9 Electron–Positron Pair Production in Superstrong Laser Fields

Considering the interaction of charged particles with strong radiation fields in vacuum we looked at the non-quantum electrodynamic (QED) properties of electromagnetic vacuum. At such consideration, vacuum stipulates only the classical dispersion properties of EM waves propagating with the speed of light c. However, the latter is valid for radiation fields that are not superstrong ($\xi_0 < 1$), otherwise the excitation of QED vacuum and production of electron–positron pairs becomes possible.

As follows from the physical meaning of the wave intensity parameter ξ_0, at values of $\xi_0 > 1$ the energy acquired by an electron over a wavelength of a coherent radiation field exceeds the electron rest energy mc^2. On the other hand, the energetic width of the vacuum gap or the threshold value for the electron–positron pair production is $2mc^2$. This means that electrons of the Dirac vacuum acquiring the energy $\mathcal{E} > 2mc^2$ at the interaction with the wave field of intensity $\xi_0 > 1$ will pass from negative energy states to positive ones (excitation of the Dirac vacuum) and electron–positron pair production becomes a fact (with the presence of a third body for the satisfaction of the conservation laws for this process).

The production of electron–positron pairs by plane EM waves of relativistic intensities ($\xi_0 >> 1$) is essentially a multiphoton process, which principally differs from the known "Klein paradox" — production of electron–positron pairs in stationary and homogeneous electric field proceeding over the electron Compton wavelength. The latter corresponds to the tunnel effect through the effective energetic barrier of finite width formed from the vacuum gap of infinite width by the presence of a uniform electric field (Schwinger mechanism). The physical mechanisms are similar to two different limits of Above Threshold Ionization of atoms in strong radiation fields — multiphoton and tunnel ionization.

This chapter considers the excitation of the Dirac vacuum in superstrong EM fields and the electron–positron pair production process in the presence of a diverse type third body.

9.1 Vacuum in Superstrong Electromagnetic Fields. Klein Paradox

It has long been well known that in the background of a stationary and homogeneous electric field the QED vacuum is unstable and electron–positron (e^-, e^+) pair production from the vacuum occurs (this mechanism is often referred to as the Schwinger mechanism). However, a measurable rate for pair production requires extraordinarily strong electric field strengths comparable to the critical vacuum field strength

$$E_c = \frac{m^2 c^3}{e\hbar}, \tag{9.1}$$

the work of which on an electron over the Compton wavelength $\lambda_c = \hbar/mc$ equals the electron rest energy. As we will see the probability of this process reaches optimal values when

$$\zeta = \frac{E_0}{E_c} \gtrsim 1, \tag{9.2}$$

where E_0 is the magnitude of a uniform electric field strength.

Fortunately, it seems possible to produce EM fields with electric field strengths of the order of the Schwinger critical field in the focus of expected X-ray FEL and consideration of this problem is theoretically important, since it requires one to go beyond perturbation theory, and its experimental observation would verify the validity of theory in the domain of strong fields.

To solve the problem of e^-, e^+ pair production in the given electric field we shall make use of the Dirac model — all vacuum negative energy states are filled with electrons and e^-, e^+ pair production by the electric field occurs when the vacuum electrons with initial negative energies $\mathcal{E}_0 < 0$ due to "acceleration" pass to the final states with positive energies $\mathcal{E} > 0$. To distinguish the free particle states we will switch on and switch off the electric field elaborating on a model which retains the main features of the spatially uniform electric field and allows one to obtain an exact solution for the Dirac equation and final expressions for the pair production rate in closed form. Thus, we will assume an electric field of the form

$$\mathbf{E}(t) = \frac{E_0}{\cosh^2\left(\frac{t}{T}\right)}\widehat{\mathbf{z}}, \tag{9.3}$$

where T is the characteristic period of the field and $\widehat{\mathbf{z}}$ is the unit vector along the field strength. The vector potential corresponding to this field may be written as

$$\mathbf{A}(t) = -c \int\limits_{-\infty}^{t} \mathbf{E}(t)dt = -cE_0T\widehat{\mathbf{z}}\left[\tanh\left(\frac{t}{T}\right)+1\right]. \tag{9.4}$$

We will solve the Dirac equation in the spinor representation (see Eqs. (1.77), (1.78)). Since the interaction Hamiltonian does not depend on the space coordinates, the generalized momentum $\mathbf{p}_0$ is conserved. Hence, the solution of Eq. (1.77) may be represented in the form

$$\Psi(\mathbf{r},t) = \Psi_{\mathbf{p}_0}(t)\, e^{\frac{i}{\hbar}\mathbf{p}_0\mathbf{r}}, \tag{9.5}$$

and from Eq. (1.77) for the function $\Psi_{\mathbf{p}_0}(t)$ we obtain the following equation:

$$i\hbar\frac{d\Psi_{\mathbf{p}_0}}{dt} = \left[c\alpha\left(\mathbf{p}_0+\frac{e}{c}\mathbf{A}(t)\right)+mc^2\beta\right]\Psi_{\mathbf{p}_0}. \tag{9.6}$$

In this section the electron charge will be assumed to be $-e$. Since $\mathbf{A}(-\infty) = 0$ the solution of Eq. (9.6) at $t \to -\infty$ should be superposition of the free particle solutions $\psi^{(\varkappa)}_{\mathbf{p}_0,\sigma}$ with negative ($\varkappa = -1$) and positive ($\varkappa = 1$) energies and polarizations $\sigma = \pm\frac{1}{2}$ (spin projections $S_z = \pm\frac{1}{2}$ in the rest frame of the particle):

$$\psi^{(\varkappa)}_{\mathbf{p}_0,1/2} = \sqrt{\frac{1}{2\mathcal{E}_0\left(\mathcal{E}_0 - \varkappa c p_{0z}\right)}}\begin{pmatrix} \varkappa mc^2 w^{(1/2)} \\ \left(\mathcal{E}_0 - \varkappa c\sigma\mathbf{p}_0\right) w^{(1/2)} \end{pmatrix} e^{-\frac{i}{\hbar}\varkappa\mathcal{E}_0 t}, \tag{9.7}$$

$$\psi^{(\varkappa)}_{\mathbf{p}_0,-1/2} = \sqrt{\frac{1}{2\mathcal{E}_0\left(\mathcal{E}_0 + \varkappa c p_{0z}\right)}}\begin{pmatrix} \left(\mathcal{E}_0 + \varkappa c\sigma\mathbf{p}_0\right) w^{(-1/2)} \\ \varkappa mc^2 w^{(-1/2)} \end{pmatrix} e^{-\frac{i}{\hbar}\varkappa\mathcal{E}_0 t}, \tag{9.8}$$

where $\mathcal{E}_0 = \sqrt{c^2\mathbf{p}_0^2 + m^2c^4}$, σ are Pauli matrices, and the spinors $w^{(\pm 1/2)}$are

$$w^{(1/2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad w^{(-1/2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

At $t \to \infty$, the electric field $\mathbf{E}(\infty) = 0$ but

$$\mathbf{A}(\infty) = -2cE_0T\widehat{\mathbf{z}}, \tag{9.9}$$

and the solution of Eq. (9.6) at $t \to \infty$ should be superposition of the free particle solutions (9.7), (9.8) where the "final momentum"

$$\mathbf{p} = \mathbf{p}_0 - e \int_{-\infty}^{\infty} \mathbf{E}(t)dt = \mathbf{p}_0 + \frac{e}{c}\mathbf{A}(\infty) \tag{9.10}$$

stands for $\mathbf{p}_0$. Equation (9.6) in the quadratic form (see Eqs. (1.82), (1.83)) for the bispinor components

$$\Psi_{\mathbf{p}_0}(t) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \tag{9.11}$$

gives the following set of equations:

$$\left\{\hbar^2 \frac{d^2}{dt^2} + \mathcal{E}_0^2 + e^2 A^2(t) + 2ecp_{0z} A(t) \mp iec\hbar E(t)\right\} f_{1,2} = 0, \tag{9.12}$$

$$\left\{\hbar^2 \frac{d^2}{dt^2} + \mathcal{E}_0^2 + e^2 A^2(t) + 2ecp_{0z} A(t) \pm iec\hbar E(t)\right\} f_{3,4} = 0. \tag{9.13}$$

Thus, solving the equation

$$\left\{\hbar^2 \frac{d^2}{dt^2} + \mathcal{E}_0^2 + e^2 A^2(t) + 2ecp_{0z} A(t) - \delta iec\hbar E(t)\right\} \Phi = 0 \tag{9.14}$$

with $\delta = \pm 1$ one can construct the whole bispinor (9.11). Passing in Eq. (9.14) to the new variable

$$z = -e^{2\frac{t}{T}},$$

and seeking the solution in the form

$$\Phi(t) = (-z)^{i\frac{\mathcal{E}_0 T}{2\hbar}} (1-z)^{i\delta\frac{eE_0 T^2 c}{\hbar}} F(z), \tag{9.15}$$

we obtain the equation for hypergeometric function $F(\alpha, \beta, \gamma, z)$:

$$z(1-z)F'' + (\gamma - (\alpha + \beta + 1)z)F' - \alpha\beta F = 0. \tag{9.16}$$

The parameters α, β, γ are defined as follows:

$$\alpha(\mathcal{E}_0, \delta) = i\frac{\mathcal{E}_0 + \mathcal{E} + 2i\delta eE_0 cT}{2\hbar} T,$$

$$\beta\left(\mathcal{E}_0,\delta\right)=i\frac{\mathcal{E}_0-\mathcal{E}+2i\delta eE_0cT}{2\hbar}T, \tag{9.17}$$

$$\gamma\left(\mathcal{E}_0\right)=1+i\frac{\mathcal{E}_0}{\hbar}T,$$

where according to Eqs. (9.10) and (9.9)

$$\mathcal{E}=\sqrt{c^2\left(\mathbf{p}_0-2eE_0T\widehat{\mathbf{z}}\right)^2+m^2c^4}.$$

The general solution for hypergeometric equation (9.16) is

$$F(z)=F\left(\alpha,\beta,\gamma,z\right)+z^{1-\gamma}F\left(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,z\right). \tag{9.18}$$

Taking into account the relations

$$\alpha\left(\mathcal{E}_0,\delta\right)-\gamma\left(\mathcal{E}_0\right)+1=\alpha\left(-\mathcal{E}_0,\delta\right),$$

$$\beta\left(\mathcal{E}_0,\delta\right)-\gamma\left(\mathcal{E}_0\right)+1=\beta\left(-\mathcal{E}_0,\delta\right),$$

$$2-\gamma=\gamma\left(-\mathcal{E}_0\right),$$

$$i\frac{\mathcal{E}_0}{2\hbar}T+1-\gamma\left(\mathcal{E}_0\right)=-i\frac{\mathcal{E}_0}{2\hbar}T,$$

the general solution for bispinor $\Psi_{\mathbf{p}_0}(t)$ can be written as follows:

$$\Psi_{\mathbf{p}_0}(t)=\begin{pmatrix} A_1\Phi\left(\mathcal{E}_0,1;z\right)+A_2\Phi\left(-\mathcal{E}_0,1;z\right) \\ B_1\Phi\left(\mathcal{E}_0,-1;z\right)+B_2\Phi\left(-\mathcal{E}_0,-1;z\right) \\ C_1\Phi\left(\mathcal{E}_0,-1;z\right)+C_2\Phi\left(-\mathcal{E}_0,-1;z\right) \\ D_1\Phi\left(\mathcal{E}_0,1;z\right)+D_2\Phi\left(-\mathcal{E}_0,1;z\right) \end{pmatrix}, \tag{9.19}$$

where

$$\Phi\left(\mathcal{E}_0,\delta;z\right)=(-z)^{i\frac{\mathcal{E}_0}{2\hbar}T}\left(1-z\right)^{i\delta\frac{eE_0c}{\hbar}T^2}$$

$$\times F\left(\alpha\left(\mathcal{E}_0,\delta\right),\beta\left(\mathcal{E}_0,\delta\right),\gamma\left(\mathcal{E}_0\right),z\right), \tag{9.20}$$

and the coefficients $A_{1,2}$, $B_{1,2}$, $C_{1,2}$, $D_{1,2}$ should be defined from the initial condition.

