On Hermitian Geometry of Complex Surfaces

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Dedicated to Professor Lieven Vanhecke

1 Introduction

The aim of this exposition is to place our recent joint work on anti-self-dual Hermitian surfaces in the more general context of *locally conformal Kähler metrics*—which literally means that the metric is conformal to a Kähler metric, locally. From now on we will adopt the standard notation $l.c.K.$ for these metrics which were introduced and studied by Vaisman in the 1970s.

We start by recalling some preliminaries. Throughout this work S will denote a *smooth* complex surface—a complex manifold of complex dimension 2—with complex structure $J \in \mathrm{A}ut(TM)$ with $J^2 = -id$. A Riemannian metric g on the underlining real four-manifold S is said to be *Hermitian*, if it is compatible with the complex structure in the sense that J acts as an isometry: for all tangent vectors X and Y in TM,

$$g(JX, JY) = g(X, Y).$$

In this situation, we can define a non-degenerate 2-form $\omega \in \Lambda^{1,1}(S)$ usually called the Kähler form of the Hermitian metric by prescribing

$$\omega(X, Y) = g(X, JY),$$

and consider the linear map from one-forms to three-forms defined by taking wedge product with ω

$$\text{Ł} : \Lambda^1(S) \longrightarrow \Lambda^3(S),$$
$$\eta \mapsto \omega \wedge \eta.$$

Using the fact that ω is non-degenerate, the linear map L is always injective and therefore is an isomorfism because S is of real dimension four. We conclude that in this dimension there always is a unique one-form $\theta \in \Lambda^1(S)$ such that,

$$d\omega = \omega \wedge \theta.$$

θ is usually called the *Lee form* of the metric and it is easily seen to satisfy the following properties:

1. $\theta = 0$ – the Lee form vanishes $\iff$ g is a Kähler metric – i.e., $d\omega = 0$.
2. $\theta = df$ – the Lee form is exact $\iff$ the metric $e^{-f}g$ is Kähler – i.e., g is globally conformal Kähler.
3. $d\theta = 0$ – the Lee form is closed (locally exact) $\iff$ g is $l.c.K.$
4. We will also consider the case of *parallel Lee form* $\nabla\theta = 0$ where ∇ is the Levi-Civita connection of g; this of course implies $d\theta = 0$ and therefore it is a special class of $l.c.K.$ metrics also called *generalized Hopf manifolds* by Vaisman [36]. Notice that such surfaces must have vanishing Euler characteristic: $\chi(S) = 0$, when S is compact.

The main purpose of this note is to address the following question of Vaisman.

Question 1.1 ([37, p.122]) *Which compact complex surfaces* (S, J) *can admit* $l.c.K.$ *metrics?*

We will take the natural approach of first reducing the problem to minimal surfaces and then look at the Enriques–Kodaira classification. The rest of the section is devoted to give a brief account of these notions.

We start by explaining the minimal model of a surface introduced by Kodaira [16]: If one applies the classical monoidal transformation of blowing up a point on S, the result is a new complex surface $\tilde{S}$ containing a smooth rational curve C of self-intersection $C^2 = -1$. The blown up surface $\tilde{S}$ is diffeomorphic to the connected sum $S\#\overline{\mathbb{CP}}_2$. Conversely, a smooth rational curve C of self-intersection $C^2 = -1$ on a complex surface $\tilde{S}$ can always be blown down to a smooth point and the resulting smooth surface S will have second Betti number $b_2(\tilde{S}) - 1$; therefore if $\tilde{S}$ is compact, after a finite number of blowing down we will obtain that S is *minimal* – i.e., without rational curves of self-intersection -1. Such an S is called a *minimal-model* for the compact complex surface $\tilde{S}$ and in general is not unique.

It is then enough to understand *minimal* complex surfaces and this is the general philosophy of the classification which however is also very suitable to address the geometrical problem of Question 1.1 because of the following result of Tricerri which generalizes the analogous result in the Kähler case:

Proposition 1.2 ([34]) *A complex manifold* M *is* $l.c.K.$ *if and only if the blow up of* M *at point is* $l.c.K.$

As noticed in [34, Remark 4.3], this reduces the above question of Vaisman to minimal surfaces, for this reason from now on we can assume that S is a *minimal compact* complex surface and heavily rely on the famous Enriques–Kodaira classification which is summarized in the following table taken from the book of Barth–Peters–Van de Ven [5, p.188]. The classification divides all minimal surfaces into ten classes belonging to four groups according to the possible values of the *Kodaira dimension*, $\mathrm{Kod}(S) = -\infty, 0, 1, 2$ which appears in the second column of the table, while in the other columns we have indicated the *algebraic dimension* $a(S)$, the *Euler characteristic* $\chi(S)$ and the *first Betti number* $b_1(S)$.

