9

Markov Processes on Banach Spaces on Cycles

The problem of defining denumerable Markov chains by a countable infinity of weighted directed cycles is solved by using suitable Banach spaces $\mathrm{l_p}$ on cycles and edges. Furthermore, it is showed that the transition probabilities of such chains may be described by Fourier series on orthonormal collections of homologic ingredients.

9.1 Banach Spaces on Cycles

9.1.1 Euclidean spaces associated with infinite graphs

Now we shall consider an irreducible and positive-recurrent Markov chain $\xi = (\xi_n)_n$, whose state space S is a denumerable set. The corresponding graph G is usually required to satisfy the local finiteness condition, that is, for each $i \in S$ there are finitely many $j \in S$ such that $p_{ij} > 0$ or $p_{ji} > 0$. We now explain that the local finiteness condition is necessary for the existence of topologies of Euclidean spaces comparable with the topology of $l_2(R)$ (according to Hilton and Wylie (1967) p.45).

Let $G = (\underline{\mathrm{N}}, E)$ be an infinite directed graph where $\underline{\mathrm{N}} = \{n_u\}$ are the vertices (nodes) of G and $E = \{e_{n_u n_k}\}$ are the oriented edges of G. To fix the ideas, we shall consider that $\underline{\mathrm{N}}$ and E are denumerable sets.
The graph G may be viewed as an infinite abstract simplicial complex, noted also by G, where

- (i) the vertices n_u of G are called 0-simplexes,
- (ii) the oriented edges $e_{n_u n_k}$ of G (which are completely determined by the ordered pairs (n_u, n_k) of vertices) are called 1-simplexes.

Accordingly, the graph G is an oriented complex of dimension 1. To the 1-dimensional complex G we may attach a topological space, symbolized by $(|G|, \Im)$ and called the polyhedron of G, as follows. First, to define the space-elements of the set $|G|$, and then the topology $\Im$, we introduce an ordering on the set $\underline{\mathrm{N}}$. This is equivalent, by a homeomorphic translation in Euclidean spaces, with a choice of a system of orthogonal axes. Since $\underline{\mathrm{N}}$ is denumerable, we may use the index set $\mathrm{I} = \{0, 1, \ldots\}$, which particularly is totally ordered. Accordingly, $\underline{\mathrm{N}} = \{n_0, n_1, \ldots\}$ becomes a totally ordered set with respect to the ordering-relation " ¡ "defined as

$$n_i < n_j \quad \text{if and only if} \quad \mathrm{i} < \mathrm{j}.$$

With this preparation we give now the definition of the polyhedron $(|G|, \Im)$ as follows. To define the set $|G|$, we first consider a family $W$of weight-functions on the vertices and edges of G in the following way:

$$\begin{aligned} W = \{ &{}^0w : \{0 - \mathit{simplexes}\} \to \{1\} : {}^0w(n_i) \equiv 1, \text{ for any } n_i \in \underline{\mathrm{N}}\} \cup \\ &\{{}^1w : \{1 - \mathit{simplexes} = (n_{i_k}, n_{i_m})\} \to [0,1] \times [0,1] : {}^1w(n_{i_k}, n_{i_m}) \\ = &({}^1w_1(n_{i_k}), {}^1w_2(n_{i_m})), \text{where} \\ &\text{(i)} \quad {}^1w_1(n_{i_k}), {}^1w_2(n_{i_m}) \text{ vary in } [0,1], \\ &\text{(ii)} \quad {}^1w_1(n_{i_k}) + {}^1w_2(n_{i_m}) = 1\}. \end{aligned}$$

Or, better we may consider the family W defined as

$$\begin{aligned} W = \{&w_i,\, i \in N \colon w_i \colon \underline{\mathrm{N}} \to [0,1],\, w_i(n_j) \equiv 1, \text{ if } j = i; \quad \text{or } 0, \text{ if } \mathrm{j} \neq \mathrm{i}\} \cup \\ &\{w_{ij}, (n_i, n_j) \in E : w_{ij} \colon \underline{\mathrm{N}} \to [0,1], w_{ij}(n_k) > 0 \text{ if } k = i, j; \\ &w_{ij}(n_k) = 0, \text{if } k \neq i, j; \text{ and } w_{ij}(n_i) + w_{ij}(n_j) = 1\}. \end{aligned}$$

Then the family W involves a weighting procedure according to which we attach to each vertex n_i of G one nonnegative real weight $\tilde{w}_i$ such that

(i) if (n_i) is a 0-simplex of G, $\tilde{w}_i$ may be chosen to be equal to $w_i(n_i) = 1$;
(ii) if n_i is a vertex of an 1-simplex (n_i, n_j), then $\tilde{w}_i$ may be chosen, along with $\tilde{w}_j$, to be the nonnegative real number given by w_{ij}, that is, $\tilde{w}_i \equiv w_{ij}(n_i) > 0, \tilde{w}_j \equiv w_{ij}(n_j) > 0$, and $\tilde{w}_i + \tilde{w}_j = 1\}$.

In this way, the images of the weight-functions of W provide a collection of sequences which have either the form

(α) $(1, 0, 0, \ldots), (0, 1, 0, \ldots), \ldots$ for the case (i) above,
or, the form
(β) $(0, \ldots, 0, \tilde{w}_i, 0, \ldots, \tilde{w}_j, 0, \ldots)$, for the case (ii) above if $i < j$, with $\tilde{w}_i, \tilde{w}_j > 0$, and with $\tilde{w}_i + \tilde{w}_j = 1$, where (n_i, n_j) varies in the set E of oriented edges of G.

Then the set $|G|$ is that whose elements are all the sequences of the form (α) and (β).

An equivalent way to describe the set $|G|$ is as follows: associate the 0-simplex n_1 to the sequence $(1,0,\ldots)$, the 0-simplex n_2 to the sequence $(0,1,0,\ldots)$, and so on.

Furthermore, to each 1-simplex (n_i,n_j), $i<j$, associate the subsets $\bar{b}_{ij}$ and b_{ij} of $|G|$ defined as

$$\bar{b}_{ij} = \{(0,\ldots,0,\tilde{w}_i,0,\ldots,\tilde{w}_j,0,\ldots) : \tilde{w}_i,\tilde{w}_j \geq 0, \tilde{w}_i+\tilde{w}_j=1\},$$
$$b_{ij} = \{(0,\ldots,0,\tilde{w}_i,0,\ldots,\tilde{w}_j,0,\ldots) : \tilde{w}_i,\tilde{w}_j > 0, \text{with}\, \tilde{w}_i+\tilde{w}_j=1\}.$$

Then

$$\begin{aligned}|G| &= \{(1,0,0,\ldots),(0,1,0,\ldots),\ldots\} \cup (\cup_{(n_i,n_j)}\bar{b}_{ij}) \\ &= \{(1,0,0,\ldots),(0,1,0,\ldots),\ldots\} \cup (\cup_{(n_i,n_j)} b_{ij}).\end{aligned}$$

Now let us see how to define the topology $\Im$ of $|G|$. Consider the projection pr_i associated to the 0-simplex n_i and which associates the sequence $(0,0,\ldots,1,0,\ldots)$ (where 1 has the rank i in the sequence) with the number 1. Analogously we may consider the projection pr_{ij}: $\bar{b}_{ij} \to R^2$ for any edge (n_i,n_j) of G, that is, pr_{ij} associates any sequence $(0,0,\ldots,0,\tilde{w}_i,0,\ldots,\tilde{w}_j,0,\ldots) \in \bar{b}_{ij}$ with the ordered pair (w_i,w_j).