To determine the probability of e^-, e^+ pair production we use the initial condition: at $t \to -\infty$ when $\mathbf{A}(-\infty) = 0$ this wave function must turn into the free Dirac equation solution with negative energy in accordance with the Dirac model. Then taking into account that at

$$t \to -\infty; \quad z \to 0,$$

$$\Phi(\mathcal{E}_0, \delta; z \to 0) = (-z)^{i\frac{\mathcal{E}_0}{2\hbar}T} = e^{\frac{i}{\hbar}\mathcal{E}_0 t},$$

we obtain

$$\Psi^{(-1)}_{\mathbf{p}_0,1/2} = \sqrt{\frac{1}{2\mathcal{E}_0(\mathcal{E}_0 + cp_{0z})}} \begin{pmatrix} -mc^2\Phi(\mathcal{E}_0, 1; z) \\ 0 \\ (\mathcal{E}_0 + cp_{0z})\Phi(\mathcal{E}_0, -1; z) \\ (cp_{0x} + icp_{0y})\Phi(\mathcal{E}_0, 1; z) \end{pmatrix}, \tag{9.21}$$

$$\Psi^{(-1)}_{\mathbf{p}_0,-1/2} = \sqrt{\frac{1}{2\mathcal{E}_0(\mathcal{E}_0 - cp_{0z})}} \begin{pmatrix} (-cp_{0x} + icp_{0y})\Phi(\mathcal{E}_0, 1; z) \\ (\mathcal{E}_0 + cp_{0z})\Phi(\mathcal{E}_0, -1; z) \\ 0 \\ -mc^2\Phi(\mathcal{E}_0, 1; z) \end{pmatrix}. \tag{9.22}$$

After the interaction at $t \to +\infty; \quad z \to -\infty$ these wave functions become the superposition of the free Dirac equation solutions. To determine the asymptotes of these functions we will use the following property of the hypergeometric function:

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)}(-z)^{-\alpha} F\left(\alpha, \alpha + 1 - \gamma, \alpha + 1 - \beta, \frac{1}{z}\right)$$

$$+\frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)}(-z)^{-\beta} F\left(\beta, \beta + 1 - \gamma, \beta + 1 - \alpha, \frac{1}{z}\right). \tag{9.23}$$

Hence, for the function Φ we obtain

$$\Phi(\mathcal{E}_0, \delta; z \to -\infty) = e^{-\frac{i}{\hbar}\mathcal{E}t} \frac{\Gamma(\gamma(\mathcal{E}_0))\Gamma(\beta(\mathcal{E}_0, \delta) - \alpha(\mathcal{E}_0, \delta))}{\Gamma(\beta(\mathcal{E}_0, \delta))\Gamma(\gamma(\mathcal{E}_0) - \alpha(\mathcal{E}_0, \delta))}$$

$$+e^{\frac{i}{\hbar}\mathcal{E}t}\frac{\Gamma\left(\gamma\left(\mathcal{E}_0\right)\right)\Gamma\left(\alpha\left(\mathcal{E}_0,\delta\right)-\beta\left(\mathcal{E}_0,\delta\right)\right)}{\Gamma\left(\alpha\left(\mathcal{E}_0,\delta\right)\right)\Gamma\left(\gamma\left(\mathcal{E}_0\right)-\beta\left(\mathcal{E}_0,\delta\right)\right)}. \tag{9.24}$$

Taking into account the relations

$$\frac{\mathcal{E}-\mathcal{E}_0+2eE_0cT}{\mathcal{E}_0-\mathcal{E}+2eE_0cT}=\frac{\mathcal{E}-cp_z}{\mathcal{E}_0+cp_{0z}},$$

$$p_{0z}-p_z=2eE_0T,$$

for the bispinor wave function (9.21) we obtain

$$\Psi^{(-1)}_{\mathbf{p}_0,1/2}\left(t\to+\infty\right)=C\left(\mathcal{E}\right)\psi^{(1)}_{\mathbf{p},1/2}+C\left(-\mathcal{E}\right)\psi^{(-1)}_{\mathbf{p},1/2}, \tag{9.25}$$

where

$$C\left(\mathcal{E}\right)=\sqrt{\frac{\mathcal{E}\mathcal{E}_0}{\left(\mathcal{E}_0-\mathcal{E}+2eE_0cT\right)\left(\mathcal{E}-\mathcal{E}_0+2eE_0cT\right)}}$$

$$\times\frac{2\Gamma\left(i\frac{\mathcal{E}_0}{\hbar}T\right)\Gamma\left(-i\frac{\mathcal{E}}{\hbar}T\right)}{\Gamma\left(i\frac{\mathcal{E}_0-\mathcal{E}+2eE_0cT}{2\hbar}T\right)\Gamma\left(i\frac{\mathcal{E}_0-\mathcal{E}-2eE_0cT}{2\hbar}T\right)}. \tag{9.26}$$

The probability of the e^-, e^+ pair production summed over the spin states is

$$W(\mathcal{E})=2\left|C\left(\mathcal{E}\right)\right|^2. \tag{9.27}$$

Taking into account that

$$\left|\Gamma\left(iy\right)\right|^2=\frac{\pi}{y\sin\pi iy},$$

for the probability (9.27) we obtain

$$W(\mathcal{E})=2\frac{\cosh\left(\pi\frac{2eE_0cT^2}{\hbar}\right)-\cosh\left(\pi\frac{\mathcal{E}-\mathcal{E}_0}{\hbar}T\right)}{\cosh\left(\pi\frac{\mathcal{E}+\mathcal{E}_0}{\hbar}T\right)-\cosh\left(\pi\frac{\mathcal{E}-\mathcal{E}_0}{\hbar}T\right)}. \tag{9.28}$$

The number of created e^-, e^+ pairs per unit space volume is

$$N=\int W(\mathcal{E})\frac{d\mathbf{p}_0}{\left(2\pi\hbar\right)^3},$$

which with Eq. (9.28) is written as

$$N = \frac{2}{(2\pi\hbar)^3} \int \frac{\cosh\left(\pi \frac{2eE_0cT^2}{\hbar}\right) - \cosh\left(\pi \frac{\mathcal{E}-\mathcal{E}_0}{\hbar} T\right)}{\cosh\left(\pi \frac{\mathcal{E}+\mathcal{E}_0}{\hbar} T\right) - \cosh\left(\pi \frac{\mathcal{E}-\mathcal{E}_0}{\hbar} T\right)} dp_{0z} dp_{0x} dp_{0y}. \quad (9.29)$$

The probability (9.28) has a maximum at $p_{0z} = eE_0T$ (the electrons and positrons are created with the same energy, i.e., $p_z = -eE_0T$). In the limit $T \to \infty$ the electric field (9.3) tends to a constant one: $\mathbf{E}(t) \to E_0\widehat{\mathbf{z}}$ and from Eq. (9.28) one can obtain the probability of the e^-, e^+ pair production in the static, spatially uniform electric field. In this case in the integral (9.29) over p_{0z} the main contribution gives the maximum point with the width $\delta p_{0z} \approx eE_0T$. Hence, at

$$(ceE_0T)^2 >> m^2c^4 + c^2p_{0\perp}^2; \qquad p_{0\perp} = \sqrt{p_{0x}^2 + p_{0y}^2},$$

we have

$$\mathcal{E}_0 \approx \mathcal{E} \approx ceE_0T + \frac{m^2c^4 + c^2p_{0\perp}^2}{2ceE_0T},$$

and for the number of e^-, e^+ pairs created per unit time and unit space volume we obtain

$$\frac{N}{T} \approx \frac{2}{(2\pi\hbar)^3} eE_0 \int \exp\left[-\pi \frac{m^2c^4 + c^2p_{0\perp}^2}{ceE_0\hbar}\right] dp_{0x} dp_{0y}. \quad (9.30)$$

Integrating in Eq. (9.30) over transversal momentum we obtain the Schwinger formula:

$$\frac{N_{Sch}}{T} = \frac{e^2E_0^2}{4\pi^3\hbar^2c} \exp\left[-\pi \frac{m^2c^3}{e\hbar E_0}\right], \quad (9.31)$$

or in the terms of critical field

$$\frac{N_{Sch}}{T} = \frac{\zeta^2}{4\pi^3\lambda_c^3} \frac{mc^2}{\hbar} \exp\left[-\frac{\pi}{\zeta}\right]. \quad (9.32)$$

If $\zeta << 1$ the probability of pair production is exponentially suppressed and reaches the optimal values when $\zeta \gtrsim 1$ at which

$$\frac{N_{Sch}}{T} \gtrsim 10^{49} \mathrm{cm}^{-3}\mathrm{c}^{-1}.$$

9.2 Electron–Positron Pair Production by Superstrong Laser Field and γ-Quantum

For the electron–positron pair production by superstrong laser fields of relativistic intensities as a third body for the satisfaction of conservation laws in physically more interesting cases can serve a γ-quantum or a nucleus/ion.

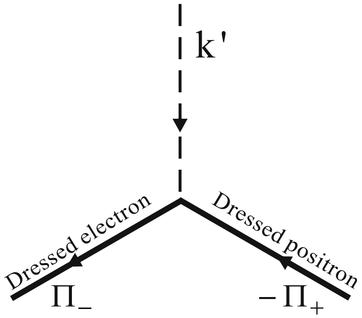

Fig. 9.1. Feynman diagram for electron–positron pair production by laser field and γ-quantum.

The e^-, e^+ pair production process by a plane monochromatic radiation field and a γ-quantum in the scope of QED is described by the first order Feynman diagram (Fig. 9.1) where wave functions (1.94) correspond to electron/positron lines. As in QED the production of electron and positron with quasimomentums Π_- and Π_+ respectively is interpreted as a transition of an electron from the vacuum state "$-\Pi_+$" to state Π_-. The Feynman diagram is topologically equivalent to that of the Compton effect. Hence, the S-matrix amplitude of this process can be obtained from the Compton-effect S-matrix amplitude (1.114) by the substitutions: $\epsilon_\mu^* \to \epsilon_\mu$, $k' \to -k'$, $\Pi \to -\Pi_+$, $\Pi' \to \Pi_-$:

$$S_{fi} = -i\,(2\pi\hbar)^4 \sqrt{\frac{\pi\alpha_0}{2\omega' c\Pi_{0+}\Pi_{0-}V^3}}\,\overline{u}_{\sigma'}(p_-)$$

$$\times \widehat{M}_{fi}^{(Compton)}\left(\epsilon^* \to \epsilon, k' \to -k', \Pi \to -\Pi_+, \Pi' \to \Pi_-\right) u_\sigma(-p_+). \quad (9.33)$$

We will assume that the γ-quantum is nonpolarized and corresponding summation over the electron and positron polarizations will be made. Taking into account that at the summation over the positron polarizations one should replace $u(-p_+)\overline{u}(-p_+)$ by $c^2(\widehat{p}_+ - mc)$ one can see that

$$\frac{1}{2}\sum_{\sigma',\sigma,\epsilon} |S_{fi}|^2$$

$$= -\frac{1}{2}\sum_{\sigma',\sigma,\epsilon,\epsilon} |S_{fi}|^2_{(Compton)} \left(k' \to -k', \Pi \to -\Pi_+, \Pi' \to \Pi_-\right). \qquad (9.34)$$

For the differential probability of e^-, e^+ pair production per unit time we have

$$dW = \frac{1}{2\Delta t}\sum_{\sigma',\sigma,\epsilon} |S_{fi}|^2 V \frac{d\mathbf{\Pi}_-}{(2\pi\hbar)^3} V \frac{d\mathbf{\Pi}_+}{(2\pi\hbar)^3}. \qquad (9.35)$$