Table 1. Table of Enriques–Kodaira classification

Class of S	Kod(S)	$a(S)$	$\chi(S)$	$b_1(S)$
1) rational surfaces		2	3,4	0
2) class VII_0 surfaces	$-\infty$	0,1	≥ 0	1
3) ruled surfaces of genus g		2	$4(1-g)$	2g
4) Enriques surfaces		2	12	0
5) Hyperelliptic surfaces		2	0	2
6) Kodaira surfaces	0	1	0	1,3
7) $K3$-surfaces		0,1,2	24	0
8) tori		0,1,2	0	4
9) properly elliptic surfaces	1	2	≥ 0	even
		1	0	odd
10) surfaces of general type	2	2	> 0	even

2 The case $b_1(S)$ even

It is well-known from Hodge theory that any compact Kähler manifold M must have odd de Rham cohomology of even dimension. Vice-versa, in the special case of surfaces, due to the fact that $H^1(S, \mathbb{C}) = H^{1,0}(S) \oplus H^{0,1}(S)$ whether b_1 is even or odd [5, p.117], we have the following result of Vaisman:

Proposition 2.1 ([35, Prop 2.3]) *Every l.c.K. metric on a compact surface with even first Betti number is actually globally conformal Kähler.*

Therefore, in the case b_1 even Vaisman's question reduces to the more classical one of finding Kähler metrics on surfaces. As conjectured by Kodaira and Morrow [17] the answer is the following:

Theorem 2.2 ([27, 31]) *A compact complex surface is Kähler if and only if $b_1(S)$ is even.*

The original proof of this result was done case by case using Enriques–Kodaira classification of minimal surfaces. We give a brief account of the proof following the table of the previous section.

Because every Moischezon surface S – i.e., of top algebraic dimension $a(S) = 2$ – is actually projective algebraic [5, p.127] it follows that surfaces in 1), 3), 4), 5), and 10) are certainly Kähler because they are submanifolds of $\mathbb{CP}_n$. Tori 8) admit flat Kähler metrics while elliptic surfaces 9) with b_1 even are Kähler by a result of Miyaoka [27]. The problem remained open for the only class left, namely for K3 surfaces, until it was solved by Siu [31] building on preliminary work of Todorov.

It is also interesting to notice that quite recently Buchdhal and Lamari found two unified proofs of this theorem—i.e., not using Kodaira's classification. Their works are independent—using different complex analytical methods – and appeared in the same issue of the same journal [6, 18].

3 The case $\mathbf{b_1(S)}$ odd and $\chi(\mathbf{S}) = \mathbf{0}$

From now on we can assume that S is a minimal compact complex surface with odd first Betti number and look for strictly $l.c.K.$ metrics on S—i.e., not globally conformal Kähler. We see from Kodaira's classification of these surfaces that the Euler characteristic $\chi(S)$ cannot be negative and in our treatment we distinguish two main cases: The first one $\chi(S) = 0$ is completely understood both from the point of view of the classification of the complex structure [2] and the existence of $l.c.K.$ metrics [1]; notice that $\chi(S) = 0$ is also a necessary condition for the metric to have parallel Lee form.

We start by presenting a brief description of the complex structure of these surfaces in order of decreasing Kodaira dimension.

Properly elliptic surfaces with $\mathbf{b_1}$ odd

A surface S is said to be elliptic if it admits a holomorphic map to a curve B with generic fiber an elliptic curve. It was shown by Kodaira [5, 16, 26] that when S is minimal with $b_1(S)$ odd, the singular fibers can only be multiple fibers; in this situation S admits an unbranched covering $\tilde{S}$ which is a (topologically non-trivial) elliptic fiber bundle over a smooth complex curve B with $b_1(\tilde{S}) = b_1(B) + 1$ and $b_2(\tilde{S}) = 2b_1(B)$. In particular, we conclude that $\chi(S) = 0$ for any minimal elliptic surface with $b_1(S)$ odd.

