On 3D-Riemannian Manifolds with Prescribed Ricci Eigenvalues

Oldřich Kowalski and Zdeněk Vlášek

Charles University Prague, Czech Republic
kowalski@karlin.mff.cuni.cz**
vlasek@karlin.mff.cuni.cz

Dedicated to Professor Lieven Vanhecke

Summary. In this paper, we deal with 3-dimensional Riemannian manifolds where some conditions are put on their principal Ricci curvatures. In Section 2 we classify locally all Riemannian 3-manifolds with prescribed distinct Ricci eigenvalues, which can be given as arbitrary real analytic functions. In Section 3 we recall, for the *constant* distinct Ricci eigenvalues, an explicit solution of the problem, but in a more compact form than it was presented in [17]. Finally, in Section 4 we give a survey of related results, mostly published earlier in various journals. Last but not least, we compare various PDE methods used for solving problems of this kind.

1 Introduction

The problem of how many Riemannian metrics exist on the open domains of R^3 with prescribed constant Ricci eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$ was completely solved in [15] and [19]. The main existence theorem says that the local isometry classes of these metrics are always parametrized by *two arbitrary functions of one variable*. Some non-trivial explicit examples were presented in [15], as well. A more elegant but less rigorous proof of the main existence theorem was given in [5].

The case of distinct constant Ricci eigenvalues is more interesting. Here, the first examples were presented by K.Yamato in [33], namely a complete (but not locally homogeneous) metric defined on R^3 for each prescribed triplet $(\varrho_1, \varrho_2, \varrho_3)$ of constant distinct Ricci eigenvalues satisfying certain algebraic inequalities. Thus, these triplets

** The authors have been partly supported by the grant GAČR 201/02/0616 and by the project MSM 113200007.
Key words: Riemannian manifold, principal Ricci curvatures, PDE methods, curvature homogeneous spaces.
Subject Classifications: 53C21, 53B20.

form an open set in $\mathbb{R}^3[\varrho_1, \varrho_2, \varrho_3]$. This open set was essentially extended by new examples in [13]. Finally, in [17], non-trivial explicit examples were constructed for *every choice* of Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$. These examples are not locally homogeneous but mostly local and not complete. (There is still an open problem for which triplets $\varrho_1 > \varrho_2 > \varrho_3$ a complete metric exists with such Ricci eigenvalues.)

The problem of '*how many* local isometry classes (or, more exactly, how many *isometry classes of germs*) of Riemannian metrics exist for prescribed constant Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$' was solved first by A. Spiro and F. Tricerri in [30], using the theory of formally integrable analytic differential systems. They proved that this "local moduli space" depends on an infinite number of parameters. This solution was not satisfactory enough for us and we succeeded to show in [26] that this local moduli space is parametrized, in fact, by (the germs of) three arbitrary functions of two variables. Moreover, the method of solution was completely "classical", based on the Cauchy–Kowalewski Theorem. Yet, for many mathematicians, this solution may be not completely satisfactory for a different reason: The partial differential equations expressing the geometric conditions are rather cumbersome (see Section 3), and one of the main steps of the proof is not transparent enough, because it depends heavily on a hard computer work (using Maple V) for the huge amount of routine symbolic manipulations with the corresponding PDE system.

In this paper, we prove the same result by a different method. Here the computer assistence (using Maple V) is also used, but in a much more transparent way. Namely, when using the new method, some cumbersome formulas occur again. Yet, for the main argument, we need only their qualitative properties and not the explicit expressions.

Moreover, by the new method, we are able to *generalize* the original result to the situation when the prescribed distinct Ricci eigenvalues are not constants but *arbitrary functions*. This is the content of Section 2.

In Section 3, we come back to the old version from [17] and [26] (with constant ϱ_i and a complicated PDE system) to show that there is a general *explicit* formula involving three parameters $\varrho_1 > \varrho_2 > \varrho_3$ and producing a Riemannian metric with the Ricci principal curvatures ϱ_i. This result was essentially proved already in [17], but now it is presented in a particularly simple form, in the spirit of the pioneering work by K. Yamato [33]. It is obvious that the new method from Section 2 is not suitable to produce such explicit examples and so one can compare the advantages and disadvantages of both (very different) methods.

The last Section 4 is mainly a survey of related results which have been published earlier (except the last subsection inspired by the work by S. Ivanov and I. Petrova [11]). The main purpose of Section 4 is to show that there are more geometric problems concerning prescribed properties of the Ricci eigenvalues for which a completely satisfactory geometric solution was found, but where "the method of the Ricci characteristic polynomial", introduced in Section 2, obviously fails. Namely, from the optics of this method, one comes to an overdetermined system of PDE. Yet, we are still able to describe "the size" of the general solutions of such systems just coming back from a known geometric result to the corresponding PDE system. This might be a useful contribution to the "philosophy of PDE methods" in Riemannian geometry.

2 The case of arbitrary distinct Ricci eigenvalues

Let $\varrho_1(x, y, z) > \varrho_2(x, y, z) > \varrho_3(x, y, z)$ be three real analytic functions defined on a domain $\mathcal{U} \subset \mathsf{R}^3[x, y, z]$. Let (M, g) be a Riemannian manifold and $\mathcal{U}' \subset M$ a coordinate neighborhood. We say that *the metric g restricted to $\mathcal{U}'$ has principal Ricci curvatures $\varrho_1, \varrho_2, \varrho_3$*, if this is valid with respect to a local chart $\varphi : \mathcal{U}' \to \mathcal{U}$, i.e., when expressing $g_{|\mathcal{U}'}$ in the local coordinates x, y, z.

The main theorem of this paper is the following:

Theorem 1. *Let $\varrho_1(x, y, z) > \varrho_2(x, y, z) > \varrho_3(x, y, z)$ be three real analytic functions defined on a domain $\mathcal{U} \subset \mathsf{R}^3[x, y, z]$. Then, the local moduli space of (local) Riemannian metrics with the prescribed principal Ricci curvatures $\varrho_1, \varrho_2, \varrho_3$ can be parametrized by three arbitrary functions of two variables.*

We shall start with the hard part of the proof, which is based on the following:

Theorem 2. *Let $\varrho_1(x, y, z) > \varrho_2(x, y, z) > \varrho_3(x, y, z)$ be three real analytic functions defined in a domain $\mathcal{U} \subset \mathsf{R}^3[x, y, z]$. Then all (local) diagonal Riemannian metrics with the principal Ricci curvatures ϱ_1, ϱ_2 and ϱ_3 depend on six arbitrary functions of two variables.*

The following Theorem should be considered as a "folklore", see e.g. [9].

Theorem 3. *Let (M, g) be a real analytic 3-dimensional Riemannian manifold. Then, in a neighborhood of each point $p \in M$, there is a system (x, y, z) of local coordinates in which g adopts a diagonal form. All coordinate transformations for which the diagonality of a metric is preserved depend on 3 arbitrary functions of two variables.*

Thus, in the sequel, we can assume that each Riemannian metric g in consideration has the matrix $\left(g_{ij}\right)$ of components written in the form

$$\left(g_{ij}\right) = \begin{pmatrix} K(x, y, z) & 0 & 0 \\ 0 & L(x, y, z) & 0 \\ 0 & 0 & M(x, y, z) \end{pmatrix}, \ (i, j = 1, 2, 3).$$

Here, of course, the functions K, L and M are positive and real analytic in the corresponding domain $\mathcal{U} \subset \mathsf{R}^3[x, y, z]$.

A routine calculation gives the following expression for the Ricci *operator* Ric in the given local coordinates x, y, z. Here, we introduce the following abbreviated notation: If G denotes K, L or M, evaluated at a general point (x, y, z), we write,

$$G_1 = \frac{\partial G}{\partial x},\ G_2 = \frac{\partial G}{\partial y},\ G_3 = \frac{\partial G}{\partial z},$$

$$G_{11} = \frac{\partial^2 G}{(\partial x)^2},\ G_{12} = \frac{\partial^2 G}{\partial x \partial y},\ \ldots,\ G_{33} = \frac{\partial^2 G}{(\partial z)^2},$$

evaluated at (x, y, z), as well. Now, in the abbreviated notation, we have the following formulas:

$$\begin{aligned}
\mathrm{Ric}_1^1 &= -\,(MK_{22} + LK_{33} + ML_{11} + LM_{11})\,/\,(2KLM) + [LM^2K_1L_1 \\
&\quad + L^2MK_1M_1 + LM^2K_2^2 + KM^2K_2L_2 - KLMK_2M_2 + L^2MK_3^2 \\
&\quad - KLMK_3L_3 + KL^2K_3M_3 + KM^2L_1^2 + KL^2M_1^2\,]/(4K^2L^2M^2), \\
\mathrm{Ric}_2^2 &= -\,(MK_{22} + ML_{11} + KL_{33} + KM_{22})\,/\,(2KLM) + [LM^2K_1L_1 \\
&\quad + LM^2K_2^2 + KM^2K_2L_2 - KLMK_3L_3 + KM^2L_1^2 - KLML_1M_1 \\
&\quad + K^2ML_2M_2 + K^2ML_3^2 + K^2LL_3M_3 + K^2LM_2^2\,]/(4K^2L^2M^2), \\
\mathrm{Ric}_3^3 &= -\,(LK_{33} + KL_{33} + LM_{11} + KM_{22})\,/\,(2KLM) + [L^2MK_1M_1 \\
&\quad - KLMK_2L_2 + L^2MK_3^2 + KL^2K_3M_3 - KLML_1M_1 + K^2ML_2M_2 \\
&\quad + K^2ML_3^2 + K^2LL_3M_3 + KL^2M_1^2 + K^2LM_2^2\,]/(4K^2L^2M^2), \\
\mathrm{Ric}_1^2 &= \mathrm{Ric}_2^1 \\
&= -(2KLMM_{12} - LMK_2M_1 - KML_1M_2 - KLM_1M_2)/(4KL^2M^2), \\
\mathrm{Ric}_1^3 &= \mathrm{Ric}_3^1 \\
&= -(2KLML_{13} - LMK_3L_1 - KML_1L_3 - KLL_3M_1)/(4KL^2M^2), \\
\mathrm{Ric}_2^3 &= \mathrm{Ric}_3^2 \\
&= -(2KLMK_{23} - LMK_2K_3 - KMK_2L_3 - KLK_3M_2)/(4K^2LM^2).
\end{aligned}$$

