8

Cycloid Markov Processes

As we have already seen, finite homogeneous Markov chains ξ admitting invariant probability distributions may be defined by collections $\{c_\kappa, w_k\}$ of directed circuits and positive weights, which provide linear decompositions for the corresponding finite-dimensional probability distributions. The aim of the present chapter is to generalize the preceding decompositions to more relaxed geometric entities occurring along almost all the sample paths of ξ such as the cycloids, which are closed chains of edges with various orientations. Then ξ is called a cycloid Markov chain. Correspondingly, the passage-functions associated with the algebraic cycloids have to express the change of the edge-direction, while the linear decompositions in terms of the cycloids provide shorter descriptions for the finite-dimensional distributions, called *cycloid decompositions*.

A further development of the cycloid decompositions to real balance functions is particularly important because of the revelation of their intrinsic homologic nature. Consequently, the cycloid decompositions enjoy a measure-theoretic interpretation expressing the same essence as the known Chapman–Kolmogorov equations for the transition probability functions. The development of the present chapter follows S. Kalpazidou (1999a, b).

8.1 The Passages Through a Cycloid

Let S be a finite set and let $G = (S, E)$ be any connected oriented graph $G = (S, E)$, where E denotes the set of all directed edges (i, j), which sometimes will be symbolized by $b_{(i,j)}$.

If $\tilde{c}$ is a sequence $(e_1, \ldots, e_m)$ of directed edges of E such that each edge $e_r, 2 \le r \le m-1$, has one common endpoint with the edge $e_{r-1}(\neq e_r)$ and a second common endpoint with the edge $e_{r+1}(\neq e_r)$, then $\tilde{c}$ is called the *chain* which joins the free endpoint u of e_1 and the free endpoint v of e_m. Both u and v are called endpoints of the chain. If any endpoint of the edges $e_1, \ldots, e_m$ appears once when we delete the orientation, then $\tilde{c}$ is called an *elementary chain.*

Definition 8.1.1. A *cycloid* is any chain of distinct oriented edges whose endpoints coincide.

From the definition of the elementary chain, we correspondingly obtain the definition of an *elementary cycloid.* Consequently, a *directed circuit or cycle* c is any cycloid whose edges are oriented in the same way, that is, the terminal point of any edge of c is the initial point of the next edge. Accordingly, we also obtain the definition of the elementary cycle.

To describe the passages along an arbitrary cycloid $\tilde{c}$, we need a much more complex approach than that given for the directed circuits in Chapter 1. It is this approach that we introduce now.

Let $\tilde{c}$ be an elementary cycloid of G. Then $\tilde{c}$ is defined by giving its edges $e_1, e_2, \ldots, e_s$, which are not necessarily oriented in the same way, that is, the closed chain $(e_1, e_2, \ldots, e_s)$ does not necessarily define a directed circuit in S. However, we may associate the cycloid $\tilde{c}$ with a unique directed circuit (cycle) c and with its opposite c_- made up by the consecutive points of $\tilde{c}$. Note that certain edges of both c and c_- may eventually be not in the graph G.

We shall call c and c_- the *directed circuits (cycles) associated with the cycloid* $\tilde{c}$. For instance, consider the cycloid $\tilde{c} = ((1,2),(3,2),(3,4),(4,1))$. Then the associated directed circuits are $c = (1,2,3,4,1)$ and $c_- = (1,4,3,2,1)$. With these preparations we now introduce the following definitions.

The passage-function associated with a cycloid $\tilde{c}$ and its associated directed circuit c is the function $J_{\tilde{c},c_-}$: E $\to \{-1,0,1\}$ defined as

$$\begin{aligned} J_{\tilde{c},c}(i,j) &= 1, && \text{if } (i,j) \text{ is an edge of } \tilde{c} \text{ and } c, \\ &= -1, && \text{if } (i,j) \text{ is an edge of } \tilde{c} \text{ and } c_-, \\ &= 0, && \text{otherwise.} \end{aligned} \tag{8.1.1}$$

Analogously, the passage-function associated with the pair $(\tilde{c}, c_-)$ is the function $J_{\tilde{c},c_-}$: E $\to \{-1,0,1\}$ defined as

$$\begin{aligned} J_{\tilde{c},c_-}(i,j) &= 1, && \text{if } (i,j) \text{ is an edge of } \tilde{c} \text{ and } c_-, \\ &= -1, && \text{if } (i,j) \text{ is an edge of } \tilde{c} \text{ and } c, \\ &= 0, && \text{otherwise.} \end{aligned}$$

Then we have

$$J_{\tilde{c},c}(i,j) = -J_{\tilde{c},c_-}(i,j), \qquad i,j \in S,$$

and

$$J_{\tilde{c},c}(i,j) \neq J_{\tilde{c},c}(j,i), \qquad i,j \in S.$$

In particular, if the cycloid $\tilde{c}$ coincides with the cycle c, then

$$J_{\tilde{c},c}(i,j) = J_c(i,j), \qquad i,j \in S,$$

where $J_c(i,j)$ is the passage-function of c, which is equal to 1 or 0 according to whether or not (i,j) is an edge of c.