Next, for any edge $(n_i,n_j) \in G$ we topologize the subset $\bar{b}_{ij}$ by requiring that pr_{ij} be a homeomorphism in R^2. Then, we topologize $|G|$ by specifying its closed sets: $A \subseteq |G|$ is closed if and only if $A \cap \bar{b}_{ij}$ is closed in $\bar{b}_{ij}$ for every 1-simplex (n_i,n_j) of G.

The topology $\Im$ of $|G|$ may be in some cases (involving conditions on the configuration of the graph G) compatible with the topology of Euclidean spaces defined by the metric $\rho((x_i),(y_i)) = \sqrt{\sum (x_i-y_i)^2}$. Such a case is given by the graphs which are locally finite (i.e., each vertex belongs only to finitely many edges) and contain denumerable sets of vertices and edges.

Let $G=(\underline{N},E)$ be such a graph. Then G can be realized in $l_2(R) = \{(x_n)_n\colon\ x_n \in R, \sum_n (x_n)^2 < \infty\ \}$ by the inclusion (see Hilton and Wylie (1967), p.45).

9.1.2 Banach spaces on cycles

Let $\underline{N}=\{n_1,n_2,\ldots\}$ and let $C=\{c_1,c_2,\ldots\}$ be a sequence of overlapping directed circuits or cycles in $\underline{N}$ as those corresponding to an irreducible and positive-recurrent Markov chain. Then the Vertex-set C and Arc-set C will symbolize the sets of all vertices and edges of C, respectively. Throughout the paragraph we shall assume the collection C of directed circuits in $\underline{N}$ such that Vertex-set $C=\underline{N}$, and we shall consider arbitrary orderings on $\underline{N}$ and Arc-set C. For instance, without any loss of generality, we shall assume that the first $p(c_1)$ points and pairs of $\underline{N}$ and Arc-set C will belong to the circuit c_1, the next $p(c_2)$ to c_2, and so on. Also we shall assume that any circuit $c=(i_1,i_2,\ldots,i_s,i_1)$ of C has all points $i_1,i_2,\ldots,i_s$ distinct each from the other.

Now let $G = (\underline{\mathrm{N}}, E)$ be the oriented graph associated with C, that is, $\underline{\mathrm{N}}$ =Vertex-set C and E = Arc-set C, and assume that G is locally finite. With every pair $(i,j) \in E$ we associate the symbol $b_{(i,j)}$. Then, since a directed circuit $c = (i_1, i_2, \ldots, i_s, i_1), s \geq 1$, is completely defined by the sequence $(i_1, i_2), (i_2, i_3), \ldots, (i_s, i_1)$ of directed edges, we may further associate c with the sequence of symbols $b_{(i_1,i_2)}, b_{(i_2,i_3)}, \ldots, b_{(i_s,i_1)}$. An equivalent version is to associate any circuit c of C with the formal expression $\underline{c} = b_{(i_1,i_2)} + b_{(i_2,i_3)} + \cdots + b_{(i_s,i_1)} = \Sigma_{(i,j)} J_c(i,j)\, b_{(i,j)}$, where $J_c(i,j)$ is the passage-function which equals 1 or 0 according to whether or not (i,j) is an adge of c.
Then the sets $B = \{b_{(i,j)}, (i,j) \in E\} = \{b_1, b_2, \ldots\}$ and $\underline{C} = \{\underline{c}_1, \underline{c}_2, \ldots\}$ will be ordered according to the chosen orderings on E and C, respectively.

With these preparations we shall now define certain Banach spaces by using the sets C, $\underline{\mathrm{N}}$ = Vertex-set C and E = Arc-set C. In this direction we first introduce the vector spaces generated by $\underline{\mathrm{N}} = \{n_1, n_2, \ldots\}, \mathrm{B} = \{b_1, b_2, \ldots\}$ and $\underline{\mathrm{C}} = \{\underline{c}_1, \underline{c}_2, \ldots\}$, respectively. Let

$$\mathcal{N} = \{\underline{\mathrm{n}} = \sum_{k=1}^{s} x_k n_k : s \in N, n_k \in \underline{\mathrm{N}}, x_k \in R\},$$

$$\mathcal{E} = \{\underline{\mathrm{b}} = \sum_{k=1}^{r} a_k b_k : r \in N, a_k \in R, b_k \in B\},$$

$$\mathcal{C} = \{\underline{\mathrm{c}} = \sum_{k=1}^{m} w_k \underline{c}_k : m \in N, w_k \in R, \underline{c}_k \in \underline{\mathrm{C}}\},$$

where $\underline{\mathrm{n}}$, $\underline{\mathrm{b}}$ and $\underline{\mathrm{c}}$ are formal expressions on $\underline{\mathrm{N}}$, B and $\underline{\mathrm{C}}$, and N and R denote as usual the sets of natural and real numbers, respectively.
Then the sets $\mathcal{N}$, $\mathcal{E}$, and $\mathcal{C}$ may be organized as real vector spaces with respect to the operations + and scalar-multiplicity defined as follows. For the formal expressions of $\mathcal{N}$, we define

$$\sum_{k=1}^{s} x_k n_k + \sum_{k=1}^{r} x_k{}'\, n_k = \sum_{k} (x_k + x_k{}')n_k,$$

$$\lambda \sum_{k=1}^{s} x_k n_k = \sum_{k=1}^{s} (\lambda\, x_k) n_k, \lambda \in R.$$

Then $\mathcal{N}$ will become, except for an equivalence relation, a real vector space, which is isomorph with

$$\sigma(\mathrm{N}) = \{(x_1, x_2, \ldots, x_s, 0, 0, \ldots) : s \in N, x_k \in R, k = 1, \ldots, s\}.$$

Analogously, the set $\mathcal{E}$ becomes, except for an equivalence relation, a real vector space whose base is B, if we shall not adhere to the notational convention: $b_{(j,i)} = -b_{(i,j)}, (i,j) \in E$.

Then $\mathcal{E}$ is isomorph with

$$\sigma(E) = \{(w(i_1, j_1), \ldots, w(i_n, j_n), 0, 0, \ldots) : \\ n \in N, w(i_k, j_k) \in R, (i_k, j_k) \in E, k = 1, \ldots, n\}.$$

Here the index k of $(i_k, j_k), k = 1, \ldots, n$, means the k-th rank according to the ordering of E, that is, $b_{(i_k, j_k)} = b_k$, $k = 1, 2, \ldots$.