We now express the above formulas in a shorter way:

$$\begin{aligned}
\mathrm{Ric}_1^1 &= -\,(MK_{22} + LK_{33} + ML_{11} + LM_{11})\,/\,(2KLM) + G_1^1, \\
\mathrm{Ric}_2^2 &= -\,(MK_{22} + ML_{11} + KL_{33} + KM_{22})\,/\,(2KLM) + G_2^2, \\
\mathrm{Ric}_3^3 &= -\,(LK_{33} + KL_{33} + LM_{11} + KM_{22})\,/\,(2KLM) + G_3^3, \\
\mathrm{Ric}_1^2 &= \mathrm{Ric}_2^1 = -M_{12}/(2LM) + G_1^2, \\
\mathrm{Ric}_1^3 &= \mathrm{Ric}_3^1 = -L_{13}/(2LM) + G_1^3, \\
\mathrm{Ric}_2^3 &= \mathrm{Ric}_3^2 = -K_{23}/(2KM) + G_2^3,
\end{aligned} \tag{1}$$

where G_i^j are rational functions of $K, L, M, K_1, K_2, \ldots, M_3$, i.e., they depend only on the functions K, L, M and their first derivatives.

Consider now the prescribed Ricci eigenvalues $\varrho_1(x,y,z)$, $\varrho_2(x,y,z)$, $\varrho_3(x,y,z)$ (which are real analytic functions defined in the same domain as K, L and M). The corresponding geometric conditions can be expressed, in the simplest way, through the characteristic polynomial $\det(\lambda I - \mathrm{Ric}) = \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0$ of the Ricci operator Ric, in the form

$$c_2 = -\sum_{i=1}^{3} \varrho_i, \quad c_1 = \sum_{1 \le i < j \le 3} \varrho_i\varrho_j, \quad c_0 = -\varrho_1\varrho_1\varrho_3. \tag{2}$$

This is a system of nonlinear PDE's of second order because

$$c_2 = -\sum_{i=1}^{3} \mathrm{Ric}^i_i, \quad c_1 = \sum_{1 \le i < j \le 3} (\mathrm{Ric}^i_i \mathrm{Ric}^j_j - (\mathrm{Ric}^i_j)^2), \quad c_0 = -\det[\mathrm{Ric}^i_j]. \tag{3}$$

We can see easily from (1) that the only "non-mixed" second partial derivatives involved in the functions Ric^i_j are K_{22}, K_{33}, L_{11}, L_{33}, M_{11} and M_{22}. Hence we cannot use the Cauchy–Kowalewski Theorem in the basic setting. We shall try to remove this defect by a linear transformation of independent variables (which is optimal in some sense), namely,

$$u = z, \; v = y, \; w = x + y + z. \tag{4}$$

The metric g, if expressed in the new variables u, v, w, is not anymore in the diagonal form. The new Ricci components $\mathrm{Ric}^\alpha_\beta$ will become linear combinations of the original components Ric^i_j. Nevertheless, because, with respect to the new variables we get

$$[\mathrm{Ric}^\alpha_\beta] \; = \; [S][\mathrm{Ric}^i_j][S^{-1}],$$

where S is a constant regular matrix, the characteristic polynomial of Ric will remain invariant and the expression (3) have the same form for the old components Ric^i_j as for the new components $\mathrm{Ric}^\alpha_\beta$. Thus, we can still use the old components Ric^i_j in our computations and all to be done is to transform all Ric^i_j to the new variables u, v, w. As the first step, the original functions K, L, M and their partial derivatives have to be transformed.

We now introduce new positive functions U, V, W of three variables u, v, w by

$$\begin{aligned} U(u, v, w) &= K(w - u - v, v, u),\\ V(u, v, w) &= L(w - u - v, v, u),\\ W(u, v, w) &= M(w - u - v, v, u). \end{aligned} \tag{5}$$

Rewriting the old coordinates, we get,

$$\begin{aligned} K(x, y, z) &= U(z, y, x + y + z),\\ L(x, y, z) &= V(z, y, x + y + z),\\ M(x, y, z) &= W(z, y, x + y + z). \end{aligned} \tag{6}$$

If F denotes U, V or W evaluated at $(u, v, w) = (z, y, x + y + z)$, we shall write

$$F_1 = \frac{\partial F}{\partial u}, \; F_2 = \frac{\partial F}{\partial v}, \; F_3 = \frac{\partial F}{\partial w},$$
$$F_{11} = \frac{\partial^2 F}{(\partial u)^2}, \; F_{12} = \frac{\partial^2 F}{\partial u \partial v}, \; \ldots, \; F_{33} = \frac{\partial^2 F}{(\partial w)^2},$$

evaluated at the point $(u, v, w) = (z, y, x + y + z)$, as well.

We get easily, in our abbreviated form,

$$
\begin{aligned}
K &= U,\ K_1 = U_3,\ K_2 = U_2 + U_3,\ K_3 = U_1 + U_3,\ K_{11} = U_{33},\\
K_{12} &= U_{23} + U_{33},\ K_{13} = U_{13} + U_{33},\ K_{22} = U_{22} + 2U_{23} + U_{33},\\
K_{23} &= U_{12} + U_{13} + U_{23} + U_{33},\ K_{33} = U_{11} + 2U_{13} + U_{33},\\
L &= V,\ L_1 = V_3,\ L_2 = V_2 + V_3,\ L_3 = V_1 + V_3,\ L_{11} = V_{33},\\
L_{12} &= V_{23} + V_{33},\ L_{13} = V_{13} + V_{33},\ L_{22} = V_{22} + 2V_{23} + V_{33},\\
L_{23} &= V_{12} + V_{13} + V_{23} + V_{33},\ L_{33} = V_{11} + 2V_{13} + V_{33},\\
M &= W,\ M_1 = W_3,\ M_2 = W_2 + W_3,\ M_3 = W_1 + W_3,\ M_{11} = W_{33},\\
M_{12} &= W_{23} + W_{33},\ M_{13} = W_{13} + W_{33},\ M_{22} = W_{22} + 2W_{23} + W_{33},\\
M_{23} &= W_{12} + W_{13} + W_{23} + W_{33},\ M_{33} = W_{11} + 2W_{13} + W_{33}.
\end{aligned}
\tag{7}
$$

Hence, we obtain, for the old components Ric^i_j evaluated at $(u, v, w) = (z, y, x + y + z)$,

$$
\begin{aligned}
\mathrm{Ric}^1_1 &= -\,((V + W)\,U_{33} + W V_{33} + V W_{33})\,/\,(2UVW) + F^1_1,\\
\mathrm{Ric}^2_2 &= -\,(W U_{33} + (U + W)\,V_{33} + U W_{33})\,/\,(2UVW) + F^2_2,\\
\mathrm{Ric}^3_3 &= -\,(V U_{33} + U V_{33} + (U + V)\,W_{33})\,/\,(2UVW) + F^3_3,\\
\mathrm{Ric}^2_1 &= \mathrm{Ric}^1_2 = -W_{33}/(2VW) + F^2_1,\\
\mathrm{Ric}^3_1 &= \mathrm{Ric}^1_3 = -V_{33}/(2VW) + F^3_1,\\
\mathrm{Ric}^3_2 &= \mathrm{Ric}^2_3 = -U_{33}/(2UW) + F^3_2,
\end{aligned}
\tag{8}
$$

where F^i_j are rational functions of U, V, W, their first partial derivatives with respect to u, v, w, and their second partial derivatives which are different from U_{33}, V_{33} and W_{33}.

