On approximate mixed Nash equilibria and average marginal functions for two-stage three-players games

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Summary. In this paper we consider a two-stage three-players game: in the first stage one of the players chooses an optimal strategy knowing that, at the second stage, the other two players react by playing a noncooperative game which may admit more than one Nash equilibrium. We investigate continuity properties of the set-valued function defined by the Nash equilibria of the (second stage) two players game and of the marginal functions associated to the first stage optimization problem. By using suitable approximations of the mixed extension of the Nash equilibrium problem, we obtain without convexity assumption the lower semicontinuity of the set-valued function defined by the considered approximate Nash equilibria and the continuity of the associate approximate average marginal functions when the second stage corresponds to a particular class of noncooperative games called antipotential games.

Key Words: mixed strategy, Radon probability measure, ε-approximate Nash equilibrium, marginal functions, noncooperative games, two-stage three-players game, antipotential game.

1 Introduction

Let X, Y_1, Y_2 be compact subsets of metric spaces and f_1, f_2 be two real valued functions defined on $X \times Y_1 \times Y_2$. Consider the parametric noncooperative two players game $\Gamma(x) = \{Y_1, Y_2, f_1(x,\cdot,\cdot), f_2(x,\cdot,\cdot)\}$ where $x \in X$ and f_1, f_2 are the payoff functions of players P_1 and P_2. Any player is assumed to minimize his own payoff function called cost function. For all $x \in X$, we denote by $N(x)$ the set of the Nash equilibria ([19]) of the game $\Gamma(x)$, i.e. the set of the solutions to the following problem $\mathcal{N}(x)$

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$$\begin{cases} \text{find } (\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2 \text{ such that} \\ f_1(x, \bar{y}_1, \bar{y}_2) = \inf_{y_1 \in Y_1} f_1(x, y_1, \bar{y}_2) \\ f_2(x, \bar{y}_1, \bar{y}_2) = \inf_{y_2 \in Y_2} f_2(x, \bar{y}_1, y_2). \end{cases}$$

When the set $N(x)$ has more than one element for at least one $x \in X$, one can investigate some continuity properties of the set-valued function defined by the set $N(x)$ for all $x \in X$. These properties could be useful, from theoretical and numerical point of view, in problems involving the so-called *marginal functions*. More precisely, let l be a real valued function defined on $X \times Y_1 \times Y_2$. For any $x \in X$, one can consider the following functions associated to optimization problems in which the constraints describe the set of Nash equilibria of the game $\Gamma(x)$:

$$w(x) = \sup_{(y_1, y_2) \in N(x)} l(x, y_1, y_2)$$

$$u(x) = \inf_{(y_1, y_2) \in N(x)} l(x, y_1, y_2)$$

These marginal functions, called respectively *sup-marginal function* and *inf-marginal function*, are concerned in many applicative situations as illustrated in the following.

• The multi-stage problem involving the marginal function $w(x)$

$$\begin{cases} \text{find } \bar{x} \in X \text{ such that} \\ \\ \inf_{x \in X} \sup_{(y_1, y_2) \in N(x)} l(x, y_1, y_2) = \sup_{(y_1, y_2) \in N(\bar{x})} l(\bar{x}, y_1, y_2) = w(\bar{x}) \end{cases} \tag{1}$$

corresponds to a two-stage game with the three players P_0, P_1, P_2 and l cost function of P_0. In the first stage, player P_0 (called the leader) chooses an optimal strategy knowing that, at the second stage, two players P_1 and P_2 (called the followers) react by playing a non cooperative game. When there exists more than one Nash equilibrium at the second stage for at least one strategy of P_0, if it is assumed that the leader cannot influence the choice of the followers, then the followers can react to a leader's strategy by choosing a Nash equilibrium which can hurt him as much as possible. Therefore, P_0 will choose a security strategy which minimizes the worst, assuming that he has no motivation to restrict his worst case design to a particular subset of the Nash equilibria. The hierarchical problem (1) is called "Weak Hierarchical Nash Equilibrium Problem", in line with the terminology used in previous papers on hierarchical problems (see, for example, [5], [11]). Economic examples of such games can be found in [21], [22], [14] where the supply side of an oligopolistic market supplying a homogeneous product non cooperatively is modelled and in [20], where in a two country imperfect competition model the firms face three different types of decisions. In the setting of transportation and telecommunications see, for example, [15] and [1].