The passage-functions associated with the cycloids enjoy a few simple, but basic properties.

Lemma 8.1.2. *The passage-functions $J_{\tilde{c},c}(i,j)$ and $J_{\tilde{c},c_-}(i,j)$ associated with the elementary cycloid $\tilde{c}$ are balanced functions, that is,*

$$\sum_{j\in S} J_{\tilde{c},c}(i,j) = \sum_{k\in S} J_{\tilde{c},c}(k,i), \tag{8.1.2}$$

$$\sum_{j\in S} J_{\tilde{c},c_-}(i,j) = \sum_{k\in S} J_{\tilde{c},c_-}(k,i), \tag{8.1.3}$$

for any $i \in S$.

Proof. We shall prove equations (8.1.2). Consider $i \in S$. If i does not lie on $\tilde{c}$, then i does not lie on both c and c_-. Then both members of (8.1.2) are equal to zero.

Now, let i be a point of $\tilde{c}$. Then i is a point of c and c_- as well. Accordingly, we distinguish four cases.

Case 1: The edges of $\tilde{c}$, which are incident at i, have the orientation of c. Then

$$\sum_{j\in S} J_{\tilde{c},c}(i,j) = J_{\tilde{c},c}(i,u) = +1,$$
$$\sum_{k\in S} J_{\tilde{c},c}(k,i) = J_{\tilde{c},c}(v,i) = +1,$$

where (i,u) and (v,i) are the only edges of $\tilde{c}$ and c, which are incident at i.

Case 2: The point i is the terminal point of both edges of $\tilde{c}$, which are incident at i. Then, we have

$$\sum_{j\in S} J_{\tilde{c},c}(i,j) = 0,$$
$$\sum_{k\in S} J_{\tilde{c},c}(k,i) = J_{\tilde{c},c}(v,i) + J_{\tilde{c},c}(u,i) = (+1) + (-1) = 0,$$

where (v,i) and (u,i) are the only edges of $\tilde{c}$, one lying on c and the other on c_-, which have i as a terminal point.

Case 3: The point i is the initial point of both edges of $\tilde{c}$ which are incident at i. Accordingly, we write

$$\sum_{j \in S} J_{\tilde{c},c}(i,j) = (+1) + (-1) = 0,$$
$$\sum_{k \in S} J_{\tilde{c},c}(k,i) = 0.$$

Case 4: The edges of $\tilde{c}$, which are incident at i, have the orientation of c_-. Then

$$\sum_{j \in S} J_{\tilde{c},c}(i,j) = -1,$$
$$\sum_{k \in S} J_{\tilde{c},c}(k,i) = -1.$$

Finally, relations (8.1.3) may be proved by similar arguments. The proof is complete. □

Now we shall investigate how to express the passages of a particle moving along the cycloids $\tilde{c}$ of G in terms of the passage-functions.

First, let us assume that the cycloid $\tilde{c}$ coincides with the directed circuit c. Then the motion along the circuit c is characterized by the direction of c, which, in turn, allows the definition of an algebraic analogue $\underline{c}$ in the real vector space C_1 generated by the edges $\{b_{(i,j)}\}$ of the graph G. Specifically, as in paragraph 4.4 any directed circuit $c = (i_1, i_2, \ldots i_s, i_1)$, occurring in the graph G, may be assigned to a vector $\underline{c} \in C_1$ defined as follows:

$$\underline{c} = \sum_{(i,j)} J_c(i,j) b_{(i,j)},$$

where J_c is equal to 1 or 0 according to whether or not (i,j) is an edge of c. Let us now consider a cycloid $\tilde{c}$, which is not a directed circuit. To associate $\tilde{c}$ with a vector $\underline{\tilde{c}}$ in C_1, we choose a priori a direction for the passages along $\tilde{c}$, that is, we shall consider either the pair $(\tilde{c}, c)$ or the pair $(\tilde{c}, c_-)$ where c and c_- are the directed circuits associated with $\tilde{c}$. Then we may assign the graph-cycloid $\tilde{c}$ with the vectors $\underline{\tilde{c}}$ and $-\underline{\tilde{c}}$ in C_1, defined as follows:

$$\begin{aligned} \underline{\tilde{c}} &= \sum_{(i,j)} J_{\tilde{c},c}(i,j) b_{(i,j)}, \\ -\underline{\tilde{c}} &= \sum_{(i,j)} J_{\tilde{c},c_-}(i,j) b_{(i,j)}. \end{aligned} \tag{8.1.4}$$