Now, we are going to prove that, in the new variables, the standard Cauchy–Kowalewski Theorem can be used for the solution of the PDE system (2). We only have to keep in mind that the prescribed Ricci eigenvalues ϱ_i mean here the functions $\overline{\varrho}_i(u, v, w) = \varrho_i(w - u - v, v, u)$, $i = 1, 2, 3$, defined in the same domain as U, V and W. The system (2) can be expressed explicitly in the new variables u, v, w. We get the first PDE in the form

$$
\begin{aligned}
c_2 &= ((V + W)\,U_{33} + (U + W)\,V_{33} + (U + V)\,W_{33})\,/\,(UVW) + H_2\\
&= -\sum_{i=1}^{3} \overline{\varrho}_i,
\end{aligned}
\tag{9}
$$

where H_2 is a rational function of U, V, W, their first derivatives and their second derivatives which are not of the form U_{33}, V_{33} or W_{33}. H_2 is defined as a function of the variables u, v, w in the whole definition domain of the functions U, V, W. From here, we express

$$W_{33} = -\left((V+W)\,U_{33} + (U+W)\,V_{33}\right)/\,(U+V) + P, \tag{10}$$

where P is a rational function of $\overline{\varrho}_1, \overline{\varrho}_2, \overline{\varrho}_3, U, V, W$, the first derivatives of U, V, W, and their second derivatives except U_{33}, V_{33} and W_{33}. Anyway P is a real analytic function of u, v, w, U, V, W and of the corresponding derivatives.

Next, we substitute the expression for W_{33} from (10) in the formulas (8) and we obtain the Ricci components in the reduced form:

$$\begin{aligned}
\mathrm{Ric}^1_1 &= -\left((V+W)\,U_{33} + (W-V)\,V_{33}\right)/\,(2(U+V)VW) + P^1_1,\\
\mathrm{Ric}^2_2 &= -\left((W-U)\,U_{33} + (U+W)\,V_{33}\right)/\,(2(U+V)UW) + P^2_2,\\
\mathrm{Ric}^3_3 &= (U_{33}+V_{33})\,/\,(2UV) + P^3_3,\\
\mathrm{Ric}^2_1 &= \mathrm{Ric}^1_2 = \left((V+W)\,U_{33} + (U+W)\,V_{33}\right)/\,(2(U+V)VW) + P^2_1,\\
\mathrm{Ric}^3_1 &= \mathrm{Ric}^1_3 = -V_{33}/\,(2VW) + P^3_1,\\
\mathrm{Ric}^3_2 &= \mathrm{Ric}^2_3 = -U_{33}/\,(2UW) + P^3_2,
\end{aligned} \tag{11}$$

where P^i_j are functions of the same type as P introduced in (10).

So, assuming that (9) is satisfied identically, we must write down the remaining two PDE's where the Ricci operator is expressed in the form (11). The second equation of (2) can be written in the form

$$f_1(U_{33})^2 + f_2U_{33}V_{33} + f_3(V_{33})^2 + g_1U_{33} + g_2V_{33} = Q, \tag{12}$$

where Q is of the same type as P and P^i_j. Moreover, we get explicitly

$$\begin{aligned}
f_1 &= \frac{U^2(3V^2+3VW+2W^2) + UV(2V^2-VW+W^2) + V^2(V^2+W^2)}{4(U+V)^2U^2V^2W^2},\\
f_2 &= \frac{U^3(V+W) + U^2(-V^2+VW+2W^2) + UVW^2 + V^2W^2}{2(U+V)^2U^2V^2W^2},\\
f_3 &= \frac{2U^3(U+V+W) + U^2(2V^2-VW+2W^2) + UVW(V+W) + V^2W^2}{4(U+V)^2U^2V^2W^2},
\end{aligned} \tag{13}$$

and g_1, g_2 are (more complicated) rational functions of the same type as H_2 in (9).

The third equation of (2) can be written in the form

$$\begin{aligned}
f_{30}(U_{33})^3 &+ f_{21}(U_{33})^2V_{33} + f_{12}U_{33}(V_{33})^2 + f_{03}(V_{33})^3 + f_{20}(U_{33})^2\\
&+ f_{11}U_{33}V_{33} + f_{02}(V_{33})^2 + f_{10}U_{33} + f_{01}V_{33} = S,
\end{aligned} \tag{14}$$

where S is of the same type as P, Q. Moreover, we get explicitly

$$\begin{aligned}
f_{30} &= (V+W)\,[(V^2-2VW-W^2)U + V^3 + VW^2]/d,\\
f_{21} &= [2(V^2-W^2)U^2 + (V^3+3V^2W-5VW^2-3W^3)U - V^4\\
&\qquad + V^3W + V^2W^2 + 3VW^3\,]/d,
\end{aligned} \tag{15}$$

$$f_{12} = [(V-W)\,U^3 + (V^2+VW-4W^2)U^2 + \left(4V^2W - VW^2 - 3W^3\right)U - V^2W^2 + 3VW^3\,]/d,$$

$$f_{03} = (U+W)\,(V-W)\left[U^2 + (V+W)\,U - VW\right]/d,$$

where $d = 8\,(U+V)^2\,U^2V^3W^3$.

The other coefficients are functions of the same type as H_2 in (9) (but occupying many pages in the explicit form).

It remains to analyze the system (12) + (14) of PDE. If this system can be solved in an explicit form

$$U_{33} = T_1,\ \ V_{33} = T_2, \tag{16}$$

where T_1 and T_2 are algebraic functions of $\overline{\varrho}_1, \overline{\varrho}_2, \overline{\varrho}_3, U, V, W$ and of the "admisible" derivatives of U, V, W, then the full system (10) + (16) can be solved by the use of the Cauchy–Kowalewski Theorem, which will prove Theorem 2. Of course, the solvability and the correctness of all calculations will depend on the initial conditions of the corresponding Cauchy problem. (Notice that a solution in the form (16) may have more branches but this is not too relevant for the proof of our Theorem).

First, let (u_0, v_0, w_0) be a point from the definition domain of the functions U, V, W. We define six functions of two variables u, v (the Cauchy initial conditions) in a neighborhood of (u_0, v_0) by the formulas,

$$F_1(u,v) = U(u,v,w_0),\ \ F_2(u,v) = V(u,v,w_0),\ \ F_3(u,v) = W(u,v,w_0), \tag{17}$$
$$G_1(u,v) = \frac{\partial U}{\partial w}(u,v,w_0),\, G_2(u,v) = \frac{\partial V}{\partial w}(u,v,w_0),\ \ G_3(u,v) = \frac{\partial W}{\partial w}(u,v,w_0).$$

Further, denote for a moment u, v, w as u_1, u_2, u_3. We shall define constants

$$a_i = F_i(u_0,v_0) > 0,\ \ a_{i,j} = \frac{\partial F_i}{\partial u_j}(u_0,v_0),\ \ a_{i,jk} = \frac{\partial F_i}{\partial u_j\,\partial u_k}(u_0,v_0),$$
$$b_{i,j} = \frac{\partial G_i}{\partial u_j}(u_0,v_0) \quad \text{for } i = 1,2,3 \text{ and } j,k = 1,2. \tag{18}$$

It is obvious that, for every choice of the constants in (18), we can still define functions $F_i(u,v)$ and $G_i(u,v)$ satisfying (18) as *arbitrary* real analytic functions in a neighborhood of (u_0, v_0). (In fact, we are fixing only a finite number of initial Taylor coefficients of such functions.)

Next, if f is any real analytic function of the variables u, u, w, U, V, W, of the first derivatives of U, V, W, and of those second derivatives which are not of the form U_{33}, V_{33} or W_{33}, we shall denote by $\tilde{f}$ the corresponding value at the point (u_0, v_0, w_0). Obviously, each constant $\tilde{f}$ depends (in a real analytic way) on the constants u_0, v_0, w_0, $a_i, a_{i,j}, a_{i,jk}$ and $b_{i,j}$. In particular, we can make the substitution $u = u_0,\ v = v_0,\ w = w_0$ in the coefficients f_i, g_i, f_{ij}, Q and S of the equations (12) and (14). Let us choose the initial constants $a_i > 0$, $a_{i,j}, a_{i,jk}$ and $b_{i,j}$ in such a way that the equation (12) at the origin defines a generic real quadratic curve and the equation (14) a generic cubic curve, both in the coordinate plane $\mathsf{R}[X = U_{33}, Y = V_{33}]$. Moreover, we can make our choice in such a way that these two curves meet transversaly at some point (X_0, Y_0).

Now, using the real analytic version of the "implicit function theorem" for more variables, we obtain easily the following

Lemma 1. *Let $P(X, Y)$ and $Q(X, Y)$ be two polynomials of two variables X, Y and with the coefficients which are arbitrary parameters. If, for a fixed choice of these parameters, the equations $P(X, Y) = 0$ and $Q(X, Y) = 0$ have a common solution (X_0, Y_0) such that the Jacobian* $\det \begin{bmatrix} \partial P/\partial x & \partial P/\partial y \\ \partial Q/\partial x & \partial Q/\partial y \end{bmatrix}$ *is nonzero at (X_0, Y_0) then, in a neighborhood of (X_0, Y_0), the variables X, Y can be expressed from the above equations in a unique way as a real analytic function of the corresponding coefficients.*

Now, consider for a moment, the coefficients f_i, g_i, f_{ij}, Q and S in the equations (12) and (14) as arbitrary parameters. Applying Lemma 1 to this situation, we see that, in a neighborhood of the point (X_0, Y_0), the quantities U_{33} and V_{33} are expressed in a unique way as real analytic functions of the above coefficients and, consequently, as real analytic functions of u, v, w, U, V, W and their admissible derivatives in the neighborhood of the set $(u_0, v_0, w_0, a_i, a_{i,j}, b_{i,j})$ of initial values.

