Contraction mapping fixed point algorithms for solving multivalued mixed variational inequalities

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Summary. We show how to choose regularization parameters such that the solution of a multivalued strongly monotone mixed variational inequality can be obtained by computing the fixed point of a certain multivalued mapping having a contraction selection. Moreover a solution of a multivalued cocoercive variational inequality can be computed by finding a fixed point of a certain mapping having nonexpansive selection. By the Banach contraction mapping principle it is easy to establish the convergence rate.

Keywords. Multivalued mixed variational inequality, cocoerciveness, contraction and nonexpansiveness, Banach fixed point method.

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1 Introduction

The contraction and nonexpansive fixed point–methods for solving variational inequalities have been developed by several authors (see e.g. [1, 2, 6, 8, 15, 16] and the references therein). In our recent paper [1] we have used the auxiliary problem-method and the Banach contraction mapping fixed point principle to solve mixed variational inequalities involving single valued strongly monotone and cocoercive operators. Then in [2] we extended our method and combined it with the proximal point algorithm to solve mixed monotone variational inequalities.

In this paper we further extend the idea in [1, 2] to mixed multivalued variational inequalities involving strongly monotone and cocoercive cost operators with respect to the Hausdorff distance. Namely, we show that a necessary and sufficient condition for a point to be the solution of a multivalued strongly

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monotone mixed variational inequality is that it is the fixed point of a certain multivalued mapping having a contractive selection. For mixed variational inequalities involving multivalued cocoercive cost operators we show that their solutions can be computed by finding fixed points of corresponding multivalued mappings having a nonexpansive selection. These results allow that the Banach contraction mapping principle and its modifications can be applied to solve strongly monotone and cocoercive multivalued mixed variational inequalities. By the Banach contraction fixed point principle it is straightforward to obtain the convergence rate of the proposed algorithms.

2 Fixed Point Formulations

Let C be a nonempty, closed, convex subset of $\mathbb{R}^n$, let $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a multivalued mapping. Throughout this paper we suppose that dom$F$ contains C and that $F(x)$ is closed, convex for every $x \in C$. We suppose further that we are given a convex, subdifferentiable function $\varphi : C \to \mathbb{R}$. We consider the following multivalued mixed variational inequality problem that we shall denote by (VIP) :

Find $x^* \in C$ and $w^* \in F(x^*)$ such that

$$\langle w^*, x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0 \quad \forall x \in C. \tag{2.1}$$

This problem has been considered by some authors (see e.g., [4, 9, 12, 13, 14] and the references quoted therein). As usual in what follows we shall refer to F as cost operator and to C as constraint set.

As an example we consider an oligopolistic Cournot market model where there are n-firms producing a common homogeneous commodity. We assume that the price p_i of firm i depends on the total quantity of the commodity. Let h_i denote the cost of firm i when its production level is x_i. Suppose that the profit of firm i is given by

$$f_i(x_1, ..., x_n) = x_i p_i(\sum_{i=1}^{n} x_i) - h_i(x_i) \ (i = 1, ..., n).$$

Let U_i denote the strategy set of firm i and $U := U_1 \times ... \times U_n$ be the strategy set of the model. In the classical Cournot model the price and the cost functions for each firm are assumed to be affine of the forms

$$p_i(\sigma) = \alpha_i - \beta_i \sigma, \quad \alpha_i \geq 0, \beta_i > 0, \sigma = \sum_{i=1}^{n} x_i,$$

$$h_i(x_i) = \mu_i x_i + \xi_i, \ \mu_i > 0, \xi_i \geq 0 \ (i = 1, ..., n).$$

The problem is to find a point $x^* = (x_1^*, ..., x_n^*) \in U$ such that

$$f_i(x_1^*, ..., x_{i-1}^*, y_i, x_{i+1}^*, ..., x_n^*) \leq f_i(x^*) \ \forall y_i \in U_i, \forall i.$$

A vector $x^* \in U$ satisfying this inequality is called a Nash-equilibrium point of the model.

It is not hard to show (see also [10], and [9] for the case $\beta_i \equiv \beta$ for all i) that the problem of finding a Nash-equilibrium point can be formulated in the form (2.1) where

$$C = U := U_1 \times, ..., \times U_n, \ \varphi(x) := \sum_{i=1}^{n} \beta_i x_i^2 + \sum_{i=1}^{n} h_i(x_i), F(x) := Bx - \alpha$$

with

$$B := \begin{pmatrix} 0 & \beta_1 & \beta_1 & ... & \beta_1 \\ \beta_2 & 0 & \beta_2 & ... & \beta_2 \\ ... & ... & ... & ... & ... \\ \beta_n & \beta_n & \beta_n & ... & 0 \end{pmatrix}$$

and $\alpha := (\alpha_1, ..., \alpha_n)^T$. Some practical problems that can be formulated in a problem of form (2.1) can be found, for example, in [6, 9, 11].

For each fixed $x \in C$ and $w \in F(x)$, we denote by $h(x, w)$ the unique solution of the strongly convex program

$$\min\{\frac{1}{2}\langle y - x, G(y - x)\rangle + \langle w, y - x\rangle + \varphi(y) \ |y \in C\}, \tag{2.2}$$

where G is a symmetric, positive definite matrix. It is well known (see e.g., [5, 9, 11]) that $h(x, w)$ is the solution of (2.2) if and only if $h(x, w)$ is the solution of the variational inequality

$$\langle w + G(h(x, w) - x) + z, y - h(x, w)\rangle \geq 0 \ \ \forall y \in C, \tag{2.3}$$

for some $z \in \partial\varphi(h(x, w))$.

Now for each $x \in C$, we define the multivalued mapping

$$H(x) := \{h(x, w)|w \in F(x)\}.$$

Clearly, H is a mapping from $I\!R^n$ to C and, since $C \subseteq \mathrm{dom} H$, we have $C \subseteq \mathrm{dom}\ H \subseteq \mathrm{dom} F$.

The next lemma shows that a point x^* is a solution to (VIP) if and only if it is a fixed point of H.

