7 Induced Channeling Process in a Crystal

It is known that due to the relativistic motion of a charged particle in a crystal an exotic situation takes place when the effective potential of the crystal planes or axes becomes a potential well for the particle in the transversal direction with respect to its initial motion, and so-called channeling of the particle occurs accompanied by spontaneous channeling radiation.

The channeling radiation of ultrarelativistic electrons and positrons in a crystal is of great interest for two major reasons: the radiation is in the short-wave region (X-ray and γ-ray domains), and its spectral intensity considerably exceeds that of other types of radiation in this range of frequencies.

Induced channeling radiation in the presence of an external coherent radiation field becomes important as a potential source for short-wave coherent radiation, which may be considered as a version of a free electron laser.

As a periodic system with high coherency and owing to the similar periodic character of particle motion, the crystal channel may be compared with an undulator — it is a "micro-undulator" with the space period much smaller than that of an undulator.

On the other hand, the particle–external coherent EM wave interaction process in the channel of a crystal proceeds with the inverse stimulated effect reducing the particle acceleration and other classical and quantum coherent effects.

Hence, this chapter will consider the induced channeling process with regard to general aspects of coherent interaction of relativistic electrons and positrons with a plane transversal EM wave in a crystal.

7.1 Positron–Strong Wave Interaction at the Planar Channeling in a Crystal

If a charged particle with relativistic velocity enters a crystal at the angle with respect to a crystal plane or crystallographic axis smaller than some specified angle (Lindhard angle)

$$\theta_\alpha = \sqrt{\frac{2U_0}{\mathcal{E}}}, \tag{7.1}$$

then the effective electrostatic field of the crystal becomes a transversal potential well related to the particle motion and the latter moves in the crystal channel — the channeling of the particle occurs. Here U_0 is the depth of the potential well and $\mathcal{E}$ is the particle energy. In the most interesting case of ultrarelativistic energies for channeling phenomenon the transversal de Broglie wavelength of the particle

$$\lambda_D = \frac{\hbar c}{\sqrt{2U_0\mathcal{E}}} \tag{7.2}$$

is much smaller than the interplanar or interaxial distance d in a crystal (U_0 is of the order of the kinetic energy of the particle transversal motion) and consequently $d/\lambda_D >> 1$. On the other hand, the quantity d/λ_D with the coefficient coincides with the number of bound states l of the particle transversal motion in the crystal channel. Hence, in the most important region of energies $l >> 1$ and the particle motion at the channeling can be described classically.

We will study the induced interaction of a charged particle channeled in a crystal with the external coherent radiation field within the scope of the classical theory. In this section the case of the planar channeling will be considered.

As is known for a positron planar channeling the effective electrostatic potential of the crystal planes within the channel is well enough described by the parabolic law

$$U(x) = 4\frac{U_0}{d^2}x^2, \tag{7.3}$$

where d is the distance between the crystal planes, and the transversal coordinate x is evaluated from the median plane. The classical relativistic equation of motion for a positron in the fields (7.3) and an external plane monochromatic EM wave

$$\mathbf{E} = \mathbf{E}_0 \cos(\omega_0 t - \mathbf{k}_0\mathbf{r}); \qquad \mathbf{k}_0 = \nu\frac{n_0\omega_0}{c} \tag{7.4}$$

($n_0 = n_0(\omega_0)$ is the refractive index of the crystal on the wave frequency) is written as

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c}[\mathbf{vH}] - \nabla U(x). \tag{7.5}$$

As for the permitted maximal values of the wave intensities in the dielectric media the characteristic interaction parameter $\xi_0 = eE_0/mc\omega_0 << 1$ (see Section 2.2), then for the ultrarelativistic energies of the channeled particles the interaction with the EM wave in a crystal with great accuracy can be described by the classical perturbation theory over the field (7.4). Consequently, in the zero order over the EM wave field from Eq. (7.5) we have the equations

$$\frac{dp_x}{dt} = -\frac{dU(x)}{dx}, \tag{7.6}$$

$$\frac{dp_y}{dt} = 0; \qquad \frac{dp_z}{dt} = 0. \tag{7.7}$$

Choosing the axis z along the initial motion of the particle from Eqs. (7.6) and (7.7) for the particle energy and momentum we obtain respectively

