A Case for Curvature: the Unit Tangent Bundle

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Dedicated to Professor Lieven Vanhecke

1 Curvature theory

In the scientific work of L. Vanhecke, the notion of curvature is never more than a step away, if not studied explicitly. This is only right, since, in the words of R. Osserman, "curvature is *the* central concept (in differential geometry and, more in particular, in Riemannian geometry), distinguishing the geometrical core of the subject from those aspects that are analytic, algebraic or topological". The reason for this can be seen as follows:

— if we equip a differentiable manifold M with a metric g, then its curvature is completely determined. If the metric g has nice properties (e.g., a large group of isometries), then this is reflected in a 'nice' curvature;
— conversely, we can often deduce information about the metric from special properties of the curvature. In some cases, knowledge about the curvature even suffices to completely determine the metric (at least locally). Locally symmetric spaces are the prime example here: they are distinguished from non-symmetric spaces by their parallel curvature and, starting from the curvature, one can reconstruct the manifold and its metric (locally).

The curvature information is contained in the Riemannian curvature tensor R. This is an analytic object, a (0, 4)-tensor which is not easy to handle, in general, despite its many symmetries. It is often very difficult to extract the geometrical information which is, as it were, encoded within. For this reason, the famous geometer M. Gromov calls the curvature tensor "a little monster of multilinear algebra whose full geometric meaning

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remains obscure". One therefore works not only with the curvature tensor R itself, but with other forms of curvature or related operators as well, which have a more direct geometric interpretation or which are easier to deal with. We mention the sectional curvature, the Ricci curvature, the scalar curvature and the Jacobi operators. However, not all of these contain the same amount of curvature information. Curvature theory has as its explicit aim the shedding of light on the interplay between the curvature of a Riemannian manifold and its geometric properties, in spite of the difficulties mentioned before.

The study of manifolds from the point of view of curvature has two complementary aspects, roughly corresponding to the two passages: from the metric (and all the geometry that it entails) to the curvature and from the curvature to the metric.

1. *Direct theory.* First, one looks at 'simple' manifolds. By this we mean Riemannian manifolds with a high degree of symmetry and hence with a relatively easy curvature tensor. In some cases, one can even write it down explicitly. As examples of such spaces, we mention locally symmetric spaces, homogeneous spaces and two-point homogeneous spaces. One studies their geometric properties, which are often generalizations of properties from classical Euclidean geometry. In particular, one also studies associated objects like small geodesic spheres, tubes about curves and submanifolds, tangent and unit tangent bundles, special transformations, ...

2. *Inverse theory.* Next, one compares more general manifolds with one of these 'simple' spaces: one takes the latter as a model and investigates which of its properties (or those of its associated objects) are *characteristic* for the model space. In other words: can one recognize the model space based on those specific properties? If not, one tries to find a complete classification of Riemannian manifolds with those properties. The technical details at this stage differ considerably from those in the direct theory. Indeed, for general manifolds, no explicit description of the curvature is available. Further, quantities such as, e.g., the volume of small geodesic spheres and balls can no longer be written down in closed form. Instead, one often uses series expansions for these quantities, the coefficients of which depend on the curvature. Geometric information concerning, e.g., the volumes of the small geodesic spheres then lead to restrictions on the curvature via the series expansions. In other situations, the geometric properties considered have natural consequences for the Jacobi operators or other forms of curvature. In this way, one collects curvature information and hopes to be able to draw conclusions from this concerning the metric. Curvature acts here as the bridge between the geometric properties and the metric itself.

The contributions of L. Vanhecke to the field of curvature theory in the above spirit are too numerous to specify and his influence on geometry and on geometers worldwide, the present one included, can readily be discerned. In this note, I only intend to illustrate the above program using the geometry of the unit tangent bundle as a showcase. On this topic, I have worked for some years now, often in collaboration with L. Vanhecke and other colleagues. For a survey of earlier results, see [5]. Here, I will concentrate on two aspects of the unit tangent bundle: its geodesics and the question of reducibility. The presentation will be rather brief. Full statements and proofs can be found in the articles [1] and [3].

2 The unit tangent bundle

We first recall a few of the basic facts and formulas about the unit tangent bundle of a Riemannian manifold. A more elaborate exposition and further references can be found in [4].