In other words, any cycloid $\tilde{c}$ of the graph G may be assigned, except for the choice of a direction, with a vector $\underline{\tilde{c}}$ in C_1. The vector $\underline{\tilde{c}}$ will be called a cycloid, as well. If $\tilde{c}$ is elementary, then $\underline{\tilde{c}}$ is called an elementary cycloid in C_1.

On the other hand, it turns out that all the cycloids $\underline{\tilde{c}}$, associated with the connected oriented graph G, generate a subspace $\widetilde{C}_1$ of C_1. The dimension

B of the vector space $\tilde{C}_1$ is called the Betti number of the graph G. One method to obtain a base for $\tilde{C}_1$ consists in considering a maximal (oriented) tree of G. A maximal tree is a connected subgraph of G without cycloids and maximal with this property. This may be obtained by deleting B suitable edges $e_1, \ldots, e_{\mathrm{B}} \in E$, which complete B uniquely determined elementary cycloids $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_B$, each of $\tilde{\lambda}_k$ being in $T \cup \{e_k\}$ and associated with the circuit λ_k orientated according to the direction of $e_k, k = 1, \ldots, B$. Then the vector-cycloids $\underline{\lambda_1}, \ldots, \underline{\lambda_B} \in \tilde{C}_1$, associated to $(\tilde{\lambda}_1, \lambda_1), \ldots, (\tilde{\lambda}_B, \lambda_B)$ as in (8.1.4), form a base for $\tilde{C}_1$ and are called Betti cycloids. Furthermore, the number B is independent of the choice of the initial maximal tree.

Now we turn back to our original point to express the dynamical status of the passages of a particle moving along a cycloid $\tilde{c}$ of G in terms of the passage-functions.

First, let us consider that the cycloid $\tilde{c}$ is an elementary directed circuit c of G. Then, if i is a point of $c = (i_1, \ldots, i_k, \ldots, i_s, i_1)$, say $i = i_k$, we have

$$J_c(i) = \sum_{j \in S} J_c(i, j) = \sum_{k \in S} J_c(k, i) \neq 0. \tag{8.1.5}$$

Specifically, there are only two edges of c that make nonzero both members of (8.1.5): (i_{k-1}, i) and (i, i_{k+1}). Then relations (8.1.5) become: $J_c(i_{k-1}, i) = J_c(i, i_{k+1}) = 1 = J_c(i)$ and consequenty we have the following simple intuitive interpretation: a particle moving along c is passing through i if and only if it is passing through the edges of c preceding and succeeding i. This interpretation allows us to say that *a directed circuit c passes through a point i if and only if the corresponding passage-function J_c satisfies relations (8.1.5).*

Now let us consider a cycloid $\tilde{c}$ that is not a directed circuit. Then it may happen that a point i belongs to $\tilde{c}$, but the last inequality of (8.1.5) may eventually be not verified by the passage-functions $J_{\tilde{c},c}(i, j)$, that is,

$$\sum_{j \in S} J_{\tilde{c},c}(i, j) = \sum_{k \in S} J_{\tilde{c},c}(k, i) = 0.$$

Consequently, to describe intuitively the passage along an arbitrary cycloid $\tilde{c}$, we have to take into account the associated directed circuit (cycle) c; namely, we say that *a cycloid $\tilde{c}$ passes through the point i if and only if the associated directed circuit c passes through the point i*, that is, relations (8.1.5) hold for c.

8.2 The Cycloid Decomposition of Balanced Functions

We present the following theorem:

Theorem 8.2.1. *Let S be a nonvoid set. Assume w is a real function defined on $S \times S$ whose oriented graph G is connected, satisfying the folowing* *balance equations:*

$$\sum_{j\in S} w(i,j) = \sum_{k\in S} w(k,i), \qquad i \in S. \tag{8.2.1}$$

Then there exists a finite collection $\mathcal{C}^* = \{\tilde{c}_1, \ldots, \tilde{c}_B\}$ *of independent elementary cycloids in* G *and a set* $\{\alpha_1, \ldots, \alpha_B\}$ *of real nonnull numbers such that*

$$w(i,j) = \sum_{k=1}^{B} \alpha_k J_{\tilde{c}_k, c_k}(i,j), \qquad i,j \in S, \quad \alpha_k \in R, \tag{8.2.2}$$

where B *is the Betti number of the graph* G, $\alpha_k \equiv w(i_k, j_k)$ *with* (i_k, j_k) *the chosen Betti edge for* $\tilde{c}_k$, *and* $J_{\tilde{c}_k, c_k}$ *are the passage-functions associated with the cycloids* $\tilde{c}_k, k = 1, \ldots, B$. *Furthermore, the decomposition* (8.2.2) *is independent of the ordering of* $\mathcal{C}^*$.