As concerns the set $\underline{\mathrm{C}}$ we define analogously the vector space operations and note that some vectors $\underline{\mathrm{c}}_k \in \underline{\mathrm{C}}$ may perhaps be linear expressions of other vectors of $\underline{\mathrm{C}}$. To avoid this, we shall assume that C contains only directed circuits c_k whose generated vectors $\underline{\mathrm{c}}_k$ in $\underline{\mathrm{C}} \subset \mathcal{C}$ are linear independent. This assumption may be always achieved by applying Zorn's lemma to any countable collection $\underline{\mathrm{C}}$, which perhaps contains linear dependent vectors. Then $\mathcal{C}$ may be correspondingly organized (except for an equivalence relation) as a real vector space whose base is $\underline{\mathrm{C}}$. Furthermore $\mathcal{C}$ is isomorph with

$$\sigma(C) = \{(w_{c_1}, \ldots, w_{c_m}, 0, 0, \ldots) : m \in N, w_{c_k} \in R, c_k \in C, k = 1, \ldots, m\}.$$

Since $\mathcal{C}$ is a vector subspace of $\mathcal{E}$, it is isomorph with the following subspace of $\sigma(E)$:

$$\mathcal{C}(E) = \{(\sum_{k=1}^{m} w_{c_k}\ J_{c_k}(i_1, j_1), \ldots, \sum_{k=1}^{m} w_{c_k}\ J_{c_k}(i_n, j_n), 0, 0, \ldots) : m \in N, w_{c_k} \in R, \\ c_k \in C, k = 1, \ldots, m; (i_u, j_u) \in \mathrm{Arcset}\{c_1, \ldots, c_m\}, u = 1, \ldots, n\}.$$

We proceed by introducing certain norms on the vector spaces $\mathcal{N}$, $\mathcal{E}$, and $\mathcal{C}$. For instance, we define the functions $|\cdot|_{\mathrm{k}} : \mathcal{E} \to R, k = 1, 2$, as follows:

$$|\sum_{k=1}^{r} a_k\, b_k|_1 = \sum_{k=1}^{r} |a_k|,$$
$$|\sum_{k=1}^{r} a_k\, b_k|_2 = \left(\sum_{k=1}^{r} {a_k}^2\right)^{1/2}. \tag{9.1.1}$$

Analogously, we define the functions $\|\cdot\|_k : \mathcal{C} \to R, k = 1, 2$, as follows:

$$\|\sum_{k=1}^{m} w_k\, \underline{\mathrm{c}}_k\|_1 = \sum_{k=1}^{m} |w_k|,$$
$$\|\sum_{k=1}^{m} w_k\, \underline{\mathrm{c}}_k\|_2 = \left(\sum_{k=1}^{m} {w_k}^2\right)^{1/2}. \tag{9.1.2}$$

In an analogous way we may define similar norms on $\mathcal{N}$. Then $\mathcal{N}$, $\mathcal{E}$, and $\mathcal{C}$ will become normed spaces with respect to the above norms, and consequently we may compare them with the following classic Banach spaces associated with the original collection C of circuits:

$$l_1(N^2) = \{\underline{w} = (w(i,j),(i,j) \in N^2) : w(i,j) \in R, \Sigma_{(i,j)}|w(i,j)| < \infty\},$$
$$l_2(N^2) = \{\underline{w} = (w(i,j),(i,j) \in N^2) : w(i,j) \in R, \Sigma_{(i,j)}(w(i,j))^2 < \infty\},$$
$$l_1(C) = \{(w_c, c \in C) : w_c \in R, \Sigma_c|w_c| < \infty\},$$
$$l_2(C) = \{(w_c, c \in C) : w_c \in R, \Sigma_c(w_c)^2 < \infty\},$$

where the corresponding norms for the spaces $l_1(N^2)$ and $l_2(N^2)$ are respectively given by:

$$/\underline{w}/_1 = \Sigma_{(i,j)}|w(i,j)|,$$
$$/\underline{w}/_2 = (\Sigma_{(i,j)}(w(i,j))^2)^{1/2},$$

and for the spaces $l_1(C)$ and $l_2(C)$, by

$$//(w_c)_c//_1 = \Sigma_c|w_c|,$$
$$//(w_c)_c//_2 = (\Sigma_c(w_c)^2)^{1/2}.$$

Consequently, the normed vector spaces $(\mathcal{E}, | \quad |_k), k = 1, 2,$ are isomorph with $(\sigma(E), //_{\mathrm{k}})$ (viewed included in $(l_k(N^2), //_k)), k = 1, 2$.
Analogously, the normed vector spaces $(\mathcal{C}, \| \quad \|_k), k = 1, 2,$ are isomorph with $(\sigma(C), // \, //_k), k = 1, 2$. Similar reasonings may be repeated for the space $\mathcal{N}$ as well.

All previous normed vector spaces are incomplete with respect to the corresponding topologies induced by the norms above. Then we may further consider the corresponding topological closures of $(\mathcal{C}(E), //_k)$ and $(\mathcal{C}, \| \quad \|_k), k = 1, 2,$ which, except for an isomorphism, provide Banach subspaces in $l_k(N^2), k = 1, 2,$ and the Banach spaces $l_k(C), k = 1, 2,$ respectively.

Let us now consider $\underline{c} = \sum_{k=1}^{m} w_{c_k} \underline{c}_k \in \mathcal{C}$. Then, the isomorph of $\underline{c}$ in $\sigma(C)$ will be denoted by $\underline{c}'$, and in $\mathcal{C}(E)$ by $\underline{c}''$. Throughout the paragraph we shall adhere to this notation for any vector of cl $\mathcal{C}$, where cl symbolizes the topological closure of $\mathcal{C}$ with respect to $\|\|_k, k = 1, 2$.
Correspondingly we have

$$//\underline{c}'//_1 = \sum_{k=1}^{m} |w_{c_k}|,$$
$$//\underline{c}'//_2 = \left(\sum_{k=1}^{m} (w_{c_k})^2\right)^{1/2},$$
$$/\underline{c}''/_1 = \Sigma_{(i,j)} \left| \sum_{k=1}^{m} w_{c_k} J_{c_k}(i,j) \right|$$
$$/\underline{c}''/_2 = \left(\Sigma_{(i,j)} \left(\sum_{k=1}^{m} w_{c_k} J_{c_k}(i,j)\right)^2\right)^{1/2}.$$

Consider the vector spaces $\mathcal{E}$ and $\mathcal{C}$. Define the function $< \cdot, \cdot >: \mathcal{E} \times \mathcal{E} \rightarrow R$ as follows :

$$< \sum_{k=1}^{r} a_k b_k, \sum_{k=1}^{m} a'_k \, b_k >= \sum_{k=1}^{\min(r,m)} a_k a'_k.$$

Then $(\mathcal{E}, < \cdot, \cdot >)$ is an inner product space. Analogously, define the inner product space $(\mathcal{C}, < \cdot, \cdot >')$. Then the corresponding norms induced by the inner products $< \cdot, \cdot >$ and $< \cdot, \cdot >'$ are given by the relations (9.1.1) and (9.1.2).

