6

Spectral Theory of Circuit Processes

Spectral theory of Markov processes was developed by D.G. Kendall (1958, 1959a, b) and W. Feller (1966a). The present chapter relies on Kendall's Fourier representation for transition-probability matrices and for transition-matrix functions defining discrete and continuous parameter Markov processes, respectively. A specialization of the spectral theory to circuit Markov processes is particularly motivated by the essential rôle of the circuit-weights when they decompose the finite-dimensional distributions. For this reason we shall be consequently interested in the spectral representation of the circuit-weights alone. This approach is due to S. Kalpazidou (1992a, b).

6.1 Unitary Dilations in Terms of Circuits

A preliminary element of our investigations is an $\mathbb{N}^*$-state irreducible Markov chain $\xi = (\xi_n)_{n\geq 0}$ whose transition matrix $P = (p_{ij}, i, j \in \mathbb{N}^*)$ admits an invariant probability distribution $\pi = (\pi_i, i \in \mathbb{N}^*)$, with all $\pi_i > 0$, where $\mathbb{N}^* = \{1, 2, \ldots\}$. That the denumerable state space is $\mathbb{N}^*$ does not restrict the generality of our approach. Let $(\mathscr{C}_\infty, w_c)$ be the probabilistic representative class of directed circuits and weights which decompose P as in Theorem 3.3.1. The typical result of the present section is that the sum of the probabilistic weights w_c of the circuits passing through the edge (i, j) has a Fourier representation.

Let $l_2 = l_2(\mathbb{N}^*)$ be as usual the Hilbert space of all sequences $x = (x_i)_{i\in\mathbb{N}^*}$ with x_i a complex number such that $\|x\|^2 = (x, x) = \sum_i |x_i|^2 < \infty$. The conjugate of any complex number z will be symbolized by $\bar{z}$. Further, let T be the linear transformation on l_2 whose kth component of $Tx, x \in l_2$, is given by the absolutely convergent series

$$(Tx)_k = \sum_i x_i (w(i)w(k))^{-1/2} \sum_{c \in \mathscr{C}_\infty} w_c J_c(i,k), \tag{6.1.1}$$

where

$$w(i) = \sum_{c \in \mathscr{C}_\infty} w_c J_c(i), \qquad i \in \mathbb{N}^*.$$

Then we may write

$$\begin{aligned} \|Tx\|^2 &= \sum_k \left| \sum_i x_i (w(i)w(k))^{-1/2} \sum_{c \in \mathscr{C}_\infty} w_c J_c(i,k) \right|^2 \\ &\leq \sum_k \left[\sum_u |x_u|^2 (1/(w(u))) \sum_{c \in \mathscr{C}_\infty} w_c J_c(u,k) \right] \\ &\qquad \cdot \left[\sum_j (1/(w(k))) \sum_{c \in \mathscr{C}_\infty} w_c J_c(j,k) \right] \\ &\leq \|x\|^2, \end{aligned}$$

so that T is a contraction on l_2.

With these preparations, we now prove

Theorem 6.1.1. *If $(\mathscr{C}_\infty, w_c)$ is the probabilistic representative class of weighted circuits for an irreducible Markov chain whose transition matrix $P = (p_{jk}, j, k \in \mathbb{N}^*)$ admits an invariant probability distribution $\pi = (\pi_j, j \in \mathbb{N}^*)$, with all $\pi_j > 0$, then*

$$\pi_j p_{jk} = \sum_{c \in \mathscr{C}_\infty} w_c J_c(j,k) = (w(j)w(k))^{1/2} \oint e^{i\theta} \mu_{jk}(d\theta),$$

where the complex-valued Borel measures μ_{jk} are supported by the circumference of unit radius and satisfy the Hermitian condition $\bar{\mu}_{jk} = \mu_{kj}$.

Proof. We shall follow D.G. Kendall's (1959a) approach to the integral representations for transition-probability matrices. Accordingly, we use a theorem of B.Sz. Nagy (see B.Sz. Nagy (1953), F. Riesz and B.Sz. Nagy (1952), and J.J. Schäffer (1955)) according to which, if T is a linear contraction on a Hilbert space H, then it is always possible to embed H as a closed subspace in an eventually larger Hilbert space H^+ in such a way that $T^m x = JU^m x$ and $(T^*)^m x = JU^{-m} x$, for all $x \in H$ and $m \geq 0$, where U

is a unitary operator on H^+ and J is the projection from H^+ onto H. P.R. Halmos called U a *unitary dilation* of T.T^* denotes as usual the adjoint operator of T.