Proof. Let $G = (S, E)$ be the oriented connected graph of w. That is, $(i,j) \in E$ if and only if $w(i,j) \neq 0$. With the graph G we associate the vector spaces C_1 and $\tilde{C}_1$ generated by the edges and cycloids of G, respectively.

Consider now an arbitrary maximal tree $\Im = (S, T)$ of G. Then there are edges of E, say $e_1 = (i_1, j_1), \ldots, e_B == (i_B, j_B)$, such that $E = T \cup \{e_1, \ldots e_B\}$. Hence, B is the Betti number G. Because $\Im$ is a tree, any two points of S may be joined by a chain in T. In addition, that $\Im$ is a maximal tree means that each directed edge of $E \backslash T = \{e_1, \ldots, e_B\}$, say $e_k = (i_k, j_k)$, determines a unique elementary cycloid $\tilde{c}_k$ in $T \cup \{e_k\}$ and a unique associated circuit c_k with the orientation of $e_k, k = 1, \ldots, B$. Then, by using (8.1.4), we may assign the unique vector-cycloid $\underline{\tilde{c}_k}$ to the pair $(\tilde{c}_k, c_k), k = 1, \ldots, B$.

Define

$$\alpha_1 = \alpha_1(e_1) \equiv w(i_1, j_1).$$

Put

$$w^1(i,j) \equiv w(i,j) - \alpha_1 J_{\tilde{c}_1, c_1}(i,j), \qquad i,j \in S.$$

Then w^1 is a new real balanced function on S. If $w^1 \equiv 0$, then equations (8.2.2) hold for $\mathcal{C}^* = \{\tilde{c}_1\}$ and $B = 1$. Otherwise, w^1 remains different from zero on fewer edges than w (because w^1 is zero at least on the edge (i_1, j_1)).

Further, we repeat the same reasonings above for all the edges $e_2 = (i_2, j_2), \ldots, e_B = (i_B, j_B)$, and define

$$w^B(i,j) \equiv w(i,j) - \sum_{k=1}^{B} \alpha_k J_{\tilde{c}_k, c_k}(i,j), \qquad i,j \in S.$$

where $\alpha_k \equiv w(i_k, j_k), k = 1, \ldots, B$. From the previous construction of the elementary cycloids $\tilde{c}_k$ and circuits $c_k, k = 1, \ldots, B$, there follows that the associated vector-cycloids $\underline{\tilde{c}_1}, \ldots, \underline{\tilde{c}_B}$ form a base for $\tilde{C}_1$.

Also, $w^B(i_k, j_k) = 0, k = 1, \ldots, B$, and the reduced function w^B remains a balance function on the tree T, as well. Then $w^B \equiv 0$ (see Lemma 4.4.1). Consequently, we may write

$$w(i,j) = \sum_{k=1}^{B} \alpha_k J_{\tilde{c}_k, c_k}(i,j), \qquad i,j \in S.$$

The proof is complete. □

Corollary 8.2.2. *Assume the oriented strongly connected graph $G = (S,E)$ associated with a positive balanced function on a finite set $S \times S$. If $\{\underline{\tilde{c}_1}, \ldots, \underline{\tilde{c}_B}\}$ is a base of elementary Betti cycloids, then for any $i \in S$ we have*

$$\sum_{j \in S} \sum_{k=1}^{B} J_{\tilde{c}_k, c_k}(i,j) = \sum_{u \in S} \sum_{k=1}^{B} J_{\tilde{c}_k, c_k}(u,i) \geq 1. \tag{8.2.3}$$