Hence, using Eqs. (1.114) for the Compton effect and taking into account relation (9.34) for the differential probability (9.35) we obtain

$$dW = \sum_{s>s_m}^{\infty} W^{(s)} \delta\left(\Pi_- + \Pi_+ - \hbar k' - s\hbar k\right) d\mathbf{\Pi}_- d\mathbf{\Pi}_+, \qquad (9.36)$$

where

$$W^{(s)} = \frac{\alpha_0 m^2 c^6}{2\pi\omega'\hbar^2 \Pi_{0+}\Pi_{0-}} \left[|G_s|^2 - \left(1 - \frac{\hbar^2 (kk')^2}{2(p_+k)(p_-k)}\right) \right.$$

$$\times \left(\frac{(1+g^2)\xi_0^2}{4} \left(|G_{s-1}|^2 + |G_{s+1}|^2 - 2|G_s|^2 \right) \right.$$

$$\left. \left. + \frac{(1-g^2)\xi_0^2}{4} Re\left[2G^*_{s-1}G_{s+1} - G^*_s\left(G_{s-2} + G_{s+2}\right)\right] \right) \right]. \qquad (9.37)$$

The arguments of the functions $G_s(\alpha, \beta, \varphi)$ in this case are

$$\alpha = \frac{eA_0}{\hbar c}\left[\left(\frac{\mathbf{e}_1\mathbf{p}_-}{p_-k} - \frac{\mathbf{e}_1\mathbf{p}_+}{p_+k}\right)^2 + g^2\left(\frac{\mathbf{e}_2\mathbf{p}_-}{p_-k} - \frac{\mathbf{e}_2\mathbf{p}_+}{p_+k}\right)^2\right]^{1/2}, \qquad (9.38)$$

$$\beta = -\frac{e^2 A_0^2}{8\hbar c^2}(1-g^2)\left(\frac{1}{p_+k} + \frac{1}{p_-k}\right), \qquad (9.39)$$

$$\tan\varphi = \frac{g\left(\frac{\mathbf{e}_2\mathbf{p}_-}{p_-k} - \frac{\mathbf{e}_2\mathbf{p}_+}{p_+k}\right)}{\left(\frac{\mathbf{e}_1\mathbf{p}_-}{p_-k} - \frac{\mathbf{e}_1\mathbf{p}_+}{p_+k}\right)}. \qquad (9.40)$$

Since the pair production is a threshold effect, the number of photons absorbed from the strong wave must exceed the threshold value

$$s_m = \frac{2m^{*2}c^2}{\hbar^2\,(k'k)}, \tag{9.41}$$

which follows from the conservation law of this process expressed by the δ-function in Eq. (9.36) and the dispersion law for quasimomentum (1.96). Note that in Eq. (9.41) the effective mass appears which depends on the laser intensity. If $s_m > 1$ (for low photon energies), production of the electron–positron pair may only proceed by nonlinear channels (even for $\xi_0 << 1$). Besides, this process does not have a classical limit and the quantum recoil is always essential.

For the concreteness we will investigate the case of circular polarization of the incident wave ($g = \pm 1$). In this case $|G_s|^2 = J_s^2(\alpha)$ and from Eq. (9.37) for the partial probabilities we have

$$W^{(s)} = \frac{e^2m^2c^5}{2\pi\omega'\hbar^3\Pi_{0+}\Pi_{0-}}\left[J_s^2(\alpha) - \xi_0^2\left(1 - \frac{\hbar^2\,(kk')^2}{2\,(p_+k)\,(p_-k)}\right)\right.$$

$$\left.\times\left(\left(\frac{s^2}{\alpha^2} - 1\right)J_s^2(\alpha) + J_s'^2(\alpha)\right)\right]. \tag{9.42}$$

Taking into account the conservation laws, as well as the relations $p_-k = \Pi_-k$ and $p_+k = \Pi_+k$, the argument of the Bessel function can be written as

$$\alpha = \xi_0\frac{mc^2}{\hbar\omega}\left|\left[\mathbf{k}\left(\frac{\mathbf{p}_-}{p_-k} - \frac{\mathbf{p}_+}{p_+k}\right)\right]\right|$$

$$= \xi_0\frac{mc}{\hbar}\left[2s\hbar\left(\frac{1}{\Pi_-k} + \frac{1}{\Pi_+k}\right) - m_*^2c^2\left(\frac{1}{\Pi_-k} + \frac{1}{\Pi_+k}\right)^2\right]^{1/2}. \tag{9.43}$$

For a weak EM wave: $\xi_0 << 1$ and $s_m < 1$ (linear theory) the argument of the Bessel function $\alpha << 1$ and the main contribution to the probability of the pair production is the one-photon process. In this case $J_1^2(\alpha_1) \simeq \alpha_1^2/4$, $J_1'^2(\alpha_1) \simeq 1/4$, $\Pi_{0+} \simeq \mathcal{E}_+$, $\Pi_{0-} \simeq \mathcal{E}_-$ and taking into account that

$$1 - \frac{(kk')^2}{2\,(p_+k)\,(p_-k)} = -\frac{1}{2}\left[\frac{p_-k}{p_+k} + \frac{p_+k}{p_-k}\right],$$

we obtain the G. Breit, A. Wheeler formula:

$$W^{(1)} = \frac{e^2m^2c^5}{8\pi\omega'\hbar^3\mathcal{E}_+\mathcal{E}_-}\xi_0^2\left[2\left(\frac{m^2c^2}{\hbar p_-k} + \frac{m^2c^2}{\hbar p_+k}\right)\right.$$

$$-\left(\frac{m^2c^2}{\hbar p_- k}+\frac{m^2c^2}{\hbar p_+ k}\right)^2+\left[\frac{p_- k}{p_+ k}+\frac{p_+ k}{p_- k}\right]\Bigg]. \tag{9.44}$$

For a strong EM wave it is more convenient to choose the quantum recoil parameter as an integration variable:

$$\rho=\frac{\hbar^2\left(kk'\right)^2}{2\left(p_+k\right)\left(p_-k\right)}=\frac{\hbar^2\left(kk'\right)^2}{2\left(\Pi_+k\right)\left(\Pi_-k\right)}. \tag{9.45}$$

Taking into account the azimuthal symmetry with respect to the wave propagation direction one can make the following replacement:

$$\delta\left(\Pi_-+\Pi_+-\hbar k'-s\hbar k\right)\frac{d\mathbf{\Pi}_-d\mathbf{\Pi}_+}{\Pi_{0+}\Pi_{0-}}=>\frac{2\pi}{c^2}\frac{1}{\rho\sqrt{\rho^2-2\rho}}d\rho, \tag{9.46}$$

and we obtain

$$W=\frac{e^2m^2c^3}{\omega'\hbar^3}\sum_{s>s_m}^{\infty}\int_2^{2s/s_m}\Bigg[J_s^2(\alpha_s(\rho))+\xi_0^2(\rho-1)$$

$$\times\left(\left(\frac{s^2}{\alpha_s^2(\rho)}-1\right)J_s^2(\alpha_s(\rho))+J_s'^2(\alpha_s(\rho))\right)\Bigg]\frac{d\rho}{\rho\sqrt{\rho^2-2\rho}}, \tag{9.47}$$

where the argument of the Bessel function is

$$\alpha_s(\rho)=\frac{\xi_0}{\sqrt{1+\xi_0^2}}s_m\left[\frac{2s}{s_m}\rho-\rho^2\right]^{1/2}. \tag{9.48}$$

The latter reaches its maximal value

$$\alpha_{s\,\max}=\frac{\xi_0}{\sqrt{1+\xi_0^2}}s \tag{9.49}$$

at $\rho=s/s_m$. This value is in the integration range when $s>2s_m$. If $s_m>>1$, which is possible for not so hard γ-quantum, and at $\xi_0\gg 1$ one can approximate the Bessel function by the Airy one (see Eq. (1.69) for Compton effect) and for the probability of the pair production we obtain

$$W\simeq\frac{e^2m^2c^3}{\omega'\hbar^3}\sum_{s>s_m}^{\infty}\int_2^{2s/s_m}\left\{\left[1+\xi_0^2(\rho-1)\left(\frac{s^2}{\alpha^2(\rho)}-1\right)\right]\left(\frac{2}{s}\right)^{2/3}Ai^2(Z)\right.$$

$$+\xi_0^2 (\rho - 1) Ai'^2 (Z) \left(\frac{2}{s}\right)^{4/3} \Bigg\} \frac{d\rho}{\rho\sqrt{\rho^2 - 2\rho}}, \tag{9.50}$$

where

$$Z = \frac{1}{1+\xi_0^2} \left(\frac{s}{2}\right)^{2/3} \left(1 + \xi_0^2 \left(1 - \frac{s_m}{s}\rho\right)^2\right). \tag{9.51}$$

As far as the Airy function exponentially decreases with increasing of the argument one can conclude that the optimal parameters for the pair production process are determined from the condition $Z_{\min} \sim 1$, where

$$Z_{\min} = \left(\frac{s}{2}\right)^{2/3} \left(1 - \frac{\alpha^2_{s\max}}{s^2}\right) \simeq \left(\frac{s}{2\xi_0^3}\right)^{2/3},$$

which gives

$$2\xi_0^3 \gtrsim s_m.$$

For $\xi_0 >> 1$, $s_m \simeq 2m^2c^2\xi_0^2/(\hbar^2 k' k)$ we obtain

$$\zeta = \frac{\hbar^2 k' k}{m^2c^2}\xi_0 \gtrsim 1. \tag{9.52}$$

The latter means that in the rest frame of created electron the electric field strength of the EM wave exceeds the critical vacuum field (9.1). Hence, ζ is the quantum parameter of interaction in the scale of the critical vacuum field.

For $Z_{\min} >> 1$ or $\zeta << 1$ (so called tunneling regime of the pair production process) one can use the following asymptotic formula for the Airy function:

$$Ai(Z) \simeq \frac{1}{2\sqrt{\pi}} Z^{-1/4} \exp\left(-\frac{2Z^{3/2}}{3}\right).$$

Hence, the probability of the electron–positron pair production

$$W \propto \exp\left(-\frac{4}{3\zeta}\right) \tag{9.53}$$

is exponentially suppressed.

For the moderate relativistic intensities $\xi_0 \sim 1$ to show the dependence of the probability on the wave intensity and quantum parameter of interaction ζ the normalized probability

$$\widetilde{W} = \frac{\omega' \hbar^3}{e^2 m^2 c^3} W \tag{9.54}$$

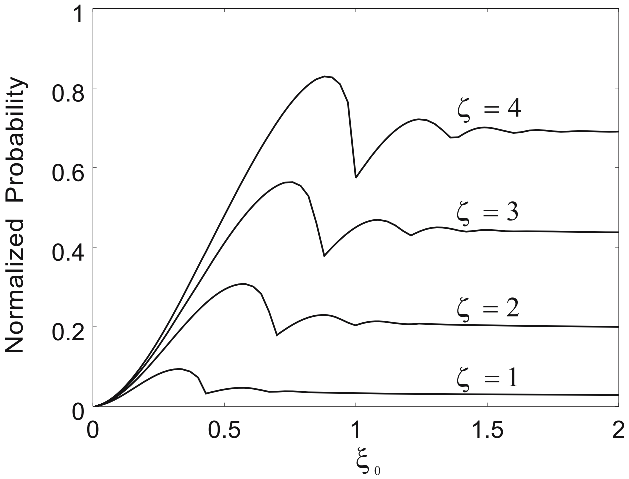

Fig. 9.2. The normalized probability $\widetilde{W} = \hbar^3\omega' W/(e^2m^2c^3)$ as a function of relativistic parameter of intensity ξ_0 for various ζ.

is displayed in Fig. 9.2 as a function of ξ_0 for various ζ.

9.3 Pair Production via Superstrong Laser Beam Scattering on a Nucleus

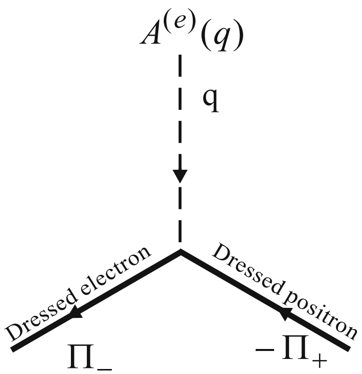

Fig. 9.3. Feynman diagram for electron–positron pair production via laser beam scattering on a nucleus.

The electron–positron pair production via superstrong laser beam scattering on a nucleus can be described again by the first-order Feynman diagram (Fig. 9.3) where wave functions (1.94) correspond to electron/positron lines.