We here apply Nagy's theorem to the contraction T defined by (6.1.1) and to the space $H = l_2$. Accordingly, there exists a unitary dilation U defined on a perhaps larger Hilbert space H^+ such that

$$JU^m J = T^m J,$$
$$JU^{-m} J = (T^*)^m J,$$

for any $m = 0, 1, 2, \dots$, where J is the orthogonal projection from H^+ onto H. From the proof of the Nagy theorem the space H^+ is defined as the direct sum of countably many copies of H.

Let us consider $u(j)$ the element of H defined by

$$(u(j))_k = \delta_{jk},$$

where δ denotes Kronecker's delta. Then we have

$$(T^m u(j), u(k)) = (U^m u(j), u(k)),$$
$$(u(j), T^m u(k)) = (U^{-m} u(j), u(k)).$$

Hence

$$\sum_{c \in \mathscr{C}_\infty} w_c J_c(j,k) = (w(j)w(k))^{1/2} (Uu(j), u(k)).$$

We now apply Wintner's theorem (see F. Riesz and B.Sz. Nagy (1952)) according to which the unitary operator U is uniquely associated with a (strongly) right-continuous spectral family of projections $\{E_\theta, 0 \le \theta \le 2\pi\}$ with $E_0 = \boldsymbol{O}$ and $E_{2\pi} = I$ such that

$$U = \int_0^{2\pi} e^{i\theta} dE_\theta.$$

Finally,

$$\begin{aligned}\sum_{c \in \mathscr{C}_\infty} w_c J_c(j,k) &= (w(j)w(k))^{1/2} \int_0^{2\pi} e^{i\theta} d(E_\theta u(j), u(k)) \\ &= (w(j)w(k))^{1/2} \oint e^{i\theta} \mu_{jk}(d\theta),\end{aligned}$$

where μ_{jk} are complex-valued measures satisfying the properties referred to in the statement of the theorem. The proof is complete. □

6.2 Integral Representations of the Circuit-Weights Decomposing Stochastic Matrices

This section is a sequel to the previous one. We shall be concerned with the same irreducible Markov chain $\xi = (\xi_n)_n$ introduced at the beginning of Section 6.1, save for the state space which is now considered to be the finite set $\mathbb{N}_v^* = \{1, 2, \ldots, v\}, v > 1$. Then the deterministic-circuit-representation theorem (Theorem 4.2.1) asserts that the transition probabilities $p_{jk}, j, k \in \mathbb{N}_v^*$, of ξ have the following decomposition in terms of the directed circuits of a finite ordered class $\mathscr{C} = \{c_1, \ldots, c_m\}, m \geq 1$, and of their positive weights w_c:

$$\pi_j p_{jk} = \sum_{c \in \mathscr{C}} w_c J_c(j,k), \qquad j, k \in \mathbb{N}_v^*, \tag{6.2.1}$$

where $\pi = (\pi_j, j \in \mathbb{N}_v^*)$ denotes the invariant probability distribution of ξ. The directed circuits $c = (i_1, \ldots, i_p, i_1), p > 1$, to be considered will have distinct points $i_1, \ldots, i_p$.

The principal theorem asserts that an integral representation can be found for the deterministic circuit weights w_c occurring in the decomposition (6.2.1). More specifically, we have

Theorem 6.2.1. *For any circuit c occurring in the decomposition* (6.2.1) *there exist a finite sequence $(j_1, k_1), \ldots, (j_m, k_m)$ in the edge-set of $\mathscr{C}$ and a Hermitian system $\{\nu_{j_1 k_1}, \ldots, \nu_{j_m k_m}\}$ of Borel measures supported by the circumference of unit radius such that $w_c = w_{c_r}$, for some $r = 1, \ldots, m$, has the expression*

$$\begin{aligned} w_{c_1} &= (w(j_1)w(k_1))^{1/2} \oint e^{i\theta} \nu_{j_1 k_1}(d\theta) \quad \textit{if} \quad r = 1, \\ w_{c_r} &= (w(j_r)w(k_r))^{1/2} \oint e^{i\theta} \nu_{j_r k_r}(d\theta) \\ &\qquad - \sum_{s=1}^{r-1} w_{c_s} J_{c_s}(j_r, k_r) \qquad \textit{if} \quad r = 2, \ldots, m \quad m > 1. \end{aligned}$$

