DIFFERENTIAL COMPLEXES AND STABILITY OF FINITE ELEMENT METHODS II: THE ELASTICITY COMPLEX

DOUGLAS N. ARNOLD*, RICHARD S. FALK†, AND RAGNAR WINTHER‡

Abstract. A close connection between the ordinary de Rham complex and a corresponding elasticity complex is utilized to derive new mixed finite element methods for linear elasticity. For a formulation with weakly imposed symmetry, this approach leads to methods which are simpler than those previously obtained. For example, we construct stable discretizations which use only piecewise linear elements to approximate the stress field and piecewise constant functions to approximate the displacement field. We also discuss how the strongly symmetric methods proposed in [8] can be derived in the present framework. The method of construction works in both two and three space dimensions, but for simplicity the discussion here is limited to the two dimensional case.

Key words. Mixed finite element method, Hellinger–Reissner principle, elasticity.

AMS(MOS) subject classifications. Primary: 65N30, Secondary: 74S05.

1. Introduction. In this paper we discuss finite element methods for the equations of linear elasticity derived from the Hellinger–Reissner variational principle. The equations can be written as a system of the form

$$A\sigma = \epsilon\, u, \qquad \operatorname{div} \sigma = f \quad \text{in } \Omega. \tag{1.1}$$

The unknowns σ and u denote the stress and displacement fields engendered by a body force f acting on a linearly elastic body that occupies a region $\Omega \subset \mathbb{R}^n$, where $n = 2$ or 3. Then σ takes values in the space $\mathbb{S} = \mathbb{R}^{n\times n}_{\text{sym}}$ of symmetric matrices and u takes values in $\mathbb{R}^n$. The differential operator ϵ is the symmetric part of the gradient, the div operator is applied row-wise to a matrix, and the compliance tensor $A = A(x) : \mathbb{S} \to \mathbb{S}$ is a bounded and symmetric, uniformly positive definite operator reflecting the properties of the body. We shall assume that the body is clamped on the boundary $\partial\Omega$ of Ω, so that the proper boundary condition for the system (1.1) is $u = 0$ on $\partial\Omega$.

Alternatively, the pair (σ, u) can be characterized as the unique critical point of the Hellinger–Reissner functional

$$\mathcal{J}(\tau, v) = \int_\Omega \big(\frac{1}{2} A\tau : \tau + \operatorname{div} \tau \cdot v - f \cdot v\big)\, dx. \tag{1.2}$$

*Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, MN 55455. The work of the first author was supported in part by NSF grant DMS-0411388.

†Department of Mathematics, Rutgers University, Hill Center, 110 Frelinghuysen Rd, Rutgers University, Piscataway, NJ 08854-8019. The work of the second author was supported in part by NSF grant DMS03-08347.

‡Centre of Mathematics for Applications and Department of Informatics, University of Oslo, P.O. Box 1053, 0316 Oslo, NORWAY. The work of the third author was supported by the Norwegian Research Council.

The critical point is sought among all $\tau \in H(\mathrm{div}, \Omega; \mathbb{S})$, the space of square-integrable symmetric matrix fields with square-integrable divergence, and all $v \in L^2(\Omega; \mathbb{R}^n)$, the space of square-integrable vector fields. Equivalently, $(\sigma, u) \in H(\mathrm{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^n)$ is the unique solution to the following weak formulation of the system (1.1)

$$\begin{aligned} \int_\Omega (A\sigma : \tau + \mathrm{div}\, \tau \cdot u)\, dx = 0, \quad & \tau \in H(\mathrm{div}, \Omega; \mathbb{S}), \\ \int_\Omega \mathrm{div}\, \sigma \cdot v\, dx = \int_\Omega f v\, dx, \quad & v \in L^2(\Omega; \mathbb{R}^n). \end{aligned} \tag{1.3}$$

A mixed finite element method determines an approximate stress field σ_h and an approximate displacement field u_h as the critical point of $\mathcal{J}$ over $\Sigma_h \times V_h$ where $\Sigma_h \subset H(\mathrm{div}, \Omega; \mathbb{S})$ and $V_h \subset L^2(\Omega; \mathbb{R}^n)$ are suitable piecewise polynomial subspaces. To ensure that a unique critical point exists and that it provides a good approximation of the true solution, the subspaces Σ_h and V_h must satisfy the stability conditions from Brezzi's theory of mixed methods [11, 12]. However, the construction of such elements has proved to be surprisingly hard, and most of the known results are limited to two space dimensions. In this case, a family of stable finite elements was presented in [8]. For the lowest order element, the space Σ_h is composed of piecewise cubic functions, while the space V_h consists of piecewise linear functions. Another approach that has proved successful in finding stable elements is the use of composite elements, in which V_h consists of piecewise polynomials with respect to one triangulation of the domain, while Σ_h consists of piecewise polynomials with respect to a different, more refined, triangulation [3, 15, 17, 23].

In the search for low order stable elements, several authors have resorted to the use of Lagrangian functionals that are modifications of the Hellinger–Reissner functional given above [1, 2, 4, 19, 20, 21, 22], in which the symmetry of the stress tensor is enforced only weakly or abandoned altogether. In order to discuss these methods, we extend the compliance tensor $A(x)$ to a symmetric and positive definite operator mapping $\mathbb{M}$ into $\mathbb{M}$, where $\mathbb{M}$ is the space of $n \times n$ matrices. In the isotropic case, the mapping $\sigma \mapsto A\sigma$ has the form

$$A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + n\lambda} \operatorname{tr}(\sigma) I\right),$$

where $\lambda(x), \mu(x)$ are positive scalar coefficients, the Lamé coefficients. A modification of the variational principle discussed above is obtained if we consider the extended Hellinger–Reissner functional

$$\mathcal{J}_e(\tau, v, q) = \mathcal{J}(\tau, v) + \int_\Omega \tau : q\, dx \tag{1.4}$$

over the space $H(\mathrm{div}, \Omega; \mathbb{M}) \times L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{K})$, where $\mathbb{K}$ denotes the space of skew symmetric matrices. We note that the symmetry condition for the space of matrix fields is now enforced through the introduction of a Lagrange multiplier. A critical point (σ, u, p) of the functional $\mathcal{J}_e$ is characterized as the unique solution of the system

$$\begin{gathered}\int_\Omega (A\sigma : \tau + \operatorname{div}\tau \cdot u + \tau : p)\, dx = 0, \quad \tau \in H(\operatorname{div}, \Omega; \mathbb{M}), \\ \int_\Omega \operatorname{div}\sigma \cdot v\, dx = \int_\Omega f v\, dx, \quad v \in L^2(\Omega; \mathbb{R}^n), \\ \int_\Omega \sigma : q\, dx = 0, \quad q \in L^2(\Omega; \mathbb{K}).\end{gathered} \tag{1.5}$$

In fact, it is clear that if (σ, u, p) is a solution of this system, then σ is symmetric, i.e., $\sigma \in H(\operatorname{div}, \Omega; \mathbb{S})$, and the pair $(\sigma, u) \in H(\operatorname{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^n)$ solves the corresponding system (1.3). In this respect, the two systems (1.3) and (1.5) are equivalent. However, the extended system (1.5) leads to new possibilities for discretization. Assume that we choose finite element spaces $\Sigma_h \times V_h \times Q_h \subset H(\operatorname{div}, \Omega; \mathbb{M}) \times L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{K})$ and consider a discrete system corresponding to (1.5). If $(\sigma_h, u_h, p_h) \in \Sigma_h \times V_h \times Q_h$ is a discrete solution, then σ_h will not necessary inherit the symmetry property of σ. Instead, σ_h will satisfy the weak symmetry condition

$$\int_\Omega \sigma_h : q\, dx = 0, \qquad \text{for all } q \in Q_h.$$

Therefore, these solutions will in general not correspond to solutions of the discrete system obtained from (1.3).

Discretizations based on the system (1.5) will be referred to as mixed finite element methods with weakly imposed symmetry. For two space dimensions, such discretizations were already introduced by Fraejis de Veubeke in [15] and further developed in [2]. In particular, the so-called PEERS element proposed in [2] used an augmented Cartesian product of the Raviart–Thomas finite element space to approximate the stress σ, piecewise constants to approximate the displacements, and continuous piecewise linear functions to approximate the Lagrange multiplier p, as suggested in the element diagram depicted in Fig. 1. In this paper we use homological

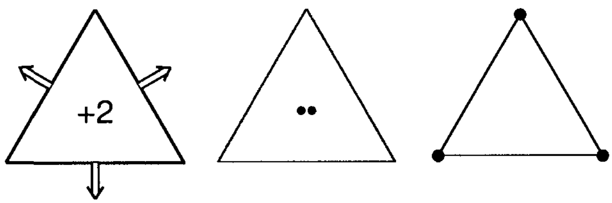

FIG. 1. *Approximation of stress, displacement, and multiplier for PEERS.*

techniques to construct a new family of stable mixed finite elements for elasticity with weakly imposed symmetry, the lowest order case of which is depicted in Fig. 2. The stresses are approximated by the Cartesian product of two copies of the Brezzi–Douglas–Marini finite element space, which means that the shape functions are simply all linear matrix fields and that there are four degrees of freedom per edge. The displacements are approximated by piecewise constants, as for PEERS, but the multipliers are as well, which means that, in contrast to PEERS, the multipliers can be eliminated by static condensation. We will also introduce a reduced version of the element with the same displacement and multiplier spaces, but only three degrees of freedom per edge for the stress. Let us also mention that there exist other mixed elements for elasticity with weakly imposed symmetry, although perhaps none as simple as those presented here. Prior to the PEERS paper, Amara and Thomas [1] developed methods with weakly imposed symmetry using a dual hybrid approach. Other elements based on the formulation in [2], including rectangular elements and elements in three space dimensions, have been developed in [20], [21], [22] and [18].