- The multi-stage problem involving the marginal function $u(x)$

$$\begin{cases} \text{find } \bar{x} \in X \text{ such that} \\ \\ \inf\limits_{x \in X} \inf\limits_{(y_1,y_2) \in N(x)} l(x, y_1, y_2) = \inf\limits_{(y_1,y_2) \in N(\bar{x})} l(\bar{x}, y_1, y_2) = u(\bar{x}) \end{cases} \tag{2}$$

corresponds to a two-stage three-players game when there exists again more than one Nash equilibrium at the second stage for at least one strategy of P_0, and it is assumed now that the leader can force the choice of the followers to choose the Nash equilibrium that is the best for him. The hierarchical problem (2) is called "Strong Hierarchical Nash Equilibrium Problem" in line with the terminology used in previous papers on hierarchical problems ([5], [11]). It is also known as a mathematical programming problem with equilibrium constraints (MPEC) in line with the terminology used in [12], [6], [7], where one can find applications and references.

Remember that, if l is a continuous real valued function defined on $X \times Y_1 \times Y_2$ and N is a sequentially lower semicontinuous and sequentially closed graph set-valued function on X, then the marginal functions w and u are continuous on X ([9]). A set-valued function T is said to be sequentially lower semicontinuous at $x \in X$ if for any sequence $(x_n)_n$ converging to x in X and for any $y \in T(x)$, there exists a sequence (y_n) converging to y in Y such that $y_n \in T(x_n)$ for n sufficiently large (see, for example, [2], [9]). The set-valued function T is said to be sequentially closed graph at $x \in X$ if for any sequence $(x_n)_n$ converging to x in X and for any sequence $(y_n)_n$ converging to y in Y such that $y_{n_k} \in T(x_{n_k})$ for a selection of integers $(n_k)_k$, we have $y \in T(x)$ (see, for example, [2], [9]). For simplicity in the following the word "sequentially" will be omitted.

Unfortunately, the set-valued function N can be non lower semicontinuous even when smooth data are present (see, for example, [19], [17]). So, in [17] a suitable approximate Nash equilibrium concept has been introduced which guarantees lower semicontinuity results under some convexity assumption on the cost functions. When these convexity assumptions are not satisfied, as in the case of zero-sum games previously investigated by the authors ([13]), one can consider mixed strategies for P_1 and P_2 and the mixed extension of the parametric Nash equilibrium problem $\mathcal{N}(x)$. More precisely, let $\overline{M}(Y_1)$, $\overline{M}(Y_2)$ be the sets of Radon probability measures on Y_1 and Y_2 ([4], [23]) and assume that the cost functions of P_1 and P_2, respectively $f_1(x,\cdot,\cdot)$, $f_2(x,\cdot,\cdot)$, are continuous functions on $Y_1 \times Y_2$ for all $x \in X$. The *average cost functions* of players P_1 and P_2 are defined by (see, for example, [3]):

$$\hat{f}_i(x, \mu_1, \mu_2) = \int_{Y_1} \int_{Y_2} f_i(x, y_1, y_2)\ d\mu_1(y_1)\ d\mu_2(y_2)$$

for $i = 1, 2$. We denote by $\hat{N}(x)$ the set of Nash equilibria of the extended game defined by $\hat{\Gamma}(x) = \{\overline{M}(Y_1), \overline{M}(Y_2), \hat{f}_1(x,\cdot,\cdot)\hat{f}_2(x,\cdot,\cdot)\}$, i.e. the set of mixed Nash equilibria of the game $\Gamma(x)$. Assuming that $l(x,\cdot,\cdot)$ is a continuous function on $Y_1 \times Y_2$, for all $x \in X$, one can consider the average cost function for P_0 defined by

$$\hat{l}(x,\mu_1,\mu_2)=\int_{Y_1}\int_{Y_2} l(x,y_1,y_2)\ d\mu_1(y_1)\ d\mu_2(y_2).$$