The Feynman diagram is topologically equivalent to that of the stimulated bremsstrahlung (SB) effect. As in the previous section the S-matrix amplitude of this process can be obtained from the S-matrix amplitude of SB (1.128) by the substitutions: $\Pi \to -\Pi_+$, $\Pi' \to \Pi_-$:

$$S_{fi} = \frac{-i\pi e}{Vc\sqrt{\Pi_{0+}\Pi_{0-}}}\overline{u}_{\sigma'}(p_-)\widehat{M}_{fi}^{(SB)}\left(\Pi \to -\Pi_+, \Pi' \to \Pi_-\right)u_\sigma(-p_+). \tag{9.55}$$

Making the summation over the electron and positron polarizations one can see that

$$\sum_{\sigma',\sigma}\left|S_{fi}\right|^2 = \sum_{\sigma',\sigma}\left|S_{fi}\right|^2_{SB}\left(\Pi \to -\Pi_+, \Pi' \to \Pi_-\right). \tag{9.56}$$

The differential probability of e^-, e^+ pair production per unit time is written as

$$dW = \frac{1}{\Delta t}\sum_{\sigma',\sigma}\left|S_{fi}\right|^2 V\frac{d\mathbf{\Pi}_-}{(2\pi\hbar)^3}V\frac{d\mathbf{\Pi}_+}{(2\pi\hbar)^3}. \tag{9.57}$$

Hence, using Eq. (1.129) for the SB process and taking into account Eq. (9.56) for the differential probability of pair production per unit time we obtain

$$dW = \sum_{s>s_m}^{\infty} W^{(s)}\delta\left(\Pi_{0+} + \Pi_{0-} - s\hbar\omega\right)d\mathbf{\Pi}_- d\mathbf{\Pi}_+, \tag{9.58}$$

where

$$W^{(s)} = \frac{4\pi}{\Pi_{0+}\Pi_{0-}}\frac{e^2\left|\varphi\left(\mathbf{q}_s\right)\right|^2}{(2\pi\hbar)^6\hbar}\left\{\frac{\hbar^2\mathbf{q}_s^2c^2}{4}\left|B_s\right|^2 + \frac{e^2\hbar^2\left[\mathbf{k}\mathbf{q}_s\right]^2}{4(kp_-)(kp_+)}\right.$$

$$\left.\times\left[\left|\mathbf{B}_{1s}\right|^2 - ReB_{2s}B_s^*\right] - \left|\mathcal{E}_+B_s + \frac{e\left(\mathbf{p}_+\mathbf{B}_{1s}\right)\omega}{(kp_+)c} + \frac{e^2\omega}{2c^2(kp_+)}B_{2s}\right|^2\right\}, \tag{9.59}$$

and

$$\hbar\mathbf{q}_s = \mathbf{\Pi}_- + \mathbf{\Pi}_+ - s\hbar\mathbf{k}.$$

The threshold value of the photon number for this process is defined as follows:

$$s_m = \frac{2m^*c^2}{\hbar\omega}. \tag{9.60}$$

The arguments α, β, φ of the functions B_s, $\mathbf{B}_{1s}$, B_{2s} are defined according to Eqs. (9.38)–(9.40).

In the case of circular polarization of an incident strong wave ($g = 1$) we have

$$G_s(\alpha, 0, \varphi) = (-1)^s J_s(\alpha) e^{is\varphi}.$$

Taking into account the azimuthal symmetry with respect to the wave propagation direction one can make the following replacement:

$$\delta\left(\Pi_{0+} + \Pi_{0-} - s\hbar\omega\right) d\mathbf{\Pi}_- d\mathbf{\Pi}_+ \to 2\pi m^* \frac{\Pi_{0-}\left|\mathbf{\Pi}_-\right| \Pi_{0+}\left|\mathbf{\Pi}_+\right|}{c^2}$$

$$\times \sin\theta_+ \sin\theta_- d\theta_- d\theta_+ d\phi d\gamma_+, \tag{9.61}$$

where $\gamma_+ = \Pi_{0+}/(m^* c^2)$, θ_+, θ_- are the scattering angles of positron and electron with respect to the EM wave propagation direction and ϕ is the angle between the planes formed by $\mathbf{\Pi}_-$, $\mathbf{k}$ and $\mathbf{\Pi}_+$, $\mathbf{k}$. Hence, for the differential probability of e^-, e^+ pair production per unit time we have

$$dW = \frac{2\pi^2 \alpha_0 m^*}{(2\pi\hbar)^6 c} \sum_{s>s_m}^{\infty} \left|\mathbf{\Pi}_-\right| \left|\mathbf{\Pi}_+\right| \left|\varphi(\mathbf{q}_s)\right|^2$$

$$\times \left\{ \left[\hbar^2 \mathbf{q}_s^2 c^2 - 4\left(\Pi_{0+} - \frac{s\hbar\omega}{(kp_+)} \frac{\varkappa\left[\mathbf{k}\mathbf{p}_+\right]}{\varkappa^2} \right)^2 \right] J_s^2(\alpha_s) \right.$$

$$+ \frac{\hbar^2 e^2 A_0^2}{(kp_-)(kp_+)} \left[\mathbf{k}\mathbf{q}_s\right]^2 \left[\left(\frac{s^2}{\alpha_s^2} - 1 \right) J_s^2(\alpha_s) + J_s'^2(\alpha_s) \right]$$

$$\left. - \frac{4e^2 A_0^2}{(kp_+)^2} \frac{\left[\varkappa\left[\mathbf{k}\mathbf{p}_+\right]\right]^2}{\varkappa^2} J_s'^2(\alpha_s) \right\} \sin\theta_+ \sin\theta_- d\theta_- d\theta_+ d\phi d\gamma_+. \tag{9.62}$$

In this equation the electron quasienergy and quasimomentum are defined via Π_{0+} according to conservation law and

$$\varkappa = \frac{\left[\mathbf{k}\mathbf{p}_+\right]}{p_+ k} - \frac{\left[\mathbf{k}\mathbf{p}_-\right]}{p_- k}. \tag{9.63}$$

The Bessel function argument in Eq. (9.62)

$$\alpha_s = \frac{eA_0}{\hbar\omega}\left|\varkappa\right|$$

can be represented in the form

$$\alpha_s = \frac{\xi_0 s_m}{2\sqrt{1+\xi_0^2}} \left[\frac{\beta_+^2 \sin^2\theta_+}{(1-\beta_+\cos\theta_+)^2} + \frac{\beta_-^2 \sin^2\theta_-}{(1-\beta_-\cos\theta_-)^2} \right.$$

$$\left. -2\frac{\beta_-\beta_+ \sin\theta_+ \sin\theta_- \cos\phi}{(1-\beta_+\cos\theta_+)(1-\beta_-\cos\theta_-)} \right]^{1/2}, \qquad (9.64)$$

where

$$\beta_\pm = \frac{c\,|\mathbf{\Pi}_\pm|}{\Pi_{0\pm}}; \quad \Pi_{0-} = s\hbar\omega - \Pi_{0+}.$$

In this particular case we utilize Eq. (9.62) in order to obtain the electron–positron pair production probability on the Coulomb potential for which the Fourier transform is

$$\varphi(\mathbf{q}_s) = \frac{4\pi Z_a e}{\mathbf{q}_s^2}. \qquad (9.65)$$

Then taking into account Eq. (9.65) for the differential probability of e^-, e^+ pair production by a strong plane monochromatic wave per unit time at the scattering on the Coulomb field we will have

$$dW = \alpha_0^2 \frac{Z_a^2 m^*}{2\pi^2\hbar} \sum_{s>s_m}^{\infty} \frac{|\mathbf{\Pi}_-|\,|\mathbf{\Pi}_+|}{\hbar^4 \mathbf{q}_s^4}$$

$$\left\{ \left[\hbar^2 \mathbf{q}_s^2 c^2 - \frac{4}{\varkappa^4}\left(\varkappa\left(\frac{\Pi_{0-}[\mathbf{k}\mathbf{\Pi}_+]}{\Pi_+ k} + \frac{\Pi_{0+}[\mathbf{k}\mathbf{\Pi}_-]}{\Pi_- k}\right)\right)^2 \right] J_s^2(\alpha_s) \right.$$

$$-\frac{4e^2A_0^2}{\varkappa^2}\left(\frac{[[\mathbf{k}\mathbf{\Pi}_-][\mathbf{k}\mathbf{\Pi}_+]]}{(k\Pi_-)(k\Pi_+)}\right)^2 J_s'^2(\alpha_s) + \frac{e^2A_0^2}{(k\Pi_-)(k\Pi_+)}\left[\mathbf{k}\left(\mathbf{\Pi}_- + \mathbf{\Pi}_+\right)\right]^2$$

$$\left. \times \left[\left(\frac{s^2}{\alpha_s^2} - 1\right) J_s^2(\alpha_s) + J_s'^2(\alpha_s) \right] \right\} \sin\theta_+ \sin\theta_- d\phi d\theta_- d\theta_+ d\gamma_+. \qquad (9.66)$$

For a weak EM wave the main contribution in this process is the one-photon process. Dividing the differential probability (9.66) by the initial flux density

$$J = \frac{1}{\hbar\omega}\frac{c}{4\pi}E_0^2$$

we obtain the H.A. Bethe, W. Heitler formula:

$$d\sigma = \alpha_0^3 \frac{Z_a^2}{2\pi} \frac{|\mathbf{p}_-|\,|\mathbf{p}_+|}{\hbar^4 \mathbf{q}_1^4} \frac{1}{\hbar\omega^3}$$

$$\times \left\{ \hbar^2 \mathbf{q}_1^2 c^2 \left(\frac{[\mathbf{k}\mathbf{p}_+]}{p_+ k} - \frac{[\mathbf{k}\mathbf{p}_-]}{p_- k} \right)^2 - 4 \left(\frac{\mathcal{E}_- [\mathbf{k}\mathbf{p}_+]}{p_+ k} + \frac{\mathcal{E}_+ [\mathbf{k}\mathbf{p}_-]}{p_- k} \right)^2 \right.$$

$$\left. + \frac{2\hbar^2\omega^2}{(kp_-)(kp_+)} [\mathbf{k}(\mathbf{p}_- + \mathbf{p}_+)]^2 \right\} \sin\theta_+ \sin\theta_- d\phi d\theta_- d\theta_+ d\mathcal{E}_+. \tag{9.67}$$

In general the expression for the differential probability of e^-, e^+ pair production by strong radiation field (9.66) is very complicated (one should perform four-dimensional integration and summation over photon numbers) but without integration one can make conclusions about optimal values of laser parameters for the measurable pair production probability using the properties of the Bessel function. The Bessel function argument in Eq. (9.66) $\alpha_s(\gamma_+, \theta_+, \theta_-, \phi)$ as a function of θ_+, θ_-, ϕ reaches its maximal value at

$$\cos\theta_+ = \beta_+, \quad \cos\theta_- = \beta_-, \quad \cos\phi = -1,$$

and is equal to

$$\overline{\alpha}_s(\gamma_+) = \frac{\xi_0 s_m}{2\sqrt{1+\xi_0^2}} \left(\sqrt{\gamma_+^2 - 1} + \sqrt{\left(\frac{2s}{s_m} - \gamma_+ \right)^2 - 1} \right). \tag{9.68}$$

The latter is always small compared with the Bessel function index. Indeed, as follows from the conservation law

$$1 \leq \gamma_+ \leq \frac{2s}{s_m} - 1,$$

and in this range $\overline{\alpha}_s(\gamma_+)$ reaches its maximal value

$$\overline{\alpha}_{s\,\max} = \frac{\xi_0}{\sqrt{1+\xi_0^2}} \sqrt{s^2 - s_m^2} < s \tag{9.69}$$

at the $\gamma_+ = s/s_m$. Hence, for $\xi_0 \gg 1$ and $s_m >> 1$ the main contribution to the differential probability will give the number of photons $s >> s_m$ and as in the previous section one can approximate the Bessel function by the Airy one (1.69). The Airy function argument for $\alpha \simeq \overline{\alpha}_{s\,\max}$ will be

$$Z(s) \simeq \frac{1}{2^{2/3}\xi_0^2} s^{2/3} \left(1 + \xi_0^2 \frac{s_m^2}{s^2} \right). \tag{9.70}$$

As the Airy function exponentially decreases with increasing of the argument one can conclude that the optimal parameters for the pair production process are determined from the condition $Z_{\min} \sim 1$, $Z_{\min}$ being the minimum value of $Z(s)$. The latter corresponds to the number of photons $s = \sqrt{2}\xi_0 s_m$ at which

$$Z_{\min} = Z\left(\sqrt{2}\xi_0 s_m\right) = 3\left(\frac{E_c}{2E_0}\right)^{2/3}, \tag{9.71}$$

where E_c is the vacuum critical field strength (9.1). Hence, at $\xi \geq 1$ the probability reaches optimal values when $\zeta \equiv E_c/E_0 \geq 1$ (at $\xi_0 << 1$ quantum effects are optimal when $\zeta \sim \xi_0$, which corresponds to linear theory, that is, the perturbation theory of QED). When $\zeta << 1$ according to Eq. (9.53) the probability is exponentially suppressed:

$$W \propto \exp(-2\sqrt{3}/\zeta), \tag{9.72}$$

as in the Schwinger mechanism for e^-, e^+ pair production in the uniform electrostatic field, where $W \propto \exp(-\pi/\zeta)$. For the available superstrong optical lasers $\zeta \sim 10^{-4}$, which practically does not allow for measurable pair creation probability. As was argued, one can achieve $\zeta \sim 10^{-1}$ at the focus of expected X-ray FEL facilities, which will allow for measurable pair creation probability by the Schwinger mechanism.

