Curvature of Contact Metric Manifolds*

David E. Blair

Department of Mathematics,
Michigan State University, East Lansing, MI 48824
blair@math.msu.edu

Dedicated to Professor Lieven Vanhecke

Summary. This essay surveys a number of results and open questions concerning the curvature of Riemannian metrics associated to a contact form.

In 1975, when the author was on sabbatical in Strasbourg, it was an open question whether or not the 5-torus carried a contact structure. The author, being interested in the Riemannian geometry of contact manifolds, proved at that time ([4]) that on a contact manifold of dimension ≥ 5, there are no flat associated metrics. Shortly thereafter, R. Lutz [31] proved that the 5-torus does indeed admit a contact structure and hence the natural flat metric on the 5-torus is not an associated metric. The non-flatness result of 1975 was generalized by Z. Olszak [35], who proved in 1978 that a contact metric manifold of constant curvature c and dimension ≥ 5 is Sasakian and of constant curvature $+1$. In dimension 3, the only constant curvature cases are of curvature 0 and 1 as we will note below. Sometimes one has an intuitive sense that the existence of a contact form tends to tighten up the manifold. The non-existence of flat associated metrics does raise the question as to whether, aside from the flat 3-dimensional case, any contact metric manifold must have some positive sectional curvature. If the manifold is compact, it is known ([7] p. 99) that there is no associated metric of strictly negative curvature. This follows from a deep result of A. Zeghib [48] on geodesic plane fields. Recall that a plane field on a Riemannian manifold is said to be *geodesic* if any geodesic tangent to the plane field at some point is everywhere tangent to it. Zeghib proves that a compact negatively curved Riemannian manifold has no C^1 geodesic plane field (of non-trivial dimension). Since for any associated metric the integral curves of the characteristic vector field, or Reeb vector field, are geodesics, the characteristic vector field determines a geodesic line field to which we can apply the theorem of Zeghib to obtain the following result.

Theorem. *On a compact contact manifold, there is no associated metric of strictly negative curvature.*

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The author conjectures that this and a bit more is true locally, namely, that except for the flat 3-dimensional case, any contact metric manifold has some positive sectional curvature.

Before giving other curvature results, we must review the structure tensors of a contact metric manifold. By a *contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$; ξ is called the *characteristic vector field* or *Reeb vector field* of the contact form η. A classical theorem of Darboux states that on a contact manifold there exist local coordinates with respect to which $\eta = dz - \sum_{i=1}^{n} y^i dx^i$. We denote the *contact subbundle* or *contact distribution* defined by the subspaces $\{X \in T_m M : \eta(X) = 0\}$ by $\mathcal{D}$. Roughly speaking the meaning of the contact condition, $\eta \wedge (d\eta)^n \neq 0$, is that the contact subbundle is as far from being integrable as possible. In fact, for a contact manifold the maximum dimension of an integral submanifold of $\mathcal{D}$ is only n, whereas a subbundle defined by a 1-form η is integrable if and only if $\eta \wedge d\eta \equiv 0$. A Riemannian metric g is an *associated metric* for a contact form η if, first of all,

$$\eta(X) = g(X, \xi), \text{ i.e. the characteristic vector field is orthogonal to } \mathcal{D}$$

and secondly, there exists a field of endomorphisms ϕ such that

$$\phi^2 = -I + \eta \otimes \xi \quad \text{and} \quad d\eta(X, Y) = g(X, \phi Y).$$

We refer to (ϕ, ξ, η, g) as a *contact metric structure* and to M^{2n+1} with such a structure as a *contact metric manifold*. The product $M^{2n+1} \times \mathbb{R}$ carries a natural almost complex structure defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

and the underlying almost contact structure is said to be *normal* if J is integrable. The normality condition can be expressed as $N = 0$ where N is defined by

$$N(X, Y) = [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi,$$

$[\phi, \phi]$ being the Nijenhuis tensor of ϕ. A *Sasakian manifold* is a normal contact metric manifold. In terms of the curvature tensor a contact metric structure is Sasakian if and only if