The following real functions defined on X by

$$\hat{w}(x) = \sup_{(\mu_1,\mu_2)\in\hat{N}(x)} \hat{l}(x,\mu_1,\mu_2)$$

$$\hat{u}(x) = \inf_{(\mu_1,\mu_2)\in\hat{N}(x)} \hat{l}(x,\mu_1,\mu_2)$$

will be called respectively *sup-average marginal function* and *inf-average marginal function*. Having in mind to obtain the continuity of the average marginal functions, now we look for the lower semicontinuity of the set-valued function defined, for all $x \in X$, by the set $\hat{N}(x)$ of Nash equilibria of the game $\hat{\Gamma}(x)$ (i.e. mixed Nash equilibria of the game $\Gamma(x)$).

Unfortunately, the following example deals with a game where the set-valued function defined by $\hat{N}(x)$ is not lower semicontinuous on X.

Example 1.1 Let $X = [0,1]$ be the set of parameters and $Y_1 = \{\alpha_1, \beta_1\}$, $Y_2 = \{\alpha_2, \beta_2\}$ be the strategy sets of P_1, P_2 respectively. For any $x \in X$, we have the following bimatrix game:

	α_2	β_2
α_1	$-1,x$	$0,2x$
β_1	$0,-1$	$0,x$

Then:

$$N(x) = \begin{cases} \{(\alpha_1,\alpha_2),(\alpha_1,\beta_2)\} & \text{if } x = 0 \\ \{(\alpha_1,\alpha_2)\} & \text{if } x \neq 0. \end{cases}$$

Here $\overline{M}(Y_i)$ is the set of the discrete probability measures on Y_i $(i = 1,2)$. The extended cost functions are:

$$\hat{f}_1(x,\mu_1,\mu_2)= - pq$$

$$\hat{f}_2(x,\mu_1,\mu_2)=xp - xq + x - q + pq$$

where $\mu_1 = p\delta(\alpha_1)+(1-p)\delta(\beta_1) \in \overline{M}(Y_1)$, $\mu_2 = q\delta(\alpha_2)+(1-q)\delta(\beta_2) \in \overline{M}(Y_2)$ and $p, q \in [0,1]$. δ is the Dirac measure and μ_1 means that the strategy α_1 is chosen with probability p and the strategy β_1 is chosen with probability $1-p$, for $p \in [0,1]$.
In this case we have:

$$\hat{N}(x) = \begin{cases} \{(1,q),\ q \in [0,1]\} & \text{if } x = 0 \\ \{(1,1)\} & \text{if } x \neq 0 \end{cases}$$

which is not a lower semicontinuous set-valued function at $x = 0$.

However, by considering suitable approximations of the mixed extension of the Nash equilibrium problem, we will prove, without any convexity assumption, that the set-valued function defined by the considered approximate Nash equilibria is lower semicontinuous and that the corresponding approximate average marginal functions are continuous functions. More precisely, in Section 2 we introduce two concepts of approximate Nash equilibria for the extended game $\hat{\Gamma}(x)$ and we investigate the properties of lower semicontinuity and closedness of the set-valued functions defined by these approximate Nash equilibria. In Section 3 continuity of the associate approximate average marginal functions is obtained.