The tangent bundle TM of a Riemannian manifold (M, g) consists of pairs (x, u) where x is a point in M and u a tangent vector to M at x. The mapping $\pi : TM \to M : (x, u) \mapsto x$ is the natural projection from TM onto M. It is well-known that the tangent space to TM at a point (x, u) splits into the direct sum of the vertical subspace $VTM_{(x,u)} = \ker \pi_{*|(x,u)}$ and the horizontal subspace $HTM_{(x,u)}$ with respect to the Levi Civita connection ∇ of (M, g): $T_{(x,u)}TM = VTM_{(x,u)} \oplus HTM_{(x,u)}$.

For $w \in T_xM$, there exists a unique horizontal vector $w^h \in HTM_{(x,u)}$ for which $\pi_*(w^h) = w$. It is called the *horizontal lift* of w to (x, u). There is also a unique vertical vector $w^v \in VTM_{(x,u)}$ for which $w^v(df) = w(f)$ for all functions f on M. It is called the *vertical lift* of w to (x, u). These lifts define isomorphisms between T_xM and $HTM_{(x,u)}$ and $VTM_{(x,u)}$ respectively. Hence, every tangent vector to TM at (x, u) can be written as the sum of a horizontal and a vertical lift of uniquely defined tangent vectors to M at x. The *horizontal* (respectively *vertical*) *lift of a vector field* X on M to TM is defined in the same way by lifting X pointwise. Further, if T is a tensor field of type $(1, s)$ on M and $X_1, \ldots, X_{s-1}$ are vector fields on M, then we denote by $T(X_1, \ldots, u, \ldots, X_{s-1})^v$ the vertical vector field on TM which at (x, w) takes the value $T(X_{1x}, \ldots, w, \ldots, X_{s-1\,x})^v$, and similarly for the horizontal lift. In general, these are *not* the vertical or horizontal lifts of a vector field on M.

The *Sasaki metric* g_S on TM is completely determined by

$$g_S(X^h, Y^h) = g_S(X^v, Y^v) = g(X, Y) \circ \pi, \quad g_S(X^h, Y^v) = 0,$$

for vector fields X and Y on M.

Our interest lies in the unit tangent bundle T_1M, which is the hypersurface of TM consisting of all tangent vectors to (M, g) of length 1. It is given implicitly by the equation $g_x(u, u) = 1$. A unit normal vector field N to T_1M is given by the vertical vector field u^v. We see that horizontal lifts to $(x, u) \in T_1M$ are tangents to T_1M, but vertical lifts in general are not. For that reason, we define the *tangential lift* w^t of $w \in T_xM$ to $(x, u) \in T_1M$ by $w^t = w^v - g(w, u)N$. Clearly, the tangent space to T_1M at (x, u) is spanned by horizontal and tangential lifts of tangent vectors to M at x. One defines the *tangential lift of a vector field* X on M in the obvious way. For the sake of notational clarity, we will use $\bar{X}$ as a shorthand for $X - g(X, u)u$. Then $X^t = \bar{X}^v$. Further, we denote by VT_1M the $(n-1)$-dimensional distribution of vertical tangent vectors to T_1M.

If we consider T_1M with the metric induced from the Sasaki metric g_S of TM, also denoted by g_S, we turn T_1M into a Riemannian manifold. Its Levi Civita connection $\bar{\nabla}$ is described completely by

$$\begin{aligned} \bar{\nabla}_{X^t} Y^t &= -g(Y, u)X^t, \\ \bar{\nabla}_{X^t} Y^h &= \frac{1}{2}\,(R(u, X)Y)^h, \end{aligned} \tag{1}$$

$$\bar{\nabla}_{X^h} Y^t = (\nabla_X Y)^t + \frac{1}{2}\,(R(u,Y)X)^h,$$
$$\bar{\nabla}_{X^h} Y^h = (\nabla_X Y)^h - \frac{1}{2}\,(R(X,Y)u)^t,$$

for vector fields X and Y on M. Its Riemann curvature tensor $\bar{R}$ is given by