$$R_{XY}\xi = \eta(Y)X - \eta(X)Y.$$

In terms of the covariant derivative of ϕ the Sasakian condition is

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X.$$

A contact metric structure for which ξ is Killing is said to be *K-contact* and it is easy to see that a Sasakian manifold is K-contact. In dimension 3, a K-contact manifold is necessarily Sasakian but this is not true in higher dimensions. In the theory of contact metric manifolds there is another tensor field that plays a fundamental role, viz. $h = \frac{1}{2}\pounds_\xi \phi$ where $\pounds$ denotes Lie differentiation. The operator h is symmetric, it anti-commutes with ϕ, $h\xi = 0$ and h vanishes if and only if the contact metric structure is K-contact. The complexification of the tangent bundle of a contact metric manifold admits a holomorphic subbundle $\mathcal{H} = \{X - i\phi|_{\mathcal{D}}X : X \in \mathcal{D}\}$ and its Levi form is given by $-d\eta(X, \phi|_{\mathcal{D}}Y)$, $X, Y \in \mathcal{D}$. In this way a contact metric manifold becomes a (non-integrable) strongly pseudo-convex CR-manifold. The CR-structure is integrable if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. Tanno [46] showed that the integrability is equivalent to $(\nabla_X\phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$. For later use, we mention briefly the idea of a *$\mathcal{D}$-homothetic deformation* of a contact metric structure. Let a be a positive constant and define a new structure by,

$$\tilde{\eta} = a\eta, \quad \tilde{\xi} = \frac{1}{a}\xi, \quad \tilde{\phi} = \phi, \quad \tilde{g} = ag + a(a-1)\eta \otimes \eta.$$

The new structure is again a contact metric structure and if the original structure is a Sasakian, a K-contact, or a strongly pseudo-convex integrable CR-structure, so is the new structure. For details and additional information on the above development, see the author's book [7].

Returning to the positivity of curvature question, we briefly mention the following. A celebrated theorem of Myers [33] states that a complete Riemannian manifold whose Ricci curvature satisfies $Ric \geq \delta > 0$ is compact. In [27] I. Hasegawa and M. Seino generalized Myers' theorem for a K-contact manifold by proving that a complete K-contact manifold for which $Ric \geq -\delta > -2$ is compact. Note however that in the K-contact case, all sectional curvatures of plane sections containing ξ are equal to 1 and hence there is a certain amount of positive curvature from the outset. In an attempt to weaken the K-contact requirement in this result, R. Sharma and the author [11] considered a contact metric manifold M^{2n+1} for which ξ is an eigenvector field of the Ricci operator. In this case if $Ric \geq -\delta > -2$ and the sectional curvatures of plane sections containing ξ are $\geq \epsilon > \delta' \geq 0$ where

$$\delta' = 2\sqrt{n(\delta - 2\sqrt{2\delta} + n + 2)} - (\delta - 2\sqrt{2\delta} + 1 + 2n),$$

then M^{2n+1} is compact. The condition that ξ be an eigenvector field of the Ricci operator is not only a natural generalization of the K-contact condition, but an important condition in its own right. D. Perrone [40] recently showed that ξ is an eigenvector field of the Ricci operator if and only if ξ is a harmonic vector field. Moreover, all complete 3-dimensional contact metric manifolds for which ξ is an eigenvector of the Ricci operator and for which the Ricci operator has constant eigenvalue are known (Koufogiorgos [29]); these are either Sasakian or particular Lie groups.

The next curvature result to discuss is the following [5].

Theorem. *A contact metric manifold M^{2n+1} satisfies $R_{XY}\xi = 0$ if and only if it is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

This structure is the standard contact metric structure on the tangent sphere bundle of Euclidean space; the standard normalizations give the curvature of the sphere factor as +4. For brevity we will not discuss the contact metric structure on the tangent sphere bundle T_1M of a Riemannian manifold M; suffice it to note that the characteristic vector field is (to within a factor of 2) the geodesic flow (again see [7], Section 9.2 for details). Now $E^{n+1} \times S^n(4)$ is a symmetric space and one can ask first when the tangent sphere bundle is locally symmetric and, more generally, whether one can classify all locally symmetric contact metric manifolds. For the first question the author proved the following result in [6].

Theorem. *The standard contact metric structure on T_1M is locally symmetric if and only if either the base manifold M is flat or 2-dimensional and of constant curvature* $+1$.

For the more general question we have the following results of Blair-Sharma [12] and A. M. Pastore [37] respectively.

Theorem. *A 3-dimensional contact metric manifold is locally symmetric if and only if it is of constant curvature* 0 *or* $+1$.