Since $\mathcal{E}$ and $\mathcal{C}$ are incomplete metric spaces, we may further consider their completions $H(\mathcal{E})$ and $H(\mathcal{C})$ along with the corresponding extensions of $< \cdot, \cdot >$ and $< \cdot, \cdot >'$. Also, since the sets $B = \{b_1, b_2, \dots\}$ and $\underline{\mathrm{C}} = \{\underline{\mathrm{c}}_1, \underline{\mathrm{c}}_2, \dots\}$ are orthonormal bases of $H(\mathcal{E})$ and $H(\mathcal{C})$, we may consequently write any $\underline{\mathrm{x}} \in H(\mathcal{E})$ and any $y \in H(\mathcal{C})$ as the following Fourier series

$$\underline{\mathrm{x}} = \sum_{k=1}^{\infty} a_k b_k,$$

$$\underline{\mathrm{y}} = \sum_{k=1}^{\infty} \alpha_k \underline{\mathrm{c}}_k,$$

where $a_k =< \underline{\mathrm{x}}, b_k >$ and $\alpha_k=<\underline{\mathrm{y}}, \underline{\mathrm{c}}_k >', k = 1, 2, \cdots$, are the corresponding Fourier coefficients.

Furthermore, according to the Riesz-Fischer representation theorem, we may write

$$H(\mathcal{E}) = \left\{ \underline{\mathrm{x}} = \sum_{k=1}^{\infty} a_k b_k : a_k \in R, \sum_{k=1}^{\infty} (a_k)^2 < \infty \right\},$$

and

$$H(\mathcal{C}) = \left\{ \underline{\mathrm{y}} = \sum_{k=1}^{\infty} \alpha_{c_k} \underline{\mathrm{c}}_k : \alpha_{c_k} \in R, \sum_{k=1}^{\infty} (\alpha_{c_k})^2 < \infty \right\}.$$

Since B and C are denumerable orthonormal bases, the Hilbert spaces $H(\mathcal{E})$ and $H(\mathcal{C})$ are, respectively, isomorph (as normed vector spaces) with $l_2(E)$ and $l_2(C)$.

Finally, a Hilbert space $H(\mathcal{N})$ may also be defined, by developing a similar approach to the vector space $\mathcal{N}$.

9.2 Fourier Series on Directed Cycles

One problem to be solved in this section has the following abstract formulation:

Find the class of all sequences $\underline{w} = (w(i,j) \in R, (i,j) \in N^2)$, $N = \{1, 2, \ldots\}$, *which satisfy the following conditions:*

(i) There is a countable collection (C, w_{c_k}) *of directed cycles in* N *and real numbers* w_{c_k} *such that the Vertex set* $C = N$ *and*

$$w(i,j) = \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j), \quad (i,j) \in Arset C, \qquad (9.2.1)$$
$$= 0, \qquad \textit{otherwise,}$$

where the series occurring in (9.2.1) is absolutely convergent for any (i,j), *and all involved sets as* N^2, C, *etc., are endowed with certain orderings.*

(ii) There is $p \geq 1$ *such that* $\underline{w} \in l_p(N^2)$.

If sequence $\underline{w} = (w(i,j), (i,j) \in N^2)$ verifies the above conditions (i) and (ii), then we shall say that $\underline{w}$ *satisfies the cycle formula for* p *and* (C, w_c). In this case, collection (C, w_c) is called a *cycle representation* for $\underline{w}$.
Throughout this paragraph we shall consider a collection $\underline{C} = \{\underline{c}_1, \underline{c}_2, \ldots\}$ of independent homologic cycles associated with a collection $C = \{c_1, c_2, \ldots\}$ of overlapping directed circuits with Vertex set $C = N$. Also, we shall assume (without any loss of generality) that the corresponding graph-sets associated with C are symbolized and ordered as mentioned in the previous section.
The spaces to be considered here are the Banach spaces $l_k(C)$ and cl $\mathcal{C}(E)$ (in $l_k(N^2)$), $k = 1, 2$, where $\mathcal{C}$ will be identified by an isomorphism of vector spaces either with $\sigma(C)$ or with $\mathcal{C}(E)$.

We shall now answer the question of whether or not the Fourier series $\sum_{k=1}^{\infty} w_{c_k}\, \underline{c}_k$ may define a sequence $(w(i,j), (i,j) \in N^2)$ which satisfies the cycle formula following Kalpazidou and Kassimatis (1998). Namely, we have

Theorem 9.2.1. *Let the Fourier series*

$$\sum_{k=1}^{\infty} w_{c_k} \underline{c}_k \in H(\mathcal{C}),$$

where $w_{c_k}, k = 1, 2, \ldots,$ *are positive numbers.*
Then the following statements are pairwise equivalent:

(i) *Except for an isomorphism of vector spaces, the sequence* $\{\sum_{k=1}^{n} w_{c_k}\, \underline{c}_k\}_n$ *converges coordinate-wise, as* $n \to \infty$, *to a sequence* $\underline{w} = (w(i,j), (i,j) \in N^2)$, *which satisfies the cycle formula for* $p = 1$ *and with respect to* (C, w_c). *Furthermore,* $/\underline{w}/_1 = \sum_{k=1}^{\infty} p(c_k)\, w_{c_k}$;

(ii) $\sum_{k=1}^{\infty} p(c_k) w_{c_k} < \infty$;

(iii) *Except for an isomorphism of vector spaces, the sequence* $\{\sum_{k=1}^{n} w_{c_k} c_k\}_n$ *converges in* $l_1(N^2)$, *as* $n \to \infty$.

Proof. First we shall prove that (i) implies (ii). Let $\underline{w}_m = \sum_{k=1}^{m} w_{c_k} \underline{c}_k, m = 1, 2, \ldots,$ with $w_{c_k} > 0, k, = 1, \ldots, m$. Then the isomorph $\underline{w}_m{}''$ of $\underline{w}_m$ in $\mathcal{C}(E)$ is given by

$$\underline{w}_m{}'' = \left(\sum_{k=1}^{m} w_{c_k} J_{c_k}(i_1, j_1), \ldots, \sum_{k=1}^{m} w_{c_k} \ J_{c_k}(i_n, j_n), 0, 0, \ldots \right); m = 1, 2, \ldots,$$

where $(i_1, j_1), \ldots, (i_n, j_n)$, are the first n edges of Arc-set $\{\underline{c}_1, \ldots, \underline{c}_m\}$ indexed according to the ordering of Arcs-set $C \equiv E$. If (i) holds, then for any $(i, j) \in N^2$ there exists a positive number $w(i, j)$ defined as follows:

$$\begin{aligned} w(i, j) &= \lim_{m \to \infty} \sum_{k=1}^{m} w_{c_k} J_{c_k}(i, j), \quad \text{if } (i, j) \in E, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Denote $\underline{w} = (w(i, j), (i, j) \in N^2)$. Then

$$\sum_{k=1}^{\infty} p(c_k) w_{c_k} = \sum_{k=1}^{\infty} \sum_{(i,j)} w_{c_k} J_{c_k}(i, j) = \sum_{(i,j)} |w(i, j)| < \infty.$$

The proof of (ii) is complete.