2 ε-approximate Nash equilibria

In line with the approximate solution concept introduced in [10] and in [17], we introduce a concept of approximate mixed Nash equilibrium:

Definition 2.1 *Let $x \in X$ and $\varepsilon > 0$; a strict ε-approximate mixed Nash equilibrium is a solution to the problem $\breve{\mathcal{N}}(x,\varepsilon)$:*

$$\begin{cases} \text{find } (\bar{\mu}_1, \bar{\mu}_2) \in \overline{M}(Y_1) \times \overline{M}(Y_2) \text{ s.t.} \\ \hat{f}_1(x, \bar{\mu}_1, \bar{\mu}_2) + \hat{f}_2(x, \bar{\mu}_1, \bar{\mu}_2) \\ \qquad < \inf_{\mu_1 \in \overline{M}(Y_1)} \hat{f}_1(x, \mu_1, \bar{\mu}_2) + \inf_{\mu_2 \in \overline{M}(Y_2)} \hat{f}_2(x, \bar{\mu}_1, \mu_2) + \varepsilon \end{cases}$$

The set of solutions to the problem $\breve{\mathcal{N}}(x,\varepsilon)$ will be denoted by $\breve{N}(x,\varepsilon)$.

Remark 2.1 *For all $x \in X$, the set of the strict ε-approximate mixed Nash equilibria $\breve{N}(x,\varepsilon)$ is not empty, differently from the set of the strict ε-approximate Nash equilibria $\tilde{N}(x,\varepsilon)$ ([18]) defined by*

$$\tilde{N}(x,\varepsilon) = \{(\bar{y}_1, \bar{y}_2) \in Y_1 \times Y_2 :\ f_1(x, \bar{y}_1, \bar{y}_2) + f_2(x, \bar{y}_1, \bar{y}_2) <$$

$$\inf_{y_1 \in Y_1} f_1(x, y_1, \bar{y}_2) + \inf_{y_2 \in Y_2} f_2(x, \bar{y}_1, y_2) + \varepsilon\}$$

which can be empty. In fact, in the matching pennies example $\tilde{N}(x,\varepsilon) = \emptyset$ but $\breve{N}(x,\varepsilon)$ is an open nonempty square. More precisely, let X be the set of parameters and $Y_1 = \{\alpha_1, \beta_1\}$, $Y_2 = \{\alpha_2, \beta_2\}$ be the strategy sets of P_1, P_2 respectively. For any $x \in X$, we have the following bimatrix game:

	α_2	β_2
α_1	1,−1	−1,1
β_1	−1,1	1,−1

Then for any $x \in X$ we have that $N(x) = \emptyset$ and $\tilde{N}(x,\varepsilon) = \emptyset$ for any $\varepsilon > 0$. If mixed strategies are considered $\overline{M}(Y_i)$ (i=1,2), $\hat{f}_1(x,\mu_1,\mu_2)=4pq - 2p - 2q + 1$, $\hat{f}_2 = -\hat{f}_1$ for $p,q \in [0,1]$ and $\hat{N}(x) = \{(1/2,1/2)\}$, $\breve{N}(x,\varepsilon) = \{(p,q) \in [0,1]^2 \ : \ p \in]1/2 - \varepsilon/2, 1/2+\varepsilon/2[, \ q \in]|p-1/2| + (1-\varepsilon)/2, -|p-1/2| + (1+\varepsilon)/2[\}$.

Obviously, the set-valued function defined by the set of the strict ε-approximate mixed Nash equilibrium of a game is not always closed graph on X. The following theorem gives sufficient conditions for its lower semicontinuity on X and will be used later on.

Theorem 2.1 *Assume that f_1, f_2 are continuous functions on $X \times Y_1 \times Y_2$. Then, for all $\varepsilon > 0$, the set-valued function $\breve{N}(\cdot,\varepsilon)$ is lower semicontinuous on X.*

Proof. We have to prove that for all $x \in X$, for all (x_n) converging to x and for all $(\mu_1,\mu_2) \in \breve{N}(x,\varepsilon)$, there exists a sequence $(\mu_{1,n},\mu_{2,n})$ converging to (μ_1,μ_2) s.t. $(\mu_{1,n},\mu_{2,n}) \in \breve{N}(x_n,\varepsilon)$ for n sufficiently large.