Theorem. *A 5-dimensional contact metric manifold is locally symmetric if and only if it is locally isometric to $S^5(1)$ or $E^3 \times S^2(4)$.*

Very early in the development of the Riemannian geometry of contact manifolds the following had been shown.

Theorem. *A locally symmetric K-contact manifold is of constant curvature* $+1$ *and Sasakian.*

This result was due to Tanno in 1967 [43] and in the Sasakian with dimension ≥ 5 case to Okumura in 1962 [34].

We now consider briefly the analog of holomorphic sectional curvature, namely ϕ-sectional curvature. A plane section in T_mM^{2n+1} is called a *ϕ-section* if there exists a vector $X \in T_mM^{2n+1}$ orthogonal to ξ such that $\{X, \phi X\}$ span the section and the sectional curvature is called *ϕ-sectional curvature*. Just as the sectional curvatures of a Riemannian manifold and the holomorphic sectional curvatures of a Kähler manifold determine the curvature completely, on a Sasakian manifold the ϕ-sectional curvatures determine the curvature completely. Moreover, on a Sasakian manifold of dimension ≥ 5, if at each point the ϕ-sectional curvature is independent of the choice of ϕ-section at the point, it is constant on the manifold and the curvature tensor is given by,

$$\begin{aligned} R_{XY}Z = {} & \frac{c+3}{4}\big(g(Y,Z)X - g(X,Z)Y\big) \\ & + \frac{c-1}{4}\big(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ & + g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z\big). \end{aligned}$$

A Sasakian manifold of constant ϕ-sectional curvature is called a *Sasakian space form*. A well-known result of Tanno [44] is that a complete simply connected Sasakian manifold of constant ϕ-sectional curvature c is isometric to one of certain model spaces depending on whether $c > -3$, $c = -3$ or $c < -3$. The model space for $c > -3$ is a sphere with a $\mathcal{D}$-homothetic deformation of the standard structure. For $c = -3$ the model space is $\mathbb{R}^{2n+1}$ with the contact form $\eta = \frac{1}{2}(dz - \sum_{i=1}^n y^i dx^i)$, the factor of $\frac{1}{2}$ being convenient for normalization reasons, together with the metric $ds^2 = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^{n} ((dx^i)^2 + (dy^i)^2)$. For the $c < -3$ case one has a canonically defined contact metric structure on the product $B^n \times \mathbb{R}$ where B^n is a simply connected bounded domain in $\mathbb{C}^n$ with a Kähler structure of constant negative holomorphic curvature. In particular, Sasakian space forms exist for all values of c. In the general context of contact metric manifolds, J. T. Cho [23] recently introduced the notion of a *contact Riemannian space form*. We get at this notion in the following way. In [47] Tanno showed that the CR-structure of the tangent sphere bundle with its standard contact metric structure is integrable if and only if the base manifold is of constant curvature. Cho first computes the covariant derivative of h in this case obtaining

$$(\nabla_X h)Y = g((h - h^2)\phi X, Y)\xi + \eta(Y)(h - h^2)\phi X - \mu\eta(X)h\phi Y,$$

where μ is a constant. He then abstracts this idea and defines the class $\mathfrak{Q}$ of contact Riemannian manifolds with integrable CR-structure for which the covariant derivative of h satisfies the above condition. We remark that in the study of contact manifolds in general, lack of control of the covariant derivative of h is often an obstacle to further results. Now for a contact metric manifold M^{2n+1} of class $\mathfrak{Q}$ for which the ϕ-sectional curvature is independent of the choice of ϕ-section, Cho shows that the ϕ-sectional curvature is constant on M^{2n+1} and computes the curvature tensor explicitly. He then defines a *contact Riemannian space form* to be a complete, simply connected contact metric manifold of class $\mathfrak{Q}$ of constant ϕ-sectional curvature. Cho also gives a number of non-Sasakian examples and shows that a contact Riemannian space form is locally homogeneous and is strongly locally ϕ-symmetric (see below).