Proof. Let $i \in S$ and let c be an elementary directed circuit of G that passes through i, that is,

$$\sum_{j \in S} J_c(i,j) = \sum_{u \in S} J_c(u,i) = 1.$$

Then we may apply the cycloid decomposition formula (8.2.2) to the balance function $J_c(\cdot,\cdot)$ on the set E of the edges of G and correspondingly we write

$$J_c(i,j) = \sum_{k=1}^{B} J_c(i_k, j_k) J_{\tilde{c}_k, c_k}(i,j), \quad i,j \in S,$$

where $(i_1, j_1), \ldots, (i_B, j_B)$ are the Betti edges of G that uniquely determine the elementary Betti cycloids $\tilde{c}_1, \ldots, \tilde{c}_B$ by the method of maximal tree. Consequently, we have

$$\begin{aligned} 1 = \sum_{j \in S} J_c(i,j) &= \sum_{j \in S} \sum_{k=1}^{B} J_c(i_k, j_k) J_{\tilde{c}_k, c_k}(i,j) \\ &= \sum_{u \in S} \sum_{k=1}^{B} J_c(i_k, j_k) J_{\tilde{c}_k, c_k}(u,i) \\ &\leq \sum_{j \in S} \sum_{k=1}^{B} J_{\tilde{c}_k, c_k}(i,j) = \sum_{u \in S} \sum_{k=1}^{B} J_{\tilde{c}_k, c_k}(u,i). \end{aligned}$$

The proof is complete. □

8.3 The Cycloid Transition Equations

Let S be a finite set. Consider the connected oriented graph $G = (S,E)$ and denote by $\mathcal{C}^*$ the collection of all overlapping cycloids occurring in G

(whose edge-set is identical to E). Then each maximal tree of G provides a collection $\mathcal{B}$ of Betti edges in E. Denote by $\mathcal{P}(E)$ the power set of E.

Define the function μ: $\mathcal{C}^* \times \mathcal{P}(E) \to R$ as follows:

$$\mu(\tilde{c}, A) = \sum_{(i,j)\in A} J_{\tilde{c},c}(i,j), \quad \text{if A } \in \mathcal{P}(E), \quad \text{A} \neq \varnothing, \quad \text{and} \quad \tilde{c} \in \mathcal{C}^*, \qquad (8.3.1)$$
$$= 0, \qquad \text{otherwise.}$$

Plainly, for each $(i,j) \in E$, the numbers $\mu(\tilde{c}, (i,j)), \tilde{c} \in \mathcal{C}^*$, are the coordinates of the algebraic cycloid $\underline{\tilde{c}}$ in C_1 defined as

$$\underline{\tilde{c}} = \sum_{(i,j)\in E} J_{\tilde{c},c}(i,j) b_{(i,j)}.$$

Furthermore, the function μ enjoys some interesting properties given by the following.

Proposition 8.3.1. *Consider $G = (S, E)$ a connected oriented graph on a finite set S, and the measurable space $(E, \mathcal{P}(E))$.*
Then the function μ: $\mathcal{C}^* \times \mathcal{P}(E) \to R$ defined by (8.3.1) enjoys the following properties:

(i) *For any $\tilde{c} \in \mathcal{C}^*$ the set function $\mu(\tilde{c}, \cdot)$: $\mathcal{P}(E) \to R$ is a signed measure;*
(ii) *For any $A \in \mathcal{P}(E)$, the function $\mu(\cdot, A)$ is $\mathcal{P}(\mathcal{C}^*)$-measurable;*
(iii) *For arbitrary $\tilde{c} \in \mathcal{C}^*$ and $A \in \mathcal{P}(E)$, the following equations hold*

$$\mu(\tilde{c}, A) = \sum\nolimits_{u\in\mathcal{B}} \mu(\tilde{c}, \{u\}) \mu(\tilde{c}_u, A), \qquad (8.3.2)$$

where $\mathcal{B}$ denotes a base of Betti edges of G, and for each $u \in \mathcal{B}, \tilde{c}_u$ denotes the unique elementary Betti cycloid associated with u by the maximal-tree-method.

Proof. (i) We have $\mu(\tilde{c}, \varnothing) = 0, \tilde{c} \in \mathcal{C}^*$, and

$$\mu(\tilde{c}, \bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(\tilde{c}, A_n), \quad \tilde{c} \in \mathcal{C}^*,$$

for all pairwise disjoint sequences $\{A_n\}_n$ of subsets of E. Hence $\mu(\tilde{c}, \cdot)$ is a signed measure on $\mathcal{P}(E)$ for any $\tilde{c} \in \mathcal{C}^*$.

(ii) That $\mu(\cdot, A)$ is $\mathcal{P}(\mathcal{C}^*)$-measurable is immediate.

