3

Cycle Representations of Recurrent Denumerable Markov Chains

This chapter deals with the cycle generating equations defined by the transition probabilities of denumerable Markov chains ξ which are recurrent. The solutions $(\mathscr{C}, w_c)$ of cycles and weights to these equations will be called *cycle representations* of ξ.

A natural idea to define a cycle (circuit) weight w_c is similar to that providing an "edge-weight" $\pi_i p_{ij}$, that is, the w_c will be the mean number of the appearances of c along almost all the sample paths. This will argue for a probabilistic criterion assuring the uniqueness of the cycle representation, that is, for a probabilistic algorithm with a unique solution of cycles and weights which decompose the finite-dimensional distributions of ξ.

An alternate method of development is a deterministic approach according to which the circuit weights are given by a sequence of nonprobabilistic algorithms.

Our exposition follows the results of the Peking school of Qians (1978–1991), S. Kalpazidou (1990a, 1992e, 1993c, 1994b), and Y. Derriennic (1993).

3.1 The Derived Chain of Qians

As we have already seen in Theorem 1.3.1, the representative collection $(\mathscr{C}, w_c)$ of circuits and weights is not, in general, unique. It depends on the choice of the ordering of the representative circuits in the algorithm of Theorem 1.3.1.

In general, there are many algorithms of cycle decompositions for the finite-dimensional distributions of Markov chains which admit invariant probability distributions. Some of them provide a unique solution $(\mathscr{C}, w_c)$ as a representative class, and some others have many solutions of representative classes (as the algorithm of Theorem 1.3.1). So, when we say that we look for the uniqueness of the representative class $(\mathscr{C}, w_c)$, we understand that we shall refer to a definite algorithm with a unique solution $(\mathscr{C}, w_c)$.

Expectedly, such an algorithm can be defined involving a probabilistic argument. It is Qians's school that first introduced probabilistic arguments to a unique cycle representation using, as a basic tool, a Markov process whose state space consists of the ordered sequences $(i_1, \ldots, i_n)$ of distinct points of a denumerable set S. Here we shall present Qians's approach in the contexts of our formalism exposed in Chapter 1. So, preliminary elements of our exposition are the directed cycles with distinct points as introduced by Definition 1.1.3. Accordingly, a cycle is an equivalence class with respect to the equivalence relation defined by (1.1.1); for instance, to the circuit $c = (i_1, \ldots, i_n, i_1)$ is assigned the cycle $\hat{c} = (i_1, \ldots, i_n)$ which represents the cycle-class $\{(i_1, \ldots, i_n), (i_2, i_3, \ldots, i_n, i_1), \ldots, (i_n, i_1, \ldots, i_{n-1})\}$. This presupposes that all further entities which rely on cycles should not depend on the choice of the representatives while the circuits to be considered will have distinct points (except for the terminals).

The idea of taking directed cycles arises from the topological property of the trajectories of certain Markov chains providing directed cycles along with directed circuits, that is, the chains pass through the states $i_1, i_2, \ldots, i_n, i_1$, or any cyclic permutation (see Figure 3.1.1).

So, the occurrence of a cycle $(i_1, \ldots, i_n)$ along a trajectory of these chains presupposes the appearance of the corresponding circuit $(i_1, \ldots, i_n, i_1)$. Such a chain is any homogeneous, irreducible, aperiodic, and positive-recurrent Markov chain $\xi = (\xi_n, n \geq 0)$ with a countable state space S. Namely, if a typical realization of a sample path $(\xi_n(\omega))_n$ is $(i_1, i_2, i_3, i_2, i_3, i_4, i_1, i_3, i_5, \ldots), i_k \in S, k = 1, 2, \ldots,$ then the sequence of the cycles is $(i_2, i_3), (i_2, i_3, i_4, i_1)$, (see Figure 3.1.1).

The interpretation of a cycle $\hat{c} = (i_1, \ldots, i_r)$ in terms of the chain ξ is that it appears on a sample path $(\xi_n(\omega))_n$ (and then on almost

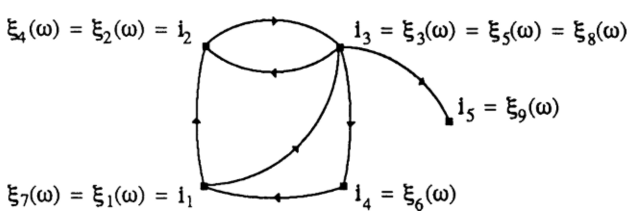

Figure 3.1.1.

all the sample paths as we shall see below), that is, the chain passes through the states $i_1, i_2, \ldots, i_r, i_1$ (or any cyclic permutation). For instance, if the values of $(\xi_n(\omega))_{n\geq 0}$ are given by $(1, 4, 2, 3, 2, 6, 7, 6, 1, \ldots)$, then the sequence of cycles occurring on this trajectory is given by $(2, 3), (6, 7), (1, 4, 2, 6), \ldots,$ while the corresponding tracks of the remaining states are $(1, 4, 2, 6, 7, 6, 1, \ldots)(1, 4, 2, 6, 1, \ldots)$ (S. Kalpazidou (1990a, 1994b)). The previous decycling procedure can be found in various fields under different versions. For instance, S. Alpern (1991) introduced a similar decycling method in game theory. This leads naturally to a new chain $y = (y_n(\omega))_{n\geq 0}$ whose value at time k is the track of the remaining states, in sequence, after discarding the cycles formed up to k along $(\xi_n(\omega))_{n\geq 0}$.

In the following table we give the trajectory (1, 4, 2, 3, 2, 6, 7, 6, 1, ...) of $(\xi_n(\omega))_n$ along with the attached trajectory $(y_n(\omega))_n$ as well as the cycles occurring along $(\xi_n(\omega))_n$:

n	0	1	2	3	4
$\xi_n(\omega)$	1	4	2	3	2
$y_n(\omega)$	[1]	[1, 4]	[1, 4, 2]	[1, 4, 2, 3]	[1, 4, 2]
Cycles					(2, 3)
n	5	6	7	8	...
$\xi_n(\omega)$	6	7	6	1	...
$y_n(\omega)$	[1, 4, 2, 6]	[1, 4, 2, 6, 7]	[1, 4, 2, 6]	[1]	...
Cycles			(6, 7)	(1, 4, 2, 6)	...

It turns out that each cycle $\hat{c} = (i_1, \ldots, i_r)$ is closed by the edge (i_r, i_1) which occurs either after $\hat{c}$, or before completing $\hat{c}$, as (i_1, i_2) in the cycle (i_2, i_3, i_4, i_1) of Figure 3.1.1, or as (1, 4) in the cycle (1, 4, 2, 6) of the table above, where the time unit is the jump-time of $(\xi_n(\omega))_n$.

Let $w_{c,n}(\omega)$ be the *number of occurrences of the cycle* $\hat{c}$ *up to time* n along the trajectory ω of ξ. The rigorous definition of $w_{c,n}(\omega)$ is due to Minping Qian et al. (1982). It is this definition that we describe further. If $t_n(\omega)$ denotes the nth jump time of $(\xi_n(\omega))_n$, then introduce

$$\tau_1(\omega) = \min\{t_n(\omega) : \exists m < n \text{ such that } \xi_{t_n(\omega)}(\omega) = \xi_{t_m(\omega)}(\omega)\},$$
$$\tau_1^*(\omega) = t_m(\omega), \text{ if } t_m(\omega) < \tau_1(\omega) \text{ and } \xi_{t_m(\omega)}(\omega) = \xi_{\tau_1(\omega)}(\omega).$$

Define

$$\xi_n^{(1)}(\omega) = \begin{cases} \xi_n(\omega), & \text{if } n < \tau_1^*(\omega) \text{ or } n > \tau_1(\omega); \\ \xi_{\tau_1(\omega)}(\omega) = \xi_{\tau_1^*(\omega)}(\omega), & \text{if } \tau_1^*(\omega) \leq n \leq \tau_1(\omega). \end{cases}$$

Further we continue the same procedure of discarding cycles by considering the nth jump-time $t_n^{(1)}(\omega)$ of $(\xi_n^{(1)}(\omega))_n$. Then we put

$$\tau_2(\omega) = \min\{t_n^{(1)}(\omega) : \exists m < n \text{ such that } \xi^{(1)}_{t_n^{(1)}(\omega)}(\omega) = \xi^{(1)}_{t_m^{(1)}(\omega)}(\omega)\}$$

and so on, obtaining the sequence:

$$\tau_1(\omega) < \tau_2(\omega) < \cdots < \tau_n(\omega) < \cdots$$

and

$$\tau_1^*(\omega) < \tau_2^*(\omega) < \cdots < \tau_n^*(\omega) < \cdots.$$

Now, denote an ordered sequence of distinct points $i_1, \ldots, i_r$ by $[i_1, \ldots, i_r]$ and identify the ordered union $[[i_1, \ldots, i_m], [i_{m+1}, \ldots, i_{m+n}]]$ with $[i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n}]$. The set $[S]$ of all finite ordered sequences $[i_1, \ldots, i_r], r \geq 1$, of points of S is denumerable.