Let (x_n) be a sequence converging to x and $(\bar{\mu}_1,\bar{\mu}_2) \in \breve{N}(x,\varepsilon)$. Then

$$\hat{f}_1(x,\bar{\mu}_1,\bar{\mu}_2)+\hat{f}_2(x,\bar{\mu}_1,\bar{\mu}_2) < \inf_{\mu_1 \in \overline{M}(Y_1)} \hat{f}_1(x,\mu_1,\bar{\mu}_2) + \inf_{\mu_2 \in \overline{M}(Y_2)} \hat{f}_2(x,\bar{\mu}_1,\mu_2) + \varepsilon. \tag{3}$$

Since $\hat{f}_1, \hat{f}_2$ are continuous, for all sequences $(\bar{\mu}_{1,n})$ converging to $\bar{\mu}_1$ and $(\bar{\mu}_{2,n})$ converging to $\bar{\mu}_2$ we have that

$$\lim_{n\to+\infty} \left(\hat{f}_1(x_n,\bar{\mu}_{1,n},\bar{\mu}_{2,n}) + \hat{f}_2(x_n,\bar{\mu}_{1,n},\bar{\mu}_{2,n})\right) = \hat{f}_1(x,\bar{\mu}_1,\bar{\mu}_2) + \hat{f}_2(x,\bar{\mu}_1,\bar{\mu}_2). \tag{4}$$

Since $\overline{M}(Y_1)$, $\overline{M}(Y_2)$ are compact, $\inf_{\mu_1 \in \overline{M}(Y_1)} \hat{f}_1(\cdot,\mu_1,\cdot)$ and $\inf_{\mu_2 \in \overline{M}(Y_2)} \hat{f}_2(\cdot,\cdot,\mu_2)$ are lower semicontinuous functions (Proposition 4.1.1 in [9]). Therefore, in light of (3) and (4)

$$\lim_{n\to+\infty} \left(\hat{f}_1(x_n,\bar{\mu}_{1,n},\bar{\mu}_{2,n}) + \hat{f}_2(x_n,\bar{\mu}_{1,n},\bar{\mu}_{2,n})\right) = \hat{f}_1(x,\bar{\mu}_1,\bar{\mu}_2) + \hat{f}_2(x,\bar{\mu}_1,\bar{\mu}_2) <$$

$$\inf_{\mu_1 \in \overline{M}(Y_1)} \hat{f}_1(x,\mu_1,\bar{\mu}_2) + \inf_{\mu_2 \in \overline{M}(Y_2)} \hat{f}_2(x,\bar{\mu}_1,\mu_2) + \varepsilon \leq$$

$$\lim_{n\to+\infty}\Big(\inf_{\mu_1\in\overline{M}(Y_1)}\hat f_1(x_n,\mu_1,\bar\mu_{2,n})+\inf_{\mu_2\in\overline{M}(Y_2)}\hat f_2(x_n,\bar\mu_{1,n},\mu_2)\Big)+\varepsilon.$$

For n sufficiently large, we can deduce:

$$\begin{aligned}&\hat f_1(x_n,\bar\mu_{1,n},\bar\mu_{2,n})+\hat f_2(x_n,\bar\mu_{1,n},\bar\mu_{2,n})\\ <&\inf_{\mu_1\in\overline{M}(Y_1)}\hat f_1(x_n,\mu_1,\bar\mu_{2,n})+\inf_{\mu_2\in\overline{M}(Y_2)}\hat f_2(x_n,\bar\mu_{1,n},\mu_2)+\varepsilon\end{aligned}$$

that is $(\bar\mu_{1,n},\bar\mu_{2,n})\in\breve N(x_n,\varepsilon)$.