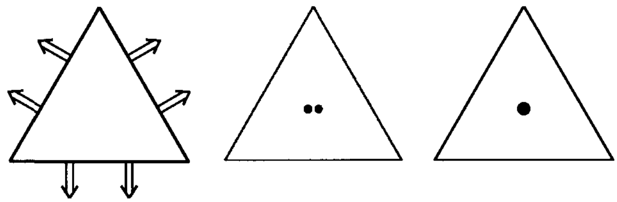

FIG. 2. *Approximation of stress, displacement, and multiplier for an element introduced below.*

Just as there is a close connection between mixed finite elements for Poisson's problem and discretization of the de Rham complex, there is also a close connection between mixed finite elements for elasticity and discretization of another differential complex, the elasticity complex. The importance of this complex was already recognized in [8], where mixed methods for elasticity in two space dimensions were discussed. However, the new ingredient here is that we utilize a close connection between the elasticity complex and the ordinary de Rham complex. This connection is described in Eastwood [13] and is based on a general construction given in [10], the Bernstein–Gelfand–Gelfand resolution. By mimicking this construction in the discrete case, we will be able to derive new mixed finite elements for elasticity in a systematic manner from known discretizations of the de Rham complex. The discussion here will be limited to two space dimensions. However, in a forthcoming paper [7], we will carry out the analogous construction and so obtain mixed finite element methods in three space dimensions.

An outline of the paper is as follows. In Section 2, we describe the notation to be used and recall some standard results about the stability of mixed finite element methods. In Section 3, we give two complexes related to the two mixed formulations of elasticity given by (1.3) and (1.5). In Section 4, we use the framework of differential forms to show how the elasticity complex can be derived from the de Rham complex (basically following the work of Eastwood [13]). In Section 5, we derive discrete analogues of these elasticity complexes beginning from discrete analogues of the de Rham complex, identify the required properties of the discrete spaces necessary for this construction, and explain how a discrete elasticity complex leads to stable finite element methods. In Section 6, we provide examples of finite element spaces that satisfy these conditions. The PEERS element is also discussed in this context. Finally, in Section 7, we show how an element with strongly imposed symmetry, previously obtained in [8], can be derived from discrete de Rham complexes using the methodology introduced in this paper.

2. Notation and preliminaries. We begin with some basic notation and hypotheses. We denote by $\mathbb{M}$ the space of all 2×2 real matrices and by $\mathbb{S}$ and $\mathbb{K}$ the subspaces of symmetric and skew symmetric matrices, respectively. The operators $\operatorname{sym} : \mathbb{M} \to \mathbb{S}$ and $\operatorname{skw} : \mathbb{M} \to \mathbb{K}$ denote the symmetric and skew symmetric parts, respectively. We assume that Ω is a simply connected domain in $\mathbb{R}^2$ with boundary Γ. We shall use the standard function spaces, like the Lebesgue space $L^2(\Omega)$ and the Sobolev space $H^s(\Omega)$. For vector-valued functions, we include the range space in the notation following a semicolon, so $L^2(\Omega; \mathbb{V})$ denotes the space of square integrable functions mapping Ω into a normed vector space $\mathbb{V}$. The space $H(\operatorname{div}, \Omega; \mathbb{R}^2)$ denotes the subspace of (vector-valued) functions in $L^2(\Omega; \mathbb{R}^2)$ whose divergence belongs to $L^2(\Omega)$. Similarly, $H(\operatorname{div}, \Omega; \mathbb{M})$ denotes the subspace of (matrix-valued) functions in $L^2(\Omega; \mathbb{M})$ whose divergence (by rows) belongs to $L^2(\Omega; \mathbb{R}^2)$.

Assuming that $\mathbb{V}$ is an inner product space, then $L^2(\Omega; \mathbb{V})$ has a natural norm and inner product, which will be denoted by $\| \cdot \|$ and $(\cdot, \cdot)$, respectively. For a Sobolev space $H^s(\Omega; \mathbb{V})$, we denote the norm by $\| \cdot \|_s$ and for $H(\operatorname{div}, \Omega; \mathbb{V})$, the norm is denoted by $\|v\|_{\operatorname{div}} := (\|v\|^2 + \|\operatorname{div} v\|^2)^{1/2}$. The space $\mathcal{P}_k(\Omega)$ denotes the space of polynomial functions on Ω of total degree $\leq k$. Usually we abbreviate this to just $\mathcal{P}_k$.

We recall that the mixed finite element approximation derived from (1.5) takes the form:

Find $(\sigma_h, u_h, p_h) \in \Sigma_h \times V_h \times Q_h$ such that

$$\begin{aligned} (A\sigma_h, \tau) + (\operatorname{div} \tau, u_h) + (\tau, p_h) = 0, &\quad \tau \in \Sigma_h, \\ (\operatorname{div} \sigma_h, v) = (f, v) &\quad v \in V_h, \\ (\sigma_h, q) = 0, &\quad q \in Q_h, \end{aligned} \tag{2.1}$$

where $\Sigma_h \subset H(\operatorname{div}, \Omega; \mathbb{M})$, $V_h \subset L^2(\Omega; \mathbb{R}^2)$, and $Q_h \in L^2(\Omega; \mathbb{K})$ are finite element spaces with h a mesh size parameter. Following the general theory of mixed methods, cf. [11, 12], the stability of the saddle–point system (2.1) is ensured by the following conditions:

(A1) $\|\tau\|_{\mathrm{div}}^2 \le c_1(A\tau, \tau)$ whenever $\tau \in \Sigma_h$ satisfies $(\operatorname{div}\tau, v) = 0 \ \forall v \in V_h$ and $(\tau, q) = 0 \ \forall q \in Q_h$,

(A2) for all nonzero $(v, q) \in V_h \times Q_h$, there exists nonzero $\tau \in \Sigma_h$ with $(\operatorname{div}\tau, v) + (\tau, q) \ge c_2\|\tau\|_{\mathrm{div}}(\|v\| + \|q\|)$,

where c_1 and c_2 are positive constants independent of h.

If we instead derive the mixed finite element method from the weak formulation (1.3), we need to construct finite element subspaces $\Sigma_h \subset H(\mathrm{div}, \Omega; \mathbb{S})$, i.e., with the symmetry condition strongly imposed, and $V_h \subset L^2(\Omega; \mathbb{R}^2)$. The discrete system then determines $(\sigma_h, u_h) \in \Sigma_h \times V_h$ by the equations

$$\begin{aligned} (A\sigma_h, \tau) + (\operatorname{div}\tau, u_h) &= 0, \quad \tau \in \Sigma_h, \\ (\operatorname{div}\sigma_h, v) &= (f, v) \quad v \in V_h. \end{aligned} \tag{2.2}$$

In this case, the stability condition is that Σ_h and V_h must satisfy (A1) and (A2) with $Q_h = 0$. As we shall see below, it is much harder to construct stable elements for elasticity with strongly imposed symmetry than it is with weakly imposed symmetry.

In the preceding paper [6], we have seen the close connection between the construction of stable mixed finite element methods for the approximation of the Poisson problem

$$\Delta p = f \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

and discrete versions of the de Rham complex. In this paper, we pursue an analogous approach for the elasticity problem.

3. The elasticity complex. We now proceed to a description of two elasticity complexes, corresponding to strongly and weakly imposed symmetry of the stress tensor. For the case of strongly imposed symmetry, corresponding to the mixed elasticity system (1.3), we require a characterization of the divergence-free symmetric matrix fields. In order to give such a characterization, define $J : C^\infty(\Omega) \to C^\infty(\Omega; \mathbb{S})$ by

$$Jq = \begin{pmatrix} \partial^2 q/\partial x_2^2 & -\partial^2 q/\partial x_1 \partial x_2 \\ -\partial^2 q/\partial x_1 \partial x_2 & \partial^2 q/\partial x_1^2 \end{pmatrix}.$$

It is easy to check that $\operatorname{div} \circ J = 0$. In other words,

$$\mathcal{P}_1 \hookrightarrow C^\infty \xrightarrow{J} C^\infty(\mathbb{S}) \xrightarrow{\operatorname{div}} C^\infty(\mathbb{R}^2) \to 0, \tag{3.1}$$

is a complex. Here, and frequently in the sequel, the dependence of the domain Ω is suppressed, i.e., $C^\infty(\mathbb{S})$ is short for $C^\infty(\Omega; \mathbb{S})$. When Ω is simply connected, then (3.1) is an exact sequence, a fact which will follow from the discussion below. The complex (3.1) will be referred to as the elasticity complex. If we followed the program that has been outlined in [6] for mixed methods for scalar second order elliptic equations, the construction of stable mixed finite elements for elasticity would be based on extending the sequence (3.1) to a complete commuting diagram of the form

$$\begin{array}{ccccccccc} \mathcal{P}_1 & \hookrightarrow & C^\infty & \xrightarrow{J} & C^\infty(\mathbb{S}) & \xrightarrow{\operatorname{div}} & C^\infty(\mathbb{R}^2) & \to & 0 \\ & & \downarrow \Pi_h^2 & & \downarrow \Pi_h^d & & \downarrow \Pi_h^0 & & \\ \mathcal{P}_1 & \hookrightarrow & W_h & \xrightarrow{J} & \Sigma_h & \xrightarrow{\operatorname{div}} & V_h & \to & 0 \end{array}$$

where $W_h \subset H^2(\Omega)$, $\Sigma_h \subset H(\operatorname{div}, \Omega; \mathbb{S})$ and $V_h \subset L^2(\Omega; \mathbb{R}^2)$ are suitable finite element spaces and Π_h^2, Π_h^d, and Π_h^0 are corresponding interpolation operators. This is exactly the construction performed in [8]. In particular, since the finite element space W_h is required to be a subspace of $H^2(\Omega)$, we can conclude that the piecewise polynomial space W_h must contain quintic polynomials, and therefore the lowest order space Σ_h will at least involve piecewise cubics. In fact, for the lowest order elements discussed in [8], W_h is the classical Argyris space, while Σ_h consists of piecewise cubic symmetric matrix fields with a linear divergence. In Section 7 we shall show how the element proposed in [8] arises naturally from the general construction outlined below.