Set $t_0(\omega) = 0$. Define

$$\begin{aligned}
y_0(\omega) &= [\xi_0(\omega)], \\
y_n(\omega) &= [\xi_0(\omega)], && \text{if } n < t_1(\omega) \\
y_n(\omega) &= [\xi_0(\omega), \xi_{t_1(\omega)}(\omega), \ldots, \xi_n(\omega)], && \text{if } t_1(\omega) \leq n < \tau_1(\omega), \\
y_{\tau_1(\omega)}(\omega) &= [\xi_0(\omega), \xi_{t_1(\omega)}(\omega), \ldots, \xi_{\tau_1^*(\omega)}(\omega)], \\
y_n(\omega) &= [y_{\tau_1(\omega)}(\omega), [\xi_{t_s(\omega)}(\omega)]_{\tau_1(\omega) < t_s(\omega) \leq n}], && \text{if } \tau_1(\omega) < n < \tau_2(\omega),
\end{aligned}$$

and so on. It is easy to see that $y = \{y_n\}_{n \geq 0}$ is an $[S]$-state Markov chain called by Minping Qian the *derived chain associated to* ξ.

Furthermore, it is seen in S. Kalpazidou (1990a) that if for a cycle $\hat{c} = (i_1, \ldots, i_r)$ the sum

$$\sum_{m=1}^{n} \sum_{k=1}^{r} 1_{\{\omega : y_{m-1}(\omega) = [y_m(\omega), [i_k, i_{k-1}, \ldots, i_{k+r-1}]]\}}(\omega)$$

is meant modulo r the cyclic permutations (i.e., it is independent of the cyclic permutations of $i_k, i_{k+1}, \ldots, i_{k+r-1}$), then it equals

$$w_{c,n}(\omega) = \sum_{m=1}^{n} 1_{\{\text{the class-cycle } \hat{c} \text{ occurs}\}}(\omega). \tag{3.1.1}$$

If $p_{jk}, j, k \in S$, denote the transition probabilities of ξ, then for $E = [k_1, k_2, \ldots, k_s]$ and $F = [j_1, j_2, \ldots, j_r]$ the transition probabilities p_{FE} of y are given as follows:

$$p_{FE} = \begin{cases} p_{j_r k_s}, & \text{if either } r \geq s \text{ and } k_1 = j_1, k_2 = j_2, \ldots, k_s = j_s, \\ & \text{or } r = s - 1 \text{ and } k_1 = j_1, k_2 = j_2, \ldots, k_r = j_r; \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.2}$$

Since ξ is recurrent, we have

$$\text{Prob}(\xi_n \text{ returns to } i/\xi_0 = i) = 1,$$

and then

$$\text{Prob}(y_n \text{ returns to } [i]/y_0 = [i]) = \text{Prob}(\xi_n \text{ returns to } i/\xi_0 = i) = 1.$$

Let now $[E]_i$ be the subset of all ordered sequences in $[S]$ whose first element is i. Then $[E]_i$ is a stochastically closed class of y. Therefore y is recurrent on each irreducible class $[E]_i$. The invariant probability distribution $\tilde{\pi}$ is given on the point sets $[i]$ by

$$\tilde{\pi}([i]) = \pi(i), \tag{3.1.3}$$

where $\pi = (\pi_i, i \in S)$ denotes the invariant probability distribution of ξ.

The general definition of $\tilde{\pi}([i_1, i_2, \ldots, i_s])$ has a much more complex algebraic expression in terms of the transition probabilities p_{ij} of ξ as we see in the following theorem due to Minping Qian and Min Qian (1982):

Theorem 3.1.1.

(i) *The invariant probability distribution of the chain y on the recurrent class $[E]_i$ is given by*

$$\begin{aligned}\tilde{\pi}([i_1, i_2, \ldots, i_s]) = p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{s-1} i_s} \cdot \pi_{i_1} N(i_2, i_2/i_1) \\ \times N(i_3, i_3/i_1, i_2) \cdots N(i_s, i_s/i_1, \ldots, i_{s-1})\end{aligned} \tag{3.1.4}$$

where $i_1 = i$ and $N(i, j/i_1, \ldots, i_k), 1 \leq k \leq s-1$, denotes the taboo Green function

$$\begin{aligned}N(i, j/i_1, \ldots, i_k) = \sum_{n=0}^{\infty} \text{Prob}(\xi_n = j, \xi_m \neq i_1, \ldots, i_k; \\ \textit{for } 1 \leq m < n/\xi_0 = i).\end{aligned}$$

(ii)

$$\tilde{\pi}([i_1, i_2, \ldots, i_s]) p_{i_s i_1} = \sum_{k=1}^{s} \sum_{j_2, \ldots, j_r} \tilde{\pi}([j_1, \ldots, j_r, i_k, i_{k+1}, \ldots, i_{k-1}]) p_{i_{\kappa-1} i_\kappa}, \tag{3.1.5}$$

where j_1 is fixed in the complement set of $\{i_1, i_2, \ldots, i_s\}$ and the inner sum is taken over all distinct choices $j_2, j_3, \ldots, j_r \in S \backslash \{j_1, i_1, \ldots, i_s\}$. The sums $k+1, k+2, \ldots, k+s-1$ are understood to be modulo s.

(iii) *For any fixed points i and j we write*

$$\pi_j = \sum_{j_2, \ldots, j_r} \tilde{\pi}([i, j_2, \ldots, j_r, j]), \tag{3.1.6}$$

where the sum is taken over all distinct choices $j_2, j_3, \ldots, j_r \in S \backslash \{i, j\}$.

Proof. According to T.E. Harris (1952) we have the following identities:

$$\pi_i N(j,j/i) = \pi_j N(i,i/j), \tag{3.1.7}$$
$$\tilde{\pi}(E_1)\,\tilde{q}(E_1,E_2) = \tilde{\pi}(E_2)\,\tilde{q}(E_2,E_1), \tag{3.1.8}$$

for any E_1, E_2 states in $[E]_i$, where $\tilde{q}(E_i,E_j)$ denotes the probability that the derived chain y starting at E_i enters E_j before returning to E_i. Then, for $E_1 = [i_1, i_2, \ldots, i_{s-1}]$ and $E_2 = [i_1, i_2, \ldots, i_{s-1}, i_s]$ we have that

$$\begin{aligned}\tilde{q}(E_1,E_2) &= p_{i_{s-1}i_s},\\ \tilde{q}(E_2,E_1) &= 1 - H(i_s, i_s/i_1, i_2, \ldots, i_{s-1}),\end{aligned}$$

where $H(i_s, i_s/i_1, i_2, \ldots, i_{s-1})$ denotes the probability that the original chain ξ starting at i_s returns to i_s before entering the states $i_1, i_2, \ldots, i_{s-1}$. Hence relation (3.1.8) becomes

$$\tilde{\pi}([i_1, i_2, \ldots, i_{s-1}])p_{i_{s-1}i_s} = \tilde{\pi}([i_1, \ldots, i_s])(1 - H(i_s, i_s/i_1, \ldots, i_{s-1})) \tag{3.1.9}$$

and

$$\tilde{\pi}([i_1, i_2, \ldots, i_s]) = \tilde{\pi}([i_1, i_2, \ldots, i_{s-1}])p_{i_{s-1}i_s} N(i_s, i_s/i_1, i_2, \ldots, i_{s-1}). \tag{3.1.10}$$

Now we may appeal to a theorem of K.L. Chung (1967) (see p. 48), and write accordingly

$$\begin{aligned}&N(i_s, i_s/i_1, \ldots, i_{s-1})N(i_{s+1}, i_{s+1}/i_1, \ldots, i_{s-1}, i_s)\\ &\quad = N(i_{s+1}, i_{s+1}/i_1, \ldots, i_{s-1})N(i_s, i_s/i_1, \ldots, i_{s-1}, i_{s+1}).\end{aligned} \tag{3.1.11}$$

Then equation (3.1.4) follows from (3.1.3) and (3.1.10). It is to be noticed that the product

$$\pi_{i_1} N(i_2, i_2/i_1) N(i_3, i_3/i_1, i_2) \ldots N(i_s, i_s/i_1, i_2 \ldots, i_{s-1}) \tag{3.1.12}$$

is unaffected by any permutation of the indices $i_1, i_2, \ldots, i_s$ because of (3.1.7) and (3.1.11).

To prove relation (3.1.5) we first show that

$$\begin{aligned}1 = \sum_{k=1}^{s} \sum_{j_2,\ldots,j_r} & N(j_1, j_1/i_1, \ldots, i_s)\\ &\cdot N(j_2, j_2/i_1, \ldots, i_s, j_1) N(j_3, j_3/i_1, \ldots, i_s, j_1, j_2) \ldots\\ &\cdot N(j_r, j_r/i_1, \ldots, i_s, j_1, \ldots, j_{r-1}) p_{j_1j_2} p_{j_2j_3} \cdots p_{j_r i_\kappa},\end{aligned} \tag{3.1.13}$$

where $j_1 \notin \{i_1, \ldots, i_s\}$ is fixed and the inner sum is taken over all distinct $j_2, \ldots, j_r \notin \{i_1, \ldots, i_s, j_1\}$. Let $p(i,j/H/n)$ be the taboo probability

$$p(i,j/H/n) = \text{Prob}(\xi_n = j, \xi_m \notin H \text{ for } 1 \le m < n/\xi_0 = i).$$

For $k, j_2, j_3, \dots, j_r$ fixed, the sum over $n_1, \dots, n_r$ of

$$p(j_1, j_1/i_1, \dots, i_s/n_1)p_{j_1 j_2} p(j_2, j_2/i_1, \dots, i_s, j_1/n_2)p_{j_2 j_3} \\ \dots p(j_r, j_r/i_1, \dots, i_s, j_1, \dots, j_{r-1}/n_r)p_{j_r i_\kappa}$$

is the probability for the chain ξ starting at j_1 to enter the set $\{i_1, \dots, i_s\}$ for the first time at the state i_k while the value of the derived chain y is $[j_1, j_2, \dots, j_r, i_k]$. Thus we get the summand of (3.1.13). Then the desired equation (3.1.5) follows by multiplying both sides of (3.1.13) with

$$p_{i_s i_1} p_{i_1 i_2} \dots p_{i_{s-1} i_s} \pi_{i_1} N(i_2, i_2/i_1) \cdot N(i_3, i_3/i_1, i_2) \dots N(i_s, i_s/i_1, \dots, i_{s-1}),$$

and using the symmetry of (3.1.12). Finally, equation (3.1.6) follows from (3.1.13) when taking $s = 1, j_1 = i$, and $i_1 = j$, and multiplying by π_j. □

3.2 The Circulation Distribution of a Markov Chain

A step closer to a probabilistic criterion for the uniqueness of the representative cycle-weights of a Markov chain ξ, under the assumptions of the previous section, is to find a definite algorithm whose quantities enjoy probabilistic interpretations in terms of the sample paths. The idea is to generalize to cycles the definition of the "edge-weight" $w(i,j) = \pi_i p_{ij}$ in terms of sample paths; namely, as is well known the $w(i,j)$ is the mean number of the consecutive passages of $(\xi_n(\omega))_n$ through the points i and j. That is, $\pi_i p_{ij}$ is the almost sure limit of

$$\frac{1}{n}\text{card}\{m \le n : \xi_{m-1}(\omega) = i, \xi_m(\omega) = j\},$$

as $n \to \infty$.