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - (\mathrm{v}_x^2 + \mathrm{v}_z^2)/c^2}} + U(x), \tag{7.8}$$

$$p_y = 0; \qquad p_z = \frac{m\mathrm{v}_z}{\sqrt{1 - (\mathrm{v}_x^2 + \mathrm{v}_z^2)/c^2}}. \tag{7.9}$$

For the transversal velocity of the particle from Eqs. (7.8) and (7.9) we have

$$\mathrm{v}_x^2 = c^2 \frac{[\mathcal{E} - U(x)]^2 - \mathcal{E}_{\shortparallel}^2}{[\mathcal{E} - U(x)]^2}, \tag{7.10}$$

where

$$\mathcal{E}_{\shortparallel} = c\sqrt{p_{\shortparallel}^2 + m^2c^2} \tag{7.11}$$

is the energy of the longitudinal motion. Equation (7.10) is the exact equation for the particle transversal motion. One can make some simplification of this equation taking into account the smallness of the potential energy related to the energy of the ultrarelativistic particle:

$$U_{\max}(x) \ll \mathcal{E}.$$

Representing the particle energy in the form

$$\mathcal{E} = \mathcal{E}_{\shortparallel} + \mathcal{E}_{\perp},$$

where $\mathcal{E}_{\perp}$ is the energy of the transversal motion, and taking into account that for the channeled particles

$$\mathcal{E}_{\perp} \lesssim U_{\max}(x) \ll \mathcal{E}_{\shortparallel},$$

then the equation for the particle transversal motion (7.10) with the accuracy of the small quantity $\mathcal{E}_{\perp}/\mathcal{E}_{\shortparallel} << 1$ will take the form

$$\mathrm{v}_x^2 = \frac{2c^2}{\mathcal{E}_{\shortparallel}} \left[\mathcal{E}_{\perp} - U(x)\right]. \tag{7.12}$$

Formally Eq. (7.10) has a nonrelativistic character where instead of particle rest mass, the relativistic mass $m_{rel} \simeq \mathcal{E}_{\shortparallel}/mc^2$ stands.

The longitudinal velocity of the particle is determined from Eq. (7.9) and has the form

$$\mathrm{v}_z(t) \simeq c\left\{1 - \frac{1}{2}\left[\frac{\mathrm{v}_x^2}{c^2} + \left(\frac{mc^2}{\mathcal{E}_{\shortparallel}}\right)^2\right]\right\}. \tag{7.13}$$

In the case of planar channeling of a positron when the effective electrostatic potential of the crystal may be approximated by Eq. (7.3), the integration of Eq. (7.12) gives the following law for the transversal motion:

$$x(t) = x_m \sin\left[\Omega(t - t_0) + \varphi\right]. \tag{7.14}$$

Here

$$\Omega = \frac{2c}{d}\sqrt{\frac{2U_0}{\mathcal{E}_{\shortparallel}}} \tag{7.15}$$

is the frequency of the positron transversal oscillations in the potential well of the crystal channel,

$$x_m = \frac{d}{2}\sqrt{\frac{\mathcal{E}_{\perp}}{U_0}} \tag{7.16}$$

is the amplitude and φ is the phase of the transversal oscillations at the moment t_0 when the positron enters into the crystal. Corresponding to Eq. (7.14) the transversal velocity of the positron is

$$\mathrm{v}_x(t) = \mathrm{v}_{xm} \cos\left[\Omega(t - t_0) + \varphi\right], \tag{7.17}$$

where

$$\mathrm{v}_{xm} = \frac{d\Omega}{2}\sqrt{\frac{\mathcal{E}_{\perp}}{U_0}} \tag{7.18}$$

is the maximal velocity of the transversal motion of the positron in the crystal channel. Then using Eq. (7.17) after the integration of Eq. (7.13) we will have

$$z(t) = \overline{\mathrm{v}}_z t - z_m \sin\left[2\Omega(t - t_0) + 2\varphi\right] + z_m \sin 2\varphi, \tag{7.19}$$

where

$$\overline{\mathrm{v}}_z = c\left\{1 - \frac{1}{2}\left[\left(\frac{mc^2}{\mathcal{E}_{\shortparallel}}\right)^2 + \frac{\mathcal{E}_{\perp}}{\mathcal{E}_{\shortparallel}}\right]\right\} \tag{7.20}$$

is the mean longitudinal velocity of the positron, and the amplitude of the longitudinal oscillations z_m is

$$z_m = \frac{c\mathcal{E}_{\perp}}{4\Omega\mathcal{E}_{\shortparallel}}. \tag{7.21}$$