Lemma 2.1 *x^* is a solution to (VIP) if and only if $x^* \in H(x^*)$.*

Proof. Let x^* solve (VIP). It means that there exists $w^* \in F(x^*)$ such that (x^*, w^*) satisfies inequality (2.1). Let $h(x^*, w^*)$ be the unique solution of Problem (2.2) corresponding to x^*, w^* and some positive definite matrix G. We replace x by $h(x^*, w^*)$ in (2.1) to obtain

$$\langle w^*, h(x^*, w^*) - x^*\rangle + \varphi(h(x^*, w^*)) - \varphi(x^*) \geq 0. \tag{2.4}$$

From (2.3) it follows that there exists z^* in $\partial\varphi(h(x^*, w^*))$ such that

$$\langle w^* + G(h(x^*, w^*) - x^*) + z^*, y - h(x^*, w^*)\rangle \geq 0 \quad \forall y \in C. \tag{2.5}$$

Replacing y by $x^* \in C$ in (2.5) we have

$$\langle w^* + G(h(x^*, w^*) - x^*) + z^*, x^* - h(x^*, w^*)\rangle \geq 0. \tag{2.6}$$

From inequalities (2.4) and (2.6) we obtain

$$\langle G(h(x^*, w^*) - x^*), x^* - h(x^*, w^*)\rangle + \langle z^*, x^* - h(x^*, w^*)\rangle$$
$$+\varphi(h(x^*, w^*)) - \varphi(x^*) \geq 0, \tag{2.7}$$

for some $z \in \partial\varphi(h(x, w))$. Since φ is convex on C, by the definition of subdifferential of a convex function, we have

$$\langle z^*, x^* - h(x^*, w^*)\rangle \leq \varphi(x^*) - \varphi(h(x^*, w^*)) \quad \forall z^* \in \partial\varphi(h(x^*, w^*)).$$

Hence

$$\langle z^*, x^* - h(x^*, w^*)\rangle - \varphi(x^*) + \varphi(h(x^*, w^*)) \leq 0 \quad \forall z^* \in \partial\varphi(h(x^*, w^*)). \tag{2.8}$$

From inequalities (2.7) and (2.8), it follows that

$$\langle G(h(x^*, w^*) - x^*), x^* - h(x^*, w^*)\rangle \geq 0.$$

Since G is symmetric, positive definite, the latter inequality implies that $h(x^*, w^*) = x^*$.

Now suppose $x^* \in H(x^*)$. Then there is w^* in $F(x^*)$ such that $x^* = h(x^*, w^*)$. But for every $x \in C, w \in F(x)$, we always have

$$\langle w + G(h(x, w) - x) + z, y - h(x, w)\rangle \geq 0 \quad \forall y \in C, \tag{2.9}$$

for some $z \in \partial\varphi(h(x, w))$. Replacing x, w, z by $x^* = h(x^*, w^*), w^*, z^*$, respectively, in inequality (2.9) we obtain

$$\langle w^* + z^*, y - x^*\rangle \geq 0 \quad \forall y \in C, \tag{2.10}$$

for some $z^* \in \partial\varphi(x^*)$. Using the definition of subdifferential of a convex function, we can write

$$\varphi(y) - \varphi(x^*) \geq \langle z^*, y - x^*\rangle \quad \forall y \in C. \tag{2.11}$$

From inequalities (2.10) and (2.11) we have

$$\langle w^*, y - x^*\rangle + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in C,$$

which means that x^* is a solution of Problem (VIP). □

Now we recall some well known definitions (see [3, 5]) about multivalued mappings that we need in the sequel.

• Let A, B be two nonempty subsets in $\mathbb{R}^n$. Let $\rho(A, B)$ denote the Hausdorff distance of A and B that is defined as

$$\rho(A, B) := \max\{d(A, B), d(B, A)\},$$

where

$$d(A, B) := \sup_{a\in A} \inf_{b\in B} ||a - b||, \quad d(B, A) := \sup_{b\in B} \inf_{a\in A} ||a - b||.$$

Let $\emptyset \neq M \subseteq \mathbb{R}^n$ and $K : \mathbb{R}^n \to \mathbb{R}^n$ be a multivalued mapping such that $M \subseteq \mathrm{dom}K$.

• K is said to be closed at x if $x^k \to x$, $y^k \in K(x^k)$, $y^k \to y$ as $k \to +\infty$, then $y \in F(x)$. We say that K is closed on M if it is closed at every point of M.

• K is said to be upper semicontinuous at x if for every open set G containing $K(x)$ there exists an open neighborhood U of x such that $K(U) \subset G$. We say that K is upper semicontinuous on M if it is upper semicontinuous at every point of M.

• K is said to be Lipschitz with a constant L (briefly L-Lipschitz) on M if

$$\rho(K(x), K(y)) \leq L||x - y|| \quad \forall x, y \in M.$$

K is called a contractive mapping if $L < 1$ and K is said to be nonexpansive if $L = 1$.

• We say that K has a L-Lipschitz selection on M if for every $x, y \in M$ there exist $w(x) \in K(x)$ and $w(y) \in K(y)$ such that

$$||w(x) - w(y)|| \leq L||x - y||.$$

If $0 < L < 1$ (resp. $L = 1$) we say that K has a contractive (resp. nonexpansive) selection on M. It is easy to check that a multivalued Lipschitz mapping with compact, convex values has a Lipschitz selection. This is why in the sequel, for short, we shall call a mapping having a Lipschitz selection a quasi-Lipschitz mapping. Likewise, a mapping having a contractive (resp. nonexpansive) selection is called quasicontractive (resp. quasinonexpansive).

• K is said to be monotone on M if

$$\langle w - w', x - x'\rangle \geq 0 \quad \forall x, x' \in M, \ \forall w \in K(x), \ \forall w' \in K(x').$$

• K is said to be strongly monotone with modulus $\beta > 0$ (briefly β-strongly monotone) on M if

$$\langle w - w', x - x'\rangle \geq \beta||x - x'||^2 \quad \forall x, x' \in M, \ \forall w \in K(x), \ \forall w' \in K(x').$$

• K is said to be cocoercive with modulus $\delta > 0$ (briefly δ-cocoercive) on M if

$$\langle w - w', x - x' \rangle \geq \delta\rho^2(K(x), K(x')) \;\; \forall x, x' \in M, \; \forall w \in K(x), \; \forall w' \in K(x'),$$

where ρ stands for the Hausdorff distance.