Accordingly, the revealing question for us will be whether or not we can analogously argue for the expression

$$\frac{1}{n}\text{card}\{m \le n : \text{ the cycle } \hat{c} \text{ occurs on } (\xi_k(\omega))_k\} = \frac{1}{n} w_{c,n}(\omega),$$

where m counts the appearances of $\hat{c}$ on $(\xi_k(\omega))_k$. (Recall that a cycle $\hat{c} = (i_1, i_2, \dots, i_r), r > 1$, appears on $(\xi_k(\omega))_k$ if the chain passes through the points $i_1, i_2, \dots, i_r, i_1$, or any cyclic permutation.)

In this direction, we first need to prove that $(1/n)w_{c,n}(\omega)$ has a limit independent of ω. Namely, we have

Theorem 3.2.1. *Let $\xi = (\xi_n)_n$ be an aperiodic, irreducible, and positive-recurrent Markov chain defined on a probability space $(\Omega, \mathscr{K}, \mathbb{P})$ and with a countable state space S, and let $\mathscr{C}_n(\omega), n = 0, 1, 2, \dots,$ be the class of all cycles occurring until n along the sample path $(\xi_n(\omega))_n$.*

Then the sequence $(\mathscr{C}_n(\omega), w_{c,n}(\omega)/n)$ of sample weighted cycles associated with the chain ξ converges almost surely to a class $(\mathscr{C}_\infty, w_c)$, that is,

$$\mathscr{C}_\infty = \lim_{n\to\infty} \mathscr{C}_n(\omega), \quad a.s. \tag{3.2.1}$$

$$w_c = \lim_{n\to\infty} (w_{c,n}(\omega)/n), \quad a.s. \tag{3.2.2}$$

Furthermore, the cycle-weights w_c are independent of the choice of an ordering on $\mathscr{C}_\infty$.

Proof. Let $\hat{p}_j \equiv \mathbb{P}(\xi_0 = j), j \in S$. Following S. Kalpazidou (1990a), we can assign to each ω the class $\lim_{n\to\infty} \mathscr{C}_n(\omega)$ of directed cycles that occur along $(\xi_n(\omega))_n$, since the sequence $(\mathscr{C}_n(\omega))$ is increasing. Denote $\mathscr{C}_\infty(\omega) \equiv \lim_{n\to\infty} \mathscr{C}_n(\omega) = \bigcup_n \mathscr{C}_n(\omega)$.

On the other hand, applying the law of large numbers to the Markov chain y we have

$$\lim_{n\to\infty} (w_{c,n}(\omega)/n) = E1_{\{\text{the class-cycle } \hat{c} \text{ occurs}\}},$$

where $\hat{c}$ is any class-cycle having the representative $(i_k, i_{k+1}, \dots, i_s, i_1, \dots, i_{k-1})$. Put

$$w_c \equiv \lim_{n\to\infty} (w_{c,n}(\omega)/n).$$

That w_c is finite and independent of ω follows from (3.1.5) and the following equalities due to Minping Qian et al. (1982):

$$\begin{aligned} w_c &= \sum_{k=1}^{s} E(1_{\{y_{n-1}=[y_n,[i_k,i_{k+1},\dots,i_s,i_1,\dots,i_{k-1}]]\}}) \\ &= \sum_{j_1} \hat{p}_{j_1} \sum_{k=1}^{s} \sum_{j_2,\dots,j_r} \tilde{\pi}([j_1, j_2, \dots, j_r, i_k, i_{k+1}, \dots, i_{k-1}]) \cdot p_{i_{k-1}i_k}, \end{aligned} \tag{3.2.2$'$}$$

where $j_1, \dots, j_r \notin \{i_1, \dots, i_s\}, r \geq 0$, are distinct from one another. From here it results that $\mathscr{C}_\infty(\omega) \equiv \mathscr{C}_\infty$ is independent of ω as well, and this completes the proof. □

We now introduce the following nomenclature:

Definition 3.2.2. The items occurring in Theorem 3.2.1 are as follows: the sequence $\{w_{c,n}(\omega)/n\}_{\hat{c}\in\mathscr{C}_\infty}$, which is called the *circulation distribution on ω up to time n*, the w_c, which is called the *cycle skipping rate* on $\hat{c}$ or c, and $\{w_c, \hat{c} \in \mathscr{C}_\infty\}$, which is called the *circulation distribution* of ξ.

Remarks

(i) Theorems 3.1.1 and 3.2.1 remain valid for periodic and positive-recurrent Markov chains as well. In general, convergence of averages along Markov chain trajectories is required (even if there is no finite-invariant measure). Recent investigations to this direction are due to Y. Derriennic (1976), and Y. Derriennic and M. Lin ((1989), (1995)).

(ii) The w_c's verify the consistency equation $w_c = w_{c \circ t_i}$, for all $i \in S$, where $\{t_i\}$ is the group of translations on Z occurring in (1.1.1).

As an immediate consequence of Theorem 3.2.1 one obtains from (3.2.2′) the exact algebraic expression for the cycle skipping rate w_c as follows:

Corollary 3.2.3. *If $\pi = (\pi_i, i \in S)$ is the invariant probability distribution of an S-state irreducible positive-recurrent Markov chain $\xi = (\xi_n)_n$ and $\hat{c} = (i_1, i_2, \ldots, i_s)$ is a cycle, then the cycle skipping rate w_c is given by equation*

$$w_c = \pi_{i_1} p_{i_1 i_2} p_{i_2 i_3} \ldots p_{i_{s-1} i_s} p_{i_s i_1} \cdot N(i_2, i_2/i_1) N(i_3, i_3/i_1, i_2) \ldots N(i_s, i_s/i_1, i_2, \ldots, i_{s-1}), \quad (3.2.3)$$

where $(p_{ij}, i, j \in S)$ is the transition matrix of ξ, and $N(i_k, i_k/i_1, \ldots, i_{k-1})$ denotes the taboo Green function introduced in (3.1.4).

3.3 A Probabilistic Cycle Decomposition for Recurrent Markov Chains

We are now prepared to answer our original question on the existence of a unique cycle decomposition, provided by a probabilistic algorithm, for the finite-dimensional distributions of the recurrent Markov chains. Namely, the probabilistic algorithm to be considered is that occurring in Theorem 3.2.1 while the desired decomposition follows from Theorem 3.1.1 (see Minping Qian and Min Qian (1982), and S. Kalpazidou (1990a)).

Consequently, we may state

Theorem 3.3.1 (The Probabilistic Cycle Representation). *Let S be any denumerable set. Then any stochastic matrix $P = (p_{ij}, i, j \in S)$ defining an irreducible and positive-recurrent Markov chain ξ is decomposed by the cycle skipping rates $w_c, \hat{c} \in \mathscr{C}_\infty$, as follows:*

$$\pi_i p_{ij} = \sum_{\hat{c} \in \mathscr{C}_\infty} w_c J_c(i, j), \quad i, j \in S, \quad (3.3.1)$$

where $\mathscr{C}_\infty$ is the class of cycles $\hat{c}$ occurring in Theorem 3.2.1, *c denotes the circuit corresponding to the cycle $\hat{c}$, $\pi = (\pi_i, i \in S)$ is the invariant probability distribution of P and $J_c(i, j) = 1$ or 0 according to whether or not (i, j) is an edge of c.*

The above cycle-weights w_c are unique, with the probabilistic interpretation provided by Theorem 3.2.1, *and independent of the ordering of $\mathscr{C}_\infty$.*

If P defines a positive-recurrent Markov chain, then a similar decomposition to (3.3.1) *holds, except for a constant, on each recurrent class.*

The representative class $(\mathscr{C}_\infty, w_c)$ provided by Theorem 3.3.1 is called the *probabilistic cycle (circuit) representation of ξ and P* while ξ is called a *circuit chain.* The term "probabilistic" is argued by the algorithm of Theorem 3.2.1 whose unique solution $\{w_c\}$ enjoys a probabilistic interpretation in terms of the sample paths of ξ.