Remark 2.2 *Let us note that Theorem 2.1 can be applied also in the case where $N(x)=\emptyset$ for some $x\in X$. In fact $\emptyset\neq\hat N(x)\subseteq\breve N(x,\varepsilon)$ for all $x\in X$ and $\varepsilon>0$.*

Having in mind to obtain closedness and lower semicontinuity simultaneously, we introduce now a suitable concept of approximate mixed Nash equilibrium.

Definition 2.2 *Let $x\in X$ and $\varepsilon>0$; an ε-approximate mixed Nash equilibrium is a solution to the problem $\hat{\mathcal N}(x,\varepsilon)$:*

$$\begin{cases}\text{find } (\bar\mu_1,\bar\mu_2)\in\overline{M}(Y_1)\times\overline{M}(Y_2)\\ \text{s.t.}\hat f_1(x,\bar\mu_1,\bar\mu_2)+\hat f_2(x,\bar\mu_1,\bar\mu_2)\\ \qquad\qquad\le\inf_{\mu_1\in\overline{M}(Y_1)}\hat f_1(x,\mu_1,\bar\mu_2)+\inf_{\mu_2\in\overline{M}(Y_2)}\hat f_2(x,\bar\mu_1,\mu_2)+\varepsilon\end{cases}$$

The set of solutions to the problem $\hat{\mathcal N}(x,\varepsilon)$ will be denoted by $\hat N(x,\varepsilon)$.

Remark 2.3 *It is easy to see that if f_1,f_2 are continuous functions on $X\times Y_1\times Y_2$, then the set-valued function $\hat N(\cdot,\varepsilon)$ is closed graph at x, for all $x\in X$.*

Example 2.1 In Example 1.1 we have that $\inf_{\mu_1\in\overline{\mathcal M}(Y_1)}\hat f_1(x,\mu_1,\mu_2)=-q$ and that $\inf_{\mu_2\in\overline{\mathcal M}(Y_2)}\hat f_2(x,\mu_1,\mu_2)=xp-1+p$. The set of the ε-approximate mixed Nash equilibria is:
for $x\le\varepsilon$

$$\hat N(x,\varepsilon)=\{(p,q)\in[0,1]^2\ s.\ t.\ p\in[1-\varepsilon+x-xq,1],q\in[0,1]\},$$

for $x>\varepsilon$

$$\hat N(x,\varepsilon)=\{(p,q)\in[0,1]^2\ s.\ t.\ p\in[1-\varepsilon+x-xq,1],q\in[1-(\varepsilon/x),1]\}.$$

Note that the set-valued function $x\in X\mapsto\hat N(x,\varepsilon)$ is closed graph and lower semicontinuous on X.

The bimatrix game in Example 1.1 has a special structure connected with the definition of exact potential games ([16]). Recall that the two players game $\{A, B, K, L\}$, where K, L are real valued functions defined on $A \times B$, is called an *exact potential game* if there is a potential function $P: A \times B \mapsto \mathcal{R}$ such that

$K(a_2, b) - K(a_1, b) = P(a_2, b) - P(a_1, b)$, for all $a_1, a_2 \in A$ and for each $b \in B$

$L(a, b_1) - L(a, b_2) = P(a, b_1) - P(a, b_2)$, for each $a \in A$ and for all $b_1, b_2 \in B$.

In exact potential games, information concerning Nash equilibria are incorporated into a real-valued function that is the potential function.

The following theorem gives a lower semicontinuity result for the set-valued function defined by the set of the ε-approximate mixed Nash equilibria.