Note that in the considered process of pair production on a nucleus one can achieve the condition $\zeta \geq 1$ (even $\zeta >> 1$) in the scheme of counterpropagating nucleus beam and X-ray FEL. Then, in the rest frame of the nucleus we will have $\zeta \simeq 2\zeta_L \gamma_L$, where γ_L is the Lorenz factor of nucleus and ζ_L is the field parameter in the laboratory frame. Since ξ_0 is the Lorenz invariant, then if $\xi_0 \geq 1$ and $\gamma_L > E_c/2E_0$ in the laboratory frame, the probability of multiphoton e^-, e^+ pair production reaches its optimal value.

9.4 Nonlinear e^-, e^+ Pair Production in Plasma by Strong EM Wave

As was shown in Chapter 6 for electron–positron pair production by a γ-quantum or a plane monochromatic EM wave, a macroscopic medium with a refractive index $n_0(\omega_0) < 1$ may serve as a third body for the satisfaction of conservation laws. In such a plasmalike medium the multiphoton production of e^-, e^+ pairs by a strong laser radiation field is possible at ordinary densities of plasma, in contrast to single-photon production $\gamma \to e^- + e^+$, which is only accessible in a superdense plasma with the electron density $\rho \gtrsim 3 \cdot 10^{34}\text{cm}^{-3}$.

In laser fields with $\xi_0 \sim 1$ when the energy of the interaction of an electron (of the Dirac vacuum) with the field over a wavelength becomes comparable to the electron rest energy ($eE_0\lambda_0 \sim mc^2$) the multiphoton pair-production process goes in through nonlinear channels. At such intensities, in general, the dispersion law of a plasma becomes nonlinear, too; i.e., the refractive index depends on the wave intensity: $n_0 = n_0(\omega_0, \xi_0^2)$. As is known, because of the intensity effect, the transparency range of a plasma widens and the dispersion law $n_0(\omega_0, \xi_0^2) < 1$, which is necessary for the production of e^-, e^+ pairs, holds all the more. But the intensities required for the appearance of a real nonlinearity in dispersion become essential when $\xi_0 >> 1$. Hence, in considering fields $\xi_0 \sim 1$ the dispersion law of a plasma can be regarded as linear ($n_0^2(\omega_0) = 1 - 4\pi\rho e^2/m\omega_0^2$).

Let a plane transverse linearly polarized EM wave with frequency ω_0 and vector potential

$$\mathbf{A}\,(\mathbf{r}, t) = \mathbf{A}_0 \cos\left(\omega_0 t - \mathbf{k}_0 \mathbf{r}\right); \quad |\mathbf{k}_0| = n_0 \frac{\omega_0}{c} \tag{9.73}$$

propagate in a plasma. The multiphoton degree s for the e^-, e^+ pair production in the light fields is defined by the condition (reaction threshold)

$$s\hbar\omega_0 \geqslant \frac{2mc^2}{\sqrt{1 - n_0^2}}. \tag{9.74}$$

To determine the multiphoton probabilities of this process it is convenient to solve the problem in the center-of-mass frame of the produced pair (C frame), in which the wave vector of the photons is $\mathbf{k}' = 0$ (the refractive index of the plasma in this frame is $n' = 0$). The velocity of the C frame with respect to the laboratory frame (L frame) is $\mathrm{v} = cn_0$. The traveling EM wave is transformed in the C frame into a varying electric field (the magnetic field $H' = 0$) with a vector potential

$$\mathbf{A}'\,(t') = \frac{\mathbf{A}_0}{2}[\exp(i\omega' t') + \exp(-i\omega' t')], \quad \omega' = \omega_0 \sqrt{1 - n_0^2}. \tag{9.75}$$

It is easily noted that with Eq. (9.75) taken into account the reaction threshold condition (9.74) is obtained from the laws of the conservation of energy $\mathcal{E}'_- + \mathcal{E}'_+ = s\hbar\omega'$ and momentum $\mathbf{p}'_- + \mathbf{p}'_+ = s\hbar\mathbf{k}' = 0$ in the C frame ($\mathcal{E}'_-$, $\mathbf{p}'_-$ and $\mathcal{E}'_+$, $\mathbf{p}'_+$ are the energy and momentum of the electron and positron, respectively, in the C frame).

To solve the problem of s-photon production of an e^-, e^+ pair in the given radiation field (9.73), we shall make use of the Dirac model (all vacuum negative-energy states are filled with electrons and the interaction of the external field proceeds only with this vacuum: on the other hand, the interaction with the plasma electrons reduces to a refraction of the wave only).

The Dirac equation in the field (9.75) has the form

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[c\widehat{\alpha}\left(\mathbf{p}' - e\mathbf{A}'\left(t'\right)\right) + \widehat{\beta}mc^2\right]\Psi, \tag{9.76}$$

where the Dirac matrices $\widehat{\alpha}$, $\widehat{\beta}$ will be chosen in the standard representation, with σ the Pauli matrices. Since in the C frame the interaction Hamiltonian does not depend on the space coordinates, the solution of Eq. (9.76) can be represented in the form of a linear combination of free solutions of the Dirac equation with amplitudes $a_i(t')$ depending only on time:

$$\Psi_{\mathbf{p}'}\left(\mathbf{r}', t'\right) = \sum_{i=1}^{4} a_i(t')\Psi_i^{(0)}\left(\mathbf{r}', t'\right). \tag{9.77}$$

Here

$$\Psi_{1,2}^{(0)}\left(\mathbf{r}', t'\right) = \sqrt{\frac{\mathcal{E}' + mc^2}{2\mathcal{E}'}}\begin{pmatrix}\varphi_{1,2}\\ \frac{c\sigma\mathbf{p}'}{\mathcal{E}'+mc^2}\varphi_{1,2}\end{pmatrix} e^{\frac{i}{\hbar}\left(\mathbf{p}'\mathbf{r}' - \mathcal{E}'t'\right)},$$

$$\Psi_{3,4}^{(0)}\left(\mathbf{r}', t'\right) = \sqrt{\frac{\mathcal{E}' + mc^2}{2\mathcal{E}'}}\begin{pmatrix}\frac{-c\sigma\mathbf{p}'}{\mathcal{E}'+mc^2}\chi_{3,4}\\ \chi_{3,4}\end{pmatrix} e^{\frac{i}{\hbar}\left(\mathbf{p}'\mathbf{r}' + \mathcal{E}'t'\right)}, \tag{9.78}$$

where

$$\mathcal{E}' = \sqrt{c^2\mathbf{p}'^2 + m^2c^4}, \qquad \varphi_1 = \chi_3 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \varphi_2 = \chi_4 = \begin{pmatrix}0\\1\end{pmatrix}. \tag{9.79}$$

The solution of Eq. (9.76) in the form Eq. (9.77) corresponds to an expansion of the wave function in a complete set of orthonormal functions of the electrons (positrons) with specified momentum (with energies $\mathcal{E}' = \pm\sqrt{c^2\mathbf{p}'^2 + m^2c^4}$ and spin projections $S_z = \pm 1/2$). The latter are normalized to one particle per unit volume. According to the assumed model only the Dirac vacuum is present prior to the turning on of the field, i.e.,

$$\left|a_3(-\infty)\right|^2 = \left|a_4(-\infty)\right|^2 = 1, \qquad \left|a_1(-\infty)\right|^2 = \left|a_2(-\infty)\right|^2 \tag{9.80}$$

(the field is turned on adiabatically at $t = -\infty$). From the condition of conservation of the norm we have

$$\sum_{i=1}^{4}\left|a_i(t')\right|^2 = 2, \tag{9.81}$$

which expresses the equality of the number of created electrons and positrons, whose creation probability is, respectively, $\left|a_{1,2}(t')\right|^2$ and $1 - \left|a_{3,4}(t')\right|^2$.

Substituting Eq. (9.77) into Eq. (9.76), multiplying by the Hermitian conjugate functions $\Psi_i^{(0)^\dagger}(\mathbf{r}', t')$, and taking into account orthogonality of the eigenfunctions (9.78) and (9.79), we obtain a set of differential equations for the unknown functions $a_i(t')$. Since in the C frame there is symmetry with respect to the direction $\mathbf{A}_0'$ (the OY axis), we can take, without loss of generality, the vector $\mathbf{p}'$ to lie in the $x'y'$ plane ($p_z' = 0$). Further, having introduced, to simplify the notation, the new symbols

$$a_1(t') \equiv b_1(t'),$$

$$a_4(t') \equiv b_4(t') \left[1 - \frac{c^2 p_y'^2}{\mathcal{E}'^2}\right]^{-1/2} \left[\frac{c^2 p_x' p_y'}{\mathcal{E}'(\mathcal{E}' + mc^2)}\right.$$

$$\left. + i\left(1 - \frac{c^2 p_y'^2}{\mathcal{E}'(\mathcal{E}' + mc^2)}\right)\right], \tag{9.82}$$

we obtain for the amplitudes $b_1(t')$ and $b_4(t')$ ($|b_4(t')| = |a_4(t')|$) the following set of equations:

$$\frac{db_1(t')}{dt'} = i\frac{ecp_y' A_y'(t')}{\hbar\mathcal{E}'} b_1(t')$$

$$+ i\frac{eA_y'(t')}{\hbar}\sqrt{1 - \frac{c^2 p_y'^2}{\mathcal{E}'^2}}\, b_4(t') \exp\left(\frac{2i\mathcal{E}'t'}{\hbar}\right),$$

$$\frac{db_4(t')}{dt'} = -i\frac{ecp_y' A_y'(t')}{\hbar\mathcal{E}'} b_4(t')$$

$$+ i\frac{eA_y'(t')}{\hbar}\sqrt{1 - \frac{c^2 p_y'^2}{\mathcal{E}'^2}}\, b_1(t') \exp\left(-\frac{2i\mathcal{E}'t'}{\hbar}\right). \tag{9.83}$$

A similar set of equations is also obtained for the amplitudes $b_2(t')$ and $b_3(t')$. To solve the system (9.83), we make the transformations

$$b_1(t') = c_1(t') \exp\left[i\frac{ecp_y'}{\hbar\mathcal{E}'} \int_{-\infty}^{t'} A_y'(\eta)d\eta\right],$$

$$b_4(t') = c_4(t') \exp\left[-i\frac{ecp_y'}{\hbar\mathcal{E}'} \int_{-\infty}^{t'} A_y(\eta)d\eta\right], \tag{9.84}$$

where $c_1(t')$ and $c_4(t')$ satisfy the initial conditions, according to Eqs. (9.80) and (9.82), $|c_1(-\infty)| = 0$ and $|c_4(-\infty)| = 0$.