The terms in the equations (3.3.1) have a natural interpretation using the sample paths of ξ as follows (S. Kalpazidou (1990a)). Consider the functions $\sigma_n(\cdot; i, j)$ defined as

$$\sigma_n(\omega; i, j) = \frac{1}{n}\text{card}\{m \le n : \xi_{m-1}(\omega) = i, \xi_m(\omega) = j\}$$

for any $i, j \in S$. Consider $\mathscr{C}_n(\omega)$ to be, as in Theorem 3.2.1, the class of all the cycles occurring up to n along the sample path $(\xi_n(\omega))_n$. We recall that a cycle $\hat{c} = (i_1, \ldots, i_r), r \ge 2$, occurs along a sample path if the chain passes through states $i_1, i_2, \ldots, i_r, i_1$ (or any cyclic permutation). Notice that the sample sequence

$$k(\omega) = (\xi_{m-1}(\omega), \xi_m(\omega))$$

occurs up to n whenever either $k(\omega)$ is passed by a cycle of $\mathscr{C}_n(\omega)$ in the sense of Definition 1.2.2 or $k(\omega)$ is passed by a circuit completed after time n on the sample path $(\xi_n(\omega))$. Therefore for $i \ne j$ and $n > 0$, great enough, we have

$$\sigma_n(\omega; i, j) = \sum_{\hat{c} \in \mathscr{C}_n(\omega)} \frac{1}{n} w_{c,n}(\omega) J_c(i, j) + \varepsilon_n(\omega; i, j)/n, \tag{3.3.2}$$

where

$$\varepsilon_n(\omega; i, j) = 1_{\substack{\{\text{the last occurrence of } (i,j) \text{ does not happen} \\ \text{together with the occurrence of a cycle of } \mathscr{C}_n(\omega)\}}}(\omega). \tag{3.3.3}$$

Then the left side of (3.3.2) converges to $\pi_i p_{ij}$ and each summand of the right side converges to $w_c J_c(i, j)$.

From the present standpoint a natural way of proving a cycle-decomposition-formula is to observe that the a.s. limit of the sums

$$\sum_{\hat{c} \in \mathscr{C}_\infty} (w_{c,n}(\omega)/n) J_c(i, j)$$

when n tends to infinity is related with the sum occurring in equations (3.3.1). This inspires a direct proof of the decomposition (3.3.1) as in the following theorem due to Y. Derriennic (1993).

Theorem 3.3.2. *Let S be a denumerable set and let $P = (p_{ij}, i, j \in S)$ be any stochastic matrix defining an irreducible and positive-recurrent Markov chain ξ. Then*

$$\begin{aligned} \pi_i p_{ij} &= \lim_{n \to \infty} \sum_{\hat{c} \in \mathscr{C}_\infty} (w_{c,n}(\omega)/n) J_c(i, j) \quad a.s. \\ &= \sum_{\hat{c} \in \mathscr{C}_\infty} w_c J_c(i, j), \end{aligned} \tag{3.3.4}$$

where $(\mathscr{C}_\infty, w_c)$ *and* $w_{c,n}(\omega)$ *have the same meaning as in Theorem* 3.2.1, $\pi = (\pi_i, i \in S)$ *is the invariant probability distribution of* P *and* $J_c(i,j) = 1$ *or* 0 *according to whether or not* (i,j) *is on edge of* c.

Proof. Consider the derived chain y associated to ξ and an arbitrarily chosen irreducible class $[E]_i$. Then the restriction of y to $[E]_i$ is a positive-recurrent chain whose invariant probability distribution is given by (3.1.4). Let $[E]_{ij}$ be the subset of $[E]_i$ which consists of all the cycles starting with the consecutive points i and j. Then, applying the Birkhoff ergodic theorem to the number of the visits of y in the set $[E]_{ij}$, one obtains relations (3.3.4). The proof is complete. □

If ξ is an irreducible null-recurrent Markov chain, then a cycle-decomposition-formula may be obtained using a similar argument where Birkhoff's theorem is replaced by the Hopf ergodic theorem for ratios. Accordingly, the limit of $(w_{c,n}(\omega)/w_{c',n}(\omega))$ exists a.s. as $n \to \infty$ for any circuits c and c'.

3.4 Weak Convergence of Sequences of Circuit Chains: A Deterministic Approach

We introduced two types of circuit representations of Markov chains according to whether or not the corresponding algorithms define the circuit-weights by a random or a nonrandom choice. In the spirit of Kolmogorov we may call such algorithms probabilistic (randomized) and deterministic (non-randomized) algorithms, respectively.

In the present section the deterministic algorithm of Theorem 1.3.1 is generalized to infinite classes of directed circuits such that the corresponding denumerable circuit Markov chain ξ can be defined as a limit of a certain sequence $({}^m\xi)_m$ of finite circuit chains. The convergence of this sequence is weak convergence in the sense of Prohorov, that is, the finite-dimensional distributions of ${}^m\xi$ converge as $m \to \infty$ to the corresponding ones of ξ.

The approach we are ready to follow will rely on the idea of circuit generating equations exposed in Section 1.3. In this direction we shall consider denumerable reversible Markov chains which are of bounded degree, that is, from each state there are finitely many passages to other states. Then a parallel to Tychonov's theorem for infinite products of compact topological spaces can be conceived along with the matching Hall-type theorem for infinite bipartite graphs (see P. Hall (1935) and K. Menger (1927)).

The preliminary element will be a stochastic matrix $P = (p_{ij}, i, j \in S)$ on a denumerable set S, that defines a reversible, irreducible, aperiodic, and positive-recurrent Markov chain $\xi = (\xi_n)_n$, whose invariant probability distribution is denoted by $\pi = (\pi_i, i \in S)$. The main theorem is that a circuit decomposition for P can be given using a deterministic algorithm according to which the directed circuits $c \in \mathscr{C}$ and their weights w_c are solutions to certain recursive balance equations where the "edge-weights" $\pi_i p_{ij}, i, j \in S$, are used without any probabilistic meaning. The representative class $(\mathscr{C}, w_c)$ will be called a *deterministic circuit representation of ξ and of P.*

One reason for choosing a deterministic algorithm is that the correspondence $P \to \mathscr{C}$ becomes nearly one-to-one, that is, the class $\mathscr{C}$ approximates the probabilistic one. It is proved below that the class $\mathscr{C}$ may be the limit of an increasing sequence ${}^n\mathscr{C}$ of finite classes of overlapping directed circuits. The one-to-one correspondences $P \to \mathscr{C}$ are particularly important for plenty of problems arising in various fields. For example, we may refer here to the so-called coding problem arising in the context of dynamical systems, that in turn leads to the problem of mapping stochastic matrices into partitions. A detailed exposition of this argument is given in Section 3.5 of Part II.

The relation $P \to (\mathscr{C}, w_c)$ for transient Markov chains is still an open problem and may be connected in particular with certain questions arising in network theory. One of them is concerned with the existence of unique cycle-currents in infinite resistive networks made up by circuits. Interesting results in this direction for edge-networks are due to H. Flanders (1971), A.H. Zemanian (1976a, 1991) and P.M. Soardi and W. Woess (1991). For instance, Flanders's condition for a current I to be the unique solution to a network-type problem (in the class of all currents with finite energy) consists of the existence of a sequence of currents in finite subnetworks approaching I.

We begin our investigations by considering a countable set S and a stochastic matrix $P = (p_{ij}, i, j \in S)$ of bounded degree, that is, for each $i \in S$ there are finitely many $j \in S$ such that $p_{ij} > 0$ or $p_{ji} > 0$. Assume P defines a reversible, irreducible, and aperiodic Markov chain ξ admitting an invariant probability distribution $\pi = (\pi_i, i \in S)$, with all $\pi_i > 0$.

We say ξ defines a directed circuit $c = (i_1, \ldots, i_n, i_1)$ where $n \geq 2$, and $i_k \neq i_m$ for distinct $k, m \leq n$, if and only if $p_{i_1 i_2}, p_{i_2 i_3}, \ldots p_{i_{n-1} i_n} P_{i_n i_1} > 0$. Throughout this section the directed circuits will be considered to have distinct points (except for the terminals). The irreducibility condition amounts to the existence for each pair (i, j), with $i \neq j$, of a directed finite sequence $\sigma(i, j)$ connecting i to j, that is,

$$\sigma(i,j) : i_0 = i, i_1, \ldots, i_n = j, n \geq 1 \text{ with } i_k \neq i_m \text{ for } k \neq m; k, m \leq n, \\ \text{such that } p_{ii_1} \ldots p_{i_{n-1}j} > o. \tag{3.4.1}$$

The following property characterizes, in general, irreducibility:

Proposition 3.4.1. *Any two points of S are cyclic-edge-connected in S.*

Proof. Let $i \neq j$. If $p_{ij} > 0$, the proof is immediate. Otherwise, there exist two directed paths $\sigma_1(i,j)$ and $\sigma_2(j,i)$ connecting i to j and j to i, respectively. If $j_1 \neq i, j$ denotes the first point of σ_1 belonging to σ_2, then there exists the directed circuit

$$c_1 = (\sigma_1(i, j_1), \sigma_2(j_1, i)),$$

such that the points j_1 and j are mutually connected by the directed paths $\sigma_1(j_1, j)$ and $\sigma_2(j, j_1)$. By repeating the previous reasonings, we obtain a sequence of directed circuits connecting i to j such that any two consecutive circuits have at least one common point. □

Consider the shortest-length-distance introduced in Section 2.2, that is,

$$d(i,j) = \begin{cases} 0, & \text{if } i = j; \\ \text{the shortest length } n & \\ \text{of the paths } \sigma(i,j) \text{ defined by (3.4.1)}, & \text{if } i \neq j; \end{cases} \tag{3.4.2}$$

where the connections are expressed by the forward–backward passage functions introduced by relation (2.2.1). Then, for any finite subgraph of P define its diameter as the maximal distance. Since any point of S is cyclic-edge-connected with all the others, we may choose an arbitrary point $\boldsymbol{O} \in S$ as the origin of the spheres $S(\boldsymbol{O}, m)$ of radius $m, m = 0, 1, \ldots$ with respect to the distance d above.

