.e.,* κ *defines a symplectic structure on* M*;*
2. $\eth_\kappa = d$*;*
3. $Q := [d,\ \Lambda] - d^\bigstar = 0$*;*
4. $[d,\ d^\bigstar] = 0$*;*
5. $\eth_\kappa$ *is a differential, i.e., it is a derivation of parity* 1 *and* $\eth_\kappa^2 = 0$.

Proof. Note first that 2. and 3. are obviously equivalent and that Q is $C^\infty(M)$-linear (cf. [4]);

1. $\Longrightarrow$ 3. It is a basic symplectic identity (cf. [4]).

3. $\Longrightarrow$ 1. From $Q = 0$, it follows

a. $0 = Q\kappa^n = [d,\ \Lambda]\kappa^n = d\Lambda\kappa^n$. Now,

$$\Lambda\kappa^n = -\bigstar L \bigstar \kappa^n = -n!\bigstar\kappa = -n\kappa^{n-1},$$

and so,

$$Q\kappa^n = 0 \Longrightarrow d\kappa^{n-1} = 0, \quad \text{i.e., } d^\bigstar\kappa = 0.$$

If $n = 2$, there is nothing else to prove, otherwise,

b. $0 = Q\kappa = [d,\ \Lambda]\kappa = -\Lambda d\kappa$.

c. From $[a.]$ we obtain,

$$Q\kappa^{n-1} = [d,\ \Lambda]\kappa^{n-1} - d^\bigstar\kappa^{n-1} = d\Lambda\kappa^{n-1} - d^\bigstar\kappa^{n-1}.$$

Now,

$$\Lambda\kappa^{n-1} = -\bigstar L \bigstar \kappa^{n-1} = -(n-1)!\bigstar\kappa^2 = -2(n-1)\kappa^{n-2}.$$

From (8), it follows

$$d^\bigstar\kappa^{n-1} = -\bigstar d \bigstar \kappa^{n-1} = -(n-1)!\bigstar d\kappa = (n-1)(n-2)d\kappa \wedge \kappa^{n-3}.$$

Finally,

$$Q\kappa^{n-1} = -2(n-1)d\kappa^{n-2} - (n-1)(n-2)d\kappa \wedge \kappa^{n-3}$$
$$= -3(n-1)(n-2)d\kappa \wedge \kappa^{n-3},$$

and thus, by Lemma 10, $Q\kappa^{n-1} = 0$ gives $d\kappa = 0$.

1. $\Longrightarrow$ 4.

$$[d,\ d^\bigstar] = [d,\ [d,\ \Lambda]] = [[d,\ d], \Lambda] - [d,\ [d,\ \Lambda]] = 0.$$

4. $\Longrightarrow$ 1. Let $f \in C^\infty(M)$. Then

$$Qdf = -d^\bigstar df$$

and so

$$[d,\ d^\bigstar] = 0 \quad \Longrightarrow \quad Q = 0 \text{ on } \wedge^1(M).$$

Let $\alpha \in \wedge^1(M)$ s.t. $d^\bigstar\alpha = 0$ (and so $\Lambda d\alpha = 0$) Thus, again using (8), we obtain:

$$d^\bigstar d\alpha = 0 = -\bigstar d \bigstar (d\alpha) = (n-2)!\bigstar(d\alpha \wedge d\kappa^{n-2}),$$

which gives $d\kappa^{n-2} = 0$ and so $d\kappa = 0$.

1. $\Longrightarrow$ 5. It is now obvious.
5. $\Longrightarrow$ 2. If $f \in C^\infty(M)$, then $\eth_\kappa f = df - fd^\bigstar\kappa$ and so

$$\eth_\kappa \quad \text{is a derivation} \quad \Longrightarrow \quad \eth_\kappa 1 = 0 \quad \Longrightarrow \quad d^\bigstar\kappa = 0.$$

Thus if $\eth_\kappa$ is a derivation, it coincides with d on functions and, since it satisfies $\eth_\kappa^2 = 0$ *it is* d.