(iii) Let $\mathcal{B}$ be the set of Betti edges associated with an arbitrarily chosen maximal tree of G. Then by applying the cycloid decomposition formula

(8.2.2) to $J_{\tilde{c},c}(i,j)$, we have

$$\begin{aligned}\sum\nolimits_{u\in\mathcal{B}} \mu(\tilde{c},\{u\})\ \mu(\tilde{c}_u,A) &= \sum\nolimits_{u\in\mathcal{B}} \sum_{(i,j)\in A} J_{\tilde{c},c}(u)J_{\tilde{c}_u,c_u}(i,j)\\ &= \sum_{(i,j)\in A} J_{\tilde{c},c}(i,j)\\ &= \mu(\tilde{c},A).\end{aligned}$$

The proof is complete. □

Remark. Conditions (i)–(iii) of Proposition (8.3.1) may be paralleled with those defining a stochastic transition function from $\mathcal{C}^*$ to $\mathcal{P}(E)$. The basic differentiations appear in property (i) where the set function $\mu(\tilde{c},\cdot)$ is a signed measure instead of a probability on $\mathcal{P}(E)$, and in (iii), where equations (8.3.2) replace the known Chapman–Kolmogorov equations. However, equations (8.3.2) keep the essence of a transition as in the classical Chapman–Kolmogorov equations: a transition from a point to a set presupposes a passage via an intermediate point. Specifically, in equations (8.3.2) the role of the intermediate is played by a Betti cycloid $\tilde{c}_u$, which is isomorphically identified with the Betti edge u. Consequently, Proposition (8.3.1) allows us to introduce the following:

Definition 8.3.2. Given an oriented connected graph $G=(S,E)$ on a finite set S and a collection $\mathcal{C}^*$ of overlapping cycloids whose edge-set is E, a *cycloid transition function* is any function π: $\mathcal{C}^*\times\mathcal{P}(E)\to R$ with the properties:

(i) For any $\tilde{c}\in\mathcal{C}^*, \pi(\tilde{c},\{(i,j)\})$ defines a balance function on $S\times S$, that is,

$$\sum_j \pi(\tilde{c},\{(i,j)\}) = \sum_k \pi(\tilde{c},\{(k,i)\}), \qquad i\in S;$$

(ii) For any $\tilde{c}\in\mathcal{C}^*, \pi(\tilde{c},\cdot)$ is a signed measure on $\mathcal{P}(E)$;

(iii) For any $\tilde{c}\in\mathcal{C}^*,\ A\in\mathcal{P}(E)$ and for any collection $\mathcal{B}$ of Betti edges, the following equation holds:

$$\pi(\tilde{c},A) = \sum\nolimits_{u\in\mathcal{B}} \pi(\tilde{c},\{u\})\,\pi(\tilde{c}_u,A). \tag{8.3.3}$$

Relations (8.3.3) are called the *cycloid transition equations.* □

Plainly, they express a homologic rule characterizing the balanced functions.

A further interpretation of the cycloid decomposition formula (8.2.2) may continue with the study of the cycloid transition equations (8.3.3) as follows.

Consider π: $\mathcal{C}^* \times \mathcal{P}(E) \to R$ the cycloid transition function introduced by (8.3.1) and assign with each $\tilde{c} \in \mathcal{C}^*$ the balanced function

$$\begin{aligned} w(i,j) &= \pi(\tilde{c},(i,j)), \quad && (i,j) \in E, \\ &= 0, && (i,j) \in S^2 \backslash E. \end{aligned}$$

Then equations (8.3.3) written for w become

$$w(i,j) = \sum\nolimits_{u \in \mathcal{B}} w(u) J_{\tilde{c}_u, c_u}(i,j), \qquad (i,j) \in S^2, \tag{8.3.4}$$

where $\mathcal{B}$ denotes the set of Betti edges of G associated with a maximal tree. Consider further the measurable space $(S^2, \mathcal{P}(S^2))$.

Denote by B the vector space of all bounded real-valued functions v on S^2 whose graphs are subgraphs of G. Then B is a Banach space with respect to the norm of supremum.

Define the linear operator U: $B \to B$ as follows:

$$(Uv)(\cdot,\cdot) = \sum\nolimits_{u \in \mathcal{B}} v(u) \;\; \pi(\tilde{c}_u, \{(\cdot,\cdot)\}).$$

Let now $\mathcal{S}$ be the space of all signed finite and aditive set-functions on the power-set $\mathcal{P}(S^2)$. A norm on $\mathcal{S}$ is given by the total variation norm.

Consider the linear operator V: $\mathcal{S} \to \mathcal{S}$ defined as follows:

$$\begin{aligned} (V\lambda)(\{u\}) &= \sum_{(i,j)\in S^2} \lambda(\{(i,j)\}) \;\; \pi(\tilde{c}_u, \{(i,j)\}), \quad && \text{if } u \in \mathcal{B}, \\ &= 0, && \text{otherwise.} \end{aligned}$$

Set

$$\langle \lambda, v \rangle = \sum_{(i,j)\in S^2} v(i,j)\lambda(\{(i,j)\}),$$

for $\lambda \in \mathcal{S}, v \in B$.