Now we can evaluate the induced channeling effect in the field of an external EM wave, by the classical perturbation theory in the first order over the field (7.4). The energy change of the channeled positron at the interaction with the plane transverse EM wave is given by

$$\Delta\mathcal{E} = e\int_{t_1}^{t_2} \mathbf{E}(t - \nu\mathbf{r}n_0/c)\mathbf{v}(t)dt, \tag{7.22}$$

where the law of motion $\mathbf{r} = \mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ of the positron in the crystal channel are determined by Eqs. (7.14), (7.19) and Eqs. (7.13), (7.17), respectively. The induced interaction time $\Delta t = t_2 - t_1$ actually will be determined by the length of the channel ($t_1$and t_2 are correspondingly the moments of the wave entrance in the crystal and exit from the channel).

For the concreteness and evaluation of the energy change (7.22) we introduce a new Cartesian coordinate system x', y', z' and assume that a quasi-monochromatic EM wave linearly polarized along the axis x' propagates along the axis z', at a small angle with respect to a crystal plane (see Eq. (7.1)). The coordinate system x', y', z' is related to the system x, y, z via Eulerian angles α, β, γ as follows:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{7.23}$$

At the motion of the positron in the crystal channel by the trajectory (7.14), (7.19), the wave phase in Eq. (7.22) corresponding to induced interaction is

$$\phi = \omega_0 t - \mathbf{k}_0\mathbf{r} = \omega t - \varkappa_1 \sin\left[\Omega\left(t - t_0\right) + \varphi\right]$$

$$+\varkappa_2 \sin 2\left[\Omega\left(t-t_0\right)+\varphi\right]+\psi, \tag{7.24}$$

where

$$\omega=\omega_0\left(1-\frac{n_0 \overline{\mathrm{v}}_z}{c} \cos \alpha \cos \beta\right) \tag{7.25}$$

is the Doppler-shifted wave frequency, and the parameters $\varkappa_1$, $\varkappa_2$, ψ are

$$\varkappa_1=n_0 \omega_0 \frac{x_m}{c} \sin \beta ; \qquad \varkappa_2=n_0 \omega_0 \frac{z_m}{c} \cos \alpha \cos \beta,$$

$$\psi=-n_0 \frac{\omega_0}{c} \cos \alpha \cos \beta\left(z_m \sin 2 \varphi-\overline{\mathrm{v}}_z t_0\right). \tag{7.26}$$

Substituting Eq. (7.24) as well as Eqs. (7.13) and (7.17) in Eq. (7.22) for the energy change of the positron due to the induced channeling effect, in the first order by the wave field we will have

$$\Delta \mathcal{E}=\sum_{s=-\infty}^{\infty} \frac{e}{\omega-s \Omega}\left\{E_{0 x} \mathrm{v}_{x m} A_1\left(s, \varkappa_1, \varkappa_2\right)+E_{0 z}\left(\overline{\mathrm{v}}_z+\mathrm{v}_{z m}\right) A_0\left(s, \varkappa_1, \varkappa_2\right)\right.$$

$$\left.-2 E_{0 z} \mathrm{v}_{z m} A_2\left(s, \varkappa_1, \varkappa_2\right)\right\}\left\{\sin \left[(\omega-s \Omega) t_2+s \Omega t_0-s \varphi+\psi\right]\right.$$

$$\left.-\sin \left[(\omega-s \Omega) t_1+s \Omega t_0-s \varphi+\psi\right]\right\}, \tag{7.27}$$

where

$$A_n(s, \alpha, \beta)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \cos ^n \varphi^{\prime} e^{i\left(\alpha \sin \varphi^{\prime}-\beta \sin 2 \varphi^{\prime}-s \varphi^{\prime}\right)} d \varphi^{\prime}$$

is the generalized Bessel function with the definitions

$$A_0(s, \alpha, \beta)=\sum_{k=-\infty}^{\infty} J_{s+2 k}(\alpha) J_k(\beta),$$