Then the Cauchy–Kowalewski Theorem can be applied to the system $\{(10), (12), (14)\}$ of PDE and the proof of Theorem 2 is completed. □

The proof of Theorem 1 now follows at once from the second part of Theorem 3. □

Remark 1. The same arguments which we used in the proof of Theorem 1 work also for the proof of the first part of Theorem 3 ! In the latter case, we are looking for a coordinate transformation $x = x(u^1, u^2, u^3)$, $y = y(u^1, u^2, u^3)$, $z = z(u^1, u^2, u^3)$ taking a general metric $g = \sum_{i,j=1}^{3} g_{ij} du^i du^j$ into a diagonal form. Here, we obtain a nonlinear PDE system of *first* order for 3 unknown functions. We need all the basic steps here as well (first a linear transformation of coordinates to ensure the applicability of the Cauchy–Kowalewski Theorem in the standard form, and, at the very end, the elementary "geometric analysis"). Instead of ensuring intersection of one quadratic curve and one cubic curve, we need in the latter case only to ensure intersection of two quadratic curves. Moreover, all the computations are much more simple and a computer aid is not needed at all.

Remark 2. The situation changes dramatically if two of the prescribed Ricci eigenvalues are asked to be equal. Consider the characteristic matrix $[\lambda I - \mathrm{Ric}]$ and substitute for λ the prescribed double Ricci eigenvalue $\varrho_1 = \varrho_2$. Then, the specified matrix has rank one and hence all sub-determinants of degree two must vanish. Because the matrix is symmetric, these conditions are obviously reduced to three independent algebraic conditions for the Ricci components Ric^i_j. We obtain three new PDE, which are of order 2 and of degree 2. Obviously, at least one of these new PDE is independent of (2). Hence we obtain an overdetermined system of PDE and the Cauchy–Kowalewski Theorem cannot be applied. We shall give a short survey about such kind of geometric problems, earlier results and corresponding methods in the last section.

Recall that we are always looking for a *geometrical* solution, i.e., we want to "parametrize" the local moduli space of Riemannian metrics for the given problem.

From this point of view, we shall see that a notion "overdetermined" and "underdetermined" PDE system has only a relative meaning, depending on the approach and method used in the particular situation.

3 The case of constant distinct Ricci eigenvalues

In [17], the first author and F. Prüfer solved the following problem: For every prescribed numbers $\varrho_1 > \varrho_2 > \varrho_3$, write down an explicit Riemannian metric g such that its Ricci eigenvalues are constant and equal to ϱ_i. A broad family of examples (so-called "generalized Yamato spaces") was constructed there. Moreover, in [18], a geometrical characterization of this family was given inside the set of all Riemannian metrics with prescribed Ricci eigenvalues as above.

In this paper, we present, for each prescribed $\varrho_1 > \varrho_2 > \varrho_3$, a particularly simple example.

Theorem 4. *Consider fixed constants $\varrho_1 > \varrho_2 > \varrho_3$ and define the new constants α, λ_i and b as follows:*

$$\begin{aligned}
&\alpha = \frac{\varrho_1 - \varrho_3}{\varrho_3 - \varrho_2} < 0,\\
&\lambda_i = (\varrho_1 + \varrho_2 + \varrho_3)/2 - \varrho_i,\ i = 1, 2, 3, \\
&b = \frac{1}{\alpha+1}\left\{-\alpha\lambda_2 + \frac{\alpha+2}{\alpha}\left((\alpha+1)\lambda_3 + \lambda_2\right)\right\} = \frac{(\varrho_3 - \varrho_2)(\varrho_1 + \varrho_3)}{\varrho_1 - \varrho_3}.
\end{aligned} \tag{19}$$

Further, define a function $a^1_{21}(w)$ as follows:

$$\begin{aligned}
&\text{(i)} \quad a^1_{21}(w) = -\frac{1}{\alpha\, w} &&\text{for } b = 0,\\
&\text{(ii)} \quad a^1_{21}(w) = \sqrt{\frac{b}{\alpha}}\tan\left(\sqrt{\alpha b}\, w\right) &&\text{for } b < 0,\\
&\text{(iii)} \quad a^1_{21}(w) = \sqrt{\frac{b}{|\alpha|}}\tanh\left(\sqrt{|\alpha| b}\, w\right) &&\text{for } b > 0.
\end{aligned} \tag{20}$$

Let $I(w)$ be a maximal open interval on which $\left(a^1_{21}(w)\right)^2 > -\lambda_2/|\alpha+1|$. Define other functions $a^i_{jk}(w)$ for $i, j, k = 1, 2, 3$ on $I(w)$ in a unique way so that $a^i_{jk}(w) + a^j_{ik}(w) = 0$ and

$$\begin{aligned}
(\alpha+1)(a^1_{21})^2 + (a^1_{23})^2 = \lambda_2, \quad a^1_{23} > 0,&\\
-\alpha\, a^1_{23} a^1_{32} = (\alpha+1)\lambda_3 + \lambda_2,&\\
a^1_{22} = 0, \quad a^1_{31} = 0, \quad a^2_{31} = -a^1_{23},&\\
a^1_{33} = 0, \quad a^2_{32} = 0, \quad a^2_{33} = (\alpha+1)\, a^1_{21}.&
\end{aligned} \tag{21}$$

Then, the metric $g = \sum_{i=1}^{3} \left(\omega^i\right)^2$ *defined on the strip* $I(w) \times \mathsf{R}^2[x, y] \subset \mathsf{R}^3[w, x, y]$ *by the orthonormal coframe*

$$\begin{aligned} \omega^1 &= \left[\left(a^1_{32} - a^1_{23}\right) y - a^1_{21} x\right] dw + dx, \\ \omega^2 &= dw, \\ \omega^3 &= dy + \left[(\alpha + 1)\, a^1_{21} y - \left(a^1_{32} + a^1_{23}\right) x\right] dw, \end{aligned} \tag{22}$$

has the following properties:

1) *The Ricci eigenvalues of g are* $\varrho_1 > \varrho_2 > \varrho_3$.
2) *The corresponding Christoffel symbols* Γ^i_{jk} *of g are the functions* $a^i_{jk}(w)$.
3) *The metric g is not locally homogeneous.*

Remark 3. The definition of the constant b in (19) is correct and the first equation (21) is always solvable because $\alpha + 1 = (\varrho_1 - \varrho_2)/(\varrho_3 - \varrho_2) < 0$. If $b > 0$ and $\lambda_2 > 0$, then we can put $I(w) = (-\infty, +\infty)$ and the metric g is defined on R^3.

Outline of the proof of Theorem 4. Instead of a direct check (which is a rather non-trivial task) we shall prove our Theorem on a broader background of "generalized Yamato spaces" as presented in [17] and [18]. (See Theorem 5 and Proposition 1 below).

Let (M, g) be a Riemannian 3-manifold of class C^∞ with distinct constant Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$. Choose an open domain $\mathcal{U} \subset M$ and a smooth orthonormal moving frame $\{E_1, E_2, E_3\}$ consisting of the corresponding Ricci eigenvectors at each point of $\mathcal{U}$. Denoting by R_{ijkl} and R_{ij} the corresponding covariant components of the curvature tensor and of the Ricci form respectively, we obtain,

$$\begin{aligned} &R_{ii} = \varrho_i \ (i = 1, 2, 3), \qquad R_{ij} = 0 \quad \text{for } i \neq j, \\ &R_{1212} = \lambda_3,\ R_{1313} = \lambda_2,\ R_{2323} = \lambda_1,\ \text{where } \lambda_i \text{ are constants}, \\ &R_{ijkl} = 0 \quad \text{if at least three indices are distinct.} \end{aligned} \tag{23}$$

Moreover, the numbers λ_i are related to the numbers ϱ_i by the middle formula of (19) and we obviously get

$$\lambda_i - \lambda_j = -(\varrho_i - \varrho_j),\ i, j = 1, 2, 3. \tag{24}$$

In a neighborhood $\mathcal{U}_m$ of any point $m \in \mathcal{U}$, one can construct a local coordinate system (w, x, y) such that

$$E_3 = \frac{\partial}{\partial y} \quad \text{on } \mathcal{U}_m. \tag{25}$$

Consider the orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ which is dual to $\{E_1, E_2, E_3\}$. Then, the coordinate expression of the coframe $\{\omega^1, \omega^2, \omega^3\}$ in $\mathcal{U}_m$ must be of the form

$$\begin{aligned} \omega^1 &= A\, dw + B\, dx, \\ \omega^2 &= C\, dw + D\, dx, \\ \omega^3 &= dy + G\, dw + H\, dx, \end{aligned} \tag{26}$$

where A, B, C, D, G, H are unknown functions to be determined.