Let us now prove the converse implication. Accordingly, assume that relation (ii) holds. Then, for any $(i, j) \in E$ the limit

$$\lim_{m \to \infty} \sum_{k=1}^{m} w_{c_k} \ J_{c_k}(i, j)$$

exists, since

$$\sum_{k=1}^{\infty} p(c_k) w_{c_k} = \sum_{(i,j)} \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i, j) < \infty.$$

Define $\underline{w} = (w(i, j), (i, j) \in N^2)$ with

$$\begin{aligned} w(i, j) &= \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i, j), \quad \text{if } (i, j) \in E, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Then $\underline{w}$ satisfies the cycle formula for $p = 1$ and with respect to (C, w_c). Furthermore, we note that $\underline{w}$ is the coordinate-wise limit of $\{\sum_{k=1}^{m} w_{c_k} \underline{c}_k\}_m$ viewed isomorphically in $\mathcal{C}(E)$. The proof of (i) is complete.

Let us now prove that (ii) implies (iii). From the relation (ii) we obtain that

$$\sum_{k=1}^{\infty} w_{c_k}\, J_{c_k}(i,j) < \infty,$$

for any $(i,j) \in E$. Then we may accordingly define the following sequence $\underline{\mathrm{w}} = (w(i,j), (i,j) \in N^2)$ in $l_1(N^2)$:

$$\begin{aligned} w(i,j) &= \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j), && \text{if } (i,j) \in E, \\ &= 0, && \text{otherwise.} \end{aligned}$$

Consider $\underline{\mathrm{w}}_n'' = (\mathrm{w}_n''(i,j), (i,j) \in N^2)$ with

$$w_n''(i,j) = \sum_{k=1}^{n} w_{c_k} J_{c_k}(i,j).$$

Then $\underline{\mathrm{w}}_n'' \in \mathcal{C}(E)$ and

$$\begin{aligned} &/\underline{\mathrm{w}} - \underline{\mathrm{w}}_n''/_1 = \\ &= \sum_{(i,j)\in E} |\sum_{k=1}^{\infty} w_{c_k} J_{c_k}\,(i,j) - \sum_{k=1}^{n} w_{c_k} J_{c_k}\,(i,j)| \\ &= \sum_{(i,j)\in E} \left(\sum_{k=n+1}^{\infty} w_{c_k} J_{c_k}(i,j) \right) = \sum_{k=n+1}^{\infty} p(c_k) w_{c_k} < \infty. \end{aligned}$$

Furthermore

$$\lim_{n\to\infty} /\underline{\mathrm{w}} - \underline{\mathrm{w}}_n''/_1 = \lim_{n\to\infty} \sum_{k=n+1}^{\infty} p(c_k) w_{c_k} = 0.$$

Therefore, the sequence of $\underline{\mathrm{w}}_n''$, $n = 1, 2, \ldots$, which are the isomorphs of $\underline{\mathrm{w}}_n = \sum_{k=1}^{n} w_{c_k} \underline{\mathrm{c}}_k$ in $\mathcal{C}(E)$, converges in $l_1(N^2)$ to $\underline{\mathrm{w}} = (w(i,j), (i,j) \in N^2)$, as $n \to \infty$. The proof of (iii) is complete.

Now we shall prove the converse, that is, from (iii) we shall obtain relation (ii). Let $\underline{\mathrm{w}} = (w(i,j), (i,j) \in N^2)$ be the

$$\lim_{n\to\infty} \sum_{k=1}^{n} w_{c_k} \underline{\mathrm{c}}_k \,\text{in}\, l_1(N^2),$$

where $\underline{\mathrm{w}}_n = \sum_{k=1}^{n} w_{c_k} \underline{\mathrm{c}}_k$ is isomorphically viewed in $\mathcal{C}(E)$.

Since for every $n \geq 1$ and any $(i,j) \in N^2 \backslash E$ we have $\sum_{k=1}^{n} w_{c_k} J_{c_k}\,(i,j) = 0$, then $w(i,j) = 0$ outside E. Therefore

$$w(i,j) = \sum_{k=1}^{\infty} w_{c_k} J_{c_k}\,(i,j),$$

for any $(i,j) \in E$ and $/w/_1 = \sum_{k=1}^{\infty} p(c_k) w_{c_k} < \infty$.

Furthermore, from convergence

$$\lim_{n\to\infty} /\underline{\mathrm{w}} - \underline{\mathrm{w}}_n''/_1 = 0,$$

where $\underline{\mathrm{w}}_n''$ is the isomorph of $\underline{\mathrm{w}}_n$ in $\mathcal{C}(E)$, we may write

$$\begin{aligned} /\underline{\mathrm{w}} - \underline{\mathrm{w}}_n''/_1 &= \sum_{(i,j)\in E} |w(i,j) - \sum_{k=1}^{n} w_{c_k} J_{c_k}(i,j)| \\ &= \sum_{(i,j)\in E} \left(\sum_{k=n+1}^{\infty} w_{c_k} J_{c_k}(i,j) \right) = \sum_{k=n+1}^{\infty} p(c_k) w_{c_k} \end{aligned}$$

and

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} p(c_k) w_{c_k} = 0.$$

The proof of Theorem is complete. □

Now we shall investigate the relations between the Hilbert spaces H($\mathcal{N}$), H($\mathcal{E}$), H($\mathcal{C}$), and the sequences that satisfy the cycle formula. We have:

Theorem 9.2.2. *Let Fourier series*

$$\sum_{k=1}^{\infty} w_{c_k} \underline{\mathrm{c}}_k \in H(\mathcal{C}),$$

with $w_{c_k} > 0, k = 1, 2, \ldots$

Then the following statements are pairwise equivalent:

(i) *Except for an isomorphism of vector spaces, the sequence* $\{\sum_{k=1}^{n} w_{c_k} \underline{\mathrm{c}}_k\}_n$ *converges coordinate-wise, as* $n \to \infty$*, to a sequence* $\underline{\mathrm{w}} = (w(i,j), (i,j) \in N^2)$*, which satisfies the cycle formula for* $p = 2$ *and with respect to* (C, w_c)*;*

(ii) $\sum_{k=1}^{\infty} (w_{c_k})^2 p(c_k) + 2 \sum_{k,s=1;k\neq s}^{\infty} w_{c_k} w_{c_s}\, card\{(i,j) : J_{c_k}(i,j) J_{c_s}(i,j) = 1\} < \infty$ *where* J_{c_k} *(i,j) is the passage-function associated with* $c_k, k = 1, 2, \ldots$*;*

(iii) *Except for an isomorphism of vector spaces, the sequence* $\{\sum_{k=1}^{n} w_{c_k} \underline{\mathrm{c}}_k\}_n$ *converges in* $H(\mathcal{E})$ *to* $\sum_{(i,j)} (\sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j))\, b_{(i,j)}$*, as* $n \to \infty$*.*