If instead we consider methods with weakly imposed symmetry, i.e., finite element methods based on the mixed formulation (1.5), we are led to study the complex

$$\mathcal{P}_1 \hookrightarrow C^\infty \xrightarrow{J} C^\infty(\mathbb{M}) \xrightarrow{(\operatorname{skw},\operatorname{div})} C^\infty(\mathbb{K} \times \mathbb{R}^2) \to 0. \tag{3.2}$$

Observe that there is a close connection between (3.1) and (3.2). In fact, (3.1) can be derived from (3.2) by performing a projection step. To see this, consider the diagram

$$\begin{array}{ccccccccc} \mathcal{P}_1 & \hookrightarrow & C^\infty & \xrightarrow{J} & C^\infty(\mathbb{M}) & \xrightarrow{(\operatorname{skw},\operatorname{div})} & C^\infty(\mathbb{K} \times \mathbb{R}^2) & \to & 0 \\ & & \downarrow id & & \downarrow \operatorname{sym} & & \downarrow \pi & & \\ \mathcal{P}_1 & \hookrightarrow & C^\infty & \xrightarrow{J} & C^\infty(\mathbb{S}) & \xrightarrow{\operatorname{div}} & C^\infty(\mathbb{R}^2) & \to & 0, \end{array} \tag{3.3}$$

where $\pi(q, u) = u - \operatorname{div} q$. The vertical maps are projections onto subspaces and the diagram commutes. It follows by a simple diagram chase that if the first row is exact, so is the second.

As we shall see below, the complexes (3.1) and (3.2) are closely connected to the standard de Rham complex. In two space dimensions, the de Rham complex is equivalent to the complex

$$\mathbb{R} \hookrightarrow C^\infty \xrightarrow{\operatorname{grad}} C^\infty(\mathbb{R}^2) \xrightarrow{\operatorname{rot}} C^\infty \to 0, \tag{3.4}$$

which is exact when Ω is simply connected. Here $\operatorname{rot} v$, where v is a vector field, is defined as the scalar field $\operatorname{rot} v = \partial v_1/\partial x_2 - \partial v_2/\partial x_1$.

An alternative identification of the de Rham complex in two space dimensions, that we shall use below, is the sequence

$$\mathbb{R} \hookrightarrow C^\infty \xrightarrow{\text{curl}} C^\infty(\mathbb{R}^2) \xrightarrow{\text{div}} C^\infty \to 0, \tag{3.5}$$

where $\operatorname{curl} \phi$ is the vector field defined by $\operatorname{curl} \phi = (-\partial\phi/\partial x_2, \partial\phi/\partial x_1)^T$. The two complexes (3.4) and (3.5) are equivalent. To see this just note that $\operatorname{curl} \phi = (\operatorname{grad} \phi)^\perp$ and $\operatorname{rot} v = \operatorname{div}(v^\perp)$, where $v^\perp$ denotes the vector perpendicular to v given by $v^\perp = (-v_2, v_1)^T$.

4. From the de Rham complex to linear elasticity. In this section we demonstrate the connection between the de Rham complex (3.4) and the elasticity complexes (3.1) and (3.2). Later, we will give an analogous construction to derive discrete elasticity complexes from corresponding discrete de Rham complexes.

We follow the notations of [6] for differential forms. Thus for Ω a domain in $\mathbb{R}^n$, $\Lambda^k = \Lambda^k(\Omega) = C^\infty(\Omega; \operatorname{Alt}^k(\mathbb{R}^n))$ denotes the space of smooth differential k-forms on Ω. Any $\omega \in \Lambda^k$ can be represented as

$$\omega_x = \sum_{i_1<i_2<\ldots<i_k} f_{i_1\ldots i_k}(x)dx^{i_1} \wedge \ldots \wedge dx^{i_k} =: \sum_I f_I(x)dx^I \tag{4.1}$$

with coefficients $f_I \in C^\infty(\Omega)$. In particular, 0–forms can be identified with scalar functions, 1–forms with vector fields under the identification $f_i dx^i \leftrightarrow f_i e_i$, and n–forms can be identified with the scalar function $f_{12\ldots n}$. The spaces $L^2\Lambda^k(\Omega)$, $H^1\Lambda^k(\Omega)$, $\ldots$, consist of those ω which can be represented as in (4.1) with the $f_I \in L^2(\Omega)$, $H^1(\Omega)$, $\ldots$.

The exterior derivative $d : \Lambda^k \to \Lambda^{k+1}$ satisfies

$$d\omega = \sum_{j,I} \frac{\partial f_I}{\partial x_j} dx^j \wedge dx^I,$$

and the de Rham complex is simply

$$\mathbb{R} \hookrightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \ldots \xrightarrow{d} \Lambda^n \to 0. \tag{4.2}$$

When $n = 2$, (4.2) becomes (3.4) under the identifications mentioned above. If we instead identify the 1–form $\omega = f_1 dx^1 + f_2 dx^2$ with the vector field $(-f_2, f_1)^T$, we obtain (3.5).

A differential k-form ω on Ω, admits a natural trace, $\operatorname{Tr} \omega$, which is a differential k-form on $\Gamma = \partial\Omega$. Namely, given k vectors $v_1, \cdots, v_k$ tangent to Γ at a point x, we have

$$(\operatorname{Tr} \omega)_x(v_1, \cdots, v_k) = \omega_x(v_1, \cdots, v_k).$$

Denoting by $d_\Gamma : \Lambda^k(\Gamma) \to \Lambda^{k+1}(\Gamma)$ the exterior derivative operator associated with Γ, we have a commuting diagram relating the de Rham complexes on Ω and Γ

$$\begin{array}{ccccccccccc} \mathbb{R} & \hookrightarrow & \Lambda^0(\Omega) & \xrightarrow{d} & \Lambda^1(\Omega) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Lambda^{n-1}(\Omega) & \xrightarrow{d} & \Lambda^n(\Omega) \to 0 \\ & & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} & & & & \downarrow{\scriptstyle \mathrm{Tr}} & & \\ \mathbb{R} & \hookrightarrow & \Lambda^0(\Gamma) & \xrightarrow{d_\Gamma} & \Lambda^1(\Gamma) & \xrightarrow{d_\Gamma} & \cdots & \xrightarrow{d_\Gamma} & \Lambda^{n-1}(\Gamma) & \to & 0. \end{array} \tag{4.3}$$

The extension to vector-valued differential forms will be important in the sequel. If $\mathbb{V}$ is a vector space, then $\Lambda^k(\mathbb{V}) = \Lambda^k(\Omega; \mathbb{V})$ refers to the k–forms with values in $\mathbb{V}$, i.e., all elements of the form (4.1), but where $f_I \in C^\infty(\Omega; \mathbb{V})$, i.e., $\Lambda^k(\mathbb{V}) = C^\infty(\Omega; \mathrm{Alt}^k(\mathbb{V}))$, where $\mathrm{Alt}^k(\mathbb{V})$ are alternating k-linear forms $\mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{V}$.

The exactness of the $\mathbb{V}$-valued de Rham complex

$$\mathbb{V} \hookrightarrow \Lambda^0(\mathbb{V}) \xrightarrow{d} \Lambda^1(\mathbb{V}) \xrightarrow{d} \ldots \xrightarrow{d} \Lambda^n(\mathbb{V}) \to 0, \tag{4.4}$$

for Ω contractible is an obvious consequence of the exactness of (4.2).

We now specialize to the case $n = 2$ and $\Omega \subset \mathbb{R}^2$, and derive the elasticity complex from the de Rham complex with values in the three-dimensional vector space $\mathbb{V} = \mathbb{R} \times \mathbb{R}^2$. Define a map K from $\Lambda^k(\mathbb{R}^2)$ to $\Lambda^k(\mathbb{R})$ by

$$\sum_I f_I(x) dx^I \mapsto \sum_I [f_I(x) \cdot x^\perp] dx^I.$$

If $(\omega, \mu) \in \Lambda^k(\mathbb{R}) \times \Lambda^k(\mathbb{R}^2) = \Lambda^k(\mathbb{V})$, then the map $\Phi(\omega, \mu) := (\omega + K\mu, \mu)$ is an automorphism of $\Lambda^k(\mathbb{V})$, with inverse $\Phi^{-1}(\omega, \mu) = (\omega - K\mu, \mu)$. Define the operator $\mathcal{A} : \Lambda^k(\mathbb{V}) \to \Lambda^{k+1}(\mathbb{V})$ by $\mathcal{A} = \Phi d \Phi^{-1}$. Then the complex

$$\Phi(\mathbb{V}) \hookrightarrow \Lambda^0(\mathbb{V}) \xrightarrow{\mathcal{A}} \Lambda^1(\mathbb{V}) \xrightarrow{\mathcal{A}} \Lambda^2(\mathbb{V}) \to 0 \tag{4.5}$$

is exact when Ω is simply connected, since (4.4) is. The operator $\mathcal{A}$ has the simple form $\mathcal{A}(\omega, \mu) = (d\omega - S\mu, d\mu)$, where $S = dK - Kd : \Lambda^k(\mathbb{R}^2) \to \Lambda^{k+1}(\mathbb{R})$. Since $d \circ d = 0$,

$$dS = d^2K - dKd = -(dK - Kd)d = -Sd. \tag{4.6}$$

Furthermore, S is purely algebraic. In fact, an easy calculation shows that if ω is represented as in (4.1) then

$$S\omega = \sum_I (f_I \cdot e_2 dx^1 \wedge dx^I - f_I \cdot e_1 dx^2 \wedge dx^I).$$