Theorem 2.2 *Assume that f_1, f_2 are continuous functions on $X \times Y_1 \times Y_2$ and that the game $\Omega(x) = \{Y_1, Y_2, f_1(x, \cdot, \cdot), -f_2(x, \cdot, \cdot)\}$ is an exact potential game for all $x \in X$ ($\Gamma(x)$ will be said to be an antipotential game for all $x \in X$). Then, for all $\varepsilon > 0$, the set-valued function $\hat{N}(\cdot, \varepsilon)$ is lower semicontinuous on X.*

Proof. Since $\Omega(x)$ is an exact potential game, according to [8], there exists a potential function P defined on $X \times Y_1 \times Y_2$ such that

$$f_1(x, y_1, y_2) = P(x, y_1, y_2) + h(x, y_2)$$

$$-f_2(x, y_1, y_2) = P(x, y_1, y_2) + k(x, y_1)$$

where h, k are real valued functions defined and continuous on $X \times Y_2, X \times Y_1$ respectively. By considering the mixed extensions of Y_1, Y_2, the function $\hat{f}_1 + \hat{f}_2 = \hat{h} - \hat{k}$ is convex on $\overline{M}(Y_1) \times \overline{M}(Y_2)$ and one can apply Corollary 3.1 in [18] to get the lower semicontinuity of $\hat{N}(\cdot, \varepsilon)$ on X. For the sake of completeness we give the proof.

Let $(\overline{\mu}_1, \overline{\mu}_2) \in \hat{N}(x, \varepsilon)$ such that $(\overline{\mu}_1, \overline{\mu}_2) \notin \breve{N}(x, \varepsilon)$. Since $\breve{N}(x, \varepsilon) \neq \emptyset$, there exists $(\breve{\mu}_1, \breve{\mu}_2) \in \breve{N}(x, \varepsilon)$ and consider the sequence $\overline{\mu}_{i,n} = (1/n)\breve{\mu}_i + (1 - 1/n)\overline{\mu}_i$ $(i = 1, 2)$ for $n \in \mathcal{N}$. We have that $\overline{\mu}_{i,n} \mapsto \overline{\mu}_i$, $i = 1, 2$ and

$$\hat{f}_1(x, \overline{\mu}_{1,n}, \overline{\mu}_{2,n}) + \hat{f}_2(x, \overline{\mu}_{1,n}, \overline{\mu}_{2,n}) < (1/n)\big[\hat{v}_1(x, \breve{\mu}_2) + \hat{v}_2(x, \breve{\mu}_1) + \varepsilon\big] +$$

$$(1 - 1/n)\big[\hat{v}_1(x, \overline{\mu}_2) + \hat{v}_2(x, \overline{\mu}_1) + \varepsilon\big] \leq \hat{v}_1(x, \overline{\mu}_{2,n}) + \hat{v}_2(x, \overline{\mu}_{1,n}) + \varepsilon$$

being $\hat{v}_1(x, \mu_2) = \inf_{\mu_1 \in \overline{M}(Y_1)} \hat{f}_1(x, \mu_1, \mu_2)$ and $\hat{v}_2(x, \mu_1) = \inf_{\mu_2 \in \overline{M}(Y_2)} \hat{f}_2(x, \mu_1, \mu_2)$. This means that $(\overline{\mu}_{1,n}, \overline{\mu}_{2,n}) \in \breve{N}(x, \varepsilon)$ and then $\hat{N}(x, \varepsilon) \subseteq cl\breve{N}(x, \varepsilon)$, where $cl\breve{N}(x, \varepsilon)$ is the sequential closure of $\breve{N}(x, \varepsilon)$. By Theorem 2.1 for all sequences $(x_n)_n$ converging to x we have $\breve{N}(x, \varepsilon) \subseteq \operatorname{Liminf}_n \breve{N}(x_n, \varepsilon)$. Therefore

$$\hat{N}(x, \varepsilon) \subseteq cl\breve{N}(x, \varepsilon) \subseteq cl \operatorname{Liminf}_n \breve{N}(x_n, \varepsilon) = \operatorname{Liminf}_n \breve{N}(x_n, \varepsilon) \subseteq \operatorname{Liminf}_n \hat{N}(x_n, \varepsilon)$$

Remark that, since $\overline{M}(Y_1)$ and $\overline{M}(Y_2)$ are first countable topological spaces, $\operatorname{Liminf}_n \breve{N}(x_n, \varepsilon)$ is a closed subset in $\overline{M}(Y_1) \times \overline{M}(Y_2)$.