Now, using the calculus of exterior forms and the standard structural equations for the connection form and the curvature form (cf. [6, 12]) one can derive the expressions for the components a^i_{jk} of the Levi-Civita connection with respect to the given frame. First, we introduce new functions $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ (where $\mathcal{D} \neq 0$) by

$$\mathcal{D} = AD - BC,\ \mathcal{E} = AH - BG,\ \mathcal{F} = CH - DG. \tag{27}$$

We also define a bracket of two functions f, g by

$$[f, g] = f'_y g - f g'_y. \tag{28}$$

Then we obtain, by a routine calculation,

$$\begin{aligned}
a^1_{21} &= \frac{1}{\mathcal{D}}(GB'_y - HA'_y + A'_x - B'_w),\ a^1_{31} = \frac{1}{\mathcal{D}}(DA'_y - CB'_y),\\
a^1_{22} &= \frac{1}{\mathcal{D}}(GD'_y - HC'_y + C'_x - D'_w),\ a^2_{32} = \frac{1}{\mathcal{D}}(AD'_y - BC'_y),\\
a^1_{33} &= \frac{1}{\mathcal{D}}(DG'_y - CH'_y),\ a^2_{33} = \frac{1}{\mathcal{D}}(AH'_y - BG'_y),\\
a^1_{23} &= \frac{1}{2\mathcal{D}}\left\{[C, D] + [A, B] - [G, H] + (G'_x - H'_w)\right\},\\
a^2_{31} &= \frac{1}{2\mathcal{D}}\left\{[C, D] - [A, B] + [G, H] - (G'_x - H'_w)\right\},\\
a^1_{32} &= \frac{1}{2\mathcal{D}}\left\{[C, D] - [A, B] - [G, H] + (G'_x - H'_w)\right\}.
\end{aligned} \tag{29}$$

From the structural equations for the connection form (ω^i_j) and for the curvature form (Ω^i_j), using the curvature conditions (23) and the subsequent exterior differentiation, we obtain the following relations for the Christoffel symbols a^i_{jk}:

$$a^2_{32} = \alpha\, a^1_{31},\ a^2_{33} = (\alpha + 1)\, a^1_{21},\ a^1_{33} = -\left(\frac{\alpha + 1}{\alpha}\right) a^1_{22}, \tag{30}$$

where α is the constant introduced in (19).

Now, assuming (30), the formulas (29) are equivalent to the following system of nine PDE for six basic Christoffel symbols a^1_{21}, a^1_{22}, a^1_{31}, a^1_{23}, a^2_{31} and a^1_{32}:

$$\begin{aligned}
A'_y &= A\, a^1_{31} + C\,(a^1_{32} - a^1_{23}),\\
B'_y &= B\, a^1_{31} + D\,(a^1_{32} - a^1_{23}),\\
C'_y &= A\,(a^1_{23} + a^2_{31}) + \alpha\, C a^1_{31},\\
D'_y &= B\,(a^1_{23} + a^2_{31}) + \alpha\, D a^1_{31},\\
G'_y &= (\alpha + 1)\, C a^1_{21} - \frac{\alpha + 1}{\alpha}\, A\, a^1_{22},\\
H'_y &= (\alpha + 1)\, D a^1_{21} - \frac{\alpha + 1}{\alpha}\, B\, a^1_{22}.
\end{aligned} \tag{31}$$

$$\begin{aligned} A'_x - B'_w &= \mathcal{D}a^1_{21} + \mathcal{E}a^1_{31} + \mathcal{F}(a^1_{32} - a^1_{23}), \\ C'_x - D'_w &= \mathcal{D}a^1_{22} + \mathcal{E}(a^1_{23} + a^2_{31}) + \alpha\mathcal{F}\,a^1_{31}, \\ G'_x - H'_w &= \mathcal{D}(a^1_{32} - a^2_{31}) - \frac{\alpha+1}{\alpha}\,\mathcal{E}\,a^1_{22} + (\alpha+1)\,\mathcal{F}\,a^1_{21}. \end{aligned} \tag{32}$$

Next, we express explicitly the geometric curvature conditions (23). Using again the structural equations for the curvature form (Ω^i_j), we obtain after lengthy but routine calculations the following system of nine PDE's for all nine Christoffel symbols, still having in mind the relations (30):

$$\begin{aligned} &A(a^1_{21})'_x - B(a^1_{21})'_w + C(a^1_{22})'_x - D(a^1_{22})'_w + G(a^1_{23})'_x - H(a^1_{23})'_w \\ &\qquad\qquad - \mathcal{D}(U_3 - \lambda_3) - \mathcal{E}V_3 - \mathcal{F}W_3 = 0, \\ &A(a^1_{21})'_y + C(a^1_{22})'_y + G(a^1_{23})'_y - (a^1_{23})'_w - AV_3 - CW_3 = 0, \\ &B(a^1_{21})'_y + D(a^1_{22})'_y + H(a^1_{23})'_y - (a^1_{23})'_x - BV_3 - DW_3 = 0, \\ &A(a^1_{31})'_x - B(a^1_{31})'_w + C(a^1_{32})'_x - D(a^1_{32})'_w + G(a^1_{33})'_x - H(a^1_{33})'_w \\ &\qquad\qquad - \mathcal{D}U_2 - \mathcal{E}(V_2 - \lambda_2) - \mathcal{F}W_2 = 0, \\ &A(a^1_{31})'_y + C(a^1_{32})'_y + G(a^1_{33})'_y - (a^1_{33})'_w - A(V_2 - \lambda_2) - CW_2 = 0, \\ &B(a^1_{31})'_y + D(a^1_{32})'_y + H(a^1_{33})'_y - (a^1_{33})'_x - B(V_2 - \lambda_2) - DW_2 = 0, \\ &A(a^2_{31})'_x - B(a^2_{31})'_w + C(a^2_{32})'_x - D(a^2_{32})'_w + G(a^2_{33})'_x - H(a^2_{33})'_w \\ &\qquad\qquad - \mathcal{D}U_1 - \mathcal{E}V_1 - \mathcal{F}(W_1 - \lambda_1) = 0, \\ &A(a^2_{31})'_y + C(a^2_{32})'_y + G(a^2_{33})'_y - (a^2_{33})'_w - AV_1 - C(W_1 - \lambda_1) = 0, \\ &B(a^2_{31})'_y + D(a^2_{32})'_y + H(a^2_{33})'_y - (a^2_{33})'_x - BV_1 - D(W_1 - \lambda_1) = 0. \end{aligned} \tag{33}$$

Here, we put (using again only the "basic" six Christoffel symbols)

$$\begin{aligned} U_1 &= \alpha a^1_{21}a^2_{31} - (\alpha-1)a^1_{22}a^1_{31} - (\alpha+2)a^1_{21}a^1_{32}, \\ V_1 &= \frac{(\alpha+1)(\alpha+2)}{\alpha}a^1_{21}a^1_{22} - (\alpha+1)a^1_{31}a^2_{31} - (\alpha-1)a^1_{31}a^1_{23}, \\ W_1 &= \frac{\alpha+1}{\alpha}(a^1_{22})^2 - (\alpha+1)^2(a^1_{21})^2 - \alpha^2(a^1_{31})^2 + a^1_{23}a^2_{31} - a^1_{32}a^2_{31} + a^1_{32}a^1_{23}, \\ U_2 &= \frac{1}{\alpha}a^1_{22}a^1_{32} + (\alpha-1)a^1_{21}a^1_{31} - \frac{2\alpha+1}{\alpha}a^1_{22}a^2_{31}, \\ V_2 &= (\alpha+1)(a^1_{21})^2 - (a^1_{31})^2 - \left(\frac{\alpha+1}{\alpha}\right)^2(a^1_{22})^2 - a^1_{32}a^1_{23} - a^1_{32}a^2_{31} - a^1_{23}a^2_{31}, \end{aligned} \tag{34}$$

$$W_2 = (1-\alpha)a_{23}^1 a_{31}^1 - (\alpha+1)a_{32}^1 a_{31}^1 + \frac{(2\alpha+1)(\alpha+1)}{\alpha} a_{22}^1 a_{21}^1,$$

$$U_3 = -(a_{21}^1)^2 - (a_{22}^1)^2 - \alpha(a_{31}^1)^2 + a_{23}^1 a_{31}^2 - a_{23}^1 a_{32}^1 + a_{32}^1 a_{31}^2,$$

$$V_3 = \frac{1}{\alpha} a_{22}^1 a_{23}^1 - (\alpha+2)a_{21}^1 a_{31}^1 - \frac{2\alpha+1}{\alpha} a_{22}^1 a_{31}^2,$$

$$W_3 = -\alpha a_{21}^1 a_{23}^1 - (\alpha+2)a_{21}^1 a_{32}^1 - (2\alpha+1)a_{22}^1 a_{31}^1.$$

By the detailed analysis of the system of 18 PDE, (31)–(33) for 12 unknown functions $A, B, \dots, H, a_{21}^1, a_{22}^1, \dots, a_{32}^1$, the following result was obtained in [17]. (Here we present the more convenient local version of the corresponding theorem.)