Note that in an important case when $G = \alpha I$ with $\alpha > 0$ and I being the identity matrix, problem (2.2) becomes

$$\min\{\frac{\alpha}{2}||y - x||^2 + \langle w, y - x \rangle + \varphi(y) \; |y \in C\}.$$

In the sequel we shall restrict our attention to this case. The following theorem shows that with a suitable value of regularization parameter α, the mapping H defined above is quasicontractive on C .

In what follows we need the following lemma:

Lemma 2.2 *Suppose that $C \subseteq I\!R^n$ is nonempty, closed, convex and $F : I\!R^n \to 2^{I\!R^n}$ is L-Lipschitz on C such that $F(x)$ is closed, convex for every $x \in C$. Then for every $x, x' \in C$ and $w \in F(x)$, there exists $w' \in F(x')$, in particular $w' = P_{F(x')}(w)$, such that $||w - w'|| \leq L||x - x'||$.*

Here, $P_{F(x')}(w)$ denotes the projection of the point w on the set $F(x')$.

Proof. Since $w \in F(x)$, by the definition of the projection and the Hausdorff distance, we have

$$\begin{aligned} ||w - w'|| = \inf_{v' \in F(x')} ||w - v'|| &\leq \sup_{v \in F(x)} \inf_{v' \in F(x')} ||v - v'|| \\ &\leq \rho\Big(F(x), F(x')\Big) \leq L||x - x'||. \end{aligned}$$

□

Theorem 2.1 *Suppose that F is β- strongly monotone and L- Lipschitz on C, and that $F(x)$ is closed, convex for every $x \in C$. Then the mapping H is quasicontractive on C with constant $\delta := \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}}$ whenever $\alpha > \frac{L^2}{2\beta}$. Namely,*

$$||h(x, w(x)) - h(x', w(x')) \leq \delta||x - x'|| \; \forall x, x' \in C \; \forall w(x) \in F(x)$$

where $w(x')$ is the Euclidean projection of $w(x)$ onto $F(x')$.

Proof. Problem (2.2) with $G = \alpha I$ can be equivalently rewritten as

$$\min_y\{\frac{1}{2}\langle \alpha||y - x||^2 \rangle + \langle w, y - x \rangle + \varphi(y) + \delta_C(y)\},$$

where δ_C is the indicator function of C. Let $h(x, w)$ be the unique solution of this unconstrained problem. Then we have

$$0 \in \alpha(h(x, w) - x) + w + N_C(h(x, w)) + \partial\varphi(h(x, w)),$$

where $N_C(h(x,w))$ is the normal cone to C at the point $h(x,w)$. Thus there are $z_1 \in N_C(h(x,w))$ and $z_2 \in \partial\varphi(h(x,w))$ such that

$$\alpha(h(x,w) - x) + w + z_1 + z_2 = 0.$$

Therefore

$$h(x,w) = x - \frac{1}{\alpha}w - \frac{1}{\alpha}z_1 - \frac{1}{\alpha}z_2. \tag{2.12}$$

Similarly for $x' \in C, w' \in F(x')$, we have

$$h(x',w') = x' - \frac{1}{\alpha}w' - \frac{1}{\alpha}z_1' - \frac{1}{\alpha}z_2', \tag{2.13}$$

where $z_1' \in N_C(h(x',w'))$ and $z_2' \in \partial\varphi(h(x',w'))$.
Since N_C is monotone, we have

$$\langle z_1 - z_1', h(x,w) - h(x',w')\rangle \geq 0. \tag{2.14}$$

Substituting z_1 from (2.12) and z_1' from (2.13) into (2.14) we obtain

$$\langle x - x' - \frac{1}{\alpha}(w - w') - \frac{1}{\alpha}(z_2 - z_2') - (h(x,w) - h(x',w')), h(x,w) - h(x',w')\rangle \geq 0,$$

which implies

$$||h(x,w) - h(x',w')||^2 \leq \langle x - x' - \frac{1}{\alpha}(w - w') - \frac{1}{\alpha}(z_2 - z_2'), h(x,w) - h(x',w')\rangle$$

$$= \langle x - x' - \frac{1}{\alpha}(w - w'), h(x,w) - h(x',w')\rangle - \frac{1}{\alpha}\langle z_2 - z_2', h(x,w) - h(x',w')\rangle. \tag{2.15}$$

Since $\partial\varphi$ is monotone on C, we have

$$\langle h(x,w) - h(x',w'), z_2 - z_2'\rangle \geq 0 \ \forall z_2 \in \partial\varphi(h(x,w)), z_2' \in \partial\varphi(h(x',w')). \tag{2.16}$$

From (2.15), (2.16) it follows that

$$||h(x,w) - h(x',w')||^2 \leq \langle x - x' - \frac{1}{\alpha}(w - w'), h(x,w) - h(x',w')\rangle$$

$$\leq ||x - x' - \frac{1}{\alpha}(w - w')|| \; ||h(x,w) - h(x',w')||.$$

Thus

$$||h(x,w) - h(x',w')||^2 \leq ||x - x' - \frac{1}{\alpha}(w - w')||^2 \tag{2.17}$$

$$= ||x - x'||^2 - \frac{2}{\alpha}\langle x - x', w - w'\rangle + \frac{1}{\alpha^2}||w - w'||^2.$$

Since F is L-Lipschitz on C and $F(x')$ is closed, for every $w(x) \in F(x)$, by Lemma 2.2, there exists $w(x') \in F(x')$ such that

$$||w(x) - w(x')|| \le L||x - x'||$$

which together with strong monotonicity of F implies

$$||x - x' - \frac{1}{\alpha}(h(x, w(x)) - h(x', w(x'))||^2 \le (1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2})||x - x'||^2. \quad (2.18)$$