$$A_1(s, \alpha, \beta)=\frac{1}{2}\left[A_0(s-1, \alpha, \beta)+A_0(s+1, \alpha, \beta)\right],$$

$$A_2(s, \alpha, \beta)=\frac{1}{4}\left[A_0(s-2, \alpha, \beta)+2 A_0(s, \alpha, \beta)+A_0(s+2, \alpha, \beta)\right],$$

and

$$\mathrm{v}_{zm} = \frac{c\mathcal{E}_{\perp}}{2\mathcal{E}_{\shortparallel}} \tag{7.28}$$

is the amplitude of the positron longitudinal velocity oscillations.

Equation (7.27) shows that the energy change of the positron after the interaction differs from zero (will have nonoscillating character in the time) if the condition

$$\omega_0 \left(1 - n_0 \frac{\overline{\mathrm{v}}_z}{c} \cos\alpha \cos\beta\right) = s\Omega; \qquad s = 0, \pm 1, \pm 2, ... \tag{7.29}$$

is satisfied for a specified s. The latter is the condition of the resonance between the transversal oscillations of the positron in the potential well of the crystal channel and EM wave. Only at the fulfillment of this condition does the coherent energy exchange of the channeled positron with the monochromatic wave become real. Then for the energy change of the positron after the interaction we have

$$\Delta\mathcal{E} = eE_0 \Delta t \left\{\mathrm{v}_{xm} \cos\beta \cos\gamma A_1(s, \varkappa_1, \varkappa_2) + (\sin\alpha \sin\gamma - \cos\alpha \sin\beta \cos\gamma) \right.$$

$$\left. \times \left[(\overline{\mathrm{v}}_z + \mathrm{v}_{zm}) A_0(s, \varkappa_1, \varkappa_2) - 2\mathrm{v}_{zm} A_2(s, \varkappa_1, \varkappa_2)\right]\right\}$$

$$\times \cos\left[s\Omega t_0 - s\varphi + n_0 \frac{\omega_0}{c} \cos\alpha \cos\beta \left(\overline{\mathrm{v}}_z t_0 - z_m \sin 2\varphi\right)\right]. \tag{7.30}$$

Expressing the functions $A_{0,1,2}(s, \varkappa_1, \varkappa_2)$ via the ordinary Bessel functions, Eq. (7.30) can be presented in the form

$$\Delta\mathcal{E} = eE_0 \Delta t \sum_{k=-\infty}^{\infty} \left\{\frac{1}{2} \mathrm{v}_{xm} \cos\beta \cos\gamma \left[J_{s-1+2k}(\varkappa_1) + J_{s+1+2k}(\varkappa_1)\right]\right.$$

$$+ \overline{\mathrm{v}}_z (\sin\alpha \sin\gamma - \cos\alpha \sin\beta \cos\gamma) J_{s+2k}(\varkappa_1)$$

$$\left. - \mathrm{v}_{zm} (\sin\alpha \sin\gamma - \cos\alpha \sin\beta \cos\gamma) \left[J_{s-2+2k}(\varkappa_1) + J_{s+2+2k}(\varkappa_1)\right]\right\} J_k(\varkappa_2)$$

$$\times \cos\left[s\Omega t_0 - s\varphi + n_0 \frac{\omega_0}{c} \left(\overline{\mathrm{v}}_z t_0 - z_m \sin 2\varphi\right) \cos\alpha \cos\beta\right]. \tag{7.31}$$

For the X-ray and γ-ray frequencies when $n_0(\omega_0) \lesssim 1$ the resonance condition (7.29) corresponds to the normal Doppler effect at which the energy absorption from the EM wave is accompanied by enhancement of the transversal oscillations of the positron (in these cases $s > 0$ in Eq. (7.31)). For the optical frequencies when $n_0(\omega_0) > 1$ the anomalous Doppler effect is possible as well:

$$1 - n_0 \frac{\overline{\mathrm{v}}_z}{c} \cos\alpha \cos\beta < 0, \qquad (7.32)$$

which corresponds to enhancement of transversal oscillations of the positron at the induced radiation (in Eq. (7.31) in this case $s < 0$). Under the condition

$$1 - n_0 \frac{\overline{\mathrm{v}}_z}{c} \cos\alpha \cos\beta = 0, \qquad (7.33)$$

that is, the Cherenkov condition in the crystal channel corresponding to $s = 0$, Eq. (7.29) expresses the real energy exchange at the positron–wave induced Cherenkov interaction.