Theorem 5. *Let a triplet $\varrho_1 > \varrho_2 > \varrho_3$ of constant Ricci eigenvalues be prescribed. Let a_{jk}^i be functions on $\mathcal{U} \subset \mathsf{R}^2[w, x]$ satisfying the following conditions:*

(N1) $a_{31}^1 = 0,\ a_{23}^1 + a_{31}^2 = 0,\ a_{22}^1 = 0,$

(N2) a_{23}^1 *is an arbitrary function of class C^∞ on $\mathcal{U} \subset \mathsf{R}^2[w, x]$ such that*

(a) $(a_{23}^1)'_w \neq 0,\ a_{23}^1 > 0,$

(b) $(a_{23}^1)^2 > \max\left\{\lambda_2, (\alpha+2)\left[(\alpha+1)\lambda_3 + \lambda_2\right]/\alpha^2\right\},$

(N3) $(\alpha+1)(a_{21}^1)^2 + (a_{23}^1)^2 = \lambda_2,\ a_{21}^1 > 0,$

(N4) $-\alpha\, a_{23}^1\, a_{32}^1 = (\alpha+1)\lambda_3 + \lambda_2.$

Then, there exist smooth functions A, B, C, D, G, H on $\mathcal{U} \times \mathsf{R}[y] \subset \mathsf{R}^3[w, x, y]$ (depending on two arbitrary functions of two variables and two arbitrary functions of one variable) such that the basic system of partial differential equations (31)–(33) *is satisfied.*

We shall now specify these functions. First, look at the function W_3 defined in (34). One can calculate explicitly from (N3) and (N4) that,

$$W_3 = f(a_{23}^1) = \sqrt{\frac{(a_{23}^1)^2 - \lambda_2}{|\alpha+1|}} \left(\alpha\, a_{23}^1 + (\alpha+2) \frac{|\alpha+1|\lambda_3 - \lambda_2}{\alpha\, a_{23}^1} \right). \tag{35}$$

We see that the inequalities in (N2)(b) just ensure that $f(a_{23}^1)$ is non-zero everywhere in our domain $\mathcal{U}$ (but this can be always assumed in our local case, because $(a_{23}^1)'_w \neq 0$).

Define now C, D as functions on $\mathcal{U}$ by,

$$C = -\frac{(a_{23}^1)'_w}{f(a_{23}^1)}, \quad D = -\frac{(a_{23}^1)'_x}{f(a_{23}^1)}. \tag{36}$$

It is shown in [17], that, if the Christoffel symbols are defined by (N1)–(N4) and the functions C, D are defined by (36), then for arbitrary choice of the functions A, B, G, H all PDE's (33) are satisfied. Further, the following was proved.

Proposition 1. *To satisfy the remaining* PDE (31) *and* (32)*, it is sufficient to define the functions* A, B, G, H *by*

$$A = C(a^1_{32} - a^1_{23})y + A_0(w, x), \ \ B = D(a^1_{32} - a^1_{23})y + B_0(w, x),$$
$$G = (\alpha + 1)\, C\, a^1_{21} y + G_0(w, x), \ \ H = (\alpha + 1)\, D\, a^1_{21} y + H_0(w, x), \tag{37}$$

where A_0, B_0, G_0, H_0 *are functions of class* C^∞ *on* $\mathcal{U} \subset \mathrm{R}^2[w, x]$ *satisfying the equations*

$$(A_0)'_x - (B_0)'_w = (DA_0 - CB_0)\, a^1_{21} + (DG_0 - CH_0)\left(a^1_{23} - a^1_{32}\right), \tag{38}$$
$$(G_0)'_x - (H_0)'_w = (DA_0 - CB_0)\left(a^1_{32} + a^1_{23}\right) - (DG_0 - CH_0)\,(\alpha + 1)\, a^1_{21}.$$

To obtain the explicit examples announced in Theorem 4, let us suppose that a^1_{23} depends on the variable w only and put $B_0 = 1, H_0 = 0$. Then, $C = -(a^1_{23})'(w)/f(a^1_{23}) \neq 0$ depends on w only and $D = 0, B = 1, H = 0$. For A and G we get explicit solutions

$$A = C\,(a^1_{32} - a^1_{23})y - C\, a^1_{21} x, \tag{39}$$
$$G = (\alpha + 1)\, C\, a^1_{21} y - C\,(a^1_{32} + a^1_{23})x.$$

It remains to verify that the formulas (19)–(22) in Theorem 4 follow from the previous ones and to make the final conclusions. First we see that if we solve the differential equation $(a^1_{23})'(w) = -f(a^1_{23})$, then the function $a^1_{23}(w)$ will be specified so that $C = 1$. If we pass from $a^1_{23}(w)$ to $a^1_{21}(w)$, we obtain a much simpler equation

$$(a^1_{21})'(w) = \alpha(a^1_{21})^2 + b. \tag{40}$$

Hence, the formulas (20) follow at once (neglecting the integration constant here). Further, we recall that the PDE system (33) is equivalent to the statement that $\varrho_1 > \varrho_2 > \varrho_3$ are corresponding Ricci eigenvalues and the PDE system (31) + (32) together with (30) says that $a^i_{jk}(w)$ defined by (21) are the corresponding Christoffel symbols. Finally, because the Christoffel symbols a^i_{jk} are calculated with respect to a Ricci-adapted frame (which is determined uniquely up to reflections at each point), and because not all a^i_{jk} are constant, the space (M, g) cannot be locally homogeneous. □

Remark 4. For the prescribed constant Ricci eigenvalues $\varrho_1, \varrho_2, \varrho_3$, (even if they are not all distinct) there is *not always* a locally homogeneous space with such Ricci eigenvalues. Some necessary conditions were given in [27] and the complete answer can be found in [16].

Remark 5. The 3-dimensional Riemannian manifolds with constant Ricci eigenvalues belong to the broader family of so-called *curvature homogeneous spaces*. See, e.g., [1, 3, 23–25, 28, 29] and, in particular, a survey in [4]. This topic was developed with strong participation of F. Tricerri and L. Vanhecke; it was originally motivated by a conjecture of M. Gromov.

We also proved the following in [26]:

Theorem 6. *The general solution of the PDE system (31)–(33) depends on six arbitrary functions of two variables and six arbitrary functions of one variable.*

The proof depends strongly on the computer aid because one has to show that all integrability conditions coming from this PDE system are consequences of the original PDE's. This is a hard computer work which is not very transparent and difficult to check by hand. After showing this, one can use the Cauchy–Kowalewski Theorem in two successive steps to obtain the result.

Now we have the following geometric existence theorem which we reproduce in full from [17], including its short proof.

Theorem 7. *The isometry classes of germs of three-dimensional (real analytic) Riemannian metrics with prescribed constant Ricci eigenvalues are parametrized by triplets of germs of arbitrary (real analytic) functions of two variables.*

Proof. Let (M, g), $(\overline{M}, \overline{g})$ be two real analytic Riemannian 3-manifolds with the same constant Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$. Let $F : \mathcal{U} \to \overline{\mathcal{U}}$ be an isometry between two open domains of M and $\overline{M}$ respectively. We construct the "Ricci adapted" orthonormal coframes $\{\omega^i\}$, $\{\overline{\omega}^i\}$ and the local coordinate systems (w, x, y), $(\overline{w}, \overline{x}, \overline{y})$ in the neighborhoods $\mathcal{U}_m \subset \mathcal{U}$ and $\overline{\mathcal{U}}_{F(m)} = F(\mathcal{U}_m) \subset \overline{\mathcal{U}}$ respectively, such that g and $\overline{g}$ are both of the form (26). We obviously have

$$F^*(\overline{\omega}^i) = \varepsilon_i \omega^i, \quad \varepsilon_i \in \{-1, 1\}, \quad i = 1, 2, 3. \tag{41}$$

Hence, we see easily that the corresponding parametrization of F in local coordinates must be of the form,

$$\overline{w} = \Phi_1(w, x), \quad \overline{x} = \Phi_2(w, x), \quad \overline{y} = \varepsilon y + \Phi_3(w, x), \tag{42}$$

where $\varepsilon = \pm 1$ and $\Phi_i(w, x)$ are arbitrary (real analytic) functions of two variables. Conversely, every local transformation F of the form (42) determines a local isometry which preserves the formulas (26) through (41). The result now follows from Theorem 6. $\square$

Let us notice that we neglect here six arbitrary functions of one variable. This is fully justified because, for the geometric conclusions, these functions are not relevant.

Remark 6. Looking at the proof carefully, we see that the same argument also works when ϱ_i are not constants but arbitrary functions! Hence, we have an alternative way to derive Theorem 1 from Theorem 2 where we don't need the second part of the "diagonalization theorem" 3.

Open problem. It is not known to the authors if an explicit construction as in Theorem 4 can be extended to non-constant Ricci eigenvalues.

4 Related problems with curvature restrictions

4.1 The Schur's theorem

Consider prescribed Ricci eigenvalues $\varrho_1(x, y, z)$, $\varrho_2(x, y, z)$, $\varrho_3(x, y, z)$ on (M, g) which are *all equal* and consider the corresponding system of partial differential equations (2). In this case, we have to add three new independent PDE's, namely $\mathrm{Ric}^i_j = 0$ for $1 \le i < j \le 3$, and the system of equations becomes strongly overdetermined. According to the Schur's Theorem, (M, g) must be a space of constant curvature. Hence the local moduli space depends only on one parameter. As a consequence of Theorem 3, the general solution of the corresponding overdetermined system depends on three arbitrary functions of two variables (and possibly, on some functions of one variable and some parameters – we shall not repeat this stipulation in the sequel).