Finally, from (2.17) and (2.18) we have

$$||h(x, w(x)) - h(x', w(x'))|| \le \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}}||x - x'||. \quad (2.19)$$

Let $\delta := \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}}$, then

$$||h(x, w(x)) - h(x', w(x'))|| \le \delta||x - x'|| \ \ \forall x, x' \in C.$$

Note that if $\alpha > \frac{L^2}{2\beta}$ then $\delta \in (0, 1)$. Thus the multivalued mapping H has a contractive selection on C with constant δ. □

Remark 2.1 *From the definition of H and Theorem 2.1 it follows that when F is single-valued, the mapping H is contractive on C.*

Note that if φ is η-strongly convex and subdifferentiable on C, then its subdifferential is η- strongly monotone on C (see e.g., [5]). This means that

$$\langle z - z', x - x'\rangle \ge \eta||x - x'||^2 \ \ \forall x, x' \in C, z \in \partial\varphi(x), x' \in \partial\varphi(x').$$

In the following theorem the strong monotonicity of F is replaced by the strong convexity of φ.

Theorem 2.2 *Suppose that F is monotone and L- Lipschitz on C, that $F(x)$ is closed, convex for every $x \in C$ and that φ is η-strongly convex and subdifferentiable on C. Then the mapping H is quasicontractive on C with constant*

$$\delta := \frac{\sqrt{L^2 + \alpha^2}}{\alpha + \eta},$$

whenever $\alpha > \frac{L^2 - \eta^2}{2\eta}$.

Proof. By the same way as in the proof of Theorem 2.1 we obtain

$$||h(x, w) - h(x', w')||^2 \le \langle x - x' - \frac{1}{\alpha}(w - w') - \frac{1}{\alpha}(z_2 - z_2'), h(x, w) - h(x', w')\rangle$$

$$= \langle x - x' - \frac{1}{\alpha}(w - w'), h(x, w) - h(x', w')\rangle - \frac{1}{\alpha}\langle z_2 - z_2', h(x, w) - h(x', w')\rangle$$

from which it follows that

$$||h(x,w) - h(x',w')||^2 \le \langle x - x' - \frac{1}{\alpha}(w - w'), h(x,w) - h(x',w')\rangle$$

$$-\frac{1}{\alpha}\langle z_2 - z_2', h(x,w) - h(x',w')\rangle, \tag{2.20}$$

for all $x, x' \in C, w \in F(x), w' \in F(x')$, for some $z_2 \in \partial\varphi(h(x,w)), z_2' \in \partial\varphi(h(x',w'))$.
Since $\partial\varphi$ is strongly monotone with modulus η on C, we have

$$\langle z_2 - z_2', h(x,w) - h(x',w')\rangle \ge \eta||h(x,w) - h(x',w')||^2$$

$$\forall z_2 \in \partial\varphi(h(x,w)), z_2' \in \partial\varphi(h(x',w'))$$

$$\Leftrightarrow -\frac{1}{\alpha}\langle z_2 - z_2', h(x,w) - h(x',w')\rangle \le -\frac{\eta}{\alpha}||h(x,w) - h(x',w')||^2. \tag{2.21}$$

Combining (2.20) and (2.21) yields

$$(1 + \frac{\eta}{\alpha})^2||h(x,w) - h(x',w')||^2 \le ||x - x' - \frac{1}{\alpha}(w - w')||^2$$

$$= ||x - x'||^2 - \frac{2}{\alpha}\langle x - x', w - w'\rangle + \frac{1}{\alpha^2}||w - w'||^2. \tag{2.22}$$

Since F is Lipschitz with constant L on C and $F(x)$ is closed, convex, it follows that for every $x, x' \in C$, $w(x) \in F(x)$, there exists $w(x') \in F(x')$ satisfying

$$||w(x) - w(x')|| \le L||x - x'||.$$

Since F is monotone, we have

$$\langle w(x) - w(x'), x - x'\rangle \ge 0,$$

which together with (2.22) implies

$$(1 + \frac{\eta}{\alpha})^2||h(x,w(x)) - h(x',w(x'))||^2 \le (1 + \frac{L^2}{\alpha^2})||x - x'||^2$$

$$\Leftrightarrow ||h(x,w(x)) - h(x',w(x'))|| \le \delta||x - x'|| \quad \forall x, x' \in C,$$

where $\delta := \frac{\sqrt{L^2+\alpha^2}}{\alpha+\eta}$. It is easy to verify that $\delta \in (0,1)$ when $\alpha > \frac{L^2-\eta^2}{2\eta}$. □

In the next theorem we weaken strong monotonicity of F by cocoercivity.

Theorem 2.3 *Suppose that F is γ-cocoercive on C, and that $F(x)$ is closed, convex for every $x \in C$. Then the mapping H is quasinonexpansive on C.*

Proof. By the same way as in the proof of Theorem 2.1, for every $x, x' \in C$, we have

$$||h(x,w) - h(x',w')||^2 \le ||x - x' - \frac{1}{\alpha}(w - w')||^2 \ \forall w \in F(x),\ \forall w' \in F(x'). \tag{2.23}$$

From the cocoercivity of F on C with modulus γ, it follows that

$$\gamma\rho^2(F(x), F(x')) \le \langle x - x', w - w'\rangle \ \ \forall x, x' \in C, w \in F(x), w' \in F(x').$$

Hence, for every $x, x' \in C$ and $w \in F(x)$, $w' \in F(x')$ we have

$$\begin{aligned} ||x - x' - \frac{1}{\alpha}(w - w')||^2 &= ||x - x'||^2 - \frac{2}{\alpha}\langle x - x', w - w'\rangle + \frac{1}{\alpha^2}||w - w'||^2 \\ &\le ||x - x'||^2 - \frac{2\gamma}{\alpha}\rho^2(F(x), F(x')) + \frac{1}{\alpha^2}||w - w'||^2. \end{aligned}$$