More specifically the action of $S = S_k : \Lambda^k(\mathbb{R}^2) \to \Lambda^{k+1}(\mathbb{R})$, $k = 0, 1$, is given by

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \xmapsto{S_0} f_2 dx^1 - f_1 dx^2,$$
$$\begin{pmatrix} f_{12} \\ f_{22} \end{pmatrix} dx^1 - \begin{pmatrix} f_{11} \\ f_{21} \end{pmatrix} dx^2 \xmapsto{S_1} (f_{12} - f_{21}) dx^1 \wedge dx^2.$$

It is important to note that S_0 is invertible (with $S_0^{-1}(f_1 dx^1 + f_2 dx^2) = (-f_2, f_1)^T$). The map S_1 is surjective but not invertible. If we identify $\Lambda^1(\mathbb{R}^2)$ with $C^\infty(\Omega, \mathbb{M})$ by

$$\begin{pmatrix} f_{12} \\ f_{22} \end{pmatrix} dx^1 - \begin{pmatrix} f_{11} \\ f_{21} \end{pmatrix} dx^2 \leftrightarrow (f_{ij}), \tag{4.7}$$

then the kernel of S_1 corresponds to the symmetric matrices.

Note that

$$\Phi(\mathbb{V}) = \{\, (\omega + \mu \cdot x^\perp, \mu) \,|\, \omega \in \mathbb{R}, \mu \in \mathbb{R}^2 \,\} = \{\, (p, S^{-1} dp) \,|\, p \in \mathcal{P}_1 \,\} \cong \mathcal{P}_1,$$

so (4.5) may be viewed as a resolution of $\mathcal{P}_1$.

We now consider a projection of (4.5) onto a subcomplex. Let

$$\Gamma^0 = \{\, (\omega, \mu) \in \Lambda^0(\mathbb{V}) : d\omega = S_0 \mu \,\}, \quad \Gamma^1 = \{\, (\omega, \mu) \in \Lambda^1(\mathbb{V}) : \omega = 0 \,\}$$

and define projections $\pi^0 : \Lambda^0(\mathbb{V}) \to \Gamma^0$, $\pi^1 : \Lambda^1(\mathbb{V}) \to \Gamma^1$ by

$$\pi^0(\omega, \mu) = (\omega, S_0^{-1} d\omega), \quad \pi^1(\omega, \mu) = (0, \mu + d S_0^{-1} \omega).$$

Then the diagram

$$\begin{array}{ccccccccc} \Phi(\mathbb{V}) & \hookrightarrow & \Lambda^0(\mathbb{V}) & \xrightarrow{\mathcal{A}} & \Lambda^1(\mathbb{V}) & \xrightarrow{\mathcal{A}} & \Lambda^2(\mathbb{V}) & \to & 0 \\ & & \Big\downarrow{\scriptstyle \pi^0} & & \Big\downarrow{\scriptstyle \pi^1} & & \Big\downarrow{\scriptstyle id} & & \\ \Phi(\mathbb{V}) & \hookrightarrow & \Gamma^0 & \xrightarrow{\mathcal{A}} & \Gamma^1 & \xrightarrow{\mathcal{A}} & \Lambda^2(\mathbb{V}) & \to & 0, \end{array} \tag{4.8}$$

commutes, and so when the first row is exact, the second is as well. Making the obvious correspondences $(\omega, S_0^{-1} d\omega) \leftrightarrow \omega$ and $(0, \mu) \leftrightarrow \mu$, we may identify Γ^0 and Γ^1 with $\Lambda^0(\mathbb{R})$ and $\Lambda^1(\mathbb{R}^2)$, respectively. Thus the bottom row of (4.8) is equivalent to

$$\mathcal{P}_1 \hookrightarrow \Lambda^0(\mathbb{R}) \xrightarrow{d \circ S_0^{-1} \circ d} \Lambda^1(\mathbb{R}^2) \xrightarrow{(-S_1, d)} \Lambda^2(\mathbb{V}) \to 0. \tag{4.9}$$

But this is just another way to write (3.2). In fact, $\Lambda^0(\mathbb{R}) = C^\infty$ and we may identify $\Lambda^1(\mathbb{R}^2)$ with $C^\infty(\mathbb{M})$ as in (4.7). Also, we may identify $\Lambda^2(\mathbb{V})$ with $C^\infty(\mathbb{K} \times \mathbb{R}^2)$ by

$$\left(f, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right) dx^1 \wedge dx^2 \leftrightarrow - \left(\begin{pmatrix} 0 & f/2 \\ -f/2 & 0 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right). \tag{4.10}$$

It is easy to check that, modulo these identifications, (4.9) coincides with (3.2).

Let us summarize the above construction. We began with the $\mathbb{V}$-valued de Rham complex (4.4) and introduced the automorphisms $\mathcal{A}$ to get (4.5). We then projected onto a subcomplex in (4.8) and made some simple identifications to obtain the elasticity complex with weakly imposed symmetry, (3.2). (Of course, we can make the further projection in (3.3) to obtain the elasticity complex with strongly imposed symmetry.)

5. The construction of a discrete elasticity complex. In this section we mimic the above construction on a discrete level to derive discretizations of the elasticity complex from discretizations of the de Rham complex, and use these to derive stable mixed finite elements for elasticity with weakly imposed symmetry.

As explained in [6], there exist a number of discrete de Rham complexes, i.e., complexes of the form

$$\mathbb{R} \hookrightarrow \Lambda_h^0 \xrightarrow{d} \Lambda_h^1 \xrightarrow{d} \Lambda_h^2 \to 0. \tag{5.1}$$

Here the spaces Λ_h^k are spaces of piecewise polynomial differential forms and there exist projections $\Pi_h = \Pi_h^k : \Lambda^k \to \Lambda_h^k$ such that the diagram

$$\begin{array}{ccccccccc} \mathbb{R} & \hookrightarrow & \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \to & 0 \\ & & \downarrow{\scriptstyle \Pi_h} & & \downarrow{\scriptstyle \Pi_h} & & \downarrow{\scriptstyle \Pi_h} & & \\ \mathbb{R} & \hookrightarrow & \Lambda_h^0 & \xrightarrow{d} & \Lambda_h^1 & \xrightarrow{d} & \Lambda_h^2 & \to & 0 \end{array} \tag{5.2}$$

commutes.

Our discrete construction begins by taking two discretizations of the de Rham complex, one scalar-valued and one vector-valued. The Cartesian product of these then gives a discretization of the $\mathbb{V}$-valued complex (4.4) which we write

$$\mathbb{V} \hookrightarrow \Lambda_h^0(\mathbb{V}) \xrightarrow{d} \Lambda_h^1(\mathbb{V}) \xrightarrow{d} \Lambda_h^2(\mathbb{V}) \to 0. \tag{5.3}$$

Next we define a discrete analog of the operator K, $K_h : \Lambda_h^k(\mathbb{R}^2) \to \Lambda_h^k(\mathbb{R})$ by $K_h = \Pi_h K$, where Π_h is the projection onto $\Lambda_h^k(\mathbb{R})$ and set $S_h = dK_h - K_h d : \Lambda_h^k(\mathbb{R}^2) \to \Lambda_h^{k+1}(\mathbb{R})$. Observe that the discrete version of (4.6),

$$dS_h = -S_h d \tag{5.4}$$

follows exactly as in the continuous case, and in light of the commutativity (5.2), we find that S_h is simply given by

$$S_h = d\Pi_h K - \Pi_h K d = \Pi_h(dK - Kd) = \Pi_h S.$$

In analogy with the continuous case, we define automorphisms Φ_h on $\Lambda_h^k(\mathbb{V})$ by $\Phi_h(\omega, \mu) = (\omega + K_h\mu, \mu)$ and obtain the exact sequence

$$\Phi_h(\mathbb{V}) \hookrightarrow \Lambda_h^0(\mathbb{V}) \xrightarrow{\mathcal{A}_h} \Lambda_h^1(\mathbb{V}) \xrightarrow{\mathcal{A}_h} \Lambda_h^2(\mathbb{V}) \to 0, \tag{5.5}$$

where $\mathcal{A}_h = \Phi_h d\Phi_h^{-1} : \Lambda_h^k(\mathbb{V}) \to \Lambda_h^{k+1}(\mathbb{V})$, so $\mathcal{A}_h(\omega, \mu) = (d\omega - S_h\mu, d\mu)$.