Equation (7.31) for the general geometry of the positron planar channeling at the arbitrary propagation and polarization directions of the wave is very bulky. It can be simplified in the case of a particular geometry of the induced interaction — if the EM wave propagates along the direction of the positron motion in the channel (axis z) with the electric field directed along the axis x — and the positron energy $\mathcal{E}_{\shortparallel} \lesssim m^2c^4/\mathcal{E}_{\perp}$. Then, for the number of harmonic s we have: $s = 0, \pm 1$ (for the coherent accumulation of energy exchange), and for the frequencies satisfying the resonance condition (7.29) one can suppose $n_0(\omega_0) \simeq 1$. The latter excepts the possibility of the induced Cherenkov effect ($s = 0$) and the anomalous Doppler effect ($s = -1$) as well. Thus, for the induced energy exchange we have a simple formula

$$\Delta\mathcal{E} = \frac{eE_0\mathrm{v}_{xm}}{2} \Delta t \cos\left[\left(\Omega + \omega_0 \frac{\overline{\mathrm{v}}_z}{c}\right) t_0 - \varphi\right]. \qquad (7.34)$$

As is seen from Eqs. (7.31) and (7.34) depending on the initial conditions- a moment t_0 when the positron enters into the crystal and a phase φ of the transversal oscillations — either the direct or the inverse induced channeling effect occurs, i.e., positron deceleration or acceleration, respectively. Hence, at the interaction of the channeled positron beam with the monochromatic EM wave the diverse particles entering into a crystal at the different moments and in the different oscillation phases will acquire or lose different energies. As a result, the modulation of the particles' velocities will take place leading to beam bunching if the longitudinal size of the latter $l_z > \pi\overline{\mathrm{v}}_z/\omega_0$.

7.2 Induced Interaction of Electrons with Strong EM Wave at the Axial Channeling

As is known, for an electron axial channeling the effective electrostatic potential of the atomic chain along the crystal axis is well enough described by the two-dimensional Coulomb potential

$$U(\rho) = -\frac{\alpha_c}{\rho}, \tag{7.35}$$

where α_c is a constant depending on the type of crystal and the particular geometry, and ρ is the distance from the crystal axis. The transversal motion of the electron in the field (7.35) with a nonzero momentum occurs by the Keplerian elliptic trajectory. If one directs the coordinate axes OX and OY correspondingly along the major and minor semiaxes of the ellipse and the axis OZ along the crystal axis, and if at the moment $t = t_0$ the electron is situated in the perihelion of the orbit of the transversal motion with the coordinate $z = z_0$, then the electron trajectory may be presented in the known parametric form

$$x = a(\cos\zeta - \epsilon); \qquad y = (-1)^{s'} b\sin\zeta,$$

$$z = \overline{\mathrm{v}}_z (t - t_0) - a^2 \frac{\epsilon\Omega}{c}\sin\zeta + z_0, \tag{7.36}$$

$$t = \frac{\zeta - \epsilon\sin\zeta}{\Omega} + t_0,$$

where for a full rotation of the electron by the elliptic orbit the parameter ζ varies from zero to 2π. Here the parameters

$$a = \frac{\alpha_c}{2|\mathcal{E}_\perp|}; \qquad b = a\sqrt{1-\epsilon^2} \tag{7.37}$$

are the major and minor semiaxes of the ellipse,

$$\epsilon = \sqrt{1 - \frac{2|\mathcal{E}_\perp| M_z^2 c^2}{\mathcal{E}_{\shortparallel}\alpha_c^2}} \tag{7.38}$$

is the eccentricity (M_z is the z-component of the orbital moment),

$$\Omega = c\frac{(2|\mathcal{E}_\perp|)^{\frac{3}{2}}}{\alpha_c\sqrt{\mathcal{E}_{\shortparallel}}} \tag{7.39}$$

is the rotation frequency, and

$$\overline{\mathrm{v}}_z = c\left(1 - \frac{m^2c^4}{2\mathcal{E}_{\shortparallel}^2}\right) - \frac{c\left|\mathcal{E}_{\perp}\right|}{\mathcal{E}_{\shortparallel}} \tag{7.40}$$

is the mean longitudinal velocity of the electron. The parameter s' in Eq. (7.36) determines the right-hand or left-hand rotation of the electron by the elliptic orbit:

$$s' = \begin{cases} 0, & \frac{M_z}{|M_z|} > 0, \\ 1, & \frac{M_z}{|M_z|} < 0. \end{cases} \tag{7.41}$$