Let $w(x) \in F(x), w(x') \in F(x')$, such that $\rho(F(x), F(x')) = ||w(x) - w(x')||$. Substituting $w(x)$ and $w(x')$ into the last inequality we obtain

$$||x - x' - \frac{1}{\alpha}(w(x) - w(x'))||^2 \le ||x - x'||^2 - (\frac{2\gamma}{\alpha} - \frac{1}{\alpha^2})||w(x) - w(x')||^2.$$

Since $\alpha \ge \frac{1}{2\gamma}$, we have

$$||x - x' - \frac{1}{\alpha}(w(x) - w(x')||^2 \le ||x - x'||^2 \ \ \forall x, x' \in C. \tag{2.24}$$

From (2.23) and (2.24) it follows that

$$||h(x, w(x)) - h(x', w(x'))|| \le ||x - x'|| \ \ \forall x, x' \in C.$$

□

Remark 2.2 *From the proof we can see that the theorem remains true if we weaken the cocoercivity of F by the following one*

$$\forall x, x' \in C, \forall w \in F(x), \exists \pi'(w) \in F(x') : \gamma\rho^2\big(F(x), F(x')\big) \le \langle w - \pi'(w), x - x'\rangle$$

Below is given a simple example for a multivalued mapping which is both monotone and Lipschitz.

Example 2.1 Let $C = \{(x,0) | x \ge 0\} \subseteq I\!R^2$, and $F : C \to 2^{I\!R^2}$ be given as

$$F(x,0) = \{(x,y) | 0 \le y \le x\}.$$

It is easy to see that F is monotone and Lipschitz on C with constant $L = \sqrt{2}$. The mapping $G := I + F$ with I identity on $I\!R^2$ is strongly monotone with modulus $\beta = 1$ and Lipschitz on C with constant $L = \sqrt{2} + 1$.

Indeed, by definition of F, it is clear that F is monotone on C. Using the definition of the Hausdorff distance we have

$$\rho(F(x,y), F(x',y')) = \sqrt{2}||(x,y) - (x',y')|| \ \ \forall (x,y), (x',y') \in C.$$

Thus F is Lipschitz on C with constant $L = \sqrt{2}$. △

3 Algorithms

The results in the preceding section lead to algorithms for solving multivalued mixed variational inequalities by the Banach contraction mapping principle or its modifications. By Theorem 2.1 and 2.2, when either F is strongly monotone or φ is strongly convex, one can choose a suitable regularization parameter α such that the solution mapping H is quasi-contractive. In this case, by the Banach contraction principle the unique fixed point of H, thereby the unique solution of Problem (2.1) can be approximated by iterative procedures

$$x^{k+1} \in H(x^k), k = 0, 1...$$

where x^0 can be any point in C.
According to the definition of H, computing x^{k+1} amounts to solving a strongly convex mathematical program. In what follows by ε-solution of (VIP) we mean a point $x \in C$ such that $||x - x^*|| \leq \varepsilon$ where x^* is an exact solution of (VIP).

The algorithm then can be described in detail as follows:
Algorithm 3.1. Choose a tolerance $\varepsilon \geq 0$.
Choose $\alpha > \frac{L^2}{2\beta}$, when F is β-strongly monotone (and choose $\alpha > \frac{L^2-\eta^2}{2\eta}$, when φ is η-strongly convex), where L is the Lipschitz constant of F.
Seek $x^0 \in C, w^0 \in F(x^0)$.
Iteration k $(k = 0, 1, 2...)$
Solve the strongly convex program

$$P(x^k): \quad \min\{\frac{1}{2}\alpha||x - x^k||^2 + \langle w^k, x - x^k\rangle + \varphi(x)|x \in C\},$$

to obtain its unique solution x^{k+1}. Find $w^{k+1} \in F(x^{k+1})$ such that $||w^{k+1} - w^k|| \leq L||x^{k+1} - x^k||$, for example $w^{k+1} := P_{F(x^{k+1})}(w^k)$ (the projection of w^k onto $F(x^{k+1})$) .
If $||x^{k+1} - x^k|| \leq \varepsilon\frac{(1-\delta)}{\delta^k}$, then terminate: x^k is an ε-solution to Problem (2.1).
Otherwise, if $||x^{k+1} - x^k|| > \varepsilon\frac{(1-\delta)}{\delta^k}$, then increase k by 1 and go to iteration k.

By Theorems 2.1 and 2.2 and the Banach contraction principle it is easy to prove the following estimation:

$$||x^{k+1} - x^*|| \leq \frac{\delta^{k+1}}{1-\delta}||x^0 - x^1|| \; \forall k,$$

where $0 < \delta < 1$ is the quasicontractive constant of h. According to Theorem 2.1 $\delta = \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}}$, when F is β-strongly monotone, and according to Theorem 2.2 $\delta = \frac{\sqrt{L^2+\alpha^2}}{\alpha+\eta}$ when φ is η-strongly convex.

Theorem 3.1 *Under the assumptions of Theorem 2.1 (or Theorem 2.2), the sequence* $\{x^k\}$ *generated by Algorithm 3.1 satisfies*

$$||x^k - x^*|| \leq \frac{\delta^{k+1}}{1-\delta}||x^0 - x^1|| \quad \forall k, \tag{3.1}$$

where x^* is the solution of (VIP). If, in addition F is closed on C, then the sequence $\{w^k\}$ converges to $w^* \in F(x^*)$ with the rate

$$||w^k - w^*|| \leq \frac{L\delta^k}{1-\delta}||x^0 - x^1|| \quad \forall k.$$

Proof. First we suppose that the assumptions of Theorem 2.1 are satisfied. Let x^* be the solution of (2.1). By Lemma 2.1,

$$x^* \in H(x^*) := \{h(x^*, w) | w \in F(x^*)\}.$$

Let $w^* \in F(x^*)$ such that $x^* = h(x^*, w^*) \in H(x^*)$. By the choice of w^{k+1} in the algorithm

$$||w^{k+1} - w^k|| \leq L||x^{k+1} - x^k|| \quad \forall k.$$

Then as shown in Theorem 2.1 we have

$$||h(x^{k+1}, w^{k+1}) - h(x^k, w^k)|| \leq \delta||x^{k+1} - x^k|| \quad \forall k,$$