Proof. We shall use the arguments of Theorems 1.3.1 and 4.2.1. In this direction, let j_0 be arbitrarily fixed in $\mathbb{N}_v^*$. Since $w(j,k) \equiv \pi_j p_{jk}$ is balanced and $\sum_k w(j_0, k) > 0$, we can find a sequence $(j_0, u_0), (u_0, u_1), \ldots, (u_{n-1}, u_n), \ldots$ of pairs, with $u_l \neq u_m$ for $l \neq m$, on which $w(\cdot, \cdot)$ is strictly positive. Choosing the $u_m, m = 0, 1, 2, \ldots$, from the finite set $\mathbb{N}_v^*$, we find that there must be repetitions of some point, say j_0. Let n be the smallest nonnegative integer such that $u_n = j_0$. Then, if $n \geq 1, c_1 : (j_0, u_0), (u_0, u_1), \ldots, (u_{n-1}, j_0)$ is a circuit, with distinct points $j_0, u_0, \ldots, u_{n-1}$ in $\mathbb{N}_v^*$, associated to w.

Let (j_1, k_1) be the pair where $w(j,k)$ attains its minimum over all the edges of c_1, that is,

$$w(j_1, k_1) = \min_{c_1} w(j,k).$$

Put

$$w_{c_1} = w(j_1, k_1)$$

and define

$$w_1(j,k) \equiv w(j,k) - w_{c_1} J_{c_1}(j,k).$$

The number of pairs (j,k) for which $w_1(j,k) > 0$ is at least one unit smaller than that corresponding to $w(i,j)$. If $w_1 \equiv 0$ on $\mathbb{N}_v^*$, then $w(j,k) \equiv w_{c_1} J_{c_1}(j,k)$. Otherwise, there is some pair (j,k) such that $w_1(j,k) > 0$. Since w_1 is balanced we may repeat the same reasoning above, according to which we may find a circuit c_2, with distinct points (except for the terminals), associated to w_1.

Let (j_2, k_2) be the edge where $w_1(j,k)$ attains its minimum over all the edges of c_2, that is,

$$w_1(j_2, k_2) = \min_{c_2} w_1(j,k).$$

Put

$$w_{c_2} = w_1(j_2, k_2)$$

and define

$$\begin{aligned} w_2(j,k) &\equiv w_1(j,k) - w_{c_2} J_{c_2}(j,k) \\ &= w(j,k) - w_{c_1} J_{c_1}(j,k) - w_{c_2} J_{c_2}(j,k). \end{aligned}$$

Then $w_2(j_1, k_1) = w_2(j_2, k_2) = 0$. Since $\mathbb{N}_v^*$ is finite, the above process will finish after a finite number $m = m(j_0)$ of steps, providing both a finite ordered class $\mathscr{C} = \{c_1, \ldots, c_m\}$ of directed circuits, with distinct points (except for the terminals), in $\mathbb{N}_v^*$ and an ordered collection of positive numbers $\{w_{c_1}, \ldots, w_{c_m}\}$ such that

$$w(j,k) = \sum_{k=1}^{m} w_{c_k} J_{c_k}(j,k), \qquad j,k \in \mathbb{N}_v^*.$$

Moreover, the strictly positive numbers w_{c_k}, called as always circuit weights, are described by a finite sequence of edges $(j_1, k_1), \ldots, (j_m, k_m)$ and the recursive equations

$$\begin{aligned} w_{c_1} &= w(j_1, k_1) \\ w_{c_2} &= w(j_2, k_2) - w(j_1, k_1) J_{c_1}(j_2, k_2), \\ &\vdots \\ w_{c_m} &= w(j_m, k_m) - \sum_{s=1}^{m-1} w_{c_s} J_{c_s}(j_m, k_m). \end{aligned} \tag{6.2.2}$$

Consider now the operator V mapping $x \in l_2(\mathbb{N}_v^*)$ into the vector Vx, where the kth component of Vx is given by the sum

$$(Vx)_k = \sum_j x_j (w(j)w(k))^{-1/2} \sum_{c \in \mathscr{C}} w_c J_c(j,k),$$

with $w(j) \equiv \sum_c w_c J_c(j)$.