$$\begin{aligned}
\bar{R}(X^t, Y^t)Z^t &= g(\bar{Y}, \bar{Z})\, X^t - g(\bar{Z}, \bar{X})\, Y^t,\\
\bar{R}(X^t, Y^t)Z^h &= (R(\bar{X}, \bar{Y})Z)^h + \frac{1}{4}([R(u,X), R(u,Y)]Z)^h,\\
\bar{R}(X^h, Y^t)Z^t &= -\frac{1}{2}(R(\bar{Y}, \bar{Z})X)^h - \frac{1}{4}(R(u,Y)R(u,Z)X)^h,\\
\bar{R}(X^h, Y^t)Z^h &= \frac{1}{2}(R(X,Z)\bar{Y})^t - \frac{1}{4}(R(X, R(u,Y)Z)u)^t\\
&\quad + \frac{1}{2}((\nabla_X R)(u,Y)Z)^h, \qquad (2)\\
\bar{R}(X^h, Y^h)Z^t &= (R(X,Y)\bar{Z})^t\\
&\quad + \frac{1}{4}(R(Y, R(u,Z)X)u - R(X, R(u,Z)Y)u)^t\\
&\quad + \frac{1}{2}((\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X)^h,\\
\bar{R}(X^h, Y^h)Z^h &= (R(X,Y)Z)^h + \frac{1}{2}(R(u, R(X,Y)u)Z)^h\\
&\quad - \frac{1}{4}(R(u, R(Y,Z)u)X - R(u, R(X,Z)u)Y)^h\\
&\quad + \frac{1}{2}((\nabla_Z R)(X,Y)u)^t,
\end{aligned}$$

for vector fields X, Y and Z on M.

From these formulas, it is clear how the curvature of the base manifold interferes in the geometry and the curvature of the unit tangent bundle. Conversely, we will be able to 'translate' information on the unit tangent bundle to the base manifold using these formulas. This should not surprise us, as the metric structure on the base manifold completely determines that of the bundle.

3 Geodesics on the unit tangent bundle

As a first illustration of the role of curvature in geometric problems, we are interested in geodesics of the unit tangent bundle. Any curve $\gamma(t) = (x(t), V(t))$ in the unit tangent bundle can be considered as a curve $x(t)$ in the base manifold M, together with a unit vector field $V(t)$ along it. The geodesic equation in (T_1M, g_S) can be readily deduced from the formulas (1) for the Levi Civita connection. We find that $\gamma(t) = (x(t), V(t))$ is a geodesic of (T_1M, g_S), if and only if

$$\nabla_{\dot x}\dot x = -R(V, \nabla_{\dot x} V)\dot x, \qquad (3)$$
$$\nabla_{\dot x}\nabla_{\dot x} V = -c^2 V,$$

where $c^2 = g(\nabla_{\dot x} V, \nabla_{\dot x} V)$ is a constant along $x(t)$. (See, e.g., [9].)

For general Riemannian manifolds, it is hopeless to try and solve the system of differential equations (3). For ‘simple’ base spaces, however, some results can be obtained. For two-dimensional base spaces, a full solution was given in [7]. When the base manifold is a space of constant curvature c, the curvature can be written as $R(X, Y)Z = c\,(g(Y, Z)X - g(X, Z)Y)$ and the equation (3) becomes much simpler. S. Sasaki ([10]) has explicitly determined all geodesics in this setting. As a side-result of his description, we state

Proposition 1. *If (M^n, g) is a space of constant curvature and $\gamma(t) = (x(t), V(t))$ is a geodesic of (T_1M, g_S), then the projected curve $x(t) = \pi(\gamma(t))$ in M^n has constant curvatures κ_1 and κ_2 and vanishing third curvature κ_3.*

For a locally symmetric base manifold, P. Nagy showed a result in the same spirit in [8].

Proposition 2. *If (M^n, g) is a locally symmetric space and $\gamma(t) = (x(t), V(t))$ is a geodesic of (T_1M, g_S), then the curve $x(t)$ in M has constant curvatures κ_i, $i = 1, \dots, n-1$.*