Since $h(x^{k+1}, w^{k+1}) = x^{k+2}$, we have

$$||x^{k+2} - x^{k+1}|| \leq \delta||x^{k+1} - x^k|| \quad \forall k,$$

from which, by the Banach contraction mapping fixed point principle, it follows that

$$||x^k - x^*|| \leq \frac{\delta^{k+1}}{1-\delta}||x^0 - x^1|| \quad \forall k.$$

Thus $x^k \to x^*$ as $k \to +\infty$. Moreover using again the contraction property we have

$$||x^{p+k} - x^k|| \leq \delta^k \frac{(1-\delta^p)}{1-\delta}||x^{k+1} - x^k|| \; \forall k, p.$$

Letting $p \to +\infty$ we obtain

$$||x^k - x^*|| \leq \frac{\delta^k}{1-\delta}||x^{k+1} - x^k|| \; \forall k.$$

Thus if $||x^{k+1} - x^k|| \leq \varepsilon \frac{(1-\delta)}{\delta^k}$, then it follows that $||x^k - x^*|| \leq \varepsilon$ which means that x^k is an ε-solution to (VIP).

On the other hand, since

$$||w^{k+1} - w^k|| \leq L||x^{k+1} - x^k||$$

we have

$$||w^{k+p} - w^k|| \leq ||w^{k+1} - w^k|| + ||w^{k+2} - w^{k+1}|| + ... + ||w^{k+p} - w^{k+p-1}||$$
$$\leq L(||x^{k+1} - x^k|| + ||x^{k+2} - x^{k+1}|| + ... + ||x^{k+p} - x^{k+p-1}||)$$
$$\leq L(\delta^k + \delta^{k+1} + ... + \delta^{k+p-1})||x^1 - x^0||.$$

Thus
$$||w^{k+p} - w^k|| < L\delta^k \frac{\delta^p - 1}{\delta - 1}||x^1 - x^0||, \tag{3.2}$$

which means that $\{w^k\}$ is a Cauchy sequence. Hence the sequence $\{w^k\}$ converges to some $w^* \in C$. Since F is closed, $w^* \in F(x^*)$. From (3.2) and letting $p \to +\infty$ we have

$$||w^k - w^*|| \leq \frac{L\delta^k}{1-\delta}||x^1 - x^0|| \ \ \forall j.$$

The proof can be done similarly under the assumptions of Theorem 2.3. □

Remark 3.1 *From* $\delta := \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L^2}{\alpha^2}}$ *(resp.* $\delta = \frac{\sqrt{L^2+\alpha^2}}{\alpha+\eta}$*) we see that the contraction coefficient* δ *is a function of the regularization parameter* α*. An elementary computation shows that* δ *takes its minimum when* $\alpha = \frac{L^2}{\beta}$ *(resp.* $\alpha = \frac{L^2-\eta^2}{\eta}$*). Therefore for the convergence, in Algorithm 3.1 the best way is to choose* $\alpha = \frac{L^2}{\beta}$ *(resp.* $\alpha = \frac{L^2-\eta^2}{\eta}$*).*

Remark 3.2 *In Algorithm 3.1, at each iteration* k*, it requires finding* $w^{k+1} \in F(x^{k+1})$ *such that* $|w^{k+1} - w^k|| \leq L||x^{k+1} - x^k||$*, which can be done when* $F(x)$ *has a special structure, for example, box, ball , simplex or a convex set given explicitly. One may ask whether the algorithm remains convergent if it takes any point from* $F(x^{k+1})$*. To our opinion, there is less hope for a positive answer to this question except cases when the set* $F(x^{k+1})$ *can be represented by any of its elements.*

Now we consider a special case that often occurs in practice.

Let $\mu = \sup\{\text{diam } F(x) | x \in C\}$, $\tau = \text{diam } C$. It is well known that if C is compact and F is upper semicontinuous on C, then μ and τ are finite.

Algorithm 3.2. Choose a tolerance $\varepsilon > 0$, $\alpha > \frac{L_0^2}{2\beta}$ when F is β-strongly monotone (and choose $\alpha > \frac{L_0^2-\eta^2}{2\eta}$ when φ is η-strongly convex), where $L_0 \geq \frac{L\tau+\mu}{\varepsilon(1-\delta)}$ and $\delta := \sqrt{1 - \frac{2\beta}{\alpha} + \frac{L_0^2}{\alpha^2}}$ when F is β-strongly monotone ($\delta = \frac{\sqrt{L_0^2+\alpha^2}}{\alpha+\eta}$ when φ is η-strongly convex).
Seek $x^0 \in C, w^0 \in F(x^0)$.
Iteration k $(k = 0, 1, 2...)$
Solve the strongly convex program

$$P(x^k): \ \ \min\{\frac{1}{2}\alpha||x - x^k||^2 + \langle w^k, x - x^k\rangle + \varphi(x) | x \in C\},$$

to obtain its unique solution x^{k+1}. Choose $w^{k+1} \in F(x^{k+1})$.
If $||x^{k+1} - x^k|| \leq \varepsilon \frac{(1-\delta)}{\delta^k}$, then terminate: x^k is an ε-solution to Problem (2.1).

Otherwise, if $||x^{k+1} - x^k|| > \varepsilon \frac{(1-\delta)}{\delta^k}$, then increase k by 1 and go to iteration k.

Theorem 3.2 *Suppose that C is compact and F is upper semicontinuous on C. Then under the assumptions of Theorem 2.1 or Theorem 2.2, the sequence $\{x^k\}$ generated by Algorithm 3.2 satisfies*

$$||x^k - x^*|| \leq \frac{\delta^{k+1}}{1-\delta}||x^0 - x^1|| \quad \forall k,$$

where x^* is the solution of (2.1). Moreover

$$||w^k - w^*|| \leq \frac{L_0 \delta^k}{1-\delta}||x^0 - x^1|| \quad \forall k.$$

Proof. By the same argument as in the proof of Theorem 3.1 we see that if $||x^{k+1} - x^k|| \leq \varepsilon \frac{(1-\delta)}{\delta^k}$, then indeed, x^k is an ε-solution.