We now make some requirements on the choice of spaces used in the discrete de Rham complexes. A minor requirement is that the global linear polynomials are contained in the space $\Lambda_h^0(\mathbb{R})$ and the constant forms dx^1 and dx^2 are contained in $\Lambda_h^1(\mathbb{R})$. The *key requirement* is that *the operator* $S_h = S_{0,h} : \Lambda_h^0(\mathbb{R}^2) \to \Lambda_h^1(\mathbb{R})$ *is onto*, and so admits a right inverse $S_h^\dagger : \Lambda_h^1(\mathbb{R}) \to \Lambda_h^0(\mathbb{R}^2)$. We can then define the subspaces Γ_h^k of $\Lambda_h^k(\mathbb{V})$, $k = 0, 1$, by

$$\Gamma_h^0 = \{ (\omega, \mu) \in \Lambda_h^0(\mathbb{V}) : d\omega = S_h\mu \}, \quad \Gamma_h^1 = \{ (\omega, \mu) \in \Lambda_h^1(\mathbb{V}) : \omega = 0 \},$$

and define projections $\pi_h^0 : \Lambda_h^0(\mathbb{V}) \to \Gamma_h^0$, $\pi_h^1 : \Lambda_h^1(\mathbb{V}) \to \Gamma_h^1$ by

$$\pi_h^0(\omega, \mu) = (\omega, \mu - S_h^\dagger S_h\mu + S_h^\dagger d\omega), \quad \pi^1(\omega, \mu) = (0, \mu + dS_h^\dagger\omega).$$

It is easy to check that these are indeed projections onto the relevant subspaces and that the following diagram commutes:

$$\begin{array}{ccccccccc} \Phi(\mathbb{V}) & \hookrightarrow & \Lambda_h^0(\mathbb{V}) & \xrightarrow{\mathcal{A}_h} & \Lambda_h^1(\mathbb{V}) & \xrightarrow{\mathcal{A}_h} & \Lambda_h^2(\mathbb{V}) & \to & 0 \\ & & \downarrow{\scriptstyle \pi_h^0} & & \downarrow{\scriptstyle \pi_h^1} & & \downarrow{\scriptstyle id} & & \\ \Phi(\mathbb{V}) & \hookrightarrow & \Gamma_h^0 & \xrightarrow{\mathcal{A}_h} & \Gamma_h^1 & \xrightarrow{\mathcal{A}_h} & \Lambda_h^2(\mathbb{V}) & \to & 0. \end{array} \tag{5.6}$$

Here we have used the fact that $\Lambda_h^0(\mathbb{R})$ contains the linears to see the $\Phi_h(\mathbb{V}) = \Phi(\mathbb{V})$ and the fact that $\Lambda_h^1(\mathbb{R})$ contains the constants to see that $\Phi(\mathbb{V}) \subset \Gamma_h^0$.

The diagram (5.6) is the desired discrete analogue of (4.8), and the bottom row is a discrete analogue of the elasticity complex with weakly imposed symmetry. Under the identification (4.7), $\Gamma_h^1 \cong \Lambda_h^1(\mathbb{R}^2)$ corresponds to a finite element space $\Sigma_h \subset H(\mathrm{div}, \Omega; \mathbb{M})$, while under the identification (4.10), $\Lambda_h^2(\mathbb{V})$ corresponds to a finite element space $Q_h \times V_h \subset L^2(\Omega; \mathbb{K}) \times L^2(\Omega; \mathbb{R}^2)$, and the mapping

$$\Gamma_h^1 \xrightarrow{\mathcal{A}_h} \Lambda_h^2(\mathbb{V})$$

corresponds to

$$\Sigma_h \xrightarrow{(-\Pi_h^Q \operatorname{skw}, \operatorname{div})} Q_h \times V_h,$$

which is the key operator for the stability of a mixed method with weakly imposed symmetry (2.1). The fact that $\operatorname{div} \Sigma_h \subset V_h$, built into our construction, ensures the stability condition (A1), since then we need only show that $\|\tau\|^2 \le c_1(A\tau, \tau)$. It is straightforward to check this condition for fixed λ and μ This condition is also true with c_1 independent of λ for τ satisfying $\operatorname{div} \tau = 0$ and $\int_\Omega \operatorname{tr}(\tau) = 0$. Note this latter condition is implied by the first equation in the mixed method (choosing $\tau = I$), and a simple reformulation of the problem and slight modification of the analysis allows this extra constraint to be easily handled (cf. [3]). The surjectivity of the operator $\mathcal{A}_h$ implies the inequality in (A2), but only for a constant c_2 depending on the mesh size h. Just as in the last section of [6], to obtain a constant independent of h requires a more technical argument, using the properties of the continuous de Rham sequence, the commuting diagram, the approximation properties of an appropriately chosen interpolation operator, and elliptic regularity results. This can be done for all the spaces we consider in the next section. A detailed proof for the three-dimensional case will be provided in a forthcoming paper [7].

Before closing this section, we establish a sufficient condition for the key requirement that $S_h = S_{0,h}$ be surjective which we shall use in the next section. First note that the surjectivity of S_h follows from the commutativity of the diagram

$$\begin{array}{ccc} \Lambda^0(\Omega, \mathbb{R}^2) & \xrightarrow{S} & \Lambda^1(\Omega, \mathbb{R}) \\ \Pi_h^0 \big\downarrow & & \Pi_h^1 \big\downarrow \\ \Lambda_h^0(\mathbb{R}^2) & \xrightarrow{S_h} & \Lambda_h^1(\mathbb{R}). \end{array}$$

Indeed, since Π_h^1 is surjective and S is surjective (even invertible), this certainly implies that S_h is surjective. Recalling that $S_h = \Pi_h^1 S$, the commutativity condition $S_h \Pi_h^0 = \Pi_h^1 S$ may be rewritten

$$\Pi_h^1 S(I - \Pi_h^0) = 0 \text{ on } \Lambda^0(\Omega, \mathbb{R}^2). \tag{5.7}$$

Now $(I - \Pi_h^0)\Lambda^0(\Omega, \mathbb{R}^2)$ is exactly the null space of Π_h^0. Thus we may summarize the condition as follows:

Whenever the projection of $\omega \in \Lambda^0(\Omega, \mathbb{R}^2)$ into $\Lambda_h^0(\mathbb{R}^2)$ vanishes, then the projection of $S\omega = \omega_2 dx^1 - \omega_1 dx^2$ into $\Lambda_h^1(\mathbb{R})$ vanishes.

We close with a summary of the main conclusion of this section. In order to construct stable mixed finite elements for the formulation (2.1), we begin with a discrete de Rham complex

$$\mathbb{R} \hookrightarrow \Lambda_h^0(\mathbb{R}) \xrightarrow{d} \Lambda_h^1(\mathbb{R}) \xrightarrow{d} \Lambda_h^2(\mathbb{R}) \to 0,$$

and a discrete vector-valued de Rham complex

$$\mathbb{R}^2 \hookrightarrow \Lambda_h^0(\mathbb{R}^2) \xrightarrow{d} \Lambda_h^1(\mathbb{R}^2) \xrightarrow{d} \Lambda_h^2(\mathbb{R}^2) \to 0.$$

If these choices satisfy the boxed condition, then the finite element spaces Σ_h corresponding to $\Lambda_h^1(\mathbb{R}^2)$, V_h corresponding to $\Lambda_h^2(\mathbb{R}^2)$, and Q_h corresponding to $\Lambda_h^2(\mathbb{R})$ can be expected to furnish a stable choice of spaces.

6. Examples of stable finite elements. In this section, we apply the construction just presented to derive stable finite element methods for the approximation of the Hellinger-Reissner formulation of linear elasticity with weakly imposed symmetry. The simplest example of such a method will require only piecewise linear functions to approximate stresses and piecewise constants to approximation displacements and multiplier.

Let $\mathcal{T}$ denote a triangular mesh of Ω, one of a shape regular family of meshes with mesh size decreasing to zero. We need to select a scalar-valued and a vector-valued discrete de Rham complex, both of which will be based on piecewise polynomials with respect to $\mathcal{T}$, for which we can verify the boxed condition of the previous section. Starting with the simplest case, we use the Whitney forms for the scalar-valued complex, i.e.,

$$\mathbb{R} \hookrightarrow \mathcal{P}_1\Lambda^0(\mathcal{T};\mathbb{R}) \xrightarrow{d} \mathcal{P}_0^+\Lambda^1(\mathcal{T};\mathbb{R}) \xrightarrow{d} \mathcal{P}_0\Lambda^2(\mathcal{T};\mathbb{R}) \to 0,$$

which is the complex (5.3) of [6] in the case $n = 2$ and $r = 0$. For the vector-valued de Rham complex, we use instead the sequence (5.4) of [6] in the case $n = 2$ and $r = 0$, i.e.,

$$\mathbb{R}^2 \hookrightarrow \mathcal{P}_2\Lambda^0(\mathcal{T};\mathbb{R}^2) \xrightarrow{d} \mathcal{P}_1\Lambda^1(\mathcal{T};\mathbb{R}^2) \xrightarrow{d} \mathcal{P}_0\Lambda^2(\mathcal{T};\mathbb{R}^2) \to 0.$$

These choices lead to the element choice $\Sigma_h \cong \mathcal{P}_1\Lambda^1(\mathcal{T};\mathbb{R}^2)$ for the stress, $V_h \cong \mathcal{P}_0\Lambda^2(\mathcal{T};\mathbb{R}^2)$ for the displacement, and $Q_h \cong \mathcal{P}_0\Lambda^1(\mathcal{T};\mathbb{R})$ for the multiplier, depicted in Fig. 2 above.

The boxed condition requires that whenever ω is a smooth vector field on Ω whose projection into the Lagrange space $\mathcal{P}_2\Lambda^0(\mathcal{T};\mathbb{R}^2)$ of continuous piecewise quadratic vector fields vanishes, then the projection of $\omega_2 dx^1 - \omega_1 dx^2$ into the Raviart–Thomas space $\mathcal{P}_0^+\Lambda^1(\mathcal{T};\mathbb{R})$ vanishes. The vanishing of the projection into the vector-valued quadratic Lagrange space implies that

$$\int_e \omega_i \, de = 0, \quad i = 1, 2, \quad e \in \Delta_1(\mathcal{T}), \tag{6.1}$$

since the edge integrals are among the degrees of freedom ($\Delta_1(\mathcal{T})$ denotes the set of edges of the mesh). We then require that

$$\int_e \operatorname{Tr}_e(\omega_2 dx^1 - \omega_1 dx^2) = 0, \quad e \in \Delta_1(\mathcal{T}),$$

since the quantities $\int_e \operatorname{Tr}_e(\tau)$ determine the projection of a 1-form τ into $\mathcal{P}_0^+\Lambda^1(\mathcal{T};\mathbb{R})$. Now, for any 1-form $\tau = \tau_1 dx^1 + \tau_2 dx^2$,

$$\int_e \operatorname{Tr}_e(\tau) = \int_e (\tau_1 t^1 + \tau_2 t^2) \, de,$$

where (t^1, t^2) is the unit tangent to e. Thus we need to show that

$$\int_e (\omega_2 t^1 - \omega_1 t^2) \, de = 0, \quad e \in \Delta_1(\mathcal{T}),$$

whenever (6.1) holds, which is obvious.