The proofs for both propositions are based on the same idea. Since both $|\dot\gamma|^2 = |\dot x|^2 + |\nabla_{\dot x} V|^2$ and $|\nabla_{\dot x} V|^2 = c^2$ are constant, we can reparametrize $\gamma(t)$ (and $x(t)$) so that $|\dot x| = 1$. Hence we can take $T = \dot x$ as the first vector in the Frenet frame $\{T, N_1, \dots, N_{n-1}\}$ along x and we have for the first three covariant derivatives of $\dot x$:

$$\begin{aligned}
\dot x^{(1)} &= \nabla_{\dot x}\dot x = \kappa_1 N_1,\\
\dot x^{(2)} &= \nabla_{\dot x}\nabla_{\dot x}\dot x = -{\kappa_1}^2\, T + \kappa_1' N_1 + \kappa_1\kappa_2 N_2, \qquad (4)\\
\dot x^{(3)} &= \nabla_{\dot x}\nabla_{\dot x}\nabla_{\dot x}\dot x = -3\kappa_1\kappa_1' T + (\kappa_1'' - \kappa_1({\kappa_1}^2 + {\kappa_2}^2))\, N_1\\
&\quad + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')\, N_2 + \kappa_1\kappa_2\kappa_3 N_3,
\end{aligned}$$

and similar expressions for the higher order derivatives of x. On the other hand, using (3), we can calculate

$$\begin{aligned}
\dot x^{(1)} &= -R(V, \dot V)\dot x,\\
\dot x^{(2)} &= -(\nabla_{\dot x} R)(V, \dot V)\dot x + R(V, \dot V)^2 \dot x, \qquad (5)\\
\dot x^{(3)} &= -(\nabla^{(2)}_{\dot x\dot x} R)(V, \dot V)\dot x + (\nabla_{R(V,\dot V)\dot x} R)(V, \dot V)\dot x\\
&\quad + 2(\nabla_{\dot x} R)(V, \dot V)R(V, \dot V)\dot x + R(V, \dot V)(\nabla_{\dot x} R)(V, \dot V)\dot x\\
&\quad - R(V, \dot V)^3 \dot x,
\end{aligned}$$

where we have put $\dot V = \nabla_{\dot x} V$ for simplicity. Again, similar expressions can be derived for higher order derivatives of x. In particular, for a locally symmetric base space, this leads to the simple formula

$$\dot{x}^{(k)} = (-1)^k R(V, \dot{V})^k \dot{x}.$$

It is easy to see from this formula that $\dot{x}^{(k)}$ has constant length for all k. Combining this with the corresponding formulas (4) for arbitrary $\dot{x}^{(k)}$, $k = 1, \ldots, n-1$, one proves by induction that all curvatures κ_i are constant. The vanishing of κ_3 for base spaces of constant curvature is a consequence of the special form of the curvature tensor.

Both propositions above are examples of *direct* results. In [1], we have looked at possible converses, at *indirect* results. We comment on the role of curvature in this context.

As concerns the converse of Proposition 2, we note that explicit expressions can be given for the curvatures κ_i in terms of the curvature tensor R and its covariant derivatives via (4) and (5). However, these expressions quickly become rather complicated and of little practical use. For this reason, we only consider the case where the first curvature κ_1 is constant. For this function, we find the expression

$$\kappa_1{}^2 = g(R(V, \dot{V})\dot{x}, R(V, \dot{V})\dot{x}). \tag{6}$$

Taking the covariant derivative along $x(t)$, we find

Proposition 3. *Let (M, g) be a Riemannian manifold. Then for any geodesic γ of (T_1M, g_S), the projected curve $x = \pi \circ \gamma$ has constant first curvature κ_1 if and only if the curvature condition*

$$g((\nabla_Y R)(V, W)Y, R(V, W)Y) = 0, \tag{7}$$

is satisfied for all vector fields Y, V and W on M.

The curvature condition (7) is the starting point for our search for a possible converse to Proposition 2. It implies several conditions on the Jacobi operators $R_\sigma = R(\cdot\,, \dot{\sigma})\dot{\sigma}$ along geodesics σ on (M, g):

1. the eigenvalues of R_σ are constant along σ for each geodesic σ of (M, g), i.e., the manifold (M, g) is a $\mathfrak{C}$-*space*;
2. the operator $R_\sigma{}^2$ is parallel along each geodesic σ of (M, g).

In the literature, a lot of results on the Jacobi operator can be found. Using those, we can obtain converse statements to Proposition 2 for several classes of Riemannian manifolds, but so far not for the general case. For the precise statements, we refer to [1].

Next, we consider a converse of Proposition 1. We will look more generally at spaces (M, g) for which projections of geodesics on (T_1M, g_S) have vanishing curvature κ_1, κ_2 or κ_3.