Now suppose $||x^{k+1} - x^k|| > \varepsilon \frac{(1-\delta)}{\delta^k}$. For every $w^{k+1} \in F(x^{k+1})$, since $L_0 \geq \frac{L\tau + \mu}{\varepsilon(1-\delta)}$, we have

$$\begin{aligned} ||w^{k+1} - w^k|| &\leq d(w^k, F(x^{k+1})) + \mathrm{diam} F(x^{k+1}) \leq L||x^{k+1} - x^k|| + \mu \\ &\leq L\tau + \mu \leq L_0 \varepsilon (1-\delta) < L_0 ||x^{k+1} - x^k||. \end{aligned}$$

where the last inequality follows from $L_0 \geq \frac{L\tau + \mu}{\varepsilon(1-\delta)}$ and $\delta^k \leq 1$ for all k. Since $||x^{k+1} - x^k|| > \varepsilon \frac{(1-\delta)}{\delta^k}$, we have

$$||w^{k+1} - w^k|| \leq L_0 ||x^{k+1} - x^k|| \; \forall k.$$

Using this inequality we can prove the theorem by the same way as in the proof of Theorem 2.1 (or Theorem 2.2 when φ is strongly convex). □

Now we return to the case when F is cocoercive. Note that in this case Problem (VIP) is not necessarily uniquely solvable. By Theorem 2.3, a solution of (VIP) can be obtained by computing a fixed point of mapping H. Since H has a nonexpansive selection, its fixed point may be computed using the following theorem.

Theorem 3.3 *Let $C \subseteq I\!R^n$ be a nonempty, closed, convex set and $S : C \to 2^C$. Suppose that $S(x)$ is compact and that S has a nonexpansive selection on C. For $0 < \lambda < 1$ define*

$$S_\lambda := (1-\lambda)I + \lambda S.$$

Then the sequences $\{x^k\}$, $\{y^k\}$ defined by $x^{k+1} \in S_\lambda(x^k)$, i.e.,

$$x^{k+1} := (1-\lambda)x^k + \lambda y^k,$$

with $y^k \in S(x^k)$ *satisfy*

$$||y^{k+1} - y^k|| \le ||x^{k+1} - x^k|| \quad \forall k = 0, 1, 2, ...$$

$$||x^k - y^k|| \to 0 \text{ as } k \to +\infty,$$

Moreover any cluster point of the sequence $\{x^k\}$ *is a fixed point of* S.

To prove this theorem we need the following lemma:

Lemma 3.1 *Under the assumptions of Theorem 3.3, for all* $i, m = 0, 1, ...,$ *we have*

$$||y^{i+m} - x^i|| \ge (1-\lambda)^{-m}[||y^{i+m} - x^{i+m}|| - ||y^i - x^i||] + (1+\lambda m)||y^i - x^i||. \tag{3.3}$$

Proof. We proceed by induction on m, assuming that (3.3) holds for a given m and for all i. Clearly, (3.3) is trivial if $m = 0$. Replacing i with $i+1$ in (3.3) yields

$$||y^{i+m+1} - x^{i+1}|| \ge (1-\lambda)^{-m}[||y^{i+m+1} - x^{i+m+1}|| - ||y^{i+1} - x^{i+1}||]$$
$$+(1+\lambda m)||y^{i+1} - x^{i+1}||. \tag{3.4}$$

Since $x^{k+1} := (1-\lambda)x^k + \lambda y^k$ with $y^k \in S(x^k)$ that

$$||y^{i+m+1} - x^{i+1}|| = ||y^{i+m+1} - [(1-\lambda)x^i + \lambda y^i]||$$
$$\le \lambda||y^{i+m+1} - y^i|| + (1-\lambda)||y^{i+m+1} - x^i||$$
$$\le (1-\lambda)||y^{i+m+1} - x^i|| + \lambda \sum_{k=0}^{m} ||x^{i+k+1} - x^{i+k}||. \tag{3.5}$$

Combining (3.4) and (3.5) we obtain

$$||y^{i+m+1} - x^i|| \ge (1-\lambda)^{-(m+1)}[||y^{i+m+1} - x^{i+m+1}|| - ||y^{i+1} - x^{i+1}||]$$
$$+(1-\lambda)^{-1}(1+\lambda m)||y^{i+1} - x^{i+1}|| - \lambda(1-\lambda)^{-1} \sum_{k=0}^{n} ||x^{i+k+1} - x^{i+k}||.$$

Since $||x^{i+k+1} - x^{i+k}|| = \lambda||y^{k+i} - x^{k+i}||$ and since the sequence $\{||y^m - x^m||\}$ is decreasing, from

$$\lambda||y^m - x^m|| = ||x^{m+1} - x^m|| = ||(1-\lambda)x^m + \lambda y^m - [(1-\lambda)x^{m-1} - \lambda y^{m-1}]||$$
$$\le (1-\lambda)||x^m - x^{m-1}|| + \lambda||y^m - y^{m-1}|| \le ||x^m - x^{m-1}|| = \lambda||y^{m-1} - x^{m-1}||$$

and $1 + m\lambda \le (1-\lambda)^{-m}$, we have

$$||y^{i+m+1} - x^i|| \geq (1-\lambda)^{-(m+1)}[||y^{i+m+1} - x^{i+m+1}|| - ||y^{i+1} - x^{i+1}||]$$

$$+ (1-\lambda)^{-1}(1+\lambda m)||y^{i+1} - x^{i+1}|| - \lambda^2(1-\lambda)^{-1}(m+1)||y^i - x^i||$$

$$= (1-\lambda)^{-(m+1)}[||y^{i+m+1} - x^{i+m+1}|| - ||y^i - x^i||]$$

$$+ [(1-\lambda)^{-1}(1+\lambda m) - (1-\lambda)^{-(m+1)}]||y^{i+1} - x^{i+1}||$$

$$+ [(1-\lambda)^{-(m+1)} - \lambda^2(1-\lambda)^{-1}(m+1)]||y^i - x^i||$$

$$\geq (1-\lambda)^{-(m+1)}[||y^{i+m+1} - x^{i+m+1}|| - ||y^i - x^i||]$$

$$+ [(1-\lambda)^{-1}(1+\lambda m) - (1-\lambda)^{-(m+1)}]||y^i - x^i||$$

$$+ [(1-\lambda)^{-(m+1)} - \lambda^2(1-\lambda)^{-1}(m+1)]||y^i - x^i||$$

$$= (1-\lambda)^{-(m+1)}[||y^{i+m+1} - x^{i+m+1}|| - ||y^i - x^i||] + [1+\lambda(m+1)]||y^i - x^i||.$$