Then, following the proof of Theorem 6.1.1 we can extend V to a unitary operator U for which there exists a Hermitian collection of spectral measures $\{\nu_{jk}\}$ such that

$$w(j,k) = (w(j)w(k))^{1/2} \oint e^{i\theta} \nu_{jk}(d\theta),$$

for all (j,k), and so, for $(j_1,k_1), \ldots, (j_m,k_m)$ occurring in (6.2.2). Accordingly, the weights given by equations (6.2.2) have the desired integral representation. The proof is complete. □

6.3 Spectral Representation of Continuous Parameter Circuit Processes

6.3.1. Consider an $\mathbb{N}^*$-state irreducible positive-recurrent Markov process $\xi = (\xi_t)_{t \geq 0}$ whose transition matrix function $P(t) = (p_{ij}(t), i,j \in \mathbb{N}^*)$ is stochastic and standard, that is,

$$p_{ij}(t) \geq 0, \quad \sum_j p_{ij}(t) = 1,$$

$$p_{ij}(t+s) = \sum_k p_{ik}(t) p_{kj}(s),$$

$$\lim_{t \to 0^-} p_{ij}(t) = p_{ij}(0) = \delta_{ij},$$

for all $i,j \in \mathbb{N}^*$ and all $t,s \geq 0$. Let $\Xi_t = (\xi_{nt})_{n \geq 0}$ be the discrete t-skeleton chain of ξ, where $t > 0$.

Consider the (weakly continuous) semigroup $\{T_t, t \geq 0\}$ of contractions associated with $P = (P(t))_{t \geq 0}$. Then this semigroup may be expressed in terms of the probabilistic circuit representative $(\mathscr{C}, w_c(t))_{t \geq 0}$, provided in Theorem 5.5.2, as follows:

$$(T_t x)_k = \sum_{i \in \mathbb{N}^*} x_i (w(i)w(k))^{-1/2} \sum_{c \in \mathscr{C}} w_c(t) J_c(i,k), \quad k \in \mathbb{N}^*, \tag{6.3.1}$$

for all $x \in l_2(\mathbb{N}^*)$, where $w(i) = \sum_{c \in \mathscr{C}} w_c(t) J_c(i)$ for any $i \in \mathbb{N}^*$.

Theorem 6.3.1. *Let $P(t) = (p_{ij}(t), i,j \in \mathbb{N}^*)$ be a standard stochastic transition matrix function defining an irreducible positive-recurrent Markov process $\xi = (\xi_t)_{t \geq 0}$ whose invariant probability distribution is denoted by*

$\pi = (\pi_i, i \in \mathbb{N}^*)$. *Then for each* $t \geq 0$ *the transition probabilities* $p_{jk}(t)$ *can be written in the form*:

$$\pi_j p_{jk}(t) = (w(j)w(k))^{1/2} \int_{-\infty}^{+\infty} e^{i\lambda t} \mu_{jk}(d\lambda),$$

where $\{\mu_{jk}, i, k \in \mathbb{N}^*\}$ *is a Hermitian collection of complex-valued totally finite Borel measures carried by the real line.*

Proof. The main argument of the proof is due to D.G. Kendall (1959b). Correspondingly, we apply a theorem of B.Sz. Nagy according to which we can embed $H \equiv l_2(\mathbb{N}^*)$ as a closed subspace in an eventually larger Hilbert space H^+ in such a way that for all $t \geq 0$

$$JU_tJ = T_tJ,$$
$$JU_{-t}J = T_t^*J,$$

where J is the orthogonal projection from H^+ onto H, T_t^* is the adjoint operator of T_t, and $\{U_t, -\infty < t < \infty\}$ is a strongly continuous group of unitary operators on H^+. (The smallest such collection $\{H^+, U_t, H\}$ is unique up to isomorphisms). Further we apply a theorem of M.H. Stone (see F. Riesz and B.Sz. Nagy (1952), p. 380) according to which there exists a right-continuous spectral family $\{E_\lambda, -\infty < \lambda < \infty\}$ of projection operators such that

$$(U_t x, y) = \int_{-\infty}^{+\infty} e^{i\lambda t} d(E_\lambda x, y), \qquad x, y \in H^+,$$

for all real t.