Finally, an elliptic surface S is called *properly elliptic* if $Kod(S) = 1$; when $b_1(S)$ is odd this amounts to say that the base B has genus $g \geq 2$. Furthermore, every surface of algebraic dimension 1 turns out to be elliptic [5, p.194].

Kodaira surfaces

By definition they are surfaces with $b_1(S)$ odd and $Kod(S) = 0$. They are divided into primary and secondary Kodaira surfaces according to whether b_1 is equal to 3 or 1. Primary Kodaira surfaces are elliptic fiber bundles over an elliptic curve and they provide interesting examples in differential geometry and topology. In fact it is shown in [30] that the complex structure J of a primary Kodaira surface anti-commutes with a symplectic structure I—generating in that way an almost hypercomplex structure on S; (S, I) was cited by Thurston as the first example of a compact symplectic manifold, which is not Kähler because $b_1 = 3$ [33]; and S also represents an interesting example in rational homotopy theory. Finally, secondary Kodaira surfaces are finite quotients of primary ones [5, p.147].

It follows from the classification table that the remaining minimal surfaces S with $b_1(S)$ odd and $\chi(S) = 0$ belong to class VII_0 – i.e., satisfy $Kod(S) = -\infty$ and $b_1(S) = 1$. The classification of surfaces in class VII_0 is known only in the special case $\chi(S) = 0$ and a theorem of Bogomolov [2] also proved by Yau et al. [22] and by Teleman [32] states that a surface in this class is either a Hopf surface or a Inoue–Bombieri surface, which we now describe briefly.

Hopf surfaces

By the work of Kodaira, a Hopf surface is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by a discrete group of biholomorphisms which is a finite extension of the infinite cyclic group generated by the contraction:

$$(z, w) \mapsto (az, bw + \lambda z^n),$$

where $a, b, \lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ satisfy $0 < |a| < |b| < 1$ and $\lambda(a - b^n) = 0$; we say that a Hopf surface is *diagonal* if $\lambda = 0$ (class 1 in the terminology used by Belgun). A Hopf surface is elliptic exactly when $\lambda = 0$ and $a^p = b^q$ for some $p, q \in \mathbb{N}$ while an elliptic surface with b_1 odd must be a Hopf surface when the base $B \cong \mathbb{CP}_1$.

Bombieri–Inoue surfaces

These surfaces were independently discovered at the same time [14] and [3], their universal cover is $\mathbb{C} \times \mathcal{H}$ where $\mathcal{H}$ denotes the upper-half plane and contrary to Hopf surfaces which always have at least one elliptic curve (namely the image of $z = 0$) Bombieri–Inoue surfaces have no complex curves at all. They come in three different families which for simplicity we denote by S_m, S_n^- and $S_{n,u}^-$ with $u \in \mathbb{C}$.

Now that we have an idea of the complex structure of minimal surfaces with odd first Betti number and zero Euler characteristic, we want to investigate which of them admit $l.c.K.$ metrics. This problem has been solved by Belgun [1] in his doctoral thesis completing the work of several authors as Vaisman, Tricerri, Gauduchon–Ornea. In fact Belgun even classified surfaces which admit metrics with parallel Lee form and his powerful results can be summarized as follows:

Theorem 3.1 ([1]) *The complete list of compact complex surfaces S with b_1 odd admitting $l.c.K.$ metrics with parallel Lee form is the following:*

1. *Properly elliptic surfaces – i.e., all surfaces with $Kod(S) = 1$.*
2. *Kodaira surfaces, primary or secondary – i.e., all surfaces with $Kod(S) = 0$.*
3. *Diagonal Hopf surfaces – i.e., Hopf surfaces with with $\lambda = 0$.*

Belgun was also able to construct $l.c.K.$ metrics on every non-diagonal Hopf surface improving therefore the previous work of Gauduchon–Ornea [10] to show that:

Theorem 3.2 ([1]) *Every Hopf surface admits a $l.c.K.$ metric.*

The only case left is that of Inoue–Bombieri surfaces whose geometry was first studied by Tricerri who constructed $l.c.K.$ metrics on all of them except for $S_{n,u}^-$ and $u \notin \mathbb{R}$ [34]. Then another remarkable theorem of Belgun is that Tricerri's result is in fact sharp.