We are now prepared to prove a deterministic circuit decomposition of P following S. Kalpazidou (1993c). As was already mentioned, we are interested in representing the chain ξ by a class $(\mathscr{C}, w_c)$ provided by a deterministic algorithm such that the correspondence $P \to \mathscr{C}$ becomes nearly one-to-one, that is, $\mathscr{C}$ will approximate the collection of all the circuits occurring along almost all the sample paths. Then the trivial case of the class containing only the circuits of period two will be avoided. We have

Theorem 3.4.2. *Consider S a denumerable set and $\xi = (\xi_n)_{n \geq 0}$ an S-state Markov chain which is irreducible, aperiodic, reversible, and positive-recurrent. Assume the transition matrix $P = (p_{ij}, i, j \in S)$ of ξ is of bounded degree.*

Then there exists a sequence $({}^m\xi)_m$ of finite circuit Markov chains, associated with a sequence of deterministic representative classes $({}^m\mathscr{C}, {}^m w_c)_m$, which converges weakly to ξ as $m \to \infty$ such that $\mathscr{C} = \lim_{m \to \infty} {}^m\mathscr{C}$ approximates the collection of all the circuits occurring along the sample paths of ξ. The chain ξ becomes a circuit chain with respect to the class $(\mathscr{C}, w_c)$ where

$$w_c = \sum_{m \to \infty} {}^m w_c.$$

Proof. Consider the balls $B(\boldsymbol{O}, n) = \bigcup_{k=0}^{n} S(\boldsymbol{O}, n), n = 0, 1, \ldots$. Then for each n and for any $i, j \in B(\boldsymbol{O}, n)$ the restriction ${}^n\xi$ of ξ to the ball $B(\boldsymbol{O}, n)$

has the transition probability

$$^{n}p_{ij} = p_{ij} / \left(\sum_{j \in B(\boldsymbol{O},n)} p_{ij} \right).$$

Correspondingly, if $\pi = (\pi_i, i \in S)$ is the invariant probability distribution of ξ then that of $^{n}\xi$ in $B(\boldsymbol{O}, n)$ is given by the sequence $^{n}\pi = (^{n}\pi_i, i \in B(\boldsymbol{O}, n))$ where

$$^{n}\pi_i = \left(\pi_i \sum_{j \in B(\boldsymbol{O},n)} p_{ij} \right) \Big/ \left(\sum_{i,j \in B(\boldsymbol{O},n)} \pi_i p_{ij} \right).$$

Put

$$^{n}p_i = \pi_i \sum_{j \in B(\boldsymbol{O},n)} p_{ij}, \quad i \in B(\boldsymbol{O},n), \qquad n = 1, 2, \ldots.$$

It is to be noticed that if $p_{ij} > 0$ there exists an n_0 such that for any $n \geq n_0$ we have $i, j \in B(\boldsymbol{O},n)$ and

$$\begin{aligned} ^{n}p_{ij} &\geq {}^{n+1}p_{ij} \geq \cdots \geq p_{ij}, \\ 0 < {}^{n}p_i &\leq {}^{n+1}p_i \leq \cdots \leq \pi_i, \end{aligned}$$

such that

$$^{n}p_i {}^{n}p_{ij} = {}^{n+1}p_i {}^{n+1}p_{ij} = \cdots = \pi_i p_{ij}. \tag{3.4.3}$$

Since any function $^{n}w(i,j) \equiv {}^{n}\pi_i {}^{n}p_{ij}, n \geq 0$, is balanced in $B(\boldsymbol{O},n)$, we can appeal to Theorem 1.3.1 and find accordingly a class $(^{n}\mathscr{C}, {}^{n}w_c)$ such that

$$^{n}\pi_i {}^{n}p_{ij} = \sum_{c \in {}^{n}\mathscr{C}} {}^{n}w_c J_c(i,j), \quad i, j \in B(\boldsymbol{O},n), \tag{3.4.4}$$

where the J_c is the backward–forward passage function given by (2.2.1). For $n = 0$, the constrained process to $B(\boldsymbol{O}, 0) = \{\boldsymbol{O}\}$ has an absorbtion state $\boldsymbol{O}$ and is represented by the class $^{0}\mathscr{C} = \{c = (\boldsymbol{O}, \boldsymbol{O})\}$ where $c = (\boldsymbol{O}, \boldsymbol{O})$ is the loop-circuit at point $\boldsymbol{O}$ and $w_c = 1$.

Let us further consider n great enough such that the ball $B(\boldsymbol{O}, n)$ comprises all the circuits with periods larger than or equal to some $k \geq 1$. Applying as above Theorem 1.3.1 to $^{n}w(i,j)$ and $B(\boldsymbol{O}, n)$ we choose a sequence $^{n}c_1, \ldots {}^{n}c_{k_1}$ of circuits such that some of them are the loops in $B(\boldsymbol{O}, n)$ and some others are certain circuits of the subgraphs in $B(\boldsymbol{O}, n)$ with diameters larger than one. Particularly, we may choose these circuits such that they occur along almost all the sample paths of $^{n}\xi$.

The irreducibility hypothesis implies that $\sum_{j \in B(\boldsymbol{O},n)} p_{bj} < 1$ for certain points $b \in B(\boldsymbol{O}, n)$. We shall call these points the boundary points of $B(\boldsymbol{O}, n)$. On the other hand, since the matrix P is of bounded degree, perhaps there are points $i \in B(\boldsymbol{O}, n)$ which satisfy equation $\sum_{j \in B(\boldsymbol{O},n)} p_{ij} = 1$. These points will be called the interior points of $B(\boldsymbol{O}, n)$.

Let us denote $n_1 = n$ and

$$^{1}\mathscr{C} \equiv {}^{n_1}\mathscr{C} = \{^{n_1}c_1, \ldots, {}^{n_1}c_{k_1}\}.$$

Then (3.4.4) becomes

$$^{n_1}\pi_i{}^{n_1}p_{ij} = \sum_{c \in {}^1\mathscr{C}} {}^{n_1}w_c J_c(i,j), \quad i,j \in B(\boldsymbol{O}, n_1). \tag{3.4.5}$$

For each boundary point $b \in B(\boldsymbol{O}, n_1)$ there is a point $j \notin B(\boldsymbol{O}, n_1)$ such that $p_{bj} > 0$. Let $n_2 > n_1$ such that all the boundary points of $B(\boldsymbol{O}, n_1)$ will become interior points in $B(\boldsymbol{O}, n_2)$.

Put

$$^{n_2}w(i,j) \equiv {}^{n_2}\pi_i{}^{n_2}p_{ij} = (\pi_i p_{ij}) / \sum_{i,j \in B(\boldsymbol{O}, n_2)} \pi_i p_{ij}, \quad i,j \in B(\boldsymbol{O}, n_2).$$

Note that because of (3.4.3) both $^{n_1}w(\cdot,\cdot)$ and $^{n_2}w(\cdot,\cdot)$ attain their minimum over the Arcset of $c_1 \equiv {}^{n_1}c_1$ at the same edge, say (i_1, j_1), that is,

$$^{n_1}w_{c_1} \equiv {}^{n_1}w(i_1, j_1) = \min_{c_1} {}^{n_1}w(i,j),$$
$$^{n_2}w(i_1, j_1) = \min_{c_1} {}^{n_2}w(i,j).$$

The latter equations enable us to choose $^{n_2}c_1 \equiv {}^{n_1}c_1 \equiv c_1$ and $^{n_2}w_{c_1} \equiv {}^{n_2}w(i_1, j_1)$. We have

$$^{n_1}w_{c_1} \geq {}^{n_2}w_{c_1} \geq \pi_{i_1} p_{i_1 j_1} > 0.$$

Further put

$$^{n_2}w_1(i,j) \equiv {}^{n_2}w(i,j) - {}^{n_2}w_{c_1} J_{c_1}(i,j), \quad i,j \in B(\boldsymbol{O}, n_2).$$

Then $^{n_2}w_1(i_1, j_1) = 0$ and the function $^{n_2}w_1(\cdot,\cdot)$ is also balanced in $B(\boldsymbol{O}, n_2)$.