We have

$$(T_t x, y) = (JU_t x, y) = (U_t x, Jy) = (U_t x, y), \qquad x, y \in H \equiv l_2.$$

Furthermore,

$$\begin{aligned} \pi_j p_{jk}(t) &= (w(j)w(k))^{1/2}(U_t u(j), u(k)) \\ &= (w(j)w(k))^{1/2} \int_{-\infty}^{+\infty} e^{i\lambda t} d(E_\lambda u(j), u(k)), \qquad t \geq 0, \end{aligned}$$

where the vector $u(j)$ lies in $l_2(\mathbb{N}^*)$ and is defined by

$$(u(j))_k = \delta_{jk}.$$

Then, by virtue of Theorem II of D.G. Kendall (1959b), we may write

$$\pi_j p_{jk}(t) = (w(j)w(k))^{1/2} \int_{-\infty}^{+\infty} e^{i\lambda t} \mu_{jk}(d\lambda), \qquad t \geq 0,$$

where the complex-valued totally finite Borel measures $\mu_{jk}, j, k \in \mathbb{N}^*$, are supported by the real line and satisfy the Hermitian condition $\mu_{kj} = \bar{\mu}_{jk}$. The proof is complete. □

6.3.2. Consider the semigroup $\{T_t, T \geq 0\}$ of contractions associated to $P = (P(t))_{t \geq 0}$ by (6.3.1), with $P(t) = \{p_{ij}(t), i, j \in \mathbb{N}^*\}$ satisfying the hypotheses of the previous paragraph. D.G. Kendall (1959b) called this semigroup *self-adjoint* if for each $t \geq 0$ the operator T_t is a self-adjoint one, that is, if the following "reversibility" condition

$$\pi_j p_{jk}(t) = \pi_k p_{kj}(t), \qquad j, k = 1, 2, \ldots, \tag{6.3.2}$$

is satisfied, where $(\pi = \pi_j, j = 1, 2, \ldots)$ denotes the invariant probability distribution of $P(t)$.

On the other hand, the existence of the probabilistic circuit-coordinates $w_c(t), c \in \mathscr{C}$, in the expression (6.3.1) of the contractions $T_t, t \geq 0$, inspires the conversion of the edge-reversibility condition (6.3.2) into a circuit-reversibility condition as follows:

Theorem 6.3.2. *The semigroup* $\{T_t, t \geq 0\}$ *of contractions defined by* (6.3.1) *is self-adjoint if and only if the probabilistic weight functions* $w_c(\cdot)$ *satisfy the consistency equation*

$$w_c(t) = w_{c_-}(t), \qquad t \geq 0,$$

for all directed circuit $c \in \mathscr{C}$, *where* c_- *denotes the inverse circuit of* c.

Proof. The proof follows combining Theorem 5.5.2, Minping Qian et al. (1979, 1982), and Corollary 6 of S. Kalpazidou (1990a) (see also Theorem 1.3.1 of Part II). □

6.3.3. An integral representation for the circuit-weight functions $w_c(t)$ that decompose the transition matrix function $P(t)$ can be found if preliminarily we express all $w_c(t)$ in terms of the $p_{ij}(t)$'s. So, applying the argument of Theorems 6.3.1 and 6.2.1 to each t-skeleton chain, we obtain

Theorem 6.3.3. *For any* $t > 0$ *and any circuit* c *occurring in the decomposition* (6.2.1) *of the matrix* $P(t)$ *indexed by* $\mathbb{N}_v^* = \{1, \ldots, v\}$ *there exist a finite sequence* $(j_1, k_1), \ldots, (j_m, k_m)$ *of edges and a Hermitian system* $\{\nu_{j_n k_n}, n = 1, \ldots, m\}$ *of complex-valued totally finite Borel measures supported by the real line such that* $w_c(t) = w_{c_r}(t)$, *for some* $r = 1, \ldots, m$, *has the expression*

$$\begin{aligned} w_{c_1}(t) &= (w(j_1)w(k_1))^{1/2} \int_{-\infty}^{+\infty} e^{i\lambda t} \nu_{j_1 k_1}(d\lambda) \qquad \text{if} \quad r = 1, \\ w_{c_r}(t) &= (w(j_r)w(k_r))^{1/2} \int_{-\infty}^{+\infty} e^{i\lambda t} \nu_{j_r k_r}(d\lambda) \\ &\qquad - \sum_{s=1}^{r-1} w_{c_s}(t) J_{c_s}(j_r, k_r) \quad \text{if} \quad r = 2, \ldots, m, \quad m > 1. \end{aligned}$$