We noted above that a locally symmetric K-contact manifold is of constant curvature $+1$ and Sasakian. For K-contact geometry this can be regarded as saying that the idea of being locally symmetric is too strong. For this reason Takahashi [41] introduced the following notion: A Sasakian manifold is said to be a *Sasakian locally ϕ-symmetric space* if

$$\phi^2(\nabla_V R)_{XY} Z = 0,$$

for all vector fields V, X, Y, Z orthogonal to ξ. It is easy to check that Sasakian space forms are locally ϕ-symmetric spaces. Note that on a K-contact manifold, a geodesic that is initially orthogonal to ξ remains orthogonal to ξ. We call such a geodesic a *ϕ-geodesic*. A local diffeomorphism s_m of M^{2n+1}, $m \in M^{2n+1}$, is a *ϕ-geodesic symmetry* if its domain contains a (possibly) smaller domain $\mathcal{U}$ such that for every ϕ-geodesic $\gamma(s)$ parametrized by arc length with $\gamma(0) \in \mathcal{U}$ and on the integral curve of ξ through m,

$$(s_m \circ \gamma)(s) = \gamma(-s),$$

for all s with $\gamma(\pm s) \in \mathcal{U}$. Takahashi defines a Sasakian manifold to be a *Sasakian globally ϕ-symmetric space* by requiring that any ϕ-geodesic symmetry can be extended to a global automorphism of the structure and that the Killing vector field ξ generates a 1-parameter group of global transformations. Among the main results of Takahashi are the following three theorems.

Theorem. *A Sasakian locally ϕ-symmetric space is locally isometric to a Sasakian globally ϕ-symmetric space and a complete, connected, simply-connected Sasakian locally ϕ-symmetric space is a globally ϕ-symmetric space.*

Theorem. *A Sasakian manifold is locally ϕ-symmetric if and only if it admits a ϕ-geodesic symmetry at every point which is a local automorphism of the structure.*

Now suppose that $\mathcal{U}$ is a neighborhood on M^{2n+1} on which ξ is a regular vector field, then since M^{2n+1} is Sasakian, the projection $\pi : \mathcal{U} \longrightarrow \mathcal{V} = \mathcal{U}/\xi$ gives a Kähler structure on $\mathcal{V}$. Furthermore if $\underline{s}_{\pi(m)}$ denotes the geodesic symmetry on $\mathcal{V}$ at $\pi(m)$, then $\underline{s}_{\pi(m)} \circ \pi = \pi \circ s_m$.

Theorem. *A Sasakian manifold is locally ϕ-symmetric if and only if each Kähler manifold which is the base of a local fibering is a Hermitian locally symmetric space.*

Recall that a Riemannian manifold is locally symmetric if and only if the local geodesic symmetries are isometries. From the Takahashi theorems we note that on a Sasakian locally ϕ-symmetric space, local ϕ-geodesic symmetries are isometries. Conversely in [13], L. Vanhecke and the author proved that if on a Sasakian manifold the local ϕ-geodesic symmetries are isometries, the manifold is a Sasakian locally ϕ-symmetric space. This was extended to the K-contact case by Bueken and Vanhecke [19] and we have the following Theorem.

Theorem. *If on a K-contact manifold the local ϕ-geodesic symmetries are isometries, the manifold is a Sasakian locally ϕ-symmetric space.*

Finally J. A. Jiménez and O. Kowalski [28] classified complete simply-connected globally ϕ-symmetric spaces.

We now ask what is the best notion of a locally ϕ-symmetric space for a general contact metric manifold? One could use the same definition, namely,

$$\phi^2(\nabla_V R)_{XY}Z = 0,$$

for all vector fields V, X, Y, Z orthogonal to ξ and this condition gives what is known as a *weakly locally ϕ-symmetric space*. Now without the K-contact property one loses the fact that a geodesic, initially orthogonal to ξ, remains orthogonal to ξ. However we have just seen that in the Sasakian case local ϕ-symmetry is equivalent to reflections in the integral curves of the characteristic vector field being isometries. E. Boeckx and L. Vanhecke [17] proposed this property as the definition for local ϕ-symmetry in the contact metric case and call a contact metric manifold with this property a *strongly locally ϕ-symmetric space*. From the work of B.-Y. Chen and L. Vanhecke [22] one can see that on a strongly locally ϕ-symmetric space,

$$g((\nabla^{2k}_{X\cdots X}R)_{XY}X, \xi) = 0,$$

$$g((\nabla^{2k+1}_{X\cdots X}R)_{XY}X, Z) = 0,$$

$$g((\nabla^{2k+1}_{X\cdots X}R)_{X\xi}X, \xi) = 0,$$

for all X, Y, Z orthogonal to ξ and all $k \in \mathbb{N}$. Conversely, in the analytic case these conditions are sufficient for the contact metric manifold to be a strongly locally ϕ-symmetric space. In particular, taking $k = 0$ in the second condition, we note that a strongly locally ϕ-symmetric space is weakly locally ϕ-symmetric. In [21], G. Calvaruso, D. Perrone and L. Vanhecke determined all 3-dimensional strongly locally