For the new amplitudes $c_1(t')$ and $c_4(t')$ from Eqs. (9.83), we obtain the set of equations

$$\frac{dc_1(t')}{dt'} = f(t')c_4(t'),$$

$$\frac{dc_4(t')}{dt'} = -f^*(t')c_1(t'), \tag{9.85}$$

where

$$f(t') = i\frac{e}{\hbar}A'_y(t')\sqrt{1 - \frac{c^2 p'^2_y}{\mathcal{E}^2}} \exp\left[\frac{2i}{\hbar}\mathcal{E}'t' - \frac{2iecp'_y}{\hbar\mathcal{E}'}\int\limits_{-\infty}^{t'} A'_y(\eta)d\eta\right]. \tag{9.86}$$

We can obtain the solution of Eqs. (9.83), which satisfies the initial conditions of the problem (9.80), with the help of successive approximations, if

$$\left|\int\limits_{-\infty}^{t'} f(\tau)d\tau\right| << 1. \tag{9.87}$$

Then, for the transition amplitude $c_1(t')$, we have

$$c_1(t') = \sum_{j=0}^{\infty} B_{2j+1}(t'), \tag{9.88}$$

where

$$B_{2j+1}(t') = (-1)^j \int\limits_{-\infty}^{t'} f(\tau_1)d\tau_1 \int\limits_{-\infty}^{\tau_1} f^*(\tau_2)d\tau_2 \int\limits_{-\infty}^{\tau_2} f^*(\tau_3)d\tau_3 \cdots$$

$$\times \int\limits_{-\infty}^{\tau_{2j-1}} f^*(\tau_{2j})d\tau_{2j} \int\limits_{-\infty}^{\tau_{2j}} f^*(\tau_{2j+1})d\tau_{2j+1}. \tag{9.89}$$

We are interested in nonlinear pair production process in the strong wave field. For that let us calculate the first term of the sum (9.88):

$$B_1(t') = \int_{-\infty}^{t'} f(\tau_1) d\tau_1,$$

substituting the concrete form of the wave vector potential $A'_y(\eta)$ from Eq. (9.75) into Eq. (9.86) and carrying out the integration. Then for $B_1(t')$ we obtain

$$B_1(t') = \frac{\mathcal{E}'}{2cp'_y}\left(1 - \frac{c^2 p'^2_y}{\mathcal{E}'^2}\right)^{1/2} \sum_{l=-\infty}^{+\infty} \frac{l\hbar\omega'}{2\mathcal{E}' - l\hbar\omega'} J_l(\alpha) e^{\frac{i}{\hbar}(2\mathcal{E}' - l\hbar\omega')t'}, \quad (9.90)$$

where $J_s(z)$ is the Bessel function,

$$\alpha \equiv 2\xi_0 \frac{mc^2}{\mathcal{E}'} \frac{cp'_y}{\hbar\omega'}, \qquad \xi_0 = \frac{eE'_0}{mc\omega'}, \qquad E'_0 = \frac{\omega'}{c} A_0.$$

As ξ_0 is a relativistic invariant parameter, in Eqs. (9.90) $\xi_0 = eE_0/mc\omega_0$, where ω_0 and E_0 are the frequency and amplitude of the electric field of the wave in the L frame.

For the considered fields, when $\xi_0 \lesssim 1$, condition (9.87) always satisfies: $|B_1(t')| << 1$, but the latter is not enough, yet, in order to be confined to that term in determination of the amplitude $c_1(t')$. Because the resonant term $l = s = 2\mathcal{E}'/(\hbar\omega')$ $(s >> 1)$ gives a real contribution in the multiphoton pair production process and in Eq. (9.90), the maximal value of the Bessel function can be shifted from the resonant value. Since $s >> 1$, that shift will be as small and negligible as possible when the argument of the Bessel function is $\alpha \sim s >> 1$. Thus, the condition, when the pair production process will have an essential nonlinear character, is

$$\alpha = 2\xi_0 \frac{mc^2}{\mathcal{E}'} \frac{cp'_y}{\hbar\omega'} >> 1. \quad (9.91)$$

If condition (9.91) is satisfied, we can be restricted to the first term of the sum (9.88) for the amplitude $c_1(t')$:

$$c_1(t') = B_1(t'). \quad (9.92)$$

The obtained approximate solution of the Dirac equation is thus applicable with such intensities of EM wave, when conditions (9.87) and (9.91) are satisfied simultaneously:

$$\frac{1}{s} << \xi_0 \lesssim 1. \quad (9.93)$$

According to Eqs. (9.82) and (9.84), for the transition amplitude of the electron from the Dirac vacuum to the state with positive energy (in a definite spinor state) in the wave field we have

$$|a_1(t')|^2 = |b_1(t')|^2 = |c_1(t')|^2 .$$

To obtain the probability amplitude for the production of electrons and positrons after the wave has been turned off we introduce a small detuning of the resonance in Eq. (9.90), corresponding to an s-photon transition: $2\mathcal{E}' = s\hbar\omega' + \hbar\Gamma$ ($\Gamma << \omega'$).

The production probability of the e^-, e^+ pair, summed over the spin states, is determined by the quantity

$$|a_1(t')|^2 + |a_2(t')|^2 = 2\,|a_1(t')|^2 \equiv 2\,|C_1(t')|^2 .$$

The differential probability of the s-photon process per unit time and phase-space volume $d\mathbf{p}'/(2\pi\hbar)^3$ (the normalization volume $V = 1$) in the center-of-mass frame of the produced particles is given by

$$dw_s^C = \frac{dW_s^C(t')}{t'} = 2 \lim_{t'\to\infty} \frac{|c_1(t')|^2}{t'} \frac{d\mathbf{p}'}{(2\pi\hbar)^3}. \tag{9.94}$$

Substituting Eq. (9.90) into Eq. (9.94) and making use of the definition of the δ-function in the form

$$\lim_{t'\to\infty} \frac{\sin^2 \Gamma t'}{\pi \Gamma^2 t'} = \delta\,(\Gamma) = \hbar\delta\,(2\mathcal{E}' - s\hbar\omega'),$$

we obtain

$$dw_s^C = \frac{s^2\omega'^2\left(\mathcal{E}'^2 - c^2 p_y'^2\right)}{16\pi^2\hbar^2 c^2 p_y'^2} J_s^2\left(\frac{2eA_0 c p_y'}{\hbar\omega'\mathcal{E}'}\right)\delta\left(\mathcal{E}' - \frac{s\hbar\omega'}{2}\right) d\mathbf{p}'. \tag{9.95}$$

Integrating Eq. (9.95) over $d\mathbf{p}'$, we obtain the total probability of the s-photon e^-, e^+ pair production in a plasma by the strong EM wave:

$$w_s^C = \frac{\hbar s^5 \omega'^5}{32\pi c^4 p'} \left\{ \left[\frac{2\alpha_s^2}{4s^2 - 1} - 1\right] J_s^2\,(\alpha_s) + \frac{\alpha_s^2 J_{s-1}^2\,(\alpha_s)}{2s(2s-1)} \right.$$

$$\left. + \frac{\alpha_s^2 J_{s+1}^2\,(\alpha_s)}{2s(2s+1)} - \frac{4c^2 p'^2}{s^2\hbar^2\omega'^2} \frac{\alpha_s^{2s}}{2^{2s}(2s+1)\,(s!)^2} \right.$$

$$\times {}_2F_3\left(s+\frac{1}{2}, s+\frac{1}{2}, s+1, 2s+1, s+\frac{3}{2}; -\alpha_s^2\right)\Bigg\}, \qquad (9.96)$$

where ${}_2F_3\left(s+\frac{1}{2}, s+\frac{1}{2}, s+1, 2s+1, s+\frac{3}{2}; -\alpha_s^2\right)$ is the generalized hypergeometric function and

$$\alpha_s = \frac{2mc^2\xi_0}{\hbar\omega'}\left(1-\frac{4m^2c^4}{s^2\hbar^2\omega'^2}\right)^{1/2}.$$

As is seen from Eq. (9.95), the pair production probability decreases highly in the directions perpendicular to the field ($p'_y = 0$), and the obtained approximate nonlinear solution describes the process behavior well at the angles not too close to $\pi/2$. Thus, Eq. (9.96), which is a result of integration over all angles, does not contain a large error.

The quantity W_s is a relativistic invariant, and so Eq. (9.96) defines the pair production probability in the L frame as well. As for the angular distribution of the probability of s-photon pair production in the L frame, it can be obtained from the expression $dW_s^C(t')$ for the differential probability in the C frame by a Lorentz transformation. Here the quantity multiplying $d\mathbf{p}'$ is the expression of $dW_s^C(t')$ (see Eq. (9.94)) transforms like the time component of the current density four-vector of the electrons in the Dirac vacuum ($\mathcal{E}' < 0$). One must here take into account that the momentum of real electrons coincides with the momentum of the vacuum electron $\mathbf{p}'$, while the momentum of a positron equals $-\mathbf{p}'$ and the vacuum phase-space volume element $d\mathbf{p}'/(2\pi\hbar)^3$ (in unit volume) goes over correspondingly into the volume element in momentum space of electrons and positrons. Further, transforming the quantities in Eq. (9.95) from the C frame to the L frame, we obtain for the differential probability of s-photon pair production per unit time in the L frame:

$$dw_s^L = \frac{dW_s^L(t)}{t} = \frac{s^2\omega_0^2\left(1-n_0^2\right)\left(\mathcal{E}-n_0cp_x\right)}{16\pi^2\hbar^2c^2p_y^2\mathcal{E}}\left[\frac{\left(\mathcal{E}-n_0cp_x\right)^2}{1-n_0^2} - c^2p_y^2\right]$$

$$\times J_s^2\left(\frac{2eA_0cp_y}{\hbar\omega_0\left(\mathcal{E}-n_0cp_x\right)}\right)\delta\left(\mathcal{E}-n_0cp_x-\frac{s\hbar\omega_0\left(1-n_0^2\right)}{2}\right)d\mathbf{p}', \qquad (9.97)$$

where $\mathcal{E}$ and $\mathbf{p}$ are the energy and momentum of the produced electron or positron. Integrating Eq. (9.97) over the electron (positron) energy, we obtain the angular distribution of the probability of the s-photon production of electrons (positrons) per solid angle element, $do = \sin\vartheta d\vartheta d\varphi$ (the azimuthal asymmetry of the probability in the L frame is due to the linear polarization of the wave: in the case of circular polarization the probability distribution has azimuthal symmetry):

$$dw_s^L = \sum_{v=1}^{2} \frac{s^3\omega_0^3\left(1-n_0^2\right)^2}{32\pi^2\hbar c^3\left(cp_v - n_0\mathcal{E}_v\cos\vartheta\right)\sin\vartheta\cos^2\varphi}$$

$$\times\left[\frac{s^2\hbar^2\omega_0^2\left(1-n_0^2\right)}{4} - c^2p_v^2\sin^2\vartheta\cos^2\varphi\right]$$

$$\times J_s^2\left[\frac{4mc^3\xi_0 p_v\sin\vartheta\cos\varphi}{s\hbar^2\omega_0^2\left(1-n_0^2\right)}\right]d\vartheta d\varphi, \tag{9.98}$$

where

$$p_{1,2} = \frac{1}{2c\left(1-n_0^2\cos^2\vartheta\right)}\left\{sn_0\hbar\omega_0\left(1-n_0^2\right)\cos\vartheta\right.$$

$$\left.\pm\left[s^2\hbar^2\omega_0^2\left(1-n_0^2\right)^2 - 4m^2c^4\left(1-n_0^2\cos^2\vartheta\right)\right]^{1/2}\right\},$$

$$\mathcal{E}_{1,2} = \frac{1}{2\left(1-n_0^2\cos^2\vartheta\right)}\left\{s\hbar\omega_0\left(1-n_0^2\right)\right.$$

$$\left.\pm n_0\cos\vartheta\left[s^2\hbar^2\omega_0^2\left(1-n_0^2\right)^2 - 4m^2c^4\left(1-n_0^2\cos^2\vartheta\right)\right]^{1/2}\right\}. \tag{9.99}$$