5.2 Starting deformation theory once more

Let (M, κ) be a compact symplectic manifold. Therefore,

$$Sym(M) := \{\text{simplectic forms on } M\},$$

is not empty. Set,

$$Sym_0^{(\kappa)}(M) := \{\tilde{\kappa} \in Sym(M) \,|\, \tilde{\kappa}^n = const.\kappa^n\}.$$

By Moser's lemma,

$$Sym(M) = \mathcal{D}\text{iff}(M) Sym_0^{(\kappa)}(M).$$

It is well known that $(\wedge^*(M), d^\bigstar, d)$ is a dGBV algebra, and so, in particular, for every $\alpha \in \wedge^*(M)$ defining,

$$\urcorner_\alpha : \wedge^*(M) \longrightarrow \wedge^*(M),$$

as

$$\urcorner_\alpha \beta := (-1)^{|\alpha|} d^\bigstar(\alpha \wedge \beta) - (-1)^{|\alpha|} d^\bigstar\alpha \wedge \beta - \alpha \wedge d^\bigstar\beta,$$

we obtain

1. $\urcorner_\alpha$ is a derivation,
2. setting $[\alpha \bullet \beta] := \urcorner_\alpha\beta$, we obtain that $(\wedge^*(M), [\,\bullet\,], d)$ is an odd dGLA.

Let $\tilde{\kappa}$ now be another almost symplectic structure on M. Write $\tilde{\kappa}(X, Y) = \kappa(AX, Y)$ and $\tilde{\kappa}^n = e^{n\lambda}\kappa^n$. Then,

$$C\eth_\kappa C^{-1} = C[\tilde{L}, \tilde{d^\bigstar}]C^{-1} = [C\tilde{L}C^{-1}, d^\bigstar].$$

Now

$$C\tilde{L}C^{-1} = \rho(A)\tilde{L}\rho(A)^{-1} = e(\rho(A)\tilde{\kappa}),$$

where, for any $\gamma \in \wedge^*(M)$, we denote by $e(\gamma)$ the left multiplication by γ, i.e., $e(\gamma)(\alpha) = \gamma \wedge \alpha$. Note also that $\rho(A)\tilde{\kappa}(X, Y) = \kappa_{A^{-1}}(X, Y) = \kappa(A^{-1}X, Y)$.

Write $\rho(A)\tilde{\kappa} = \kappa - \epsilon$ and assume $d^\bigstar\rho(A)\tilde{\kappa} = 0$, i.e., $d^\bigstar\epsilon = 0$. Thus

$$C\tilde{L}C^{-1} = [L, d^\bigstar] - [e(\epsilon), d^\bigstar] = d + \urcorner_\epsilon.$$

Consequently, defining $\mathcal{MC} \,:\, \wedge^*(M) \longrightarrow \wedge^*(M)$ as

$$\mathcal{MC}(\alpha) := d\alpha + \frac{1}{2}[\alpha \bullet \alpha],$$

we obtain

$$\mathfrak{d}_{\tilde{\kappa}}^2 = 0 \iff \mathcal{MC}(\epsilon) = 0 \iff d\tilde{\kappa} = 0.$$

Note also that

$$\begin{cases} d^{\bigstar}\rho(A)\tilde{\kappa} = 0 \\ d^{\bigstar}\tilde{\kappa} = 0 \end{cases} \qquad \Longrightarrow \qquad \lambda(A) = const.$$

Vice versa, given $\epsilon \in \mathrm{Ker}\, d^{\bigstar} \cap \wedge^2(M)$, with $det(\natural_\kappa^{-1}(\kappa - \epsilon)) \neq 0$, let $\tilde{\kappa}$ be defined by the equation,

$$\rho(A)\tilde{\kappa} = \kappa - \epsilon.$$

If $\mathcal{MC}(\epsilon) = 0$, then, again, $\mathfrak{d}_{\tilde{\kappa}}^2 = 0$ and so $\tilde{\kappa} \in Sym_0^{(\kappa)}(M)$.