Let $\mathcal{E}(1)$ be the subspace of all eigenvectors v of U corresponding to the eigenvalue 1, that is, $Uv = v$. Then we have the following theorem.

Theorem 8.3.3.

(i) The functions $J_{\tilde{c}_1,c_1}, \ldots, J_{\tilde{c}_B,c_B}$, associated with the elementary Betti cycloids $\tilde{c}_1, \ldots, \tilde{c}_B$ of the connected graph G, form a base for the space $\mathcal{E}(1)$.

(ii) The space of all solutions to the cycloid formula (8.2.2) coincides with $\mathcal{E}(1)$.

(iii) For any $v \in B$ and for any $\lambda \in \mathcal{S}$, we have

$$\langle \lambda, Uv \rangle = \langle V\lambda, v \rangle.$$

Proof. (i) From Proposition 8.3.1, we have that the passage-functions $J_{\tilde{c}_1,c_1},\dots,J_{\tilde{c}_B,c_B}$ belong to $\mathcal{E}(1)$. In addition, these functions are independent. Also, if $v\in\mathcal{E}(1)$, then v satisfies equation (8.3.4), that is, $J_{\tilde{c}_1,c_1},\dots,J_{\tilde{c}_B,c_B}$ are generators for $\mathcal{E}(1)$.
(ii) This property is an immediate consequence of the definition of U.
(iii) For any $\lambda\in\mathcal{S}$ and any $v\in B$ we have

$$\begin{aligned}\langle\lambda,Uv\rangle &= \sum_{(i,j)\in S^2}\lambda(\{(i,j)\})\sum\nolimits_{u\in\mathcal{B}} v(u)\,\pi(\tilde{c}_u,\{(i,j)\})\\ &= \sum\nolimits_{u\in\mathcal{B}} v(u)(V\lambda)(\{u\}),\end{aligned}$$

and

$$\begin{aligned}\langle V\lambda,v\rangle &= \sum_{(i,j)\in S^2} v(i,j)\,(V\lambda)(\{(i,j)\})\\ &= \sum\nolimits_{u\in\mathcal{B}} v(u)\,(V\lambda)(\{u\}).\end{aligned}$$

The proof is complete. □

8.4 Definition of Markov Chains by Cycloids

Let S be a finite set and let $G=(S,E)$ be an oriented strongly connected graph. Let B be the Betti number of G, and consider a base of elementary Betti algebraic cycloids $\mathcal{C}^*=\{\underline{\tilde{c}_1},\dots,\underline{\tilde{c}_B}\}$, which correspond to a maximal tree in G and to a set of Betti edges $(i_1,j_1),\dots,(i_B,j_B)$. Consider also B strictly positive numbers $w_1,\dots,w_B$ such that the following relations hold

$$w(i,j)\equiv\sum_{k=1}^{B} w_k J_{\tilde{c}_k,c_k}(i,j)>0,\qquad (i,j)\in E, \tag{8.4.1}$$

$$w(i)\equiv\sum_{j\in S} w(i,j)=\sum_{m\in S} w(m,i)>0,\qquad i\in S, \tag{8.4.2}$$

where $J_{\tilde{c}_k,c_k}(\cdot,\cdot), k=1,\dots,B$, denote the passage-functions of the Betti cycloids $\tilde{c}_1,\dots,\tilde{c}_B$.

If we denote

$$J_{\tilde{c}_k,c_k}(i)\equiv\sum_{j\in S} J_{\tilde{c}_k,c_k}(i,j)=\sum_{m\in S} J_{\tilde{c}_k,c_k}(m,i),\quad i\in S,$$

then

$$w(i)=\sum_{k=1}^{B} w_k J_{\tilde{c}_k,c_k}(i),\quad i\in S.$$

Define

$$p_{ij} = \frac{\sum_{k=1}^{B} w_k J_{\tilde{c}_k, c_k}(i,j)}{\sum_{k=1}^{B} w_k J_{\tilde{c}_k, c_k}(i)}, \qquad \text{if } (i,j) \in E,$$
$$= 0, \qquad \text{if } (i,j) \in S^2 \backslash E. \tag{8.4.3}$$

Then $P = (p_{ij}, i, j \in S)$ is the stochastic matrix of an irreducible Markov chain on S whose invariant probability distribution $p = (p_i, i \in S)$ has the entries

$$p_i = \frac{w(i)}{\sum\limits_{i \in S} w(i)}, \quad i \in S.$$

Conversely, given a homogeneous irreducible Markov chain ξ on a finite set S, the cycloid decomposition formula applied to the balance function $w(i,j) = \text{Prob}(\xi_n = i, \xi_{n+1} = j), i,\ j \in S, n = 1, 2, \dots,$ provides a unique collection $\{\{\tilde{c}_k\}, \{w_k\}\}$ of cycloids and positive numbers, so that, except for a choice of the maximal tree the correspondence $\xi \to \{\{\tilde{c}_k\}, \{w_k\}\}$ is one-to-one.
Then we may summarize the above results in the following statement.