4.2 The pseudo-symmetric spaces of constant type

A 3-dimensional *pseudo-symmetric space of constant type* is characterized by the following properties (cf. [7], [8], [20]–[22] and [4], Chap. 11): One of the Ricci eigenvalues is prescribed as a constant and the other two Ricci eigenvalues are required to be equal but arbitrary. (If the constant eigenvalue is zero, the space is said to be *semi-symmetric* (see [2], [3], [14], [31]–[32], and, in particular, [4] for more information). Then, we have only two PDE for the coefficients c_i of the Ricci characteristic polynomial but there are additional three quadratic equations for the Ricci components Ric^i_j involving an *arbitrary* function. Eliminating this arbitrary function, we are left with two additional PDEs, which are biquadratic. This system is not easy to analyse. Yet, using a different approach, we come to some satisfactory and surprising results.

Let us start with a 3-dimensional Riemannian manifold (M, g) whose Ricci tensor has the eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$ with constant ϱ_3. One can construct easily, in a neighborhood of any fixed point $m \in M$, a Ricci adapted orthonormal moving frame $\{E_1, E_2, E_3\}$ and a system (w, x, y) of local coordinates such that $E_3 = \partial/\partial y$. We shall also consider the dual coframe $\{\omega^1, \omega^2, \omega^3\}$.

The Ricci tensor $\widehat{R}$ expressed with respect to $\{E_1, E_2, E_3\}$ has the form $\widehat{R}_{ij} = \varrho_i \delta_{ij}$. Because each ϱ_i is expressed through the sectional curvature K_{ij} by the formula $\varrho_i = \widehat{R}_{ii} = \sum_{j \neq i} K_{ij}$, there exist a function $k = k(w, x, y)$ of the variables w, x and y, and a constant $\tilde{c}$ such that

$$\begin{aligned} K_{12} = k,\ K_{13} = K_{23} = \tilde{c}, \\ \varrho_1 = \varrho_2 = k + \tilde{c},\ \varrho_3 = 2\,\tilde{c}. \end{aligned} \tag{43}$$

From the structural equations for the connection form (ω^i_j) and for the curvature form (Ω^i_j), using the curvature conditions (43), we obtain after a simple manipulation with the corresponding exterior differential forms ω^i, ω^i_j the following results:

Proposition 2. *In a normal neighborhood of any point $m \in M$ there exist an orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ and a local coordinate system (w, x, y) such that*

$$\begin{aligned}\omega^1 &= f\,dw,\\ \omega^2 &= A\,dx + C\,dw,\\ \omega^3 &= dy + H\,dw.\end{aligned} \tag{44}$$

Here f, A and C are smooth functions of the variables w, x and y, $fA \neq 0$, and H is a smooth function of the variables w and x.

Moreover, $fA = \sigma/(k - \tilde{c})$ for some non-zero function $\sigma = \sigma(w, x)$.

Next, we obtain easily the following expression for the components of the connection form:

$$\begin{aligned}\omega_2^1 &= -A\alpha\,dx + R\,dw + \beta\,dy,\\ \omega_3^1 &= A\beta\,dx + S\,dw,\\ \omega_3^2 &= A'_y\,dx + T\,dw,\end{aligned} \tag{45}$$

where

$$\begin{aligned}\alpha &= \chi(A'_w - C'_x - HA'_y),\\ \beta &= \frac{\chi}{2}(H'_x + AC'_y - CA'_y),\end{aligned} \tag{46}$$

and

$$\begin{aligned}R &= \chi f f'_x - C\alpha + H\beta,\\ S &= f'_y + C\beta,\\ T &= C'_y - f\beta,\end{aligned} \tag{47}$$

putting here χ for $1/fA$. The curvature conditions (43) (when used in the structural equations for the curvature form) then give a system of nine PDE's for our problem:

$$\begin{aligned}(A\alpha)'_y + \beta'_x &= 0,\\ R'_y - \beta'_w &= 0,\\ (A\alpha)'_w + R'_x + SA'_y - A\beta T &= -fAk,\\ A''_{yy} - A\beta^2 &= -\tilde{c}A,\\ -A''_{yw} + T'_x + A(\beta R + \alpha S) &= \tilde{c}AH,\\ T'_y - S\beta &= -\tilde{c}C,\\ (A\beta)'_y + A'_y\beta &= 0,\\ S'_x - (A\beta)'_w - (A\alpha T + A'_y R) &= 0,\\ S'_y + T\beta &= -\tilde{c}f.\end{aligned} \tag{48}$$

This is a reasonable PDE system, because two of the equations are consequences of the others and for the remaining equations we obtain a number of nice "first integrals" (like formulas (49)–(51) below).

Now, an important tool how to simplify the system (48) is the notion of *asymptotic leaf*. It is defined as a surface $N \subset M$ generated by the principal ϱ_3-lines and such that its tangent distribution is parallel along each principle ϱ_3-line in (M, g). (Here, naturally, principal ϱ_3-lines are integral curves of the local vector field E_3. They are known to be geodesic lines in (M, g).)

Now, the following result can be proved with some effort:

Proposition 3. *For any point $p \in M$ there are just four possibilities:*

a) *There is no asymptotic leaf through p ("elliptic point").*
b) *There are just two asymptotic leafs through p ("hyperbolic point").*
c) *There is just one asymptotic leaf through p ("parabolic point").*
d) *There are infinitely many asymptotic leafs through p ("planar point").*

We call a (local) space (M, g) to be *of elliptic type* if it consists of elliptic points only. Similarly, we define spaces of *hyperbolic, parabolic* and *planar type*. Thus, on such kind of spaces we can consider *asymptotic foliations*. If the space is not elliptic, at least one asymptotic foliation exists and one can define a new local coordinate system (w, x, y) such that, in addition, the integral manifolds of the equation $dw = 0$ are asymptotic leafs. Then a dramatic simplification of the system (48) takes place, enabling to write down the general solution in the explicit form!

One has the following main results ([10], [14], [20]–[22] and [4]) proved by the first author and M. Sekizawa:

A) The local moduli space of all spaces of elliptic type (or of hyperbolic type, or of parabolic type, or of planar type respectively) is parametrized by 3 arbitrary functions of 2 variables (or by 3, or by 2, or by 1 arbitrary functions of 2 variables respectively). Hence the corresponding "overdetermined" system of PDE for the Ricci components Ric^i_j is not really overdetermined because it has a general solution depending on 6 arbitrary functions of 2 variables — the same result as for the system (2) with distinct prescribed Ricci eigenvalues.
B) The local moduli space of all spaces of non-elliptic types can be expressed by a finite number of explicit formulas involving only algebraic operations, elementary functions, differentiation, integration, and depending explicitly on the corresponding number of arbitrary functions of two variables.
This is a rare phenomenon in the theory of nonlinear PDE systems.
C) The double Ricci eigenvalue, which was supposed to be arbitrary, is in fact not arbitrary! It must be of the form

$$\varrho_1 = \varrho_2 = \frac{1}{k_1 y^2 + k_2 y + k_3} \quad \text{for } \varrho_3 = 0, \tag{49}$$

$$\varrho_1 = \varrho_2 = \frac{1}{k_1 \cos(\lambda y) + k_2 \sin(\lambda y) + k_3} + 2\lambda^2 \quad \text{for } \varrho_3 = 2\lambda^2 > 0, \tag{50}$$

$$\varrho_1 = \varrho_2 = \frac{1}{k_1 \cos(\lambda y) + k_2 \sin(\lambda y) + k_3} - 2\lambda^2 \quad \text{for } \varrho_3 = -2\lambda^2 > 0, \tag{51}$$

where k_1, k_2, k_3 are arbitrary functions of 2 variables w, x. We are somehow on the “halfway” to the Schur’s Theorem.

4.3 The semi-symmetric spaces of elliptic type with the *prescribed* non-constant double Ricci eigenvalue

The prescribed eigenvalue must be of the form (49). The problem was investigated in [14], pp. 471–474. The local moduli space depends here on one arbitrary function of 2 variables.

The corresponding system of PDE’s for the Ricci components is again overdetermined and its general solution depends on 4 arbitrary functions of two variables.

4.4 The case of constant Ricci eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$

Here we have a specialized PDE system (48) in which k is a constant. As we mentioned in the Introduction (see [15], [5], [19]), the local moduli space of all possible metrics depends on 2 arbitrary functions of 1 variable.

The PDE system for the Ricci components is again “strongly overdetermined” and the general solution depends only on 3 arbitrary functions of 2 variables.

Notice that the local moduli space here is “much smaller” than in the case of three distinct constant Ricci eigenvalues! This is obviously due to the fact that the corresponding PDE system (2) gets overdetermined by adding new equations.

4.5 The 3-dimensional Riemannian manifolds with two zero Ricci eigenvalues and one arbitrary Ricci eigenvalue

The corresponding PDE system for the Ricci characteristic polynomial is here *rather special*. In fact, we get the conditions $c_1 = 0$, $c_0 = 0$ and the additional equations saying that the 2-dimensional sub-determinants of the matrix $[\mathrm{Ric}^i_j]$ are zero. It might be an interesting problem to solve the corresponding PDE system in order to obtain the information about general solution.