A similar argument can be used to verify the boxed condition for the choice of discrete de Rham sequences

$$\mathbb{R} \hookrightarrow \mathcal{P}_{r+1}\Lambda^0(\mathcal{T};\mathbb{R}) \xrightarrow{d} \mathcal{P}_r^+\Lambda^1(\mathcal{T};\mathbb{R}) \xrightarrow{d} \mathcal{P}_r\Lambda^2(\mathcal{T};\mathbb{R}) \to 0,$$

and

$$\mathbb{R}^2 \hookrightarrow \mathcal{P}_{r+2}\Lambda^0(\mathcal{T};\mathbb{R}^2) \xrightarrow{d} \mathcal{P}_{r+1}\Lambda^1(\mathcal{T};\mathbb{R}^2) \xrightarrow{d} \mathcal{P}_r\Lambda^2(\mathcal{T};\mathbb{R}^2) \to 0,$$

for any $r \geq 0$. Thus we obtain a family of stable finite element methods with $\Sigma_h \cong \mathcal{P}_{r+1}\Lambda^1(\mathcal{T};\mathbb{R}^2)$, $V_h \cong \mathcal{P}_r\Lambda^2(\mathcal{T};\mathbb{R}^2)$, and $Q_h \cong \mathcal{P}_r\Lambda^2(\mathcal{T};\mathbb{R})$.

We also remark that it is possible to reduce the space Σ_h without changing V_h or Q_h and still maintain stability. Returning to the case $r = 0$, we see that we did not use the vanishing of the edge integrals of both components ω_i, but only of the combination $\omega_2 t^1 - \omega_1 t^2$ (the normal component). Hence, instead of the vector-valued quadratic Lagrange space $\mathcal{P}_2\Lambda^0(\mathcal{T};\mathbb{R}^2)$ we can use the reduced space obtained from it by imposing the constraint that the tangential component on each edge vary only linearly on that edge. This space of vector fields, which we denote $\mathcal{P}_2^-\Lambda^0(\mathcal{T};\mathbb{R}^2)$, is well-known as a possible discretization of the velocity field for Stokes flow [9, 14]; see also [16, p. 134 ff., 153 ff.]. An element in it is determined by its vertex values and the integral of its normal component on each edge. In order to complete the construction, we must provide a vector-valued discrete de Rham complex in which the space of 0-forms is $\mathcal{P}_2^-\Lambda^0(\mathcal{T};\mathbb{R}^2)$. This will be the complex

$$\mathbb{R}^2 \hookrightarrow \mathcal{P}_2^-\Lambda^0(\mathcal{T};\mathbb{R}^2) \xrightarrow{d} \mathcal{P}_1^-\Lambda^1(\mathcal{T};\mathbb{R}^2) \xrightarrow{d} \mathcal{P}_0\Lambda^2(\mathcal{T};\mathbb{R}^2) \to 0,$$

where it remains to define $\mathcal{P}_1^-\Lambda^1(\mathcal{T};\mathbb{R}^2)$. This will be the set of $\tau \in \mathcal{P}_1\Lambda^1(\mathcal{T};\mathbb{R}^2)$ for which $\operatorname{Tr}_e(\tau)\cdot t$ is constant on any edge e with unit tangent t and unit normal n. (In more detail: for $\tau \in \mathcal{P}_1\Lambda^1(\mathcal{T};\mathbb{R}^2)$, $\operatorname{Tr}_e(\tau)$ is a vector-valued 1-form on e of the form $g\,ds$ with $\mu : e \to \mathbb{R}^2$ linear and ds the volume form—i.e., length form—on e. If $\mu \cdot t$ is constant, then $\tau \in \mathcal{P}_1^-\Lambda^1(\mathcal{T};\mathbb{R}^2)$.) The natural degrees of freedom for this space are the integral and first moment of $\operatorname{Tr}_e(\tau)\cdot n$ and the integral of $\operatorname{Tr}_e(\tau)\cdot t$. It is straightforward to verify the commutativity of the diagram

$$\begin{array}{ccccccc}
\mathbb{R}^2 \hookrightarrow & \Lambda^0(\Omega;\mathbb{R}^2) & \xrightarrow{d} & \Lambda^1(\Omega;\mathbb{R}^2) & \xrightarrow{d} & \Lambda^2(\Omega;\mathbb{R}^2) & \to 0 \\
 & \downarrow \Pi_h & & \downarrow \Pi_h & & \downarrow \Pi_h & \\
\mathbb{R}^2 \hookrightarrow & \mathcal{P}_2^-\Lambda^0(\mathcal{T};\mathbb{R}^2) & \xrightarrow{d} & \mathcal{P}_1^-\Lambda^1(\mathcal{T};\mathbb{R}^2) & \xrightarrow{d} & \mathcal{P}_0\Lambda^2(\mathcal{T};\mathbb{R}^2) & \to 0
\end{array}$$

and so the construction may precede. If we use (4.7) to identify vector-valued 1-forms and matrix fields, then the condition for a piecewise linear matrix field F to correspond to an element of $\mathcal{P}_1^-\Lambda^1(\mathcal{T};\mathbb{R}^2)$ is that on each edge e with tangent t and normal n, $Fn \cdot t$ must be constant on e. This defines the reduced space Σ_h, with three degrees of freedom per edge. Together with piecewise constant for displacements and multipliers, this furnishes a stable choice of elements.

We end this section by outlining how the original PEERS element, described in Section 1, cf. Fig. 1, can be derived from a slightly modified version of the theory outlined in Section 5. For this element, the scalar sequence is chosen to be a discrete de Rham sequence with reduced smoothness. The subscript in the spaces defined below indicates this reduced smoothness. Consider the sequence

$$\mathbb{R} \hookrightarrow \mathcal{P}_1\Lambda^0_-(\mathcal{T};\mathbb{R}) \xrightarrow{d} \mathcal{P}_0\Lambda^1_-(\mathcal{T};\mathbb{R}) \xrightarrow{d} \mathcal{P}_1\Lambda^0(\mathcal{T};\mathbb{R})^* \to 0. \tag{6.2}$$

Here $\mathcal{P}_1\Lambda^0_-(\mathcal{T};\mathbb{R})$ is the space of piecewise linear 0-forms with continuity requirement only with respect to the zero order moment on each edge, i.e., $\mathcal{P}_1\Lambda^0_-(\mathcal{T};\mathbb{R})$ is the standard nonconforming $\mathcal{P}_1$ space. Similarly, $\mathcal{P}_0\Lambda^1_-(\mathcal{T};\mathbb{R})$ consists of piecewise constant 1-forms, while the space of 2-forms $\mathcal{P}_1\Lambda^0(\mathcal{T};\mathbb{R})^*$ is the dual of $\mathcal{P}_1\Lambda^0(\mathcal{T};\mathbb{R})$ with respect to the pairing $\int_\Omega \omega \wedge \mu$. The operator $d = d_0 : \mathcal{P}_1\Lambda^0_-(\mathcal{T};\mathbb{R}) \to \mathcal{P}_0\Lambda^1_-(\mathcal{T};\mathbb{R})$ is defined locally on each triangle, and $d = d_1 : \mathcal{P}_0\Lambda^1_-(\mathcal{T};\mathbb{R}) \to \mathcal{P}_1\Lambda^0(\mathcal{T};\mathbb{R})^*$ is defined by $\int_\Omega d\omega \wedge \mu = -\int_\Omega \omega \wedge d\mu$ for $\omega \in \mathcal{P}_0\Lambda^1_-(\mathcal{T};\mathbb{R})$ and $\mu \in \mathcal{P}_1\Lambda^0(\mathcal{T};\mathbb{R})$. The orthogonal decomposition implied by the exact sequence (6.2) has been used previously (e.g., see [5]).

The corresponding vector-valued sequence needed for the PEERS element is dictated by the element itself. We consider the sequence

$$\mathbb{R}^2 \hookrightarrow \mathcal{P}_1\Lambda^0(\mathcal{T};\mathbb{R}^2) + B \xrightarrow{d} \mathcal{P}_0^+\Lambda^1(\mathcal{T};\mathbb{R}^2) + dB \xrightarrow{d} \mathcal{P}_0\Lambda^2(\mathcal{T};\mathbb{R}^2) \to 0,$$

which is exact. Here B denotes the space of vector-valued cubic bubbles, i.e., piecewise cubic vector fields which vanish on the element edges. Note the spaces $\mathcal{P}_0^+\Lambda^1(\mathcal{T};\mathbb{R}^2) + dB$, $\mathcal{P}_0\Lambda^2(\mathcal{T};\mathbb{R}^2)$, and $\mathcal{P}_1\Lambda^0(\mathcal{T};\mathbb{R})^*$ can be identified with the finite element spaces used in PEERS. If we choose the interpolation operator Π_h onto $\mathcal{P}_0\Lambda^1_-(\mathcal{T};\mathbb{R})$ to be the L^2 projection, then clearly

$$S_{0,h} = \Pi_h S_0 : \mathcal{P}_1\Lambda^0(\mathcal{T};\mathbb{R}^2) + B \to \mathcal{P}_0\Lambda^1_-(\mathcal{T};\mathbb{R})$$

is onto. Hence, the theory from Section 5 can be applied.