Thus (3.5) holds for $m+1$. □

Proof of Theorem 3.3. Let $d := \sup\{\text{diam } S(x) | x \in C\}$, and suppose that $\lim\limits_{m\to\infty} ||y^m - x^m|| = r > 0$. Select $m \geq \frac{d}{r\lambda}$ and ε is a sufficiently small positive number such that $\varepsilon(1-\lambda)^{-m} < r$. Since $\{||y^m - x^m||\}$ is decreasing, there exists an integer i such that

$$0 \leq ||y^i - x^i|| - ||y^{m+i} - x^{m+i}|| \leq \varepsilon.$$

Therefore, using (3.3) we arrive at the contradiction

$$d + r \leq (1+m\lambda)r \leq (1+m\lambda)||y^i - x^i||$$

$$\leq ||y^{m+i} - x^i|| + (1-\lambda)^{-m}[||y^i - x^i|| - ||y^{m+i} - x^{m+i}||]$$

$$\leq ||y^{m+i} - x^i|| + (1-\lambda)^{-m}\varepsilon < d + r.$$

Consequently $r = 0$, thus $\lim\limits_{m\to\infty} ||x^m - y^m|| = 0$. Since S is a bounded-valued mapping on C and S is closed, we have that any cluster point of convergent sequences $\{x^m\}$ is a fixed point of S. □

Now applying Theorem 3.3 to H we can solve Problem (2.1) with F being cocoercive on C by finding a fixed point of H.

Algorithm 3.3. *Step 0.* Choose a tolerance $\varepsilon \geq 0$ and $\lambda \in (0,1)$, $\alpha \geq \frac{1}{2\gamma}$ and seek $x^0 \in C, w^0 \in F(x^0)$. Let $k = 0$.

Step 1. Solve the strongly convex program

$$P(x^k): \quad \min\{\frac{1}{2}\alpha||y - x^k||^2 + \langle w^k, y - x^k\rangle + \varphi(y) | y \in C\}$$

to obtain its unique solution y^k.
If $||y^k - x^k|| \leq \varepsilon$, then the algorithm terminates.
Otherwise go to Step 2.

Step 2. Take

$$x^{k+1} := (1-\lambda)x^k + \lambda y^k.$$

Find $w^{k+1} := P_{F(x^{k+1})}(w^k)$.
Let $k \leftarrow k+1$ and return to Step 1.

Theorem 3.4 *In addition to the assumptions of Theorem 2.3, suppose that C is compact, and F is upper semicontinous on C. Then, if Algorithm 3.3 does not terminate, the sequence $\{x^k\}$ is bounded and any cluster point is a solution of Problem (VIP). In addition, it holds $d(x^k, H(x^k)) \to 0$ as $k \to \infty$.*

Proof. In Algorithm 3.3, we have $w^{k+1} := P_{F(x^{k+1})}(w^k)$ with $w^k \in F(x^k)$. From Lemma 2.2 and the definition of $\rho(F(x^k), F(x^{k+1}))$ it follows that

$$||w^{k+1} - w^k|| \leq \rho(F(x^k), F(x^{k+1})).$$

From the cocoercivity of F on C with modulus γ, we have

$$\gamma\rho^2(F(x^k), F(x^{k+1})) \leq \langle x^k - x^{k+1}, w^k - w^{k+1}\rangle.$$

Thus

$$\begin{aligned} &||x^k - x^{k+1} - \frac{1}{\alpha}(w^k - w^{k+1})||^2 \\ &= ||x^k - x^{k+1}||^2 - \frac{2}{\alpha}\langle x^k - x^{k+1}, w^k - w^{k+1}\rangle + \frac{1}{\alpha^2}||w^k - w^{k+1}||^2 \\ &\leq ||x^k - x^{k+1}||^2 - \frac{2\gamma}{\alpha}||w^k - w^{k+1}||^2 + \frac{1}{\alpha^2}||w^k - w^{k+1}||^2 \\ &= ||x^k - x^{k+1}||^2 - (\frac{2\gamma}{\alpha} - \frac{1}{\alpha^2})||w^k - w^{k+1}||^2. \end{aligned}$$

Since $\alpha > \frac{1}{2\gamma}$, we have

$$||x^k - x^{k+1} - \frac{1}{\alpha}(w^k - w^{k+1})||^2 \leq ||x^k - x^{k+1}||^2$$

which together with quasinonexpansiveness of H implies

$$||y^{k+1} - y^k|| \leq ||x^{k+1} - x^k||,$$

where

$$y^k = h(x^k, w^k) \in H(x^k), y^{k+1} = h(x^{k+1}, w^{k+1}) \in H(x^{k+1}).$$

By Theorem 3.3, every cluster point of the sequence $\{x^k\}$ is the fixed point x^* of H which is also a solution to Problem (2.1).

Furthermore, since C is compact and F is upper semicontinous on C, it follows from $w^k \in F(x^k)$ that the sequence $\{w^k\}$ is bounded. Thus, without loss of generality, we may assume that the sequence $\{w^k\}$ converges to some w^*. Since F is closed at x^*, we have $w^* \in F(x^*)$ and $x^* \in C$.

To prove $d(x^k, H(x^k)) \to 0$ we observe that $y^k \in H(x^k)$, and therefore

$$d(x^k, H(x^k)) \leq ||x^k - y^k|| \ \forall k.$$

By Theorem 3.3, we have $d(x^k, H(x^k)) \to 0$ as $k \to +\infty$. □

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