Theorem 3.3 ([1]) *The Inoue–Bombieri surfaces $S_{n,u}^-$ with $u \notin \mathbb{R}$ do not admit $l.c.K.$ metrics at all.*

An interesting consequence is that, contrary to the Kähler case, $l.c.K.$ metrics are not stable under small deformations [1].

4 Anti-self-dual Hermitian metrics on surfaces of class VII_0 with $b_2 > 0$

As seen in the previous section, the work of Belgun completely answered the question of Vaisman in the case of zero Euler characteristic. It follows from the classification that the only other possible case is that of surfaces of class VII_0 with $0 < \chi = b_2$, because $b_1 = 1$. There is no classification of these surfaces but only several examples due to Inoue, Hirzebruch, Enoki, Kato, Nakamura, and Dloussky. These examples all turn out to have small deformations which are not minimal. They are blown-up Hopf surfaces.

On the *topology* of these surfaces we can therefore say that all known examples of S are diffeomorphic to $(S^1 \times S^3)\#m\overline{\mathbb{CP}}_2$ where $m = b_2(S) \geq 1$.

There are also some very basic open questions about the *complex structures*; for example it is not known whether every surface $S \in VII_0$ with $b_2(S) \geq 1$ admits a curve [26].

As far as the *Hermitian geometry* of these surfaces is concerned, very little is known. We only have examples by LeBrun [19] who constructed anti-self-dual Hermitian metrics with semi-free S^1-action on parabolic Inoue surfaces using his hyperbolic ansatz. The action must in fact be holomorphic by [29] and this fits well with a result of Hausen [11] asserting that the only surfaces in this class admitting a 1-dimensional group of biholomorphisms with fixed points are parabolic Inoue surfaces.

The crucial link here is that LeBrun's metrics are automatically $l.c.K.$ by the following result of Boyer; see also [28] for an alternative twistor proof.

Theorem 4.1 ([4]) *Let S be a compact surface with $b_1(S)$ odd admitting an anti-self-dual Hermitian metric g. Then g is $l.c.K.$ and S belongs to class VII.*

In what follows, we present a new twistor construction of anti-self-dual Hermitian metrics on class VII surfaces; by Boyer's result these metrics are automatically $l.c.K.$ and notice that all known examples of $l.c.K.$ metrics on surfaces of class VII_0 with $b_2 > 0$ are indeed anti-self-dual Hermitian. The details and the proofs of our construction will appear elsewhere [8].

4.1 Surfaces with positive b_2, according to Nakamura

Although it is still an open question whether all the class VII_0 surfaces with $b_2 > 0$ must have a curve, it is known for example that they can only have elliptic or rational curves; in fact at most one-elliptic curve and at most $b_2(S)$ rational curves some of them forming a cycle C, there can be at most two cycles of rational curves in S. More precisely, some of these surfaces can be characterized by the configuration of curves that they contain. This is the case for Inoue and Enoki surfaces which always have $b_2(S)$ rational curves and can be identified by the presence of an elliptic curve or by the number of cycles and their self-intersection numbers. Rather than giving the original definition of each specific class we will simply refer to the excellent exposition in [26] from which we extract the useful table 2.

Our construction is very much inspired by the work of Nakamura [23, 25] on rational degenerations of class VII surfaces. In what follows, we briefly explain how, Inoue and Enoki surfaces can be constructed starting from a completely different class of surfaces, namely toric surfaces which are blow-ups of $\mathbb{CP}_2$ over a fixed point of the action.

Let $p \in \mathbb{CP}_2$ be a fixed point of a standard $(\mathbb{C}^* \times \mathbb{C}^*)$-action and let $H \subset \mathbb{CP}_2$ denote the hyperplane class. We have $-K = 3H$ for the anti-canonical class which

Table 2. Table of Enoki and Inoue surfaces with $b_2 > 0$.

curves	surfaces
an elliptic curve on a cycle	parabolic Inoue surfaces
two cycles	hyperbolic Inoue surfaces
a cycle C with $C^2 < 0$ and $b_2(S) = b_2(C)$	half Inoue surfaces
a cycle C with $C^2 = 0$	Enoki surfaces

can therefore be represented by a cycle of three rational curves—each of them having self-intersection number $+1$—and let p be one of the three corners. Blowing up $\mathbb{CP}_2$ at the point p yields the Hirzebruch surface Σ_1 with anti-canonical divisor $-K$ which is a cycle of four rational curves with self-intersection numbers $-1, 0, +1, 0$.