7. An element with strongly imposed symmetry. In this section, we shall discuss finite elements with strongly imposed symmetry, i.e., we consider the system (2.2). A family of stable elements was derived in [8], where, in the lowest degree case, the stress space $\Sigma_h \subset H(\mathrm{div}, \Omega; \mathbb{S})$ consists of piecewise cubics with linear divergence, while the space $V_h \subset L^2(\Omega; \mathbb{R}^2)$ consists of discontinuous linears. The purpose here is to show how this element can be derived from discrete de Rham complexes using the methodology introduced above.

As in the previous section, we start with one scalar-valued and one vector-valued discrete de Rham complex, which we denote here

$$\mathbb{R} \hookrightarrow \mathcal{P}_5\Lambda^0_\sharp(\mathcal{T};\mathbb{R}) \xrightarrow{d} \mathcal{P}_4\Lambda^1_\sharp(\mathcal{T};\mathbb{R}) \xrightarrow{d} \mathcal{P}_3\Lambda^2_\sharp(\mathcal{T};\mathbb{R}) \to 0 \tag{7.1}$$

and

$$\mathbb{R}^2 \hookrightarrow \mathcal{P}_4\Lambda^0_\flat(\mathcal{T};\mathbb{R}^2) \xrightarrow{d} \mathcal{P}_3\Lambda^1_\flat(\mathcal{T};\mathbb{R}^2) \xrightarrow{d} \mathcal{P}_2\Lambda^2_\flat(\mathcal{T};\mathbb{R}^2) \to 0. \tag{7.2}$$

On a single triangle, the scalar-valuded complex will be simply

$$\mathbb{R} \hookrightarrow \mathcal{P}_5\Lambda^0(T) \xrightarrow{d} \mathcal{P}_4\Lambda^1(T) \xrightarrow{d} \mathcal{P}_3\Lambda^2(T) \to 0,$$

but the degrees of freedom we use will impose extra smoothness on the assembled spaces. This extra smoothness appears to be necessary for the final construction.

For the quintic 0–form space, $\mathcal{P}_5\Lambda^0_\sharp(\mathcal{T};\mathbb{R})$, we determine a form on a triangle T by the following 21 values:

$$\phi(x),\ \operatorname{grad}\phi(x),\ \operatorname{grad}^2\phi(x), \quad x \in \Delta_0(T), \quad \int_e \frac{\partial\phi}{\partial n}, \quad e \in \Delta_1(T). \tag{7.3}$$

The resulting space, $\mathcal{P}_5\Lambda^0_\sharp(\mathcal{T};\mathbb{R})$, is then the well-known Argyris space, a subspace of $C^1(\Omega)$.

An element $\omega \in \mathcal{P}_4\Lambda^1(T)$ of the form $\omega = -g_2dx^1+g_1dx^2$ is determined by the 30 degrees of freedom given as

$$g_i(x),\ \operatorname{grad} g_i(x), \quad x \in \Delta_0(T), \quad \int_e g_i,\ \int_e p \operatorname{div} g,\ p \in \mathcal{P}_1(e), e \in \Delta_1(T),$$

and these determine the assembled space $\mathcal{P}_4\Lambda^1_\sharp(\mathcal{T};\mathbb{R})$. Here $\operatorname{div} g$ is the divergence of the vector field $g = (g_1, g_2)$. It is straightforward to check that these conditions determine an element of $\mathcal{P}_4\Lambda^1(T)$ uniquely. For if all of them are zero, then the cubic polynomial $\operatorname{div} g$ is zero on the boundary, and by the divergence theorem, the mean value of $\operatorname{div} g$ over T is zero. Hence, $\operatorname{div} g$, or $d\omega$, is zero, and therefore $\omega = d\phi$, where $\phi \in \mathcal{P}_5(T)$, and where we can assume that ϕ is zero at one of the vertices. However, it now follows that all the degrees of freedom for ϕ given by (7.3) vanish, and hence $\omega = d\phi$ is zero. If $\omega \in \mathcal{P}_4\Lambda^1_0(\mathcal{T};\mathbb{R})$, then ω is continuous, and, moreover, $d\omega = \operatorname{div} g$ is also continuous.

We complete the description of the desired scalar discrete de Rham complex, by letting $\mathcal{P}_3\Lambda^2_\sharp(\mathcal{T};\mathbb{R})$ denote the space of continuous piecewise cubic 2–forms, with standard Lagrange degrees of freedom, i.e., if $\omega = gdx_1 \wedge dx_2$, we specify

$$g(x), \quad x \in \Delta_0(T), \quad \int_e gp, \quad p \in \mathcal{P}_1(e),\ e \in \Delta_1(T), \text{ and } \int_T g.$$

It is easy to check that $d[\mathcal{P}_5\Lambda^0_\sharp(\mathcal{T};\mathbb{R})] \subset \mathcal{P}_4\Lambda^1_\sharp(\mathcal{T};\mathbb{R})$ and $d[\mathcal{P}_4\Lambda^1_\sharp(\mathcal{T};\mathbb{R})] = \mathcal{P}_3\Lambda^2_\sharp(\mathcal{T};\mathbb{R})$. Further, the complex (7.1) is exact. To check this, it is enough to show that

$$\dim \mathcal{P}_5\Lambda^0_\sharp(\mathcal{T};\mathbb{R}) + \dim \mathcal{P}_3\Lambda^2_\sharp(\mathcal{T};\mathbb{R}) = \dim \mathcal{P}_4\Lambda^1_\sharp(\mathcal{T};\mathbb{R}) + 1,$$

and this is a direct consequence of Euler's formula.

We now turn to the description of the spaces entering the vector-valued de Rham complex (7.2). The space $\mathcal{P}_4\Lambda^0_\flat(\mathcal{T};\mathbb{R}^2)$ consists of continuous piecewise quartic vector valued 0–forms $\omega = (f_1, f_2)^T$. The degrees of freedom are taken to be

$$f_i(x),\ \operatorname{grad} f_i(x), \quad x \in \Delta_0(T), \quad \int_e f_i,\ \int_e p \operatorname{div} f,\ p \in \mathcal{P}_1(e),\ e \in \Delta_1(T).$$

Note that the space $\mathcal{P}_4\Lambda^0_\flat(\mathcal{T};\mathbb{R}^2)$ is not simply the Cartesian product of two copies of a space of scalar-valued of 0–forms. However, the spaces are constructed exactly such that the operator S_0 (defined in Section 4) maps $\mathcal{P}_4\Lambda^1_\sharp(\mathcal{T};\mathbb{R})$ isomorphically onto $\mathcal{P}_4\Lambda^0_\flat(\mathcal{T};\mathbb{R}^2)$. Thus $S_{0,h}$ is simply the restriction of S_0 in this case. It is invertible, and, certainly the key requirement of Section 5, that it is surjective, is satisfied.

The space $\mathcal{P}_3\Lambda^1_\flat(\mathcal{T};\mathbb{R}^2)$ corresponds to a non-symmetric extension of the stress space used in [8]. On each triangle, the elements consist of cubic 1–forms

$$\omega = \begin{pmatrix} f_{12} \\ f_{22} \end{pmatrix} dx^1 - \begin{pmatrix} f_{11} \\ f_{21} \end{pmatrix} dx^2 \tag{7.4}$$

such that $\operatorname{div} F$ is linear, where $F = (f_{ij})$. This space has dimension $40 - 6 = 34$. In fact, 34 unisolvent degrees of freedom are given by $F(x)$ for $x \in \Delta_0(T)$, $\int_T F$ and basis elements for the spaces of moments

$$\int_e (Fn) \cdot p, \quad p \in \mathcal{P}_1(e;\mathbb{R}^2), \quad \int_e p \operatorname{skw}(F), \quad p \in \mathcal{P}_1(e;\mathbb{K}),\ e \in \Delta_1(T).$$

If all these degrees of these degrees of freedom vanish, then $\operatorname{skw}(F) = 0$ on the triangle T, and the corresponding unisolvence argument given in [8] implies $\omega = 0$ on T.

Finally, the space $\mathcal{P}_1\Lambda^2_\flat(\mathcal{T};\mathbb{R}^2) = \mathcal{P}_1\Lambda^2(\mathcal{T};\mathbb{R}^2)$ is the standard space of discontinuous linear vector-valued 2–forms, with degrees of freedom $\int_T \omega \wedge \mu$ for μ in a basis for $\mathcal{P}_1\Lambda^0(T,\mathbb{R}^2)$. By definition, we have the inclusion $d[\mathcal{P}_3\Lambda^1_\flat(\mathcal{T};\mathbb{R}^2)] \subset \mathcal{P}_1\Lambda^2_\flat(\mathcal{T};\mathbb{R}^2)$, and from [8] we know that the symmetric subspace of $\mathcal{P}_3\Lambda^1_\flat(\mathcal{T};\mathbb{R}^2)$ is mapped onto $\mathcal{P}_1\Lambda^2_\flat(\mathcal{T};\mathbb{R}^2)$ by d. Therefore, $d[\mathcal{P}_3\Lambda^1_\flat(\mathcal{T};\mathbb{R}^2)] = \mathcal{P}_1\Lambda^2_\flat(\mathcal{T};\mathbb{R}^2)$. Furthermore, clearly $d[\mathcal{P}_4\Lambda^0_\flat(\mathcal{T};\mathbb{R}^2)] \subset \mathcal{P}_3\Lambda^1_\flat(\mathcal{T};\mathbb{R}^2)$. Hence, as above we can use a dimension count to show that the complex (7.2) is exact.