Note once more that, given $\tilde{\kappa}$ almost symplectic, by Moser's lemma, there exists $\phi \in \mathcal{D}\mathrm{iff}(M)$ such that $\hat{\kappa} := \phi^*(\tilde{\kappa})$ satisfies $\hat{\kappa}^n = e^{n\lambda}\kappa^n$ with $\lambda = const.$

Summarizing, let $(M,\ \kappa)$ be a symplectic manifold and let

$$\mathcal{A} = \left(\mathrm{Ker}\, d^{\bigstar} \cap \bigoplus_{p>0} \wedge^p(M)\right)[1],$$

where, as usual, [1] is the degree -1 shift. Consequently,

$$\mathcal{A}_p = \begin{cases} \wedge^{p+1}(M) \cap \mathrm{Ker}\, d^{\bigstar}, & \text{if } 0 \leq p \leq 2n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $(\mathcal{A},\ [\,\bullet\,],\ d)$ is the dGLA that governs the deformation theory of the symplectic structure κ. In particular, if

$$\mathfrak{MC}(\mathcal{A}) := \{\epsilon \in \mathcal{A}_1 \,|\, \mathcal{MC}(\epsilon) = 0\},$$

and

$$\mathfrak{MC}^*(\mathcal{A}) = \{\epsilon \in \mathfrak{MC}(\mathcal{A}) \,|\, det(\natural_\kappa^{-1}(\kappa - \epsilon)) \neq 0\},$$

then,

$$A \mapsto I - A^{-1},$$

induces a bijection,

$$Sym_0^{(\kappa)}(M) \longleftrightarrow \mathfrak{MC}^*(\mathcal{A}).$$

Note that, if

$$\mathcal{D}\mathrm{iff}_0^\kappa(M) = \{\phi \in \mathcal{D}\mathrm{iff}(M) \,|\, \phi^*(\kappa^n) = \kappa^n\ ,\ \phi \text{ is isotopic to the identity}\},$$

then the action of $\mathcal{D}\mathrm{iff}_0^\kappa(M)$ on $\wedge^2(M)$ corresponds to the action of $\exp(\mathcal{A}^{(0)})$ on $\mathfrak{MC}^*(\mathcal{A})$. In fact, given $X \in \mathfrak{H}(M)$, then

1.
$$d^{\bigstar}\#_{\kappa}(X) = \operatorname{div} X;$$

2. on $\wedge^r(M)$, we have
$$\iota_X = (-1)^r \bigstar e(\#_{\kappa}(X))\bigstar$$
and, consequently, on $\mathcal{A}$
$$\bigstar L_X \bigstar = \urcorner_{\#_{\kappa}(X)};$$

3. $\exp(X) \in \mathcal{D}\mathrm{iff}_0^{\kappa}(M)$ sends d to $ad(\bigstar\rho((\exp(X)_*)\bigstar)d$ and so the infinitesimal action is
$$\alpha \mapsto \bigstar L_X \bigstar\alpha = \urcorner_{\#_{\kappa}(X)}\alpha = [\#_{\kappa}(X) \bullet \alpha].$$

Consequently,
$$\mathfrak{MC}^*(\mathcal{A})/\exp(\mathcal{A}_0)$$
is the moduli space of (infinitesimal) constant volume deformations of the symplectic structure κ.

We want to show now that the theory is totally unobstructed.

Let (M, κ) be a compact symplectic manifold, and assume
$$\int_M \kappa^n = 1.$$
Let $\tilde{\kappa}$ be an almost symplectic form, and let
$$e^{nc} := \int_M \tilde{\kappa}^n > 0.$$
Then,
$$\int_M (e^c\kappa)^n = \int \tilde{\kappa}^n,$$
and so, by Moser's lemma, there exists $\phi \in \mathcal{D}\mathrm{iff}(M)$ s.t.
$$[\phi^*(\tilde{\kappa})]^n = e^{nc}\kappa^n.$$

Let now $\alpha \in \wedge^2(M)$, $d\alpha = 0$. Set $\kappa_t := \kappa + t\alpha$. Let $t \mapsto \phi_t$ be a smooth curve in $\mathcal{D}\mathrm{iff}(M)$ s.t.