Appealing to the algorithm of Theorem 1.3.1 in $B(\boldsymbol{O}, n_2)$, we find an edge (i_2, j_2) of $c_2 \equiv {}^{n_1}c_2$ $(n_1 = n)$ where both $^{n_1}w_1$ and $^{n_2}w_1$ attain their minimum, that is,

$$^{n_1}w_{c_2} = ^{n_1} w_1(i_2, j_2) \equiv \min_{c_2} {}^{n_1}w_1(i,j)$$
$$= \left(1 / \left(\sum_{i,j \in B(\boldsymbol{O}, n_1)} \pi_i p_{ij}\right)\right) (\pi_{i_2} p_{i_2 j_2} - \pi_{i_1 p_{i_1 j_1}} J_{c_1}(i_2, j_2)),$$

and

$$^{n_2}w_1(i_2, j_2) \equiv \min_{c_2} {}^{n_2}w_1(i,j)$$
$$= \left(1 / \left(\sum_{i,j \in B(\boldsymbol{O}, n_2)} \pi_i p_{ij}\right)\right) (\pi_{i_2} p_{i_2 j_2} - \pi_{i_1} p_{i_1 j_1} J_{c_1}(i_2, j_2)).$$

Then we may choose ${}^{n_2}c_2 \equiv c_2$ and ${}^{n_2}w_{c_2} \equiv {}^{n_2}w_1(i_2, j_2)$. Hence

$$^{n_1}w_{c_2} > {}^{n_2}w_{c_2} \geq \pi_{i_2}p_{i_2j_2} - \pi_{i_1}p_{i_1j_1}J_{c_1}(i_2, j_2) > 0.$$

Repeating the same reasonings above, we conclude that all the circuits in $B(\boldsymbol{O}, n_1)$ are circuits in $B(\boldsymbol{O}, n_2)$ as well, that is,

$$\begin{aligned} {}^{n_1}c_1 &= {}^{n_2}c_1 \equiv c_1, \\ {}^{n_1}c_2 &= {}^{n_2}c_2 \equiv c_2, \\ &\vdots \\ {}^{n_1}c_{k_1} &= {}^{n_2}c_{k_1} \equiv c_{k_1}. \end{aligned}$$

Then the ${}^{n_2}w(i, j)$ is decomposed in $B(\boldsymbol{O}, n_2)$ by a class $({}^2\mathscr{C}, {}^{n_2}w_c)$ where

$$^2\mathscr{C} = \{c_1, \ldots, c_{k_1}, c_{k_1+1}, \ldots, c_{k_2}\}, \quad k_2 > k_1,$$

may particularly contain circuits which occur along the sample paths of the restriction ${}^{n_2}\xi$ of ξ to $B(\boldsymbol{O}, n_2)$.

Hence

$$^{n_2}w(i, j) \equiv \ {}^{n_2}\pi_i \ {}^{n_2}p_{ij} = \sum_{c \in {}^2\mathscr{C}} {}^{n_2}w_c J_c(i, j).$$

Continuing the previous reasonings, we shall find a sequence $\{{}^s\mathscr{C}\}_{s \geq 1}$ of finite classes of directed circuits which is increasing. Then there exists the limiting class

$$\mathscr{C} \equiv \lim_{s \to \infty} {}^s\mathscr{C} = \{c_1, c_2, \ldots, c_{k_1}, \ldots\}.$$

On the other hand, for any circuit $c \in \mathscr{C}$, we find a sequence $\{{}^{n_s}w_c\}_{s \geq 1}$ of positive numbers which is decreasing, and so convergent to a number $w_c \in [0, 1]$, that is, $\lim_{s \to \infty}, {}^{n_s}w_c = w_c$. Moreover, there is some $\sigma \geq 1$ such that $c \in {}^\sigma\mathscr{C}$. Then

$$^{n_s}w_c \geq \pi_{i_r}p_{i_rj_r} - \sum_{k=1}^{r-1} \pi_{i_\kappa}p_{i_\kappa j_\kappa}J_{c_\kappa}(i_r, j_r) > 0,$$

for all $s \geq \sigma$ and some $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$ where $r = 1, \ldots, k_\sigma$. Thus, $w_c > 0$, for all $c \in \mathscr{C}$.

Now consider any i, j in S such that $p_{ij} > 0$. Then there exists $\sigma \geq 1$ such that i, j are interior points of $B(\boldsymbol{O}, n_\sigma)$ and $(i, j) \in \text{Arcset}\ {}^\sigma\mathscr{C}$. Hence

$$^{n_\sigma}w(i, j) \equiv {}^{n_\sigma}\pi_i \ {}^{n_\sigma}p_{ij} = \sum_{r=1}^{k_\sigma} {}^{n_\sigma}w_{c_r}J_{c_r}(i, j),$$

and

$$^{n_s}w(i, j) = \sum_{r=1}^{k_\sigma} {}^{n_s}w_{c_r}J_{c_r}(i, j), \quad \text{for all } s \geq \sigma.$$

Finally, we have

$$\begin{aligned}\pi_i p_{ij} &= \lim_{s\to\infty} {}^{n_s}\pi_i {}^{n_s}p_{ij}\\ &= \lim_{s\to\infty} \sum_{r=1}^{k_\sigma} {}^{n_s}w_{c_r} J_{c_r}(i,j)\\ &= \sum_{r=1}^{k_\sigma} w_{c_r} J_{c_r}(i,j)\\ &= \sum_{c\in\mathscr{C}} w_c J_c(i,j).\end{aligned}$$

The proof is complete. □

Remark. As was shown in the previous proof, there is a definite algebraic-topological property of a directed circuit $c = (i_1, \ldots, i_s, i_1)$ defined by $w(i,j) = \pi_i p_{ij}, i, j \in S$. Namely, we have

Lemma 3.4.3. *Let f_1 and f_2 be two positive functions defined on S^2. In order that equations*

$$\sum_j f_1(i,j) = \sum_j f_2(j,i), \quad i \in S,$$

be circuit-generating ones it is necessary that for some $i_1, \ldots, i_s \in S$ the inequalities

$$\begin{aligned}f_1(i_1,i_2)f_1(i_2,i_3)\cdots f_1(i_{s-1},i_s)f_1(i_s,i_1) > 0,\\ f_2(i_1,i_2)f_2(i_2,i_3)\cdots f_2(i_{s-1},i_s)f_2(i_s,i_1) > 0,\end{aligned}$$

imply each other.

3.5 Weak Convergence of Sequences of Circuit Chains: A Probabilistic Approach

A denumerable reversible positive-recurrent Markov chain is a weak limit of finite circuit Markov chains whose representative circuits and weights are algorithmically given according to Theorem 3.4.2. It might be interesting to investigate the same asymptotics when the representatives enjoy probabilistic interpretations. For instance, we may consider that the cycle-weights are provided by the probabilistic algorithm of Theorem 3.3.1. In this section we give a more detailed argument following S. Kalpazidou (1992a, b, e) and Y. Derriennic (1993).

Consider S a denumerable set and $\xi = (\xi_n)_{n\geq 0}$ an irreducible and positive-recurrent Markov chain (not necessarily reversible) whose transition matrix and invariant probability distribution are, respectively,

$P = (p_{ij}, i, j \in S)$ and $\pi = (\pi_i, i \in S)$. Let $(\xi_m(\omega))_{m \geq 0}$ be a sample path of ξ and let n be any positive integer chosen to be a sufficiently great number. Put

$$\begin{aligned} \mathscr{C}_n(\omega) &= \text{the collection of all circuits with distinct points} \\ &\quad\ \text{(except for the terminals) occurring along } (\xi_m(\omega))_m \\ &\quad\ \text{until time } n; \\ S_n(\omega) &= \text{the set of the points of } \mathscr{C}_n(\omega). \end{aligned}$$

Throughout this section the circuits will be considered to have distinct points (except for the terminals).

Consider

$$\begin{aligned} w_{c,n}(\omega) = &\text{ the number of occurrences of the circuit } c \text{ along} \\ &(\xi_m(\omega))_m \text{ up to time } n, \end{aligned}$$

and the functions

$$w_n(i,j) = {}_\omega w_n(i,j) \equiv \sum_{c \in \mathscr{C}_n(\omega)} (w_{c,n}(\omega)/n) J_c(i,j),$$

$$w_n(i) = {}_\omega w_n(i) \equiv \sum_{c \in \mathscr{C}_n(\omega)} (w_{c,n}(\omega)/n) J_c(i),$$

for all $i, j \in S_n = S_n(\omega)$. Since the constrained passage-function $J_c(\cdot,\cdot)$, with $c \in \mathscr{C}_n(\omega)$, to the set S_n is still balanced, the function $w_n(\cdot,\cdot)$ does as well. Therefore the collection $\{w_n(i), i \in S_n\}$ plays the rôle of an invariant measure for the stochastic matrix ${}^nP = {}^nP_\omega \equiv ({}_\omega w_n(i,j)/{}_\omega w_n(i), i, j \in S_n), n = 1, 2, \ldots$.

Accordingly, we may consider a sequence $({}^n\xi)_n$ of Markov chains ${}^n\xi = {}^n_\omega\xi = \{{}^n_\omega\xi_m, m = 1, 2, \ldots\}$ whose transition probabilities in S_n are defined as

$${}^np_{ij} = {}^n_\omega p_{ij} \equiv \begin{cases} ({}_\omega w_n(i,j))/({}_\omega w_n(i)), & \text{if } (i,j) \text{ is an edge of a circuit in } \mathscr{C}_n(\omega); \\ 0, & \text{otherwise.} \end{cases}$$

Put

$${}^n\pi_i = {}^n_\omega\pi_i = c_n(\omega)\, {}_\omega w_n(i), \qquad i \in S_n,$$

where $c_n(\omega) = 1/(\sum_i {}_\omega w_n(i))$.