The angle φ varies from 0 to 2π, while ϑ varies from 0 to $\vartheta_{\max}$, which is determined from the energy and momentum conservation laws (9.99). Further, depending on the value of the plasma refractive index n_0, the electron (positron) production at the given angle is possible for a particular momentum or for one of two momenta with different magnitude. For values

$$n_0 < \sqrt{1 - \frac{2mc^2}{s\hbar\omega_0}}$$

(in this case the threshold condition (9.74) for the process is certainly satisfied), we should take in Eqs. (9.99) only the upper sign, corresponding to the fact that in the probability (9.98) only $\nu = 1$ (p_1) remains and $\vartheta_{\max} = \pi$; i.e., particles are produced in all directions for the given angle ϑ with definite momentum. In the opposite case we must also take into account the reaction threshold condition in the region of values of the index of refraction,

$$\sqrt{1 - \frac{2mc^2}{s\hbar\omega_0}} < n_0 < \sqrt{1 - \frac{4m^2c^4}{s^2\hbar^2\omega_0^2}},$$

and an electron (positron) is produced in a given direction with one of two different values of momentum p_1 and p_2 in a cone, opened forward, whose opening angle is

$$\vartheta_{\max} = \arcsin\left\{\left[\left(1-n_0^2\right)\left(s^2\hbar^2\omega_0^2\left(1-n_0^2\right)-4m^2c^4\right)\right]^{1/2}/2mc^2n_0\right\}.$$

The problem of e^-, e^+ pair production by the photon field is solved in the C frame and the probability expressions (9.94)–(9.96) in that frame are adduced with express purpose. This is of independent physical interest, since Eqs. (9.94)–(9.96) describe the process of pair production in vacuum by a uniform periodic electric field (electric undulator)

$$\mathbf{E}(t) = \mathbf{E}_0 \cos\omega_0 t, \tag{9.100}$$

with the reaction threshold (see Eq. (9.74) when $n' = 0$)

$$s\hbar\omega_0 \geqslant 2mc^2. \tag{9.101}$$

By integrating over the electron (positron) energy, we obtain the angular distribution of the nonlinear production of electrons (positrons) in the periodic electric field (in contrast to the pair production by the photon field (9.98), here there is azimuthal symmetry):

$$dw_s = \frac{s^3\omega_0^3}{32\pi\hbar c^3}\frac{4m^2c^4\cos^2\vartheta + \hbar^2s^2\omega_0^2\sin^2\vartheta}{\left(\hbar^2s^2\omega_0^2-4m^2c^4\right)^{1/2}\cos^2\vartheta}$$

$$\times J_s^2\left[\frac{2ceE_0\left(\hbar^2s^2\omega_0^2-4m^2c^4\right)^{1/2}\cos\vartheta}{s\hbar^2\omega_0^3}\right]\sin\vartheta d\vartheta, \tag{9.102}$$

where ϑ is the angle between the directions of the momentum of produced electrons (positrons) and the electric field.

Finally, we consider the case of weak fields, $eA/\left(\hbar\omega_0\right) << 1$ ($\xi_0 << 1/s$), when perturbation theory is applicable. In this case, as was noted above, we cannot be confined to the first term of the sum (9.88), since every term $B_{2l+1}(t')$ of the sum at $\alpha << 1$ (see Eq. (9.90) for the expression of α) includes a resonant multiplier $\sim \xi_0^s$ (at $2l+1 \leqslant s$) in the lowest order of perturbation theory. Then from Eq. (9.88) we obtain the formula of perturbation theory for the pair production probability in the C frame, which has a more compact analytical form (here we could get free of the sum of unwieldy products):

$$dw_s^C = 2\pi\hbar\Phi^2\delta\left(2\mathcal{E}' - s\hbar\omega'\right)\frac{d\mathbf{p}'}{\left(2\pi\hbar\right)^3}, \tag{9.103}$$

where

$$\Phi = \beta \left(\frac{\alpha}{2}\right)^s \omega' \left[\frac{1}{(s-1)!} + \sum_{K=1}^{[(s-1)/2]} \sum_{S_1=1}^{s-2K}\right.$$

$$\ldots \sum_{S_j=1}^{s-1-(S_1+\ldots+S_{j-1})-2K+j} \ldots \sum_{S_{2K}=1}^{s-1-(S_1+\ldots+S_{2K-1})} \tag{9.104}$$

$$\left\{\frac{(-1)^{S_2+S_4+\ldots+S_{2K}}}{(s-S_1)(S_1+S_2)\ldots[s-(S_1+S_2+\ldots+S_{2K-1})](S_1+S_2+\ldots+S_{2K})}\right.$$

$$\left.\left.\times \frac{\beta^{2K}}{(S_1-1)!(S_2-1)!\ldots(S_{2K}-1)![s-1-(S_1+S_2+\ldots+S_{2K})]!}\right\}\right].$$

Here $s \geqslant 3$, and parameters

$$\beta = \frac{\mathcal{E}'}{2cp'_y}\left(1-\frac{c^2 p'^2_y}{\mathcal{E}'^2}\right)^{1/2}, \qquad \alpha = s\xi_0 \frac{mc^3 p'_y}{\mathcal{E}'^2}; \qquad \xi_0 << \frac{1}{s}.$$

9.5 Pair Production by Superstrong EM Waves in Vacuum

As we saw in the previous section the conservation laws for the pair production in the field of a plane monochromatic wave can be satisfied in a plasmalike medium where EM waves propagate with a phase velocity larger than the speed of light in vacuum. In this case

$$\frac{\omega^2}{c^2} - \mathbf{k}^2 > 0, \tag{9.105}$$

which means that we have a "photon with nonzero rest mass" providing the creation of the particles with the rest masses. The satisfaction of conservation laws for the e^-, e^+ pair production process in the EM field is equivalent to the satisfaction of the condition

$$\mathbf{E}^2 - \mathbf{H}^2 > 0, \tag{9.106}$$

where **E**, **H** are the electric and magnetic strengths of the field. The latter is obvious in the frame of reference where there is only an electric field that provides the pair creation (in the opposite case we would have only a magnetic field that cannot produce a pair). The condition (9.106) can be satisfied in the stationary maxima of a standing wave being formed by two counterpropagating waves (opposite laser beams) of the same frequencies. It can also be satisfied in the field of a plane monochromatic wave in a wiggler. Thus, these processes of multiphoton pair production via nonlinear channels in vacuum by superstrong laser fields are of special interest.

Let plane transverse linearly polarized EM waves with frequency ω and amplitude of vector potential $\mathbf{A}_0$

$$\mathbf{A}_1 = \mathbf{A}_0 \cos(\omega t - \mathbf{kr}), \quad \mathbf{A}_2 = \mathbf{A}_0 \cos(\omega t + \mathbf{kr}), \tag{9.107}$$

propagate in opposite directions in vacuum. To solve the problem of s-photon production of an e^-, e^+ pair in the given radiation fields (9.107) we shall make use of the Dirac model for electron–positron vacuum. The Dirac equation in the field (9.107) has the form

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[c\widehat{\alpha}(\widehat{\mathbf{p}} - \frac{e}{c}\mathbf{A}_0 \cos(\omega t - \mathbf{kr}) - \frac{e}{c}\mathbf{A}_0 \cos(\omega t + \mathbf{kr})) + \widehat{\beta} mc^2 \right] \Psi. \tag{9.108}$$

Then we have stationary maxima of a standing wave and Eq. (9.108) may be rewritten in the form

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[c\widehat{\alpha}(\widehat{\mathbf{p}} - 2\frac{e}{c}\mathbf{A}_0 \cos \mathbf{kr} \cos \omega t) + \widehat{\beta} mc^2 \right] \Psi. \tag{9.109}$$

According to the Dirac model the electron–positron pair production by the EM wave field occurs when the vacuum electrons with initial negative energies $\mathcal{E}_0 < 0$ due to s-photon absorption pass to the final states with positive energies $\mathcal{E} = \mathcal{E}_0 + s\hbar\omega > 0$. Since we study the case of superstrong laser fields in which the pairs are essentially produced at the length $l << \lambda$ (λ is the wavelength of laser radiation) and on the other hand the Hamiltonian of the interaction $H_{int} \sim \mathbf{p}(\mathbf{A}_1 + \mathbf{A}_2)$, then the significant contribution in the process of e^-, e^+ pair creation will be conditioned by the areas of stationary maxima in the direction along the electric field strength of the standing wave. Consequently, we can neglect the inhomogeneity of the field in the considered problem, i.e., Eq. (9.109) will reduce to the following equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[c\widehat{\alpha}(\widehat{\mathbf{p}} - 2\frac{e}{c}\mathbf{A}_0 \cos \omega t) + \widehat{\beta} mc^2 \right] \Psi. \tag{9.110}$$

In this approximation the magnetic fields of the counterpropagating waves cancel each other. In the case of e^-, e^+ pair production in a plasma we had a similar equation in the center-of-mass frame of created particles (9.76). Thus, we will follow the approach developed in the previous section. Since the interaction Hamiltonian does not depend on the space coordinates, the solution of Eq. (9.110) can be represented in the form of a linear combination of free solutions of the Dirac equation with amplitudes $a_i(t)$ depending only on time (9.77). The application of the unitarian transformations (9.82) and (9.84) yields the set of equations

$$\frac{dc_1(t)}{dt} = f(t)c_4(t), \tag{9.111}$$

$$\frac{dc_4(t)}{dt} = -f^*(t)c_1(t). \tag{9.112}$$

Here the function $f(t)$ (see Eq. (9.86)) is expanded into series

$$f(t) = i \sum_{s'=-\infty}^{\infty} f_{s'} \exp\left[\frac{i}{\hbar}(2\mathcal{E} - s'\hbar\omega)t\right], \tag{9.113}$$

where

$$f_{s'} = \frac{\mathcal{E}}{2cp_y}\left(1 - \frac{c^2p_y^2}{\mathcal{E}^2}\right)^{\frac{1}{2}} s'\omega J_{s'}\left(4\xi_0 \frac{mc^2}{\mathcal{E}} \frac{p_y c}{\hbar\omega}\right), \tag{9.114}$$

and J_s is the ordinary Bessel function. The new amplitudes $c_1(t)$ and $c_4(t)$ satisfy the initial conditions

$$|c_1(-\infty)| = 0, |c_4(-\infty)| = 1.$$

Because of space homogeneity the generalized momentum of a particle is conserved so that the real transitions in the field occur from a $-\mathcal{E}$ negative energy level to positive $+\mathcal{E}$ energy level (in the assumed approximation) and, consequently, the multiphoton probabilities of e^-, e^+ pair production will have maximal values for the resonant transitions $2\mathcal{E} \simeq s\hbar\omega$. The latter just is the conservation law of the pair production process at which both electrons and positrons will be created back-to-back according to zero total momentum: $\mathbf{p}_{e^-} + \mathbf{p}_{e^+} = 0$, since the considered field is only time dependent. Thus, we can utilize the resonant approximation, as in a two-level atomic system in the monochromatic wave field.