The case $\kappa_1 \equiv 0$ is easily dealt with. From (4) and (5) we see that the base manifold must necessarily be flat.

Next, suppose that $\kappa_2 \equiv 0$ for every projected geodesic. Comparing the two different descriptions of $\dot{x}^{(2)}$, we find

Proposition 4. *Let (M, g) be a Riemannian space. Then any geodesic γ of (T_1M, g_S) projects to a curve x of M for which $\kappa_2 \equiv 0$ if and only if*

$$R(V, W)^2 Y = -|R(V, W)Y|^2 Y, \tag{8}$$

$$|R(V, W)Y|^2 (\nabla_Y R)(V, W)Y = g((\nabla_Y R)(V, W)Y, R(V, W)Y) R(V, W)Y, \tag{9}$$

for all vector fields V, W and Y on M with $|Y| = 1$.

In this way, we have again translated the original geometric data about geodesics of $(T_1 M, g_S)$ into a curvature condition on (M, g). In particular, it follows from (9) that every Jacobi operator R_σ on (M, g) has parallel eigenspaces along the geodesic σ, i.e., (M, g) is a $\mathfrak{P}$*-space.* Since the only Riemannian manifolds which are both $\mathfrak{C}$- and $\mathfrak{P}$-spaces are the locally symmetric ones (see [2]), we find

Proposition 5. *Let (M, g) be a Riemannian space and suppose that any geodesic γ of $(T_1 M, g_S)$ projects to a curve x of M with constant κ_1 and vanishing κ_2. Then (M, g) is locally symmetric.*

Restricting now to locally symmetric base spaces, we can prove

Theorem 6. *Let (M, g) be a non-flat locally symmetric space and suppose that any geodesic γ of $(T_1 M, g_S)$ projects to a curve x in M with vanishing second curvature κ_2. Then (M, g) is two-dimensional.*

The proof of this result uses different techniques. First, one shows that the rank of the universal covering $(\tilde{M}, \tilde{g})$ of (M, g) must be one. For this, we use the root space decomposition of the Lie algebra corresponding to a representation G/H of $\tilde{M}$ as a homogeneous space. The condition (8) is fundamental here. In a second step, we show easily that no four-dimensional locally irreducible symmetric spaces exist which satisfy (8). Finally, we use the classification by B.-Y. Chen and T. Nagano of maximal totally geodesic submanifolds of rank-one symmetric spaces ([6]) to finish the proof.

To treat the case of vanishing third curvature $\kappa_3 \equiv 0$, we restrict at once to locally symmetric spaces.

Proposition 7. *Let (M, g) be a locally symmetric space. Then any geodesic γ of $(T_1 M, g_S)$ projects to a curve x in M for which $\kappa_3 \equiv 0$, if and only if*

$$R(V, W)^3 Y + ({\kappa_1}^2 + {\kappa_2}^2) R(V, W)Y = 0, \tag{10}$$

for all vector fields V, W and Y on M. The coefficient ${\kappa_1}^2 + {\kappa_2}^2$ only depends on V and W, not on Y. Its value is given by

$${\kappa_1}^2 + {\kappa_2}^2 = |R(V, W)^2 Y|^2 / |R(V, W)Y|^2,$$

for any Y such that $R(V, W)Y \neq 0$.

Again using a mixture of Lie group theory and results on totally geodesic submanifolds in symmetric spaces, we are able to prove from this curvature condition the following converse to Proposition 1.

Theorem 8. *Let (M^n, g), $n \geq 3$, be a locally symmetric space such that the projection $x = \pi \circ \gamma$ of any geodesic γ of $(T_1 M, g_S)$ has vanishing third curvature κ_3. Then, (M^n, g) is either a space of constant curvature or a local product of a flat space and a space of constant curvature.*

4 Reducibility of the unit tangent bundle

As a second illustration of the programme set out in the first section, we consider the question: when is the unit tangent bundle of a Riemannian manifold reducible? In answering this question, the curvature tensor will again be the main actor, even if completely different techniques are needed compared to the ones used in the preceding section. On the whole, the answer to the above question requires a lot of calculations, but the underlying ideas are very simple. We will outline the argument and refer to [3] for the technical details.