ϕ-symmetric spaces. In [18] E. Boeckx, P. Bueken and L. Vanhecke gave an example of a non-unimodular Lie group with a weakly locally ϕ-symmetric contact metric structure which is not strongly locally ϕ-symmetric.

As a generalization of both $R_{XY}\xi = 0$ and the Sasakian case, $R_{XY}\xi = \eta(Y)X - \eta(X)Y$, Th. Koufogiorgos, B. Papantoniou and the author [10] considered the (κ, μ)-*nullity condition*,

$$R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

where κ and μ are constants and gave several reasons for studying it. We refer to a contact metric manifold satisfying this condition as a (κ, μ)-*manifold*. On a (κ, μ)-manifold, $\kappa \leq 1$. If $\kappa = 1$, the structure is Sasakian and if $\kappa < 1$, the (κ, μ)-nullity condition determines the curvature of M^{2n+1} completely. When $\kappa < 1$, the non-zero eigenvalues of h are $\pm\sqrt{1-\kappa}$ each with multiplicity n. Th. Koufogiorgos and C. Tsichlias [30] considered this condition where κ and μ are functions; they showed that in dimensions ≥ 5, κ and μ must be constants but that in dimension 3 these "generalized (κ, μ)-manifolds" exist. The standard contact metric structure on the tangent sphere bundle T_1M satisfies the (κ, μ)-nullity condition if and only if the base manifold M is of constant curvature. In particular if M has constant curvature c, then $\kappa = c(2-c)$ and $\mu = -2c$. A $\mathcal{D}$-homothetic deformation destroys a condition like $R_{XY}\xi = 0$ or

$$R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y).$$

However, the form of the (κ, μ)-nullity condition is preserved under a $\mathcal{D}$-homothetic deformation with

$$\tilde{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \tilde{\mu} = \frac{\mu + 2a - 2}{a}.$$

Given a non-Sasakian (κ, μ)-manifold M, E. Boeckx [15] introduced an invariant

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-\kappa}},$$

and showed that for two non-Sasakian (κ, μ)-manifolds $(M_i, \phi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, we have $I_{M_1} = I_{M_2}$ if and only if up to a $\mathcal{D}$-homothetic deformation, the two spaces are locally isometric as contact metric manifolds. Thus we know all non-Sasakian (κ, μ)-manifolds locally as soon as we have, for every odd dimension $2n + 1$ and for every possible value of the invariant I, one (κ, μ)-manifold (M, ϕ, ξ, η, g) with $I_M = I$. For $I > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have $I = (1 + c)/|1 - c|$. Boeckx also gives a Lie algebra construction for any odd dimension and any value of $I \leq -1$.

Returning to the strongly locally ϕ-symmetric spaces, we note that the non-Sasakian (κ, μ)-spaces are strongly locally ϕ-symmetric as was shown by E. Boeckx [14]. Special cases of these are the non-Abelian 3-dimensional unimodular Lie groups with left-invariant contact metric structures. To see these examples, we first note the classification of simply connected homogeneous 3-dimensional contact metric manifolds as given by D. Perrone in [39]. Let τ denote the scalar curvature and

$$w = \frac{1}{8}(\tau - Ric(\xi) + 4),$$

the Webster scalar curvature. The classification of 3-dimensional Lie groups and their left invariant metrics was given by Milnor [32].

Theorem. *Let* (M^3, η, g) *be a simply connected homogeneous contact metric manifold. Then* M^3 *is a Lie group G and both g and* η *are left-invariant. More precisely we have the following classification:* (1) *If G is unimodular, then it is one of the following Lie groups:*

1. *The Heisenberg group when* $w = |\pounds_\xi g| = 0$*;*
2. $SU(2)$ *when* $4\sqrt{2}w > |\pounds_\xi g|$*;*
3. *the universal covering of the group of rigid motions of the Euclidean plane when* $4\sqrt{2}w = |\pounds_\xi g| > 0$*;*
4. *the universal covering of* $SL(2, \mathbb{R})$ *when* $-|\pounds_\xi g| \neq 4\sqrt{2}w < |\pounds_\xi g|$*;*
5. *the group of rigid motions of the Minkowski plane when* $4\sqrt{2}w = -|\pounds\xi g| < 0$.