Since we have already noted that $S_{0,h}$ is surjective, it follows from the general theory of Section 5, that the bottom row of diagram (5.6) is exact. Furthermore, since $S_{0,h}$ is invertible, we can identify the space Γ_h^0 with $\Lambda_h^0(\mathbb{R})$. Now, if ω given by (7.4) belongs to $\mathcal{P}_3\Lambda_\flat^1(\mathcal{T};\mathbb{R}^2)$, then $S_1\omega = (f_{12} - f_{21})dx^1 \wedge dx^2$ belongs to $\mathcal{P}_3\Lambda_\sharp^2(\mathcal{T};\mathbb{R})$. Hence $S_{1,h}$ is just the restriction to $\mathcal{P}_3\Lambda_\flat^1(\mathcal{T};\mathbb{R}^2)$ of S_1 in this case, and the bottom row of (5.6) can be identified with

$$\mathcal{P}_1 \hookrightarrow \Lambda_h^0(\mathbb{R}) \xrightarrow{d\circ S_0^{-1}\circ d} \Lambda_h^1(\mathbb{R}^2) \xrightarrow{(-S_1,d)} \Lambda_h^2(\mathbb{V}) \to 0, \tag{7.5}$$

which, in the present case and notation, takes the form

$$\begin{aligned}\mathcal{P}_1 \hookrightarrow \mathcal{P}_5\Lambda_\sharp^0(\mathcal{T};\mathbb{R}) &\xrightarrow{d\circ S_0^{-1}\circ d} \mathcal{P}_3\Lambda_\flat^1(\mathcal{T};\mathbb{R}^2)\\ &\xrightarrow{(-S_1,d)} \mathcal{P}_3\Lambda_\sharp^2(\mathcal{T};\mathbb{R}) \times \mathcal{P}_1\Lambda^2(\mathcal{T};\mathbb{R}^2) \to 0.\end{aligned} \tag{7.6}$$

Identifying the spaces of differential forms with spaces of piecewise polynomial scalar, vector, and matrix fields as usual, the form space $\mathcal{P}_3\Lambda_\sharp^2(\mathcal{T};\mathbb{R})$ corresponds to the space Q_h of all continuous piecewise cubic skew matrix fields, $\mathcal{P}_1\Lambda^2(\mathcal{T};\mathbb{R})$ corresponds to the space V_h of all piecewise linear vector fields, and $\mathcal{P}_5\Lambda_\sharp^0(\mathcal{T};\mathbb{R})$ corresponds to the Argyris space of piecewise quintic scalar fields. The space $P_3\Lambda_\flat^1(\mathcal{T};\mathbb{R}^2)$ corresponds to a space Ξ_h consisting of all piecewise cubic matrix fields in $H(\mathrm{div},\Omega;\mathbb{M})$ which have piecewise linear divergence, are continuous at the vertices, and for which the skew part is continuous. With these identifications, the sequence (7.6) is equivalent to

$$\mathcal{P}_1 \hookrightarrow W_h \xrightarrow{J} \Xi_h \xrightarrow{(\mathrm{skw},\mathrm{div})} Q_h \times V_h \to 0,$$

which is a discrete version of (3.2).

In order to derive the desired discrete version of (3.1), we develop a discrete analogue of the projection done in (3.3). Observe that of the 34 degrees of freedom determining an element $F \in \Xi_h$ on a given triangle T, there are 10 that only involve $\mathrm{skw}(F)$, i.e., $\mathrm{skw}(F)$ at each vertex, $\int_T \mathrm{skw}(F)$, and $\int_e p\,\mathrm{skw}(F)$ for $p \in \mathcal{P}_1(e;\mathbb{K})$. Moreover, these are exactly the degrees of freedom of $\mathrm{skw}(F)$ in Q_h. Let L_h denote this set of degrees of freedom, and L_h^c the remaining 24 degrees of freedom. Then we can define an injection $i_h : Q_h \to \Xi_h$, determining $i_h q$ on T by

$$l(i_h q) = l(q), \quad l \in L_h, \qquad l(i_h q) = 0, \quad l \in L_h^c.$$

By construction, $\mathrm{skw}\, i_h q = q$ for all $q \in Q_h$. The operator i_h may be considered a discrete analogue of the inclusion of $C^\infty(\Omega;\mathbb{K}) \hookrightarrow C^\infty(\Omega,\mathbb{M})$. (However Q_h is not contained in Ξ_h, and $i_h q$ need not be skew-symmetric.) The operator $\mathrm{sym}_h := I - i_h\,\mathrm{skw}$ is a projection of Ξ_h onto the subspace Σ_h consisting of the symmetric matrix fields in Ξ_h. That is,

$$\Sigma_h := \mathrm{sym}_h(\Xi_h) = \Xi_h \cap H(\mathrm{div},\Omega;\mathbb{S}).$$

A discrete version of the diagram (3.3) is now given by

$$\begin{array}{ccccccccc} \mathcal{P}_1 & \hookrightarrow & W_h & \xrightarrow{J} & \Xi_h & \xrightarrow{(\text{skw},\text{div})} & Q_h \times V_h & \to & 0 \\ & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle \text{sym}_h} & & \big\downarrow{\scriptstyle \Pi_h} & & \\ \mathcal{P}_1 & \hookrightarrow & W_h & \xrightarrow{J} & \Sigma_h & \xrightarrow{\text{div}} & V_h & \to & 0 \end{array}$$

where $\Pi_h(q, v) = v - \operatorname{div} i_h q$. It is straightforward to check this diagram commutes and hence the bottom row is exact. This is exactly the discrete sequence utilized in [8].

REFERENCES

[1] MOHAMED AMARA AND JEAN-MARIE THOMAS, *Equilibrium finite elements for the linear elastic problem*, Numer. Math. **33** (1979), no. 4, 367–383.

[2] DOUGLAS N. ARNOLD, FRANCO BREZZI, AND JIM DOUGLAS, JR., *PEERS: a new mixed finite element for plane elasticity*, Japan J. Appl. Math. **1** (1984), no. 2, 347–367.

[3] DOUGLAS N. ARNOLD, JIM DOUGLAS, JR., AND CHAITAN P. GUPTA, *A family of higher order mixed finite element methods for plane elasticity*, Numer. Math. **45** (1984), no. 1, 1–22.

[4] DOUGLAS N. ARNOLD AND RICHARD S. FALK, *A new mixed formulation for elasticity*, Numer. Math. **53** (1988), no. 1-2, 13–30.

[5] DOUGLAS N. ARNOLD AND RICHARD S. FALK, *A uniformly accurate finite element method for the Reissner-Mindlin plate*, SIAM J. on Numer. Anal. **26** (1989), 1276–1290.

[6] DOUGLAS N. ARNOLD, RICHARD S. FALK, AND RAGNAR WINTHER, *Differential complexes and stability of finite element methods. I. The de Rham complex*, this volume.

[7] DOUGLAS N. ARNOLD, RICHARD S. FALK, AND RAGNAR WINTHER, *Mixed finite element methods for linear elasticity with weakly imposed symmetry*, preprint.

[8] DOUGLAS N. ARNOLD AND RAGNAR WINTHER, *Mixed finite elements for elasticity*, Numer. Math. **92** (2002), no. 3, 401–419.

[9] CHRISTINE BERNARDI AND GENEVIÉVE RAUGEL, *Analysis of some finite elements for the Stokes problem*, Math. Comp. **44** (1985), 71–79.

[10] I.N. BERNSTEIN, I.M. GELFAND, AND S.I. GELFAND, *Differential operators on the baseaffine space and a study of g–modules*, Lie groups and their representation, I.M. Gelfand (ed) (1975), 21–64.

[11] FRANCO BREZZI, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **8** (1974), no. R-2, 129–151.

[12] FRANCO BREZZI AND MICHEL FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[13] MICHAEL EASTWOOD, *A complex from linear elasticity*, Rend. Circ. Mat. Palermo (2) Suppl. (2000), no. 63, 23–29.

[14] MICHEL FORTIN, *Old and new finite elements for incompressible flows*, Int. J. Numer. Methods Fluids **1** (1981), 347–354.

[15] BAUDOIUN M. FRAEJIS DE VEUBEKE, *Stress function approach*, World Congress on the Finite Element Method in Structural Mechanics, Bornemouth, 1975.

[16] V. GIRAULT AND P.-A. RAVIART, *Finite element methods for Navier-Stokes equations. Theory and algorithms*, Springer Series in Computational Mathematics, 5. Springer-Verlag, Berlin, 1986.

[17] Claes Johnson and Bertrand Mercier, *Some equilibrium finite element methods for two-dimensional elasticity problems*, Numer. Math. **30** (1978), no. 1, 103–116.
[18] Mary E. Morley, *A family of mixed finite elements for linear elasticity*, Numer. Math. **55** (1989), no. 6, 633–666.
[19] Erwin Stein and Raimund Rolfes, *Mechanical conditions for stability and optimal convergence of mixed finite elements for linear plane elasticity*, Comput. Methods Appl. Mech. Engrg. **84** (1990), no. 1, 77–95.
[20] Rolf Stenberg, *On the construction of optimal mixed finite element methods for the linear elasticity problem*, Numer. Math. **48** (1986), no. 4, 447–462.
[21] ———, *A family of mixed finite elements for the elasticity problem*, Numer. Math. **53** (1988), no. 5, 513–538.
[22] ———, *Two low-order mixed methods for the elasticity problem*, The mathematics of finite elements and applications, VI (Uxbridge, 1987), Academic Press, London, 1988, pp. 271–280.
[23] Vernon B. Watwood Jr. and B.J. Hartz, *An equilibrium stress field model for finite element solution of two-dimensional elastostatic problems*, Internat. Jour. Solids and Structures **4** (1968), 857–873.