Theorem 8.4.1.

(i) Let S be any finite set and let $G = (S, E)$ be an oriented strongly connected graph on S. Then for any choice of the Betti base $\mathcal{C}^* = \{\underline{\tilde{c}_1}, \dots, \underline{\tilde{c}_B}\}$ of elementary cycloids and for any collection $\{w_1, \dots, w_B\}$ of strictly positive numbers such that relations (8.4.1) and (8.4.2) hold, there exists a unique irreducible S-state Markov chain ξ whose transition probability matrix $P = (p_{ij}, i, j \in S)$ is defined as

$$p_{ij} = \frac{\sum_{k=1}^{B} w_k J_{\tilde{c}_k, c_k}(i,j)}{\sum_{k=1}^{B} w_k J_{\tilde{c}_k, c_k}(i)}, \qquad \text{if } (i,j) \in E.$$

(ii) Given a finite set S and an irreducible homogeneous S-state Markov chain $\xi = (\xi_n)$, for any choice of the maximal tree in the graph of ξ there exists a unique minimal collection of elementary cycloids $\{\tilde{c}_1, \dots, \tilde{c}_B\}$ and strictly positive numbers $\{w_1, \dots, w_B\}$ such that we have the following cycloid decomposition:

$$Prob(\xi_n = i, \xi_{n+1} = j) = \sum_{k=1}^{B} w_k J_{\tilde{c}_k, c_k}(i,j), \quad i, j \in S.$$

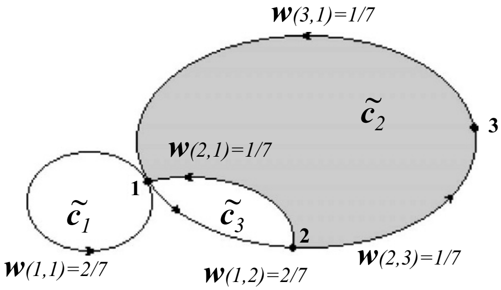

Figure 8.4.1.

Example. We now apply the cycloid representation formula of Theorem (8.4.1) to the stochastic matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix},$$

whose invariant distribution is the row-vector $\pi = (4/7, 2/7, 1/7)$. The graph of P is given in Figure 8.4.1 below.
Consider the vector $\underline{w} = \sum w(i,j) b_{(i,j)}$, with $w(i,j) = \pi_i p_{ij}, i, j \in \{1,2,3\}$. The set of edges of the graph is $\{(1,1),(3,1),(2,1)(1,2),(2,3)\}$.

Consider the maximal tree $T = \{(2,1),(2,3)\}$ associated with the Betti edges $\mathcal{B} = \{(1,1),(3,1),(1,2)\}$. Accordingly, the base of Betti algebraic cycloids is as follows:

$$\underline{\tilde{c}_1} = 1 \cdot b_{(1,1)}, \quad \underline{\tilde{c}_2} = 1 \cdot b_{(3,1)} + (-1) \cdot b_{(2,1)} + 1 \cdot b_{(2,3)},$$
$$\underline{\tilde{c}_3} = 1 \cdot b_{(2,1)} + 1 \cdot b_{(1,2)},$$

and they correspond to the graph-cycloids $\tilde{c}_1 = ((1,1))$, $\tilde{c}_2 = ((3,1),(2,1),(2,3))$, and $\tilde{c}_3 = ((2,1),(1,2))$ associated with the directed circuits $c_1 = (1,1), c_2 = (3,1,2,3)$, and $c_3 = (2,1,2)$.

Then according to Theorem 8.4.1 (ii), the cycloid decomposition of P corresponding to the maximal tree T is as follows:

$$\pi_i p_{ij} = \frac{2}{7} J_{\tilde{c}_1, c_1}(i,j) + \frac{1}{7} J_{\tilde{c}_2, c_2}(i,j) + \frac{2}{7} J_{\tilde{c}_3, c_3}(i,j), \quad i, j \in \{1,2,3\}.$$