(2) *If G is non-unimodular, its Lie algebra is given by*

$$[e_1, e_2] = \alpha e_2 + 2\xi, \quad [e_1, \xi] = \gamma e_2, \quad [e_2, \xi] = 0,$$

where $\alpha \neq 0$, $e_1, e_2 = \phi e_1 \in \mathcal{D}$ *and* $4\sqrt{2}w < |\pounds_\xi g|$*. Moreover, if* $\gamma = 0$*, the structure is Sasakian and* $w = -\alpha^2/4$.

The structures on the unimodular Lie groups in this theorem satisfy the (κ, μ)-nullity condition and hence they are strongly locally ϕ-symmetric. The weak locally ϕ-symmetric contact metric structure which is not the strong locally ϕ-symmetric given by Boeckx, Bueken and Vanhecke [18] is the non-unimodular case with $\gamma = 2$. Notice also, in the unimodular case, the role played by the invariant $p = (4\sqrt{2}w)/|\pounds_\xi g|$. Moreover $w = (2 - \mu)/4$ and $|\pounds_\xi g| = 2\sqrt{2}\sqrt{1 - \kappa}$; thus $p = (2 - \mu)/(2\sqrt{1 - \kappa})$ which is the above invariant I_M of Boeckx.

A special case of the (κ, μ)-spaces is, of course, the case where ξ belongs to the κ-nullity distribution, i.e. $\mu = 0$ and we call such a contact metric manifold an *$N(\kappa)$-contact metric manifold*. Using the Boeckx invariant we construct an example of a $(2n + 1)$-dimensional $N(1 - (\frac{1}{n}))$-manifold, $n > 1$.

Example. Since the Boeckx invariant for a $(1 - (\frac{1}{n}), 0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n + 1)$-dimensional manifold of constant curvature c so chosen that the resulting $\mathcal{D}$-homothetic deformation will be a $(1 - (1/n), 0)$-manifold. That is, for $\kappa = c(2 - c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{\kappa + a^2 - 1}{a^2}, \quad 0 = \frac{\mu + 2a - 2}{a},$$

for a and c. The result is

$$c = \frac{(\sqrt{n} \pm 1)^2}{n-1}, \quad a = 1 + c,$$

and taking c and a to be these values we obtain a $N(1 - (\frac{1}{n}))$-manifold.

Now as a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold is said to be semi-symmetric if its curvature tensor satisfies $R_{XY} \cdot R = 0$, where R_{XY} acts on R as a derivation. In [45] Tanno showed that a semi-symmetric K-contact manifold is locally isometric to $S^{2n+1}(1)$. In [38] D. Perrone began the study of semi-symmetric contact metric manifolds and in [36] B. Papantoniou showed that a semi-symmetric (κ, μ)-space of dimension ≥ 5 is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$. Similarly Ch. Baikoussis and Th. Koufogiorgos [1] showed that an $N(\kappa)$-contact metric manifold satisfying $R_{\xi X} \cdot W = 0$, W being the Weyl conformal curvature tensor, is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$. In [16] E. Boeckx and G. Calvaruso showed that the tangent sphere bundle is semi-symmetric if and only if it is locally symmetric and therefore the base manifold is either flat or 2-dimensional and of constant curvature $+1$. With this in mind it is surprising that the concircular curvature tensor,

$$Z_{XY}V = R_{XY}V - \frac{\tau}{2n(2n+1)}(g(Y,V)X - g(X,V)Y),$$

leads to other cases. Recently J.-S. Kim, M. Tripathi and the author [8] proved the following.

Theorem. *A $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold M satisfies*

$$Z_{\xi X} \cdot Z = 0,$$

if and only if M is 3-dimensional and flat, or locally isometric to the sphere $S^{2n+1}(1)$, or M is locally isometric to the above example of an $N(1 - \frac{1}{n})$-manifold.