One can go on like this by always blowing up *one of the two corners of the last exceptional divisor*. After m times the result is again a toric surface $\tilde{D}$ diffeomorphic to $\mathbb{CP}_2\#m\overline{\mathbb{CP}}_2$ with a unique $+1$-rational curve denoted by H which is disjoint from the exceptional divisor of the last blow-up, denoted by E. They are part of a cycle of $(m+2)$-rational curves which represents the anti-canonical class of the surface $\tilde{D}$,

$$-K = E + B_1 + \cdots + B_i + H + B_{i+1} + \cdots + B_m,$$

the important point here is that by always blowing up one of the two corners of the (-1)-curve E we produced an anti-canonical cycle $-K$, whose -1 components always intersect E—in other words $B_j^2 = -1$ implies $j = 1$ or $j = m$. This is the property that makes this construction produce *minimal* surfaces with $b_1 = 1$.

From this smooth toric surface $\tilde{D}$, we now construct a singular surface D': Take $\phi : H \to E$ to be a biholomorphism of the complex projective line sending the two corners of H to those of E and consider the rational surface with ordinary double curve given by the quotient

$$D' = \tilde{D}/\phi.$$

Notice that D' is a singular surface with normal crossings along the double curve $F = \phi(H) = \phi(E)$ satisfying the d-semistable condition $\nu_H \otimes \nu_E \cong \mathcal{O}(+1) \otimes \mathcal{O}(-1) = \mathcal{O}_{\mathbb{CP}_1}$.

In this setting we know from a more general result of Nakamura [23, 24] that the Kuranishi family of D' is unobstructed, the general element D_t is a smooth surface in class VII containing a global spherical shell and diffeomorphic to $(S^1 \times S^3)\#m\overline{\mathbb{CP}}_2$ with $m = b_2(\tilde{D}) - 2$. In fact he shows that every class VII surface with global spherical shell admits a rational degeneration (not necessarely toric).

Because we want to obtain VII_0 surfaces with a particular configuration of curves (as described in the table), we consider deformations of the singular pair (D', B') where $\tilde{D}$ is toric and $B' = \phi(B_1 + \cdots B_m) \subset D'$ is the normal crossing divisor given by the image of the divisor $-K - H - E$ in $\tilde{D}$. We then have the following result:

Theorem 4.2 *The Kuranishi family of the singular pair (D', B') is unobstructed, the general member D_t is either an Inoue or an Enoki surface of class VII_0 with $b_2 = m$.*

In fact more precisely, one obtains a half Inoue surface if B'/ϕ consists of just one cycle. In other cases, ϕ identifies the four end-points of B' in order to form two cycles of rational curves and we obtain a hyperbolic Inoue surface when $i \geq 2$, or a parabolic Inoue surface when $i = 1$, because in this case one of the cycles in B'/ϕ consists of just one rational curve with a double point which is deformed to a smooth elliptic curve.

Finally, in order to obtain Enoki surfaces we need $i = 1$ and to actually neglect B_1 so that the general member D_t has only a cycle of rational curves with zero self-intersection number and no elliptic curve.

4.2 Twistor construction

Now that we understand the complex structure of our surfaces as smooth deformations of the singular pair (D', B'), we are going to produce anti-self-dual Hermitian metrics by imbedding (D', B') into a singular twistor space Z'. The construction of Z' is suggested by the work of Donaldson–Friedman [7] which for our purposes fits very well with Nakamura's construction of surfaces in class VII.

The starting point is a result of Joyce [13] who constructed self-dual metrics on the connected sum of m copies of $\mathbb{CP}_2$ (denoted by $m\mathbb{CP}_2$ from now on) with isometry group $S^1 \times S^1$ and their twistor spaces were studied by Fujiki in [9]. Let $t : Z \to m\mathbb{CP}_2$ be the twistor fibration from a Joyce twistor space to a Joyce metric, as usual each fiber $t^{-1}(p) \cong \mathbb{CP}_1$ is a complex submanifold of Z with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ called *twistor line*; these fibers are invariant with respect to the *real* structure $\sigma : Z \to Z$ which is an anti-holomorphic involution which restricts to the antipodal map on each twistor line, and is therefore fixed-point free.