One can also proceed like in the subsection 4.2, and to write down a system of 9 PDE’s of second order. But this system is very hard to solve and the “parametrization problem” for the moduli space still remains open.

The problem was raised, in fact, for general dimension by S. Ivanov and I. Petrova in [11] when the authors studied “the spaces with pointwise constant curvature eigenvalues” (in fact, eigenvalues of the skew-symmetric curvature operator $R(X, Y)$. The classification problem was solved completely by the above authors in dimension 4 and later by P. Gilkey, J. Leahy and H. Sadofsky in the higher dimensions except $n = 7$ and $n = 8$. Yet, it still remains open in dimension 3 (which is just the case described in the title of this paragraph – see Remark 2 and Remark 3 in the Introduction of [11]).

The only known results are isolated examples of the above spaces:

A) The group $SU(3)$ with a special left-invariant metric (see [27] and Remark 2 in [11]).

B) The metrics of the form

$$g = \frac{1}{p^2}\ e^{-2\lambda y} dw^2 + [p\ e^{\lambda y} dx + (r\ e^{\lambda y} + s\ e^{-\lambda y}) dw]^2 + dy^2, \tag{52}$$

where $p = p(w)$, $s = s(w)$, $r = -\lambda^2 p^2(w) s(w) x^2 + p'(w) x + \psi(w)$, and $p(w), s(w), \psi(w)$ are arbitrary functions. Here, $\varrho_1 = \varrho_2 = 0$, $\varrho_3 = -2\lambda^2$. These metrics are not locally homogeneous. (See [15], Example 5.8.)

C) The example by Ivanov–Petrova: (M, g) is a warped product $M^3 = B^1 \times_f N^2$, where $B^1 = B^1(y)$ is a real line, N^2 is a space form of constant curvature K, and the warping function $f(y)$ is $\sqrt{Ky^2 + Cy + D}$ with constant C, D such that $C^2 - 4KD \neq 0$. The Ricci eigenvalues are $\left(0,\ 0,\ \frac{1}{2}(C^2 - 4KD)/(Ky^2 + Cy + D)^2\right)$.

D) The new example found by V. Hájková and O. Kowalski:

$$g = y^{2p} dw^2 + y^{2(1-p)} dx^2 + dy^2, \text{ where } p \text{ is a parameter.} \tag{53}$$

Here, $\varrho_1 = \varrho_2 = 0$ and $\varrho_3 = 2p(1-p)/y^2$. Further, $p(1-p)$ is a Riemannian invariant and the case $p = 1/2$ corresponds to the example C) for the particular choice $K = 0,\ C = 1,\ D = 0$.

E) (Added in proof). See Y. Nikolayevsky, On Riemannian manifolds cohose skew-symmetric curvature operator has constant curvature, preprint, to appear in *Bull. Austral. Math. Soc.*, 2004.

Acknowledgement

The authors are obliged to E. V. Ferapontov for bringing Theorem 3 to their attention and suggesting a fruitfull approach based on the Ricci characteristic polynomial.

References

[1] Ambrose, W., Singer, I.: On homogeneous Riemannian manifolds. *Duke Math. J.*, **25**, 647–669 (1958).

[2] Boeckx, E.: Foliated semi-symmetric spaces. Doctoral Thesis, University Leuven, Leuven (1994).

[3] Boeckx, E., Kowalski, O., Vanhecke, L.: Non-homogeneous relatives of symmetric spaces. *Diff. Geom. Appl.*, **4**, 45–69 (1994).

[4] Boeckx, E., Kowalski, O., Vanhecke, L.: Riemannian Manifolds of Conulity Two. World Scientific, Singapore (1996).

[5] Bueken, P.: Three-dimensional Riemannian manifolds with constant principal Ricci curvatures $\varrho_1 = \varrho_2 \neq \varrho_3$. *J. Math. Phys.*, **37**, 4062–4075 (1996).

[6] Cartan, E.: Les systèmes différentiels extérieurs et leurs applications geometriques, 2nd edition. Hermann, Paris (1971).

[7] Deprez, J., Deszcz, R., Verstraelen, L.: Examples of pseudo-symmetric conformally flat warped products. *Chinese J. Math.*, **17**, 51–65 (1989).

[8] Deszcz, R.: On pseudo-symmetric spaces. *Bull. Soc. Math. Belgium, Série A*, **44**, 1–34 (1992).

[9] Darboux, G.: Leçons sur les systèmes orthogonaux et les coordinées curvilignes. Gauthier-Villars, Paris (1898), 1–2.

[10] Hájková, V.: Foliated semi-symmetric spaces in dimension 3 (in Czech). Doctoral Thesis, Charles University Prague, Prague (1995).
[11] Ivanov, S., Petrova, I.: Riemannian manifolds in which the skew-symmetric curvature operator has pointwise constant eigenvalues. *Geom. Dedicata*, **70**, 269–282 (1998).
[12] Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry I. (Interscience Publishers: New York, London, Sydney) (1963).
[13] Kowalski, O.: Nonhomogeneous Riemannian 3-manifolds with distinct constant Ricci eigenvalues. *Comment. Math. Univ. Carolinae*, **34**, 451–457 (1993).
[14] Kowalski, O.: An explicit classification of 3-dimensional Riemannian spaces satisfying $R(X,Y)\cdot R = 0$. *Czech. Math. J.*, **46** (121), 427–474 (1996). Preprint: October 1991.
[15] Kowalski, O.: A classification of Riemannian 3-manifolds with constant pricipal Ricci curvatures $\varrho_1 = \varrho_2 \neq \varrho_3$. *Nagoya Math. J.*, **132**, 1–36 (1993).
[16] Kowalski, O., Nikčević, S. Ž.: On Ricci eigenvalues of locally homogeneous Riemannian 3-manifolds. *Geom. Dedicata*, **62**, 65–72 (1996).
[17] Kowalski, O., Prüfer, F.: On Riemannian 3-manifolds with distinct constant Ricci eigenvalues. *Math. Ann.*, **300**, 17–28 (1994).
[18] Kowalski, O., Prüfer, F.: A classification of special Riemannian 3-manifolds with distinct constant Ricci eigenvalues. *Z. Anal. Anwendungen*, **14**, 43–58 (1995).
[19] Kowalski, O., Sekizawa, M.: Local isometry classes of Riemannian 3-manifolds with constant Ricci eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$. *Archivum Math. (Brno)*, **32**, 137–145 (1996).
[20] Kowalski, O., Sekizawa, M.: Riemannian 3-manifolds with c-conullity two. Bolletino U.M.I. (7), **11**-B, Suppl. fasc. 2, 161–184 (1997).
[21] Kowalski, O., Sekizawa, M.: Pseudo-symmetric spaces of constant type in dimension tree – non elliptic spaces. Bull. Tokyo Gakugei Univ., Sect. IV, *Mathematics and Natural Sciences*, **50**, 1–28 (1998).
[22] Kowalski, O., Sekizawa, M.: Pseudo-symmetric spaces of constant type in dimension tree – elliptic spaces. *Rendiconti di Mathematica, Serie VII*, **17**, 477–512 (1997).
[23] Kowalski, O., Tricerri, F., Vanhecke, L.: New examples of nonhomogeneous Riemannian manifolds whose curvature tensor is that of a Riemannian symmetric space. *C. R. Acad. Sci.* Paris, Série I, **311**, 355–360 (1990).
[24] Kowalski, O., Tricerri, F., Vanhecke, L.: Curvature homogeneous Riemannian manifolds. *J. Math. Pures Appl.*, **71**, 471–501 (1992).
[25] Kowalski, O., Tricerri, F., Vanhecke, L.: Curvature homogeneous spaces with a solvable Lie group as homogeneous model. *J. Math. Soc. Japan*, **44**, 461–484 (1992).
[26] Kowalski, O., Vlášek, Z.: Classification of Riemannian 3-manifolds with distinct constant principal Ricci curvatures. *Bull. Belgian Math. Soc.*, **5**, 59–68 (1998).
[27] Milnor, J.: Curvature of left invariant metrics on Lie groups. *Advances in Math.*, **21**, 293–329 (1976).
[28] Nicolodi, L., Tricerri, F.: On two theorems of I. M. Singer about homogeneous spaces. *Ann. Global Anal. Geom.*, **8**, 193–209 (1990).
[29] Singer, I. M.,: Infinitesimally homogeneous spaces. *Comm. Pure Appl. Math.*, **13**, 685–697 (1960).
[30] Spiro, A., Tricerri, F.: 3-dimensional Riemannian metrics with prescribed Ricci principal curvatures. *J. Math. Pures. Appl.*, **74**, 253–271 (1995).
[31] Szabó, Z. I.: Structure theorems on Riemannian manifolds satisfying $R(X,Y)\cdot R = 0$, I, Local version. *J. Diff. Geom.*, **17**, 531–582 (1982).
[32] Szabó, Z. I.: Structure theorems on Riemannian manifolds satisfying $R(X,Y)\cdot R = 0$, II, Global version. *Geom. Dedicata*, **19**, 65–108 (1985).
[33] Yamato, K.: A characterization of locally homogeneous Riemannian manifolds of dimension 3. *Nagoya Math. J.*, **123**, 77–90 (1991).