The existence of a local decomposition implies some special behavior of the Riemann curvature tensor. Indeed, any curvature operator $\bar{R}(U, V)$ acting on a vector tangent to one of the components gives again a vector tangent to the same component. In particular, if in the expression $\bar{R}(U, V)W$, one of the vectors U, V, W is tangent to one component and another vector to the other component, then $\bar{R}(U, V)W$ will necessarily be zero. This is a very simple consequence of reducibility which is by no means equivalent to the existence of a local product decomposition. Still, it will bring us very far, as we will see. An additional advantage is that the curvature condition is a pointwise condition and no knowledge about covariant derivatives is needed.

Suppose first that, at a point (x, u) of $T_1 M$, the tangent space to one of the factors, say to M_1, contains a non-zero vertical vector X^t, $X \in T_x M$ and X orthogonal to u. Then it holds

$$\bar{R}(Y^t, X^t)X^t = g(X, X)\, Y^t - g(X, Y)\, X^t \in T_{(x,u)} M_1$$

for all vectors $Y \in T_x M$. As a consequence, $V T_1 M_{(x,u)} \subset T_{(x,u)} M_1$, and M_1 is at least $(n-1)$-dimensional. Hence, *if at a point of $T_1 M$ one of the factors contains a non-zero vertical vector, it contains the complete vertical distribution at that point.* We call the decomposition *vertical at* (x, u) in such a situation. Note that this is the case as soon as $\max\{\dim M_1, \dim M_2\} > n$. So, the only possibility for the decomposition not to be vertical at (x, u) is that $\dim M_1 = n$, $\dim M_2 = n - 1$ (or conversely) and neither factor is tangent to a vertical vector. We call this a *diagonal decomposition at* (x, u).

4.1 Diagonal decomposition

Suppose for now that we have a diagonal decomposition $T_1 M \simeq M_1 \times M_2$ at (x, u) with $\dim M_1 = n$ and $\dim M_2 = n - 1$. The following technical result allows us to work with suitable bases for $T_{(x,u)} M_1$ and $T_{(x,u)} M_2$. Its proof uses the symmetries of the curvature tensor.

Lemma 9. *If $T_1 M \simeq M_1 \times M_2$ is a diagonal decomposition at (x, u) with $\dim M_1 = n$ and $\dim M_2 = n - 1$, then there exist two orthonormal bases $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_{n-1}, u\}$ of $T_x M$ and $\lambda > 0$, such that an orthogonal basis for $T_{(x,u)} M_1$ is given by*

$$X_1{}^h + \lambda Y_1{}^t, \dots, X_{n-1}{}^h + \lambda Y_{n-1}{}^t, X_n{}^h$$

and an orthogonal basis for $T_{(x,u)} M_2$ is given by

$$\lambda X_1{}^h - Y_1{}^t, \dots, \lambda X_{n-1}{}^h - Y^t_{n-1}.$$

Note that the decomposition at (x, u) gives rise to *two* special orthonormal bases of $T_x M$.

Next, we express that $\bar{R}(U, V)W = 0$, if U is one of the vectors in the above basis for $T_{(x,u)}M_1$ and W one of the vectors in the basis for $T_{(x,u)}M_2$. This gives a list of curvature conditions on (M, g). The ones we will need further on are given by

$$R(u, Y_j)R(u, Y_l)X_i + R(u, Y_l)R(u, Y_j)X_i = 4(\delta_{il}X_j - 2\delta_{jl}X_i + \delta_{ij}X_l), \tag{11}$$

$$\begin{aligned} R(u, Y_j)R(u, Y_l)X_n &+ R(u, Y_l)R(u, Y_j)X_n \\ &= -2g(R(u, Y_j)X_n, R(u, Y_l)X_n)\, X_n, \end{aligned} \tag{12}$$

$$\begin{aligned} 4R(Y_l, Y_j)X_i &= R(u, Y_j)R(u, Y_l)X_i - R(u, Y_l)R(u, Y_j)X_i \\ &\quad - 4(\delta_{il}X_j - \delta_{ij}X_l), \end{aligned} \tag{13}$$

$$4R(Y_l, Y_j)X_n = R(u, Y_j)R(u, Y_l)X_n - R(u, Y_l)R(u, Y_j)X_n, \tag{14}$$

$$\begin{aligned} 4R(X_i, X_j)X_l &= \frac{4(\lambda^4 - \lambda^2 + 1)}{\lambda^2}\,(\delta_{jl}X_i - \delta_{il}X_j) \\ &\quad + R(u, R(X_j, X_l)u)X_i - R(u, R(X_i, X_l)u)X_j \\ &\quad - 2R(u, R(X_i, X_j)u)X_l, \end{aligned} \tag{15}$$