What is important for our purposes is that every Joyce twistor space contains a pair of *degree*-1 divisors D and $\bar{D}$ (in fact a generic Z contains exactly $(m+3)$ such pairs) by which we mean the following: D is an effective divisor in Z with intersection number 1 with a twistor line and $\bar{D} = \sigma(D)$. The generic twistor line intersects D at one point and there is exactly one twistor line $L_1 \subset D$, by reality it is also contained in $\bar{D}$ so that $L_1 = D \cap \bar{D}$. The restriction of the twistor map $t : D \to M$ is orientation reversing and shows that D is diffeomorphic to a blow-up of $\mathbb{CP}_2$: $D \cong \mathbb{CP}_2 \# m\overline{\mathbb{CP}}_2$ [20, prop.6].

In fact it is shown in [9] that each of this degree-1 divisors are toric surfaces with respect to a holomorphic $\mathbb{C}^* \times \mathbb{C}^*$-action on Z which is a complexification of the isometric action on M, given by the twistor correspondence.

L_1 is the component of self-intersection $+1$ in the anti-canonical cycle $-K$ of the toric surface $D \subset Z$, and in order to apply the Donaldson–Friedman construction let L_2 be the twistor line passing through one of the two corners of a (-1)-component of anti-canonical cycle $-K$. We can then follow the prescription of [7] and blow up the twistor space Z at L_1 and L_2 to obtain a smooth 3-fold $\tilde{Z}$ containing two exceptional quadrics Q_1 and Q_2 each with normal bundle $\mathcal{O}(-1, 1)$ and finally produce a singular twistor space Z' by using a biholomorphism $\psi : Q_1 \to Q_2$ which extends ϕ switching the two $\mathbb{CP}_1$-factors of the quadrics and taking the quotient space

$$Z' = \tilde{Z}/\psi.$$

According to general theory [7], Z' is a complex 3-fold with only normal crossing singularities along the smooth quadric $\psi(Q_1) = \psi(Q_2)$ satisfying the d-semistable condition and we can prove that its deformation theory is unobstructed so that it always admits smooth deformations which are twistor spaces of anti-self-dual metrics on the self-connected sum of $m\mathbb{CP}_2$ (with reversed orientation) which is $(S^1 \times S^3)\#m\mathbb{CP}_2$ – i.e., exactly what we want, topologically.

However, our construction gives us for free a lot more geometrical structure: The proper transform $\tilde{D}$ of D in the blown up twistor space $\tilde{Z}$ is exactly one of the toric surfaces considered in the previous section and is now *disjoint* from the proper transform $\tilde{\bar{D}}$ of $\bar{D}$. The divisors $\tilde{D}$ and $\tilde{\bar{D}}$ are isomorphic as toric surfaces and intersect transversely the two exceptional quadrics Q_1 and Q_2. The biholomorphism $\psi : Q_1 \to Q_2$ extends the identification ϕ so that the singular surface D' of the previous section is contained inside the singular twistor space Z' together with $\bar{D}'$ because the construction is compatible with real structures. In fact D' and $\bar{D}'$ are disjoint Cartier divisors in Z' with chains of rational curves $B' \subset D'$ and $\bar{B}' \subset \bar{D}'$. We then set $S' = D' + \bar{D}'$ and $C' = B + \bar{B}'$ and consider the triple of singular complex spaces with real structure. The deformation theory of such triples was studied by Honda [12] and we are able to prove the following result.

Theorem 4.3 *The Kuranishi family of the singular triple (Z', S', C') is unobstructed, the general member Z_t is smooth and contains a class VII$_0$ surface D_t with curves B_t.*

Because the triple (Z', S', C') has a real structure we know from general theory [7, 12, 15] that for t generic and real, Z_t is a twistor space with a degree-1 divisor D_t which is disjoint from $\bar{D}_t$ and isomorphic to one of the surfaces of 4.2. This is the key to prove the following result.

Theorem 4.4 *Every minimal hyperbolic or half Inoue surface with $b_2 = m$ admits an m-dimensional family of anti-self-dual Hermitian metrics. The same result holds on some Enoki and some parabolic Inoue minimal surfaces with $b_2 = m$.*

Altough it is not yet clear which parabolic Inoue surfaces admit anti-self-dual Hermitian metrics, let us notice that our metrics on these surfaces admit an S^1-action and should therefore be conformally isometric to LeBrun's by the general result of [21].

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