Proof. Let us assume that (i) holds. We shall now prove that relation (ii) is valid. Let $\underline{\mathrm{w}}_m = \{\sum_{k=1}^{m} w_{c_k} \underline{\mathrm{c}}_k\}$, $m = 1, 2, \ldots$ Then $\underline{\mathrm{w}}_m \in \mathcal{C}$ and sequence $\{\underline{\mathrm{w}}_m\}_m$ converges in H($\mathcal{C}$) to $\sum_{k=1}^{\infty} w_{c_k} \underline{\mathrm{c}}_k$. Consider the isomorph $\underline{\mathrm{w}}_m''$ of $\underline{\mathrm{w}}_m$ in $\mathcal{C}(E)$. Then

$$\underline{\mathrm{w}}_m'' = \left(\sum_{k=1}^{m} w_{c_k} J_{c_k}(i_1, j_1), \ldots, \sum_{k=1}^{m} w_{c_k} J_{c_k}(i_n, j_n), 0, 0, \ldots \right), m = 1, 2, \ldots,$$

where $(i_1, j_1), \ldots, (i_n, j_n)$ are the first n edges of Arcset$\{c_1, \ldots, c_m\}$ according to the ordering of E. Since (i) holds, for any $(i,j) \in N^2$ there exists a positive number $w(i,j)$ given by

$$\begin{aligned} w(i,j) &= \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j), \quad \text{if } (i,j) \in E, \\ &= 0, \qquad\qquad\qquad\qquad \text{otherwise,} \end{aligned}$$

and sequence $\underline{\mathrm{w}} = (w(i,j), (i,j) \in N^2)$ belongs to $l_2(N^2)$.

On the other hand, we have

$$\begin{aligned} \sum_{(i,j)} w^2(i,j) &= \sum_{(i,j)} \left(\sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j) \right)^2 \\ &= \sum_{(i,j)} \left(\sum_{k=1}^{\infty} (w_{c_k})^2 J_{c_k}(i,j) + 2 \sum_{k,s=1;k\neq s}^{\infty} w_{c_k} w_{c_s} J_{c_k}(i,j) J_{c_s}(i,j) \right) \\ &= \sum_{k=1}^{\infty} (w_{c_k})^2 p(c_k) + 2 \sum_{k,s=1;k\neq s}^{\infty} w_{c_k} w_{c_s} \operatorname{card}\{(i,j) : J_{c_k}(i,j) J_{c_s}(i,j) = 1\}. \end{aligned}$$

The relation (ii) holds.

Let us now prove the converse: assuming (ii), we shall prove that (i) holds. First, we have

$$\sum_{(i,j)} \left(\sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j) \right)^2 < \infty.$$

Define the sequence $\underline{\mathrm{w}} = (w(i,j), (i,j) \in N^2)$ as follows:

$$\begin{aligned} w(i,j) &= \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j), \qquad \text{if } (i,j) \in E, \\ &= 0, \qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

Then sequence $\underline{\mathrm{w}}$ satisfies the cycle formula for $p = 2$ and with respect to (C, w_c). Furthemore $\underline{\mathrm{w}}$ is the coordinate-wise-limit of the sequence $\{\underline{\mathrm{w}}_m''\}$ of isomorphs of $\underline{\mathrm{w}}_m = \sum_{k=1}^{m} w_{c_k} \underline{\mathrm{c}}_k$ in $\mathcal{C}(E)$, given by

$$\underline{\mathrm{w}}_m'' = \left(\sum_{k=1}^{m} w_{c_k} J_{c_k}(i_1, j_1), \ldots, \sum_{k=1}^{m} w_{c_k} J_{c_k}(i_n, j_n), 0, 0, \ldots \right), m = 1, 2, \ldots,$$

where $(i_1, j_1), \ldots, (i_n, j_n)$ are the edges of $c_1, \ldots, c_m$. The proof of (i) is complete. □

Let us now prove (iii) from (ii). In this direction, we define by using (ii) the sequence $\underline{\mathrm{w}} = (w(i,j), (i,j) \in N^2)$ in $l_2(N^2)$ with

$$w(i,j) = \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j), \qquad \text{if } (i,j) \in \text{Arcset}\, C,$$
$$= 0, \qquad \text{otherwise.}$$

Furthermore, the sequence

$$\underline{\mathrm{w}}_m{}'' = \left(\sum_{k=1}^{m} w_{c_k} J_{c_k}(i_1, j_1), \ldots, \sum_{k=1}^{m} w_{c_k} J_{c_k}(i_n, j_n), 0, 0, \ldots \right), m = 1, 2, \ldots,$$

where $(i_1, j_1), \ldots, (i_n, j_n)$ are the edges of $c_1, \ldots, c_m$, converges coordinatewise to $\underline{\mathrm{w}}$. Now we prove that we have more: namely, sequence $\{\underline{\mathrm{w}}_m{}''\}_m$ converges in $l_2(N^2)$ to $\underline{\mathrm{w}}$, as $m \to \infty$. In this direction, we first write

$$/\underline{\mathrm{w}} - \underline{\mathrm{w}}_m{}''/_2 = \left[\sum_{(i,j)\in E} (w(i,j) - \sum_{k=1}^{m} w_{c_k} J_{c_k}(i,j))^2 \right]^{1/2}$$
$$= \left[\sum_{(i,j)\in E} \left(\sum_{k=m+1}^{\infty} w_{c_k} J_{c_k}(i,j) \right)^2 \right]^{1/2}$$
$$= \left[\sum_{k=m+1}^{\infty} (w_{c_k})^2 p(c_k) + 2 \right.$$
$$\left. \times \sum_{k,s=m+1; k\neq s}^{\infty} w_{c_k} w_{c_s} \text{card}\{(i,j) : J_{c_k}(i,j) J_{c_s}(i,j) = 1\} \right]^{1/2}.$$

Since (ii) holds, both last series occurring in the expression of $/\underline{\mathrm{w}} - \underline{\mathrm{w}}_m{}''/_2$ converge to zero, as $m \to \infty$. Finally, the isomorphs of $\underline{\mathrm{w}}_m{}''$ and $\underline{\mathrm{w}}$ in $\mathrm{H}(\mathcal{E})$ are, respectively, $\sum_{(i,j)} (\sum_{k=1}^{m} w_{c_k} J_{c_k}(i,j))\, b_{(i,j)}$ and $\sum_{(i,j)} (\sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j))\, b_{(i,j)}$.

The proof of (iii) is complete.

To prove the converse, assume that the sequence of isomorphs of $\sum_{k=1}^{m} w_{c_k} \underline{\mathrm{c}}_k, m = 1, 2, \ldots$, in $l_2(N^2)$ converge to $\underline{\mathrm{w}} = (w(i,j), (i,j) \in N^2)$, as $m \to \infty$, where $w(i,j) = \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j)$, for any $(i,j) \in N^2$. Then, since series occurring in (ii) is related to the norm $/\underline{\mathrm{w}}/_2$, relation (ii) holds. The proof of theorem is complete. □

9.3 Orthogonal Cycle Transforms for Finite Stochastic Matrices

Let $S = \{1, 2, \ldots, n\}, n > 1$, and let $P = (p_{ij}, i, j = 1, 2, \ldots, n)$ be an irreducible stochastic matrix whose probability row-distribution is $\pi = (\pi_i, i = 1, \ldots, n)$. Let $G = G(P) = (S, E)$ be the oriented graph attached to P, where $E = \{b_1, \dots, b_\tau\}$ denotes the set of directed edges endowed with an ordering. The orientation of G means that each edge b_k is an ordered pair (i, j) of points of S such that $p_{ij} > 0$, where i is the initial point and j is the endpoint. Sometimes we shall prefer the symbol $b_{(i,j)}$ for b_k when we need to point out the terminal points.