It is to be noticed that, since

$${}^n\pi_i {}^np_{ij} = c_n(\omega)_\omega\ w_n(i,j) \neq (\pi_i p_{ij})/\left(\sum_{i,j \in S_n} \pi_i p_{ij}\right),$$

the above chain ${}^n\xi, n = 1, 2, \ldots$, is not the restriction of ξ to S_n. So, the investigations up to this point disclose differences between the weak convergence of $({}^n_\omega\xi)$, as $n \to \infty$, and that of deterministic circuit representations

occurring in Theorem 3.4.2. It is the following theorem that shows a special nature of the weak convergence of $({}^n_\omega\xi)$ to ξ, as $n \to \infty$ (S, Kalpazidou (1992e)).
Namely, we have

Theorem 3.5.1. *For almost all ω the sequence $({}^n_\omega\xi)_n$ converges weakly, as $n \to \infty$, to the chain ξ. Moreover the sequence of the circuit representations associated with $({}^n_\omega\xi)_n$ converges, as $n \to \infty$, to the probabilistic circuit representation $(\mathscr{C}, w_c)$ of ξ, where $\mathscr{C}$ is the collection of the directed circuits occurring along almost all the sample paths.*

Proof. First note that we can regard the process ${}^n_\omega\xi$ in S_n as a circuit chain with respect to the collection $(\mathscr{C}_n(\omega), w_{c,n}(\omega)/n)$. Accordingly, we have

$$^n\pi_i{}^n p_{ij} = {}^n_\omega\pi_i\ {}^n_\omega p_{ij} = c_n(\omega) \sum_{c\in\mathscr{C}_n(\omega)} (w_{c,n}(\omega)/n) J_c(i,j),$$

when (i,j) is an edge of a circuit of $\mathscr{C}_n(\omega)$, where $J_c(i,j) = 1$ or 0 according to whether or not (i,j) is an edge of c. Then, as in Theorem 3. 2.1 we may find a limiting class $(\mathscr{C}, w_c)$ defined as

$$\begin{aligned} \mathscr{C} &= \lim_{n\to\infty} \mathscr{C}_n(\omega), \quad \text{a.s.,} \\ w_c &= \lim_{n\to\infty} (w_{c,n}(\omega)/n), \quad \text{a.s.} \end{aligned}$$

The equations (3.3.2) and the same argument of Theorem 3.3.1 enables us to write

$$\begin{aligned} \pi_i p_{ij} &= \lim_{n\to\infty} {}^n_\omega\pi_i\,{}^n_\omega p_{ij} \quad \text{a.s.} \\ &= \sum_{c\in\mathscr{C}} w_c J_c(i,j), \end{aligned}$$

since $\lim_{n\to\infty} c_n(\omega) = 1$ a.s.

(Here we have replaced the index-set $\mathscr{C}_\infty$, which contains all the cycles, in Definition 3.2.2 of the circulation distribution by the set $\mathscr{C}$ of the corresponding circuits.) This completes the proof. □

3.6 The Induced Circuit Chain

Y. Derriennic (1993) has defined the denumerable circuit Markov chains as limits of weakly convergent sequences of induced chains. In particular, it is seen that the induced chain of a circuit chain is a new type of "circuit chain."

To this direction, let S be any denumerable set and let $\xi = (\xi_n)_n$ be an S-state irreducible and positive-recurrent Markov chain defined on a probability space $(\Omega, \mathscr{F}, P)$. For a given nonvoid subset A of S, the *induced chain* of ξ on the set A, denoted by ${}_A\xi$, is the Markov chain whose transition probabilities ${}_A p_{ij}, i, j \in A$, are defined as follows:

$$
\begin{aligned}
{}_A p_{ij} &= P(\xi \text{ enters first } A \text{ at state } j, \text{ if } \xi \text{ starts at } i) \\
&= \sum_{n=1}^{\infty} \left(\sum_{j_i,\ldots,j_{n-1} \in S\setminus A} p_{ij_1}\, p_{j_1 j_2} \cdots p_{j_{n-1} j} \right).
\end{aligned}
$$

Therefore, the *induced transition probability* ${}_A p_{ij}, i, j \in A$, is the expected number of times that the Markov chain ξ is in the state j before being in the set S/A, given that ξ starts from the state i:

$$
{}_A p_{ij} = \sum_{n=0}^{\infty} {}_A p_{ij}^{(n)}, \qquad i, j \in A, \tag{3.6.1}
$$

where ${}_A p_{ij}^{(0)} \equiv 0$, and ${}_A p_{ij}^{(n)}, n = 1, 2, \ldots$, is the n-step transition probability with taboo set of states A, that is,

$$
{}_A p_{ij}^{(n)} = P(\xi_n(\omega) = j, \xi_{n-1}(\omega) \notin A,\ \xi_{n-2}(\omega) \notin A, \ldots, \xi_1(\omega) \notin A / \xi_0(\omega) = i).
$$

We have

Proposition 3.6.1. *If* $\xi = (\xi_n)_n$ *is a positive-recurrent Markov chain then* ${}_A P = ({}_A p_{ij}, i, j \in A)$ *is a stochastic matrix.*

Proof. Following Chung's Theorem 3 (1967, p. 45) when $j \in A$ we have

$$
{}_A p_{ij}^{(n)} \le {}_j p_{ij}^{(n)} = f_{ij}^{(n)} \equiv P(\xi_n(\omega) = j, \xi_{n-1} \ne j, \ldots, \xi_1 \ne j | \xi_0(\omega) = i) \le 1.
$$

Hence ${}_A p_{ij} \le {}_j p_{ij} = f_{ij} \le 1$, where $f_{ij} \equiv \sum_{n \ge 1} f_{ij}^{(n)}, \quad i, j \in A.$

Also, if ξ is positive-recurrent then $f_{ii} = \sum_{n \ge 1} f_{ii}^{(n)} = 1$, for any $i \in A$. Therefore

$$
\begin{aligned}
\sum_{j \in A} {}_A p_{ij} &= \sum_{j \in A} P(\xi \text{ enters first } A \text{ at state } j / \xi_0(\omega) = i) \\
&= \sum_{n \ge 1} P(\xi \text{ enters first } A \text{ at time } n / \xi_0(\omega) = i) \\
&= P\left(\bigcup_{n \ge 1} \{\xi_n \in A\} / \xi_0(\omega) = i \right) \\
&\ge P\left(\bigcup_{n \ge 1} \{\xi_n = i\} / \xi_0(\omega) = i \right) = f_{ii} \equiv 1.
\end{aligned}
$$

The proof is complete. □

Furthermore, we prove

Proposition 3.6.2. *If ξ is an irreducible and positive-recurrent Markov chain, then ${}_A\xi$ is irreducible.*

Proof. We first write

$$ {}_Ap_{ij} = p_{ij} + \sum_{n\geq 2} \sum_{j_1,\ldots,j_{n-1}\in S\setminus A} p_{ij_1}\ p_{j_1j_2}\ \ldots p_{j_{n-1}j}, \tag{3.6.2}$$

for any $i, j \in A$. Then

$$ {}_Ap_{ij} > 0 \tag{3.6.3}$$

if either $p_{ij} > 0$, or there are $j_1, \ldots, j_{n-1} \in S\setminus A, n \geq 2$, such that

$$ p_{ij_1}\ p_{j_1j_2}\ \ldots p_{j_{n-1}j} > 0. $$

To prove that the induced Markov chain ${}_A\xi$ is irreducible, we have to show that for any pair $(i, j) \in A \times A$ either

(i)

$$ {}_Ap_{ij} > 0, \tag{3.6.4}$$

(ii) or, there exist $k_1, \ldots, k_m \in A$, $m \geq 1$, such that

$$ {}_Ap_{ik_1}\ {}_Ap_{k_1k_2}\ \ldots {}_Ap_{k_mj} > 0. $$

So, let us consider an arbitrary pair (i, j) of states in A. Then irreducibility of ξ allows us to write that either $p_{ij} > 0$, or, there exist $k_1, \ldots, k_m \in S, m \geq 1$, such that

$$ p_{ik_1}\ p_{k_1k_2}\ \ldots p_{k_mj} > 0. \tag{3.6.5}$$

If $p_{ij} > 0$ then ${}_Ap_{ij} > 0$. Otherwise we may distinguish the following cases:

Case 1: Relations (3.6.5) hold with all $k_1, \ldots, k_m \in A$. Then, according to (3.6.2), we have

$$ {}_Ap_{ik_1}\ {}_Ap_{k_1k_2}\ \cdots\ {}_Ap_{k_mj} > 0, $$

and therefore relation (3.6.4)(ii) holds.
Case 2: Relations (3.6.5) hold with all $k_1, \ldots, k_m \in S\setminus A$. Then by using (3.6.3), we have

$$ \sum_{j_i,\ldots,j_{n-1}\in S\setminus A} p_{ij_1}\ p_{j_1j_2}\ \ldots p_{j_{n-1}j} > 0, $$

with $n = m + 1$ and for $j_1 = k_1, \ldots, j_{n-1} = k_m$. Accordingly,

$$ {}_Ap_{ij} = \sum_{n\geq 1} \sum_{j_i,\ldots,j_{n-1}\in S\setminus A} p_{ij_1}\ p_{j_1j_2}\ \ldots p_{j_{n-1}j} > 0, $$

and relation (3.6.4)(i) holds.

Case 3: Relations (3.6.5) hold with some $k_t \in S \backslash A$ and some others $m_k \in A$.
For the sake of simplicity, let us consider all $k_1, \ldots, k_m \in S \backslash A$ except for some $k_t \in A$, $1 \leq t \leq m$. Then $i, k_t, j \in A$ and we may apply case 1 to the pairs $(i, k_t), (k_t, j)$ of states in A. Hence

$$_A p_{ik_t} \ _A p_{k_t j} > 0,$$

and relation (3.6.4)(ii) holds for $m = 2$.