$$\begin{aligned} 4R(X_n, X_j)X_l &= \frac{1}{\lambda^2}\, g(R(u, Y_j)X_n, R(u, Y_l)X_n)\, X_n \\ &\quad + R(u, R(X_j, X_l)u)X_n - R(u, R(X_n, X_l)u)X_j \\ &\quad - 2R(u, R(X_n, X_j)u)X_l, \end{aligned} \tag{16}$$

where $i, j, k \in \{1, \ldots, n-1\}$.

Two remarks are important here. First, if we can determine the operators $R(u, Y_l)$, $l = 1, \ldots, n-1$, satisfying (11) and (12), then we can compute consecutively the operators $R(Y_l, Y_j)$, $l, j = 1, \ldots, n-1$, from (13) and (14) and $R(X_i, X_j)$, $i, j = 1, \ldots, n-1$, from (15) and (16). The operators $R(u, Y_l)$ are therefore the most fundamental. (Note also that this gives *two* descriptions for the curvature operators $R(V, W)$: one in the basis $\{Y_1, \ldots, Y_{n-1}, u\}$ and another in the basis $\{X_1, \ldots, X_n\}$.) Second, the conditions (11) and (12) remind one of the Clifford relations $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}$, though they are not quite right. Both remarks inspire us to study the operators $R(u, Y_l)$ in some more detail.

From conditions (11) and (12), it follows readily that

$$\begin{aligned} R(u, Y_l)^2 X_l &= 0, \\ R(u, Y_l)^2 X_i &= -4X_i, \quad i \neq l, \\ R(u, Y_l)^2 X_n &= -|R(u, Y_l)X_n|^2\, X_n. \end{aligned}$$

Since $R(u, Y_j)$ is skew-symmetric, the non-zero eigenvalues of $R(u, Y_j)^2$ must have even multiplicity. Hence,

— if n is even, the eigenvalue -4 has even multiplicity $n-2$ on $\{X_j, X_n\}^\perp$. Hence, the eigenvalue corresponding to X_n must be zero. This implies $R(u, Y_j)X_n = 0$

for $j = 1, \ldots, n-1$. By (14), also $R(Y_j, Y_k)X_n = 0$ for $j, k = 1, \ldots, n-1$. We conclude that X_n belongs to the nullity distribution of the curvature tensor R_x. In this case, the conditions (12), (14) and (16) are trivially satisfied;

— if n is odd, the eigenvalue -4 has odd multiplicity $n-2$ on $\{X_j, X_n\}^\perp$. The eigenvalue corresponding to X_n must then be -4 as well. So, it holds, $|R(u, Y_j)X_n|^2 = 4$, for $j = 1, \ldots, n-1$. It even holds, $|R(u, Y)X_n|^2 = 4$, for every unit vector Y orthogonal to u and $g(R(u, Y)X_n, R(u, Z)X_n) = 4g(Y, Z)$, for all vectors Y and Z orthogonal to u. In particular, the right-hand side of (12) equals $-8\delta_{jl}X_n$. In this case, conditions (12) and (14) are included in (11) and (13) if we allow the index i to be n.

This indicates that the cases where n is even and those where n is odd will have to be treated separately.

When n is even, consider the operators $\mathcal{R}_i$, $i = 1, \ldots, n-1$, acting on $V^n = T_x M$ by

$$\mathcal{R}_i = \frac{1}{2} R(u, Y_i) - \langle X_n, \cdot \rangle X_i + \langle X_i, \cdot \rangle X_n,$$

where $\langle \cdot, \cdot \rangle = g_x$. One can show that they satisfy the Clifford relations

$$\mathcal{R}_i \circ \mathcal{R}_j + \mathcal{R}_j \circ \mathcal{R}_i = -2\delta_{ij}\,\mathrm{id}. \tag{17}$$

Hence, they correspond to a Clifford representation of an $(n-1)$-dimensional Clifford algebra on an n-dimensional vector space.