As we have already mentioned in section 4.2.1, irreducibility of P means that the graph G is strongly connected, that is, for any pair (i, j) of states there exists a sequence $b_{(i,i_1)}, b_{(i_1,i_2)}, \dots, b_{(i_s,j)}$ of edges of G connecting i to j. When $i = j$ then such a sequence is called a directed circuit of G. Throughout this chapter, we shall consider directed circuits $c = (i, i_1, i_2, \dots, i_s, i)$ where the points $i, i_1, i_2, \dots, i_s$ are all distinct.

Let C denote the collection of all directed circuits of G. Then according to Theorem 4.1.1 the matrix P is decomposed by the circuits $c \in C$ as follows:

$$\pi_i p_{ij} = \sum_{c \in C} w_c J_c(i, j), \tag{9.3.1}$$

where each w_c is uniquely defined by a probabilistic algorithm and J_c is the passage-matrix of c introduced in the previous section. Furthermore, equations (9.3.1) are independent of the ordering of C.

Now we shall look for a suitable Hilbert space where the cycle decomposition (9.3.1) is equivalent with a Fourier-type decomposition for P. In this direction we shall consider as in section 4.4 two-vector spaces C_0 and C_1 generated by the collections S and E, respectively. Then any two elements $\underline{c}_0 \in C_0$ and $\underline{c}_1 \in C_1$ have the following expressions:

$$\underline{c}_0 = \sum_{h=1}^{n} x_h n_h = \underline{x}'\underline{n}, \qquad x_h \in R, \quad n_h \in S,$$

$$\underline{c}_1 = \sum_{k=1}^{\tau} y_k b_k = \underline{y}'\underline{b}, \qquad y_k \in R, \quad b_k \in E,$$

where R denotes the set of reals. The elements of C_0 and C_1 are, respectively, called the zero-chains and the one-chains associated with the graph G.

Let $\delta \colon C_1 \to C_0$ be the boundary linear transformation defined as

$$\delta \underline{c}_1 = \underline{y}' \eta \, \underline{n},$$

where

$$\eta_{b_j n_s} = \begin{cases} +1, & \text{if } n_s \text{ is the endpoint of the edge } b_j; \\ -1, & \text{if } n_s \text{ is the initial point of the edge } b_j; \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$\tilde{C}_1 \equiv \operatorname{Ker} \delta = \{\underline{z} \in C_1 : \underline{z}' \eta = \underline{0}\},$$

where $\underline{0}$ is the neutral element of C_1.

Then $\tilde{C}_1$ is a linear subspace of C_1 whose elements are called one-cycles. One subset of $\tilde{C}_1$ is given by all the elements $\underline{c} = b_{i_1} + \cdots + b_{i_k} \in C_1$ whose edges $b_{i_1}, \ldots, b_{i_k}$ form a directed circuit c in the graph G. In general, the circuits occurring in the decomposition (9.3.1) of P determine linearly dependent one-cycles in $\tilde{C}_1$. In Lemma 4.4.1, it is proved that there are B one-cycles $\underline{\gamma}_1, \ldots, \underline{\gamma}_B$, which form a base for the linear subspace $\tilde{C}_1$, where B is the Betti number of G. When $\underline{\gamma}_1, \ldots, \underline{\gamma}_B$, are induced by genuine directed circuits $\gamma_1, \ldots, \gamma_B$ of the graph G, then we call $\gamma_1, \ldots, \gamma_B$ the Betti circuits of G.

With these preparations, we now prove

Lemma 9.3.1. *The vector space $\tilde{C}_1 = Ker\delta$ of one-cycles is a Hilbert space whose dimension is the Betti number of the graph.*

Proof. Let $\Gamma = \{\underline{\gamma}_1, \ldots, \underline{\gamma}_B\}$ be the set of Betti one-cycles of G, endowed with an ordering. Then

$$\tilde{C}_1 = \left\{ \sum_{k=1}^{B} a_k \underline{\gamma}_k, a_k \in R \right\}.$$

Consider the inner product $<,>: \tilde{C}_1 \times \tilde{C}_1 \to R$ as follows:

$$< \sum_{k=1}^{B} a_k \underline{\gamma}_k, \sum_{k=1}^{B} b_k \underline{\gamma}_k >= \sum_{k=1}^{B} a_k b_k.$$

Then $\tilde{C}_1$ is metrizable with respect to the metric

$$d\left(\sum_{k=1}^{B} a_k \underline{\gamma}_k, \sum_{k=1}^{B} b_k \underline{\gamma}_k \right) = \sqrt{\sum_{k=1}^{B} (a_k - b_k)^2}.$$

Therefore $(\tilde{C}_1, <,>)$ is an inner product space where Γ is an orthonormal base. Accordingly, to any one-cycle $\underline{z} = \sum_{k=1}^{B} a_k \underline{\gamma}_k$ there correspond the Fourier coefficients $a_k =<\underline{z}, \underline{\gamma}_k>, k = 1, \ldots, B$, with respect to the orthonormal base Γ.

Define the mapping $f : \tilde{C}_1 \to R^B$ as follows:

$$f\left(\sum_{k=1}^{B} a_k \underline{\gamma}_k \right) = (a_1, \ldots, a_B).$$

Then f preserves inner-product-space structures, that is, f is a linear bijection which preserves inner products. In particular, f is an isometry. Then $(\tilde{C}_1, <,>)$ is a Hilbert space, whose dimension is B. The proof is complete. □

The previous result may be generalized to any finite connected graph G. Now we shall focus on graphs $G(P)$ associated with irreducible stochastic matrices P. Denote by B the Betti number of $G(P)$. Consider the collection

C of cycles occurring in the decomposition (9.3.1), endowed with an ordering, that is, $C = \{c_1, \ldots, c_s\}, s > 0$. Then we have

Theorem 9.3.2. *Let $P = (p_{ij}, i, j = l, \ldots, n)$ be an irreducible stochastic matrix whose invariant probability row-distribution is $\pi = (\pi_1, \ldots, \pi_n)$. Assume that $\{\gamma_1, \ldots, \gamma_B\}$ is a collection of Betti circuits. Then πP has a Fourier representation with respect to $\Gamma = \{\underline{\gamma}_1, \ldots, \underline{\gamma}_B\}$, where the Fourier coefficients are identical with the probabilistic-homologic cycle-weights $w_{\gamma_1}, \ldots, w_{\gamma_B}$, that is,*

$$\sum_{(i,j)} \pi_i p_{ij} b_{(i,j)} = \sum_{k=1}^{B} w_{\gamma_k} \underline{\gamma}_k, \qquad w_{\gamma_k} \in R, \tag{9.3.2}$$

with

$$w_{\gamma_k} = < \pi P, \underline{\gamma}_k >, \qquad k = 1, \ldots, B\ .$$

In terms of the (i,j)-coordinate, equations (9.3.2) are equivalent to

$$\pi_i p_{ij} = \sum_{k=1}^{B} w_{\gamma_k} J_{\gamma_k}(i,j), \qquad w_{\gamma_k} \in R; i, j \in S. \tag{9.3.3}$$

If P is a recurrent stochastic matrix, then a similar representation to (9.3.2) holds, except for a constant, on each recurrent class.