We close this essay with the question of conformally flat contact metric manifolds, a question in which there has recently been renewed interest. Early on, Okumura [34] had shown that a conformally flat Sasakian manifold of dimension ≥ 5 is of constant curvature $+1$ and in [42] and [43] Tanno extended this result to the K-contact case and for dimensions ≥ 3. Thus a conformally flat K-contact manifold is of constant curvature $+1$ and Sasakian. Recently Ghosh, Koufogiorgos and Sharma [24] have shown that a conformally flat contact metric manifold of dimension ≥ 5 with a strongly pseudo-convex integrable CR-structure is of constant curvature $+1$. As we have seen, in dimension ≥ 5, a contact metric structure of constant curvature must be of constant curvature $+1$ and is Sasakian; and in dimension 3, a contact metric structure of constant curvature must be of constant curvature 0 or $+1$, the latter case being Sasakian. For simplicity set $lX = R_{X\xi}\xi$, then l is a symmetric operator. K. Bang [2] showed that in dimension ≥ 5 there are no conformally flat contact metric structures with $l = 0$, even though there are many contact metric manifolds satisfying $l = 0$, ([2] or see

[7] p. 153). Bang's result was extended to dimension 3 and generalized by F. Gouli-Andreou and Ph. Xenos [26] who showed that in dimension 3 the only conformally flat contact metric structures satisfying $\nabla_\xi l = 0$ (equivalently $\nabla_\xi h = 0$, Perrone [38]) are those of constant curvature 0 or 1. In [25] F. Gouli-Andreou and N. Tsolakidou showed that a conformally flat contact metric manifold M^{2n+1} with $l = -\kappa\phi^2$ for some function κ is of constant curvature. In the case of the standard contact metric structure on the tangent sphere bundle, Th. Koufogiorgos and the author [9] showed that the metric is conformally flat, if and only if the base manifold is a surface of constant Gaussian curvature 0 or 1. The (κ, μ)-spaces are conformally flat only in the constant curvature cases. In dimension 3, this was shown by F. Gouli-Andreou and Ph. Xenos [26], even when κ and μ are functions. In higher dimensions the proof is straightforward: Let W denote the Weyl conformal curvature tensor. $W_{X\xi}\xi = 0$ with $X \perp \xi$ yields $[2(n-1)(\mu-1)/2n-1]hX = 0$; if $n = 1$ we have the case studied by Gouli-Andreou and Xenos and if $h = 0$ we have the K-contact case. If $\mu = 1$, $h \neq 0$ and $n > 1$, we can choose two orthogonal unit eigenvectors X and Y of h with eigenvalue $\lambda > 0$ and set $Z = \phi Y$. Then using Theorem 1 of [10], $W_{XY}Z = 0$ yields $\kappa = 1$ $(\lambda = 0)$, contradicting $\lambda > 0$. In [9] Th. Koufogiorgos and the author showed that a conformally flat contact metric manifold on which the Ricci operator commutes with ϕ is of constant curvature. Then in [21] G. Calvaruso, D. Perrone and L. Vanhecke showed that in dimension 3 the only conformally flat contact metric structures, for which ξ is an eigenvector of the Ricci operator, are those of constant curvature 0 or 1. An attempt was made in [24] to generalize this to higher dimensions by assuming another condition in addition to ξ being an eigenvector of the Ricci operator. However ξ being an eigenvector of the Ricci operator is the essential condition and we now have a recent result of K. Bang and the author [3] generalizing the Calvaruso, Perrone and Vanhecke result to higher dimensions.

Theorem. *A conformally flat contact metric manifold for which the characteristic vector field is an eigenvector of the Ricci operator is of constant curvature.*

In view of these strong curvature results, one may ask if there are any conformally flat contact metric structures which are not of constant curvature. In [7] (pp. 108–110), the author shows that 3-dimensional conformally flat contact metric manifolds of non-constant curvature do exist. These examples were studied further by Calvaruso [20]; he showed that these examples satisfy $\nabla_\xi h = ah\phi$, where a is a non-constant function. He also showed that if a is a constant $\neq 2$, then a 3-dimensional conformally flat contact metric manifold satisfying $\nabla_\xi h = ah\phi$ has constant curvature. It is not known if there exist conformally flat contact metric manifolds of dimension ≥ 5 which are not locally isometric to the standard Sasakian structure on the unit sphere.

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