Next, if $k_1, \ldots, k_m \in S \backslash A$ except for some $k_t, k_{t+s} \in A$, $1 \leq t, t+s \leq m$, then we may apply again case 1 to the pairs $(i, k_t), (k_t, k_{t+s})$, and (k_{t+s}, j) of states in A. Accordingly, we get

$$_A p_{ik_t} \ _A p_{k_t k_{t+s}} \ _A p_{k_{t+s} j} > 0,$$

and relation (3.6.4)(ii) holds for $m = 3$. Finally, case 3 may be extended for general situations $m \geq 3$ by repeating the previous reasonings. Then, we conclude that the irreducibility of the original chain ξ implies the same property for $_A\xi$. The proof is complete. □

Now we are prepared to prove the following:

Theorem 3.6.3. *Let S be a denumerable set and let $\xi = (\xi_n)_{n \geq 0}$ be an S-state irreducible and positive-recurrent Markov chain. Then there exists a sequence $(^n\eta)_n$ of finite induced circuit chains, which converges weakly to ξ, as $n \to \infty$.*

Proof. Let $(A_n)_n$ be an increasing sequence of finite subsets of S such that $\lim A_n = S$, as $n \to \infty$. Then $^1\eta$, $^2\eta, \ldots, {}^n\eta, \ldots$ are taken to be the induced chain of ξ on $A_1, A_2, \ldots, A_n, \ldots$, that is, $^n\eta \equiv {}_{A_n}\xi, n = 1, 2, \ldots$. Then, following Propositions 3.6.1 and 3.6.2, any induced chain $^n\eta, n = 1, 2, \ldots$, is an irreducible finite Markov chain. Therefore, the induced transition probability $_{A_n}p_{ij}$ of any $^n\eta$ accepts a circuit representation $\{C_n, w_{c_n}\}$, that is,

$$_{A_n}p_{ij} = \frac{\sum\limits_{c_n \in C_n} w_{c_n} J_{c_n}(i,j)}{\sum\limits_{c_n \in C_n} w_{c_n} J_{c_n}(i)}, \quad i, j \in A_n, n = 1, 2, \ldots,$$

where $J_c(i,j) = 1$ or 0 according to whether or not (i,j) is an edge of c, and $(^n\eta)_n$ converges weakly to ξ. The proof is complete. □

Further, it will be interesting to define a natural procedure of inducing a circuit representation $\{C_A, w_A\}$ for the induced chain $_A\xi$ on the finite subset $A \subset S$, starting from an original circuit representation C of ξ.

Note that, $_Ap_{ij} > 0$ if and only if $_Ap_{ij}^{(n)} > 0$, for certain $n = 1, 2, \ldots$. Then a natural procedure of inducing the circuits of C into A is due to Derriennic

(1993) and consists in the following: any circuit $c = (i_1, i_2, \ldots, i_s, i_1) \in C$, which contains at least one point in A may induce a circuit c_A in A as the track of the remaining points of c in A, written with the same order and cyclically, after discarding the points of c which do not belong to A. In this manner the representative collection C of directed circuits in S determines a finite collection $C_A = \{c_1, c_2, \ldots, c_N\}$ of induced circuits into the finite subset $A \subset S$.

Furthermore, by choosing suitably the circuits in C, we may use the induced circuits $c_1, \ldots, c_N$ of C_A to partition the original collection C into the subcollections $C_0, C_1, \ldots, C_N$ defined as

$$C_k = \{c \in C\colon c \text{ induces the circuit } c_k \text{ in } A\}, \quad k = 1, \ldots, N,$$
$$C_0 = \{c \in C\colon c \text{ induces no circuit in } A\},$$

such that no circuit of C_0 passes through A.
Then

$$C = C_0 \bigcup \left(\bigcup_{k=1}^{N} C_k \right). \tag{3.6.6}$$

Let us now consider a collection of circuit-weights $\{w_c\}$ which decomposes ξ, that is,

$$P(\xi_n = i,\ \xi_{n+1} = j) = \sum_{c \in C} w_c\ J_c(i,j), \qquad i, j \in S, \tag{3.6.7}$$

for any $n = 0, 1, \ldots$
Then we may decompose the induced transition probability ${}_Ap_{ij}$ by using the induced circuits of C_A. Specifically, we may write

$$\begin{aligned} {}_Ap_{ij} &= p_{ij} + {}_Ap_{ij}^{(2)} + \cdots + {}_Ap_{ij}^{(n)} + \cdots \\ &= \frac{P\{\xi_1 = j, \xi_0 = i\}}{P\{\xi_0 = i\}} \\ &\quad + \frac{P\{\xi_2 = j, \xi_1 \in S \backslash A, \xi_0 = i\}}{P\{\xi_0 = i\}} + \cdots \\ &\quad + \frac{P\{\xi_n = j, \xi_{n-1} \in S \backslash A, \ldots, \xi_1 \in S \backslash A, \xi_0 = i\}}{P\{\xi_0 = i\}} + \cdots \end{aligned}$$

The denumerator $P(\xi_0 = i), i \in A$, occurring in the expression of ${}_Ap_{ij}$ is decomposed by the representative class C_A as follows:

$$\begin{aligned} P(\xi_0 = i) &= \sum_{c \in C} w_c J_c(i) = \sum_{k=1}^{N} \sum_{c \in C_k} w_c J_c(i) \\ &= \sum_{k=1}^{N} (\sum_{c \in C_k} w_c) J_{c_k}(i), \end{aligned}$$

where $\{w_c, c \in C\}$ are the weights occurring in 3.6.7. Then by defining the "induced" circuit-weights $\nu_{c_A}, c_A \in C_A$, as

$$\nu_{c_k} \equiv \sum_{c \in C_k} w_c, \qquad k = 1, \dots, N,$$

we have

$$P(\xi_0 = i) = \sum_{k=1}^{N} \nu_{c_k}, J_{c_k}(i), \qquad i \in A.$$

Let us now calculate the numerator of ${}_A p_{ij}, i, j \in A$, in terms of C_A:

$$\begin{aligned} P(\xi_1 = j, \xi_0 = i) + \sum_{j_1 \in S \setminus A} & P(\xi_2 = j, \xi_1 = j_1, \xi_0 = i) \\ & + \sum_{j_1 j_2 \in S \setminus A} P(\xi_3 = j, \xi_2 = j_2, \xi_1 = j_1, \xi_0 = i) \\ & + \cdots + \sum_{j_1, \dots, j_{n-1} \in S \setminus A} P(\xi_n = j, \xi_{n-1} = j_{n-1}, \dots, \xi_1 = j_1, \xi_0 = i) \\ & + \cdots \end{aligned}$$

We have

$$\begin{aligned} P(\xi_0 = i,\ \xi_1 = j) &= \sum_{c \in C} w_c J_c(i, j) \\ &= \sum_{k=1}^{N} \left(\sum_{c \in C_k} w_c J_c\ (i, j) \right) J_{c_k}(i, j) \\ &= \sum_{k=1}^{N} {}^1\nu_{c_k}(i, j) J_{c_k}(i, j), \end{aligned}$$

for any $i, j \in A$, where

$${}^1\nu_{c_k}(i, j) = \sum_{c \in C_k} w_c J_c(i, j), \qquad i, j \in A.$$

Let

$${}^2w(i, j) \equiv \sum_{j_1 \in S \setminus A} P(\xi_2 = j, \xi_1 = j_1, \xi_0 = i), \qquad i, j \in A.$$

Then, if ξ is reversible then ${}^2w(i, j)$ is symmetric. Also, ${}^2w(i, j) > 0$ implies ${}_A p_{ij} > 0$. Accordingly, the representative circuits of ${}^2w$ will belong to C_A, and we may find ${}^2\nu_{c_k} \geq 0, k = 1, \dots, N$, such that

$${}^2w(i, j) = \sum_{k=1}^{N} {}^2\nu_{c_k}\ J_{c_k}(i, j), \qquad i, j \in A.$$

By repeating the same reasoning for any

$$^n w(i,j) \equiv \sum_{j_1,\ldots,j_{n-1} \in S \setminus A} P(\xi_n = j, \xi_{n-1} = i_{n-1}, \ldots, \xi_1 = i_1, \xi_0 = i_0),$$

where $i, j \in A$, $n = 3, 4, \ldots$ we may find ${}^n\nu_{c_k} \geq 0, k = 1, \ldots, N$, such that

$$^n w(i,j) = \sum_{k=1}^{N} {}^n\nu_{c_k} \; J_{c_k}(i,j), \qquad i, j \in A.$$

Then the numerator of ${}_A p_{ij}$ is given by

$$\sum_{k=1}^{N} \nu_{c_k}(i,j) \; J_{c_k}(i,j)$$

where

$$\nu_{c_k}(i,j) = {}^1\nu_{c_k}(i,j) + \tilde{\nu}_{c_k}$$

with

$$\tilde{\nu}_{c_k} = \sum_{n \geq 2} {}^n\nu_{c_k}, \qquad k = 1, \ldots, N.$$

Therefore,

$${}_A p_{ij} = \frac{\sum_{k=1}^{N} \nu_{c_k}(i,j) J_{c_k}(i,j)}{\sum_{k=1}^{N} \nu_{c_k} J_{c_k}(i)}, \qquad i, j \in A. \tag{3.6.8}$$

In conclusion, when the positive-recurrent chain ξ is irreducible then the induced chain ${}_A\xi$ is also irreducible with respect to the invariant probability distribution ${}_A\pi = ({}_A\pi_i, i \in A)$ defined as

$${}_A\pi_i = \frac{\sum\limits_{c_A \in C_A} \nu_{c_A} \; J_{c_A}(i)}{\sum\limits_{c_A \in C_A} p(c_A) \nu_{c_A}}, \qquad i \in A,$$

where $p(c_A)$ denotes as always the period of the circuit c_A in A.

Finally, if ξ is reversible then ${}_A\xi$ is also reversible and the corresponding induced transition probability ${}_A p_{ij}$ admits a "circuit representation" given by (3.6.8).