Proof. Denote $w(i,j) = \pi_i p_{ij}, i, j = 1, \ldots, n$. Then πP may be viewed as a one-chain $\underline{\mathrm{w}} = \Sigma_{(i,j)} w(i,j) b_{(i,j)}$.
Since πP is balanced, $\underline{\mathrm{w}}$ is a one-cycle, that is, $\underline{\mathrm{w}} \in \tilde{C}_1 = \operatorname{Ker} \delta$. Then, according to Lemma 9.3.1, $\underline{\mathrm{w}}$ may be written as a Fourier series with respect to an orthonormal base $\Gamma = \{\underline{\gamma}_1, \ldots, \underline{\gamma}_B\}$ of Betti circuits of G, that is,

$$\underline{\mathrm{w}} = \sum_{k=1}^{B} < \underline{\mathrm{w}}, \underline{\gamma}_{\mathrm{k}} > \underline{\gamma}_{\mathrm{k}}, \tag{9.3.4}$$

where $< \underline{\mathrm{w}}, \underline{\gamma}_k >, k = 1, \ldots, B$, are the corresponding Fourier coefficients.

On the other hand, the homologic-cycle-formula proved by Theorem 4.5.1 asserts that $\underline{\mathrm{w}}$ may be written as

$$\underline{\mathrm{w}} = \sum_{k=1}^{B} w_{\gamma_k} \underline{\gamma}_k, \tag{9.3.5}$$

where $w_{\gamma_{\mathrm{k}}}, k = 1, \ldots, B$, are the probabilistic-homologic cycle-weights given by a linear transformation of the probabilistic weights $w_c, c \in C$, occurring in (9.3.1), that is,

$$w_{\gamma_k} = \sum_{c \in C} A(c, \underline{\gamma}_k)\, w_c, \qquad A(c, \underline{\gamma}_k) \in Z,$$

where Z denotes the set of integers.

Since representation (9.3.5) is unique, it follows that it coincides with the Fourier representation (9.3.4), that is,

$$w_{\gamma_k} =< \underline{\mathrm{w}}, \underline{\gamma}_k >, \qquad k = 1, 2, \ldots, B.$$

Accordingly, since $\underline{c} = \Sigma_\kappa A(c, \underline{\gamma}_k)\, \underline{\gamma}_k$, then

$$A(c, \underline{\gamma}_k) =< \underline{c}, \underline{\gamma}_k >, \qquad k = 1, \ldots, B,$$

and therefore

$$w_{\gamma_k} = \sum_{c \in C} < \underline{c}, \underline{\gamma}_k > w_c. \tag{9.3.6}$$

Let us now suppose that P has more than one recurrent class e in $S = \{1, \ldots, n\}$. Then we may apply the previous reasonings to each recurrent class e and to each balanced expression

$$\pi_e(i) p_{ij} = \sum_{k=1}^{B} w_{\gamma_k} J_{\gamma_k}(i,j), \qquad i, j \in e,$$

where $B = B_e$ is the Betti number of the connected component of the graph $G(P)$ corresponding to e, and $\pi_e = \{\pi_e(i)\}$ (with $\pi_e(i) > 0$, for $i \in e$, and $\pi_e(i) = 0$ outside e) is the invariant probability distribution associated to each recurrent class e. The proof is complete. □

Remark. Let $w = (w(k), k = 1, 2, \ldots, B)$ be defined as

$$w(k) = w_{\gamma_k}, \qquad k = 1, \ldots, B,$$

where $w_{\gamma_k}, k = 1, \ldots, B$, are the probabilistic-homologic weights occurring in (9.3.5). Then equations

$$w(k) = \sum_{c \in C} < \underline{c}, \underline{\gamma}_k > w_c$$

may be interpreted as the inverse Fourier transform of the probabilistic weight-function $w_c, c \in C$, associated with P.

9.4 Denumerable Markov Chains on Banach Spaces on Cycles

Now we are prepared to show how to define a denumerable Markov chain from a countable infinity of directed cycles by using the Banach spaces on cycles investigated in the previous sections. Namely we have

Theorem 9.4.1. *Let $C = \{c_1, c_2, \ldots\}$ be a countable set of overlapping directed circuits in N that verify the assumptions mentioned in section 9.2.*

If sequence $\underline{\mathrm{w}} = (w(i,j), (i,j) \in N^2)$ satisfies the cycle formula for $p = 1$ and with respect to (C, w_c), with $w_c > 0, c \in C$, then $p_{ij} \equiv w(i,j)/(\Sigma_j w(i,j)), i, j \in N$, define a stochastic matrix of an N-state *cycle Markov chain* $\xi = (\xi_n)_n$, *that is,*

$$p_{ij} = \frac{\sum\limits_{c \in C} w_c J_c(i,j)}{\sum\limits_{c \in C} w_c J_c(i)}, \quad \text{if } (i,j) \in \text{ Arcset}\, C,$$
$$= 0, \qquad \text{otherwise,}$$

where $J_c(i) = \Sigma_j J_c(i,j), i \in N, c \in C$. *Furthermore,* $\mu = \left(\sum\limits_{k=1}^{\infty} w_{c_k} J_{c_k}(i), \; i = 1, 2, \ldots \right)$ *is an invariant finite measure for the Markov chain* ξ.

Proof. Let $\underline{\mathrm{w}} = (w(i,j), (i,j) \in N^2)$ be a sequence of $l_1(N^2)$, which satisfies the cycle formula with respect to a collection (C, w_c), with $w_c > 0$, that is,

$$w(i,j) = \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j), \quad \text{if } (i,j) \in \text{ Arcset}\, C,$$
$$= 0, \qquad \text{otherwise.}$$

We may always find such a sequence if we choose the sequence $\{w_{c_k}, k = 1, 2, \ldots\}$ of positive numbers such that $\sum\limits_{k=1}^{\infty} p(c_k) w_{c_k} < \infty$ (as in condition (ii) of Theorem 9.2.1).

Define

$$w(i) = \sum_j w(i,j), \qquad i \in N.$$

Then $w(i) > 0, i \in N$, and

$$w(i) = \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i),$$

where $J_{c_k}(i) = \sum\limits_j J_{c_k}(i,j)$ for any $i \in N$.

Define

$$p_{ij} = \frac{w(i,j)}{w(i)}, \qquad i, j \in N.$$

Then $P = (p_{ij},\, i, j \in N)$ is a stochastic matrix that defines an N-state cycle Markov chain $\xi = (\xi_n)_n$ whose cycle representation is (C, w_c). Also,

$$\sum_i w(i) = \sum_{k=1}^{\infty} p(c_k) w_{c_k} < \infty$$

and

$$\sum_i w(i) p_{ij} = \sum_i w(i,j) = \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(j) = w(j),$$

for any $j \in N$. Then $\mu = (w(i), i = 1, 2, \ldots)$ is an invariant finite measure for the Markov chain ξ. The proof is complete. □