When n is odd, define the operators $\mathcal{R}_i$, $i = 1, \ldots, n-1$, acting on $V^{n+1} = T_x M \oplus \mathbb{R}X_0$ by

$$\mathcal{R}_i = \frac{1}{2} R(u, Y_i) - \langle X_0, \cdot \rangle X_i + \langle X_i, \cdot \rangle X_0,$$

where $\langle \cdot, \cdot \rangle = g_x \oplus g_0$ with $g_0(aX_0, bX_0) = ab$. Again these satisfy the relations (17) and we obtain a Clifford representation of an $(n-1)$-dimensional Clifford algebra on an $(n+1)$-dimensional vector space.

It is well-known, however, that the dimension of a Clifford algebra and that of a module over it are closely related. (See, e.g., the table in [3].) In particular, it follows that Clifford representations as above can only exist for dimensions $n = 1, 2, 3, 4, 7$ and 8. So, only for those dimensions for the base manifold (M, g) can a diagonal decomposition exist for the unit tangent bundle. Moreover, the case $n = 1$ is irrelevant, since a one-dimensional manifold is never reducible.

Finally, treating these remaining cases separately, one can show that the two descriptions for the curvature tensor mentioned higher, one in the basis $\{Y_1, \ldots, Y_{n-1}, u\}$ and the other in the basis $\{X_1, \ldots, X_n\}$, are incompatible, except when $n = 2$. Then, the base manifold is necessarily flat. We conclude that diagonal decompositions for the unit tangent bundle exist only for a flat surface as base space.

4.2 Vertical decomposition

Suppose now that we have a vertical decomposition $T_1 M \simeq M_1 \times M_2$ such that $VT_1M_{(x,u)} \subset T_{(x,u)}M_1$ everywhere. In this situation, if $(x, u) \in M_1 \times \{q\}$, for some

$q \in M_2$, it holds that $\pi^{-1}(x) \subset M_1 \times \{q\}$. Consequently, we have $M_1 \times \{q\} = \pi^{-1}(\pi(M_1 \times \{q\}))$. So, the leaves $M_1 \times \{q\}$, corresponding to the product, project under π to a foliation $\mathcal{L}_1$ on (M, g) and $\pi^{-1}(\mathcal{L}_1) = \{M_1 \times \{q\}, q \in M_2\}$. Let L_1 be the distribution on M tangent to $\mathcal{L}_1$. Define the distribution L_2 to be the orthogonal distribution to L_1 on M. Then,

$$T_{(x,u)}(M_1 \times \{q\}) = VT_rM_{(x,u)} \oplus h(L_{1x}), \quad T_{(x,u)}(\{p\} \times M_2) = h(L_{2x}),$$

where h denotes the horizontal lift. In particular, we can describe the tangent spaces to the factors of the (local) product using horizontal and vertical lifts. From the expressions (1) for the Levi Civita connection, it is easy to deduce that also L_2 is integrable, with associated foliation $\mathcal{L}_2$ with flat leaves, and that (M, g) is locally isometric to a Riemannian product $M \simeq M' \times \mathbb{R}^k$ where $k = \dim L_2 \leq n$. Conversely, it is almost immediate that a (local) decomposition $M \simeq M' \times \mathbb{R}^k$ with $k > 0$ gives rise to a (local) decomposition of (T_1M, g_S). This proves

Theorem 10 (Local version). *The unit tangent bundle (T_1M, g_S) of a Riemannian manifold (M^n, g), $n \geq 2$, is locally reducible if and only if (M, g) has a flat factor, i.e., (M, g) is locally isometric to a product $(M', g') \times (\mathbb{R}^k, g_0)$ where $1 \leq k \leq n$ and g_0 denotes the standard Euclidean metric on $\mathbb{R}^k$.*

With a little more effort (not involving curvature), we can even show the corresponding global result.

Theorem 11 (Global version). *Let (M^n, g), $n \geq 3$, be a Riemannian manifold and suppose that (T_1M, g_S) is a global Riemannian product. Then, (M, g) is either flat or it is also a global Riemannian product, with a flat factor.*

Conversely, if (M, g) is a global product space $(M', g') \times (F^k, g_0)$ where $1 \leq k \leq n$ and F is a connected and simply connected flat space, then (T_1M, g_S) is a global Riemannian product, also with (F, g_0) as a flat factor.

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