Example 2.2 Note that in Example 1.1,

$$\Omega(x) = \{Y_1, Y_2, f_1(x, \cdot, \cdot), -f_2(x, \cdot, \cdot)\}$$

is an exact potential game with potential

	α_2	β_2
α_1	$-x$	$-2x$
β_1	$1-x$	$-2x$

Remark 2.4 *Theorem 2.2 extends Theorem 3.1 in [13] where existence of approximate mixed strategies for zero-sum games is obtained without convexity assumptions.*

3 Continuity properties of the approximate average marginal functions

By using the concepts of approximate mixed Nash equilibria given in Section 2, we give the continuity results for the following approximate average marginal functions.

Definition 3.1 *Let $x \in X$ and $\varepsilon > 0$; the following real functions defined on X:*

$$\hat{w}(x, \varepsilon) = \sup_{(\mu_1,\mu_2)\in \hat{N}(x,\varepsilon)} \hat{l}(x, \mu_1, \mu_2)$$

$$\hat{u}(x, \varepsilon) = \inf_{(\mu_1,\mu_2)\in \hat{N}(x,\varepsilon)} \hat{l}(x, \mu_1, \mu_2)$$

will be called ε-approximate sup-average marginal function and ε-approximate inf-average marginal function respectively.

So, we have the following theorem.

Theorem 3.1 *Assume that l, f_1, f_2 are continuous functions on $X \times Y_1 \times Y_2$ and that $\Gamma(x)$ is an antipotential game for all $x \in X$. Then, for all $\varepsilon > 0$, the ε-approximate average marginal functions $\hat{w}(\cdot, \varepsilon)$ and $\hat{u}(\cdot, \varepsilon)$ are continuous on X.*

Proof. In light of the assumptions $\hat{l}$ is continuous on $X \times \overline{M}(Y_1) \times \overline{M}(Y_2)$, the set-valued function $\hat{N}(\cdot, \varepsilon)$ is lower semicontinuous and closed graph on X. We obtain the proof by using the results given in [9] on the inf-marginal function in a sequential setting.

Example 3.1 In Example 1.1, let l be defined as follows:

	α_2	β_2
α_1	$x-1$	x
β_1	0	x

In this case the inf-marginal function $u(x) = x - 1$ is continuous on $[0, 1]$, while the sup-marginal function

$$w(x) = \begin{cases} 0 & \text{if } x = 0 \\ x - 1 & \text{if } x \neq 0 \end{cases}$$

is not lower semicontinuous at $x = 0$. Even if we use mixed Nash equilibria of the game $\Gamma(x)$, the sup-average marginal function may be not continuous. In fact $\hat{l}(x, \mu_1, \mu_2) = (x-1)pq + x(1-q)$ and

$$\hat{w}(x) = \sup_{(\mu_1,\mu_2)\in\hat{N}(x)} \hat{l}(x, \mu_1, \mu_2) = w(x) = \begin{cases} 0 & \text{if } x = 0 \\ x - 1 & \text{if } x \neq 0 \end{cases}$$

so $\hat{w}$ is not continuous at $x = 0$.

However, by considering for $\varepsilon > 0$ the set of the ε-approximate mixed Nash equilibria, the ε-approximate inf-average marginal function

$$\hat{w}(x, \varepsilon) = \begin{cases} x & \text{if } 0 \leq x \leq \varepsilon \\ x - 1 + \varepsilon/x & \text{if } x > \varepsilon \end{cases}$$

is continuous on $[0, 1]$.

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