The probabilities of multiphoton e^-, e^+ pair production will have maximal values for the resonant transitions

$$2\mathcal{E} - s\hbar\omega \simeq 0. \tag{9.115}$$

In this case the function $f(t)$ can be represented in the following form:

$$f(t) = F_s + \Phi(t), \tag{9.116}$$

where

$$F_s = i f_s e^{i\delta_s t} \tag{9.117}$$

is the slowly varying function on the scale of the wave period and

$$\Phi(t) = i e^{i\delta_s t} \sum_{s' \neq s, s'=-\infty}^{\infty} f_{s'} e^{i(s-s')\omega t} \tag{9.118}$$

is the rapidly oscillating function. Here we have introduced resonance detuning

$$\hbar\delta_s = 2\mathcal{E} - s\hbar\omega. \tag{9.119}$$

As a consequence of this separation the probability amplitudes can be represented in the form

$$c_1(t) = c_1^{(s)}(t) + \beta_1(t), \tag{9.120}$$

$$c_4(t) = c_4^{(s)}(t) + \beta_4(t), \tag{9.121}$$

where $c_1^{(s)}(t)$ and $c_4^{(s)}(t)$ are the slowly varying amplitudes corresponding to $c_1(t)$ and $c_4(t)$. The functions $\beta_1(t)$ and $\beta_4(t)$ are rapidly oscillating functions. Substituting Eqs. (9.120), (9.121) into Eqs. (9.111), (9.112) and separating slow and rapid oscillations, taking into account Eq.(9.116), we will obtain the following set of equations for the slowly varying amplitudes $c_{1,4}^{(s)}(t)$:

$$\frac{dc_1^{(s)}}{dt} = F_s c_4^{(s)} + \overline{\Phi\,(t)\,\beta_4(t)}, \tag{9.122}$$

$$\frac{dc_4^{(s)}}{dt} = -F_s c_1^{(s)} - \overline{\Phi^*\,(t)\,\beta_1(t)}, \tag{9.123}$$

and for the rapidly oscillating functions $\beta_{1,4}$:

$$\frac{d\beta_1}{dt} = \Phi\,(t)\,c_4^{(s)}, \tag{9.124}$$

$$\frac{d\beta_4}{dt} = -\Phi^*\,(t)\,c_1^{(s)}. \tag{9.125}$$

In Eqs. (9.122) and (9.123) the bar denotes averaging over time much larger than wave period. In the set of Eqs. (9.124) and (9.125) we have neglected the terms $\sim F_s\ \beta_{1,4}(t)$ due to the rapid oscillations

$$|F_s \beta_\eta(t)| << \left| \frac{d\beta_1}{dt} \right| . \tag{9.126}$$

Solving the set of Eqs. (9.124) and (9.125), taking into account that $c_{1,4}^{(s)}$ are slowly varying functions, we obtain

$$\beta_1 = c_4^{(s)} \int_0^t \Phi(t')\, dt',$$

$$\beta_4 = -c_1^{(s)} \int_0^t \Phi^*(t')\, dt'.$$

Then substituting $\beta_{1,4}(t)$ into Eqs. (9.122) and (9.123), we will have the following equations for the functions $c_{1,4}^{(s)}$:

$$\frac{dc_1^{(s)}}{dt} = F_s c_4^{(s)} - i\frac{\delta_f}{2} c_1^{(s)}, \tag{9.127}$$

$$\frac{dc_4^{(s)}}{dt} = -F_s c_1^{(s)} + i\frac{\delta_f}{2} c_4^{(s)}, \tag{9.128}$$

where

$$\delta_f = \overline{-2i\Phi(t) \int_0^t \Phi^*(t')\, dt'} = \frac{2}{\omega} \sum_{s' \neq s, s' = -\infty}^{\infty} \frac{|f_{s'}|^2}{s - s'}. \tag{9.129}$$

The set of Eqs. (9.127) and (9.128) can be solved in the general case of arbitrary wave envelope $A_0(t)$ only numerically. But it admits an exact solution for a monochromatic wave describing "Rabi oscillations" of the Dirac vacuum. In this case the set of Eqs. (9.127) and (9.128) for the phase transformed amplitudes $c_1^{(s)} \exp(-i\delta_s t/2)$ and $c_4^{(s)} \exp(i\delta_s t/2)$ is a set of ordinary linear differential equations with fixed coefficients. The general solution of the latter is given by a superposition of two linearly independent solutions which with the initial condition is

$$c_1^{(s)}(t) = i\frac{|f_s|}{\Omega_s} e^{i\frac{\delta_s}{2}t} \sin(\Omega_s t), \tag{9.130}$$

$$c_4^{(s)} = e^{-i\frac{\delta_s}{2}t}\left[\cos\left(\Omega_s t\right) + \frac{i\Delta_s}{2\Omega_s}\sin\left(\Omega_s t\right)\right], \tag{9.131}$$

where

$$\Delta_s = \delta_f + \delta_s \tag{9.132}$$

is the resulting detuning and

$$\Omega_s = \sqrt{\left|f_s\right|^2 + \frac{\Delta_s^2}{4}} \tag{9.133}$$

is the "Rabi frequency" of the Dirac vacuum at the interaction with a periodic EM field. As is seen from Eq. (9.130) with this frequency the probability amplitude of e^-, e^+ pair production oscillates in the standing wave field during the whole interaction time similar to Rabi oscillations in two-level atomic systems. In this case the "Rabi frequency" has a nonlinear dependence on the amplitudes of the opposite EM wave fields. Considerable number of electron–positron pairs can be produced by a proper choice of intensity and duration of laser pulses.

The set of Eqs. (9.127) and (9.128) has been derived using the assumption that the amplitudes $c_{1,4}^{(s)}(t)$ are slowly varying functions on the scale of the EM wave period, i.e.,

$$\left|\frac{dc_{1,4}^{(s)}(t)}{dt}\right| << \left|c_{1,4}^{(s)}(t)\right|\omega. \tag{9.134}$$

These conditions with Eq.(9.126) define the condition of applicability of the applied resonant approximation which is equivalent to the condition

$$\Omega_s << \omega. \tag{9.135}$$

The probability of the s-photon e^-, e^+ pair production with the certain energy $\mathcal{E}$, summed over the spin states, is

$$W_s = 2\left|c_1^{(s)}(t)\right|^2 = \frac{2\left|f_s\right|^2}{\Omega_s^2}\sin^2(\Omega_s t). \tag{9.136}$$

Hence, from Eq. (9.114) we have

$$W_s = \frac{s^2\omega^2\left(p^2\sin^2\vartheta + m^2c^2\right)}{2p^2\cos^2\vartheta} J_s^2\left(4\xi_0\frac{mc^3p\cos\vartheta}{\hbar\omega\mathcal{E}}\right)\frac{\sin^2(\Omega_s t)}{\Omega_s^2}, \tag{9.137}$$

where ϑ is the angle between the directions of the momentum of produced electrons (positrons) and the amplitude of the total field electric strength.

Let us consider the case of short interaction time when

$$\Omega_s t << 1. \tag{9.138}$$

In this case we can determine a probability of multiphoton pair production per unit time according to the following definition of the Dirac δ-function:

$$\frac{\sin^2(\Omega_s t)}{\Omega_s^2} \to 2\pi\hbar t\delta(2\mathcal{E} - s\hbar\omega).$$

The differential probability of an s-photon e^-, e^+ pair production process per unit time and unit space volume, summed over the spin states, is given by the following formula:

$$dw_s = \frac{s^2\omega^2(p^2\sin^2\vartheta + m^2c^2)}{16\hbar^2\pi^2 p^2\cos^2\vartheta}$$

$$\times J_s^2\left(4\xi_0\frac{mc^3 p\cos\vartheta}{\hbar\omega\mathcal{E}}\right)\delta\left(\mathcal{E} - \frac{s\hbar\omega}{2}\right)d\mathbf{p}. \tag{9.139}$$

By integrating over the electron (positron) energy we obtain the angular distribution of the s-photon differential probability density of created electrons (positrons):

$$\frac{dw_s}{do} = \frac{s^3\omega^3}{64\pi^2\hbar c^3}\frac{4m^2c^4 + \hbar^2 s^2\omega^2\tan^2\vartheta}{(\hbar^2 s^2\omega^2 - 4m^2c^4)^{1/2}}$$

$$\times J_s^2\left(\frac{4ceE_0\left(\hbar^2 s^2\omega^2 - 4m^2c^4\right)^{1/2}\cos\vartheta}{s\hbar^2\omega^3}\right), \tag{9.140}$$

where $do = \sin\vartheta d\vartheta d\varphi$ is the differential solid angle.

Analogously one can describe the multiphoton pair production process in a wiggler by a superstrong laser pulse of relativistic intensities. Thus, as we saw in Section 5.4 at the induced interaction of a charged particle with a plane EM wave in an undulator, or with the counterpropagating waves of different frequencies (Section 5.3) the two interference waves are formed which propagate with the phase velocities $\mathrm{v}_{ph} > c$ and $\mathrm{v}_{ph} < c$. According to the conditions (9.105) and (9.106) the wave propagating with the phase velocity $\mathrm{v}_{ph} > c$ will be responsible for the pair production process. By the appropriate transformations the processes of e^-, e^+ pair production in these EM field configurations can be reduced to the considered pair production process (as in the case of plasma) in this section. Namely, one should solve the problem in the center-of-mass frame of the produced pair moving with respect to the laboratory frame with the velocity $v = c^2/v_{ph}$.

Bibliography

J. Schwinger, Phys. Rev. **82**, 664 (1951)
V.P. Yakovlev, Zh. Eksp. Teor. Fiz. **49**, 318 (1965)
T. Erber, Rev. Mod. Phys. **38**, 626 (1966)
F.V. Bunkin, I.I. Tugov, Dokl. Akad. Nauk SSSR **187**, 541 (1969) [in Russian]
A.I. Nikishov, Zh. Eksp. Teor. Fiz. **57**, 1210 (1969)
N.B. Narojni, A.I. Nikishov, Yadernaya Fizika **11**, 1072 (1970) [in Russian]
A.I. Nikishov, Nucl. Phys. B **21**, 346 (1970)
V.S. Popov, Pis'ma Zh. Eksp. Teor. Fiz. **11**, 254 (1970)
E. Brezin, C. Itzykson, Phys. Rev. D **2**, 1191 (1970)
A.I. Nikishov, V.I. Ritus, Usp. Fiz. Nauk **100**, 724 (1970) [in Russian]
F.V. Bunkin, A.E. Kazakov, Dokl. Akad. Nauk SSSR **193**, 1274 (1970) [in Russian]
Ya.B. Zeldovich, V.S. Popov, Usp. Fiz. Nauk **105**, 403 (1971) [in Russian]
V.S. Popov, Pis'ma Zh. Eksp. Teor. Fiz. **13**, 261 (1971)
V.S. Popov, Zh. Eksp. Teor. Fiz. **61**, 1334 (1971)
A.I. Nikishov, Problemi Teoreticheskoi. Fiziki, Moscow (1972) [in Russian]
V.S. Popov, Zh. Eksp. Teor. Fiz. **62**, 1248 (1972)
V.S. Popov, Zh. Eksp. Teor. Fiz. **63**, 1586 (1972)
V.I. Ritus, Nucl. Phys. B **44**, 236 (1972)
N.B. Narojni, A.I. Nikishov, Zh. Eksp. Teor. Fiz. **63**, 1135 (1972)
A.A. Grib, V.M. Mostepanenko, V.M. Frolov, Teor. Math. Fiz. **13**, 377 (1972) [in Russian]
A.M. Perelomov, Phys. Lett. A **39**, 353 (1972)
H.L. Berkowitz, R. Rosen, Phys. Rev. D **5**, 1308 (1972)
N.B. Narojni, A.I. Nikishov, Zh. Eksp. Teor. Fiz. **63**, 862 (1973)
V.S. Popov, Yadernaya Fizika **19**, 1140 (1974) [in Russian]
A.I. Nikishov, V.I. Ritus, Tr. Fiz. Inst. Akad. Nauk SSSR **111**, 5 (1979)
V.N. Radionov, Zh. Eksp. Teor. Fiz. **78**, 105 (1980)
H.K. Avetissian, A.K. Avetissian, Kh.V. Sedrakian, Zh. Éksp. Teor. Fiz. **99**, 50 (1991)
H. K. Avetissian et al., Phys. Rev. D **54**, 5509 (1996)
D. Burke et al., Phys. Rev. Lett. **79**, 1626 (1997)
Y. Kluger, E. Mottola, Phys. Rev. D **58**, 125015 (1998)
J.C.R. Bloch et al., Phys. Rev. D **60**, 116011 (1999)
A. Ringwald, Phys. Lett. B **510**, 107 (2001)
V.S. Popov, JETP Lett. **74**, 133 (2001)
H.M. Fried et al., Phys. Rev. D **63**, 125001 (2001)
R. Alkofer et al., Phys. Rev. Lett. **87**, 193902 (2001)
H. K. Avetissian et al., Phys. Rev. E **66**, 016502 (2002)

C.D. Roberts et al., Phys. Rev. Lett. **89**, 153901 (2002)
A. Di Piazza, G. Calucci, Phys. Rev. D **65**, 125019 (2002)
H.K. Avetissian et al., Nucl. Instrum. Methods. A **507**, 582 (2003)
A. Di Piazza, Phys. Rev. D **70**, 053013 (2004)
P. Krekora, Q. Su, R. Grobe, Phys. Rev. Lett. **92**, 040406 (2004)
P. Krekora, Q. Su, R. Grobe, Phys. Rev. Lett. **93**, 043004 (2004)