1. $\phi_0 = id_M$,
2. $\phi_t*(\kappa_t)$ has constant volume density, i.e.,
$$\phi_t^*(\kappa_t^n) = e^{nc(t)}\kappa^n.$$

Now,

$$e^{nc(t)} = \int_M \phi_t^*(\kappa_t^n) = \int_M \kappa_t^n = 1 + nt \int \alpha \wedge \kappa^{n-1} \ + o(t).$$

Write

$$\alpha = -\frac{1}{n}\Lambda\alpha\kappa + \beta, \qquad \text{with } \ \Lambda\beta = 0 \quad \text{i.e., } \ \beta \wedge \kappa^{n-1} = 0.$$

Therefore,

$$e^{nc(t)} = 1 - t \int_M \Lambda\alpha\kappa^n \ + o(t).$$

Now let $X \in \mathfrak{H}(M)$ s.t. its associated flow $\{\psi_t^X\}$ satisfies

$$\frac{d}{dt}(\psi_t^X)^*(\kappa_t)_{|t=0} = \frac{d}{dt}\phi_t^*(\kappa_t)_{|t=0}.$$

Consequently,

$$\frac{d}{dt}\phi_t^*(\kappa_t^n)_{|t=0} = L_X\kappa^n + n\alpha \wedge \kappa^{n-1} = -q\kappa^n,$$

where

$$q = \int_M \Lambda\alpha\kappa^n,$$

and thus, if $\gamma = \#_\kappa(X)$,

$$n\left(\alpha + d\gamma + \frac{1}{n}q\kappa\right) \wedge \kappa^{n-1} = 0,$$

i.e.,

$$\Lambda\left(\alpha + \frac{1}{n}q\kappa + d\gamma\right) = 0,$$

and so

$$d^\bigstar(\alpha + d\gamma) = 0.$$

Note that, if $\Lambda\alpha = const$ (i.e., $d^\bigstar\alpha = 0$), then

$$\Lambda\left(\alpha + \frac{1}{n}q\kappa\right) = \Lambda\alpha - \int_M \Lambda\alpha\kappa^n = 0,$$

and so $\Lambda d\gamma = 0$ and $d^\bigstar\gamma = 0$. Finally,

$$\frac{d}{dt}(\psi_t^X)^*(\kappa_t)_{|t=0} = \alpha + d\gamma,$$

and so $t \ \mapsto \ \kappa_t$ corresponds to a curve in $\mathfrak{MC}(\mathcal{A})$, with tangent $\alpha + d\gamma$ at 0.

It is clear that, if we consider the underlying $\mathbb{Z}_2$-deformation theory, we are led to the notion of *supersymplectic structure*.

Definition 10. A *supersymplectic structure* on the $2n$-dimensional differentiable manifold M is the datum of

$$\kappa \in \wedge^{\text{even}}(M) \ , \ \kappa = \sum_{p=1}^{n} \kappa_p \ , \ \kappa_p \in \wedge^{2p}(M) \ , \ 1 \leq p \leq n$$

such that:

1. $\kappa^n \neq 0$, i.e., $\kappa_1^n \neq 0$,
2. $d\kappa = 0$, i.e., $d\kappa_p = 0$, $1 \leq p \leq n$,
3. $d^\bigstar \kappa = 0$, i.e., $d^\bigstar \kappa_p = 0$, $1 \leq p \leq n$, where $\bigstar$ is computed with respect to κ_1.

Therefore, if κ is a supersymmetric structure on M, then κ_1 is a simplectic structure. Vice versa, from a symplectic structure κ_1, we can always construct a supersymplectic structure, just setting

$$\kappa := \sum_{p=1}^{n} \kappa_1^p .$$

Note that, in general, the DSLA $(\mathcal{A},\ d)$ does not satisfy the condition of lemma 4 (because, in general, the dGBV algebra $(\mathcal{A},\ d^\bigstar,\ d)$ does not satisfy the $dd^\bigstar$-lemma). Therefore, in contrast with the $\mathbb{Z}$-case, we cannot conclude that the theory is totally (formally) unobstructed; this is true whenever the symplectic manifold $(M,\ \kappa_1)$ satisfies the Hard Lefschetz Condition (cf. [2], [9], [10], [4], [13]).

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