As the electron trajectory at the axial channeling is of helical type from the point of view of the symmetry in this issue we will suppose that an EM wave has a circular polarization:

$$E_{x'} = E_0 \cos\left(\omega_0 t - \mathbf{k}_0\mathbf{r}\right); \quad E_{y'} = E_0 \left(-1\right)^{s''} \sin\left(\omega_0 t - \mathbf{k}_0\mathbf{r}\right) \tag{7.42}$$

correspondingly with the left-hand and right-hand rotations:

$$s'' = \begin{cases} 0, & \text{left-hand,} \\ 1, & \text{right-hand.} \end{cases}$$

The coordinate system $x'y'z'$ relates to the xyz one in accordance with Eq. (7.23) and in the case of the wave circular polarization one can assume that the Eulerian angle $\gamma = 0$.

We will evaluate the induced effect at the axial channeling by Eq. (7.22) again in the first order by the EM wave field. As far as the particle velocity and law of motion in the channel in this case are determined in parametric form (Eq. (7.36)) it is necessary to pass in Eq. (7.22) from the variable t to ζ. Then the induced energy exchange between the channeled electron and EM wave will be written in the form

$$\Delta\mathcal{E} = e \int\limits_{\zeta(t_1)}^{\zeta(t_2)} \mathbf{E}(\phi\left(\zeta\right)) \frac{d\mathbf{r}(\zeta)}{d\zeta} d\zeta, \tag{7.43}$$

where $\Delta t = t_2 - t_1$ is the duration of electron–wave coherent interaction at the axial channeling. In the first-order approximation for the wave phase in the integral (7.43) with the help of Eqs. (7.36)–(7.41) we have

$$\phi\left(\zeta\right) = \omega_0 t - \mathbf{k}_0\mathbf{r} = \frac{\omega_0 - k_{0z}\overline{\mathrm{v}}_z}{\Omega}\zeta - \varkappa_1 \sin\zeta - \varkappa_2 \cos\zeta + \psi, \tag{7.44}$$

where

$$\mathbf{k}_0 = n_0 \frac{\omega_0}{c} \left(\sin\beta, -\sin\alpha\cos\beta, \cos\alpha\cos\beta \right),$$

and the parameters $\varkappa_1$, $\varkappa_2$, ψ in this case are

$$\varkappa_1 = \frac{\epsilon}{\Omega} \left(\omega_0 - k_{0z}\overline{\mathrm{v}}_z \right) + (-1)^{s'} k_{0y} b - k_{0z} a^2 \epsilon \frac{\Omega}{c}; \quad \varkappa_2 = a k_{0x},$$

$$\psi = \omega_0 t_0 + k_{0x} a\epsilon - k_{0z} z_0.$$

Performing integration in Eq. (7.43) with the help of Eqs. (7.36) and (7.44) we obtain the following ultimate equation for the coherent energy exchange between the electron and external strong EM wave at the axial channeling:

$$\Delta\mathcal{E} = -eE_0\Omega\Delta t \Bigg\{ J_s(\varkappa) \left[(-1)^{s''} \sin\alpha\sin\varphi - \cos\alpha\sin\beta\cos\varphi \right] \frac{\overline{\mathrm{v}}_z}{\Omega}$$

$$+ \frac{s}{\varkappa} J_s(\varkappa) \Bigg[a\cos\beta\sin\varphi_1\cos\varphi + (-1)^{s'} b\sin\alpha\sin\beta\cos\varphi\cos\varphi_1$$

$$+ (-1)^{s'+s''} b\cos\alpha\sin\varphi\cos\varphi_1 + \left(1 + \frac{2c\left|\mathcal{E}_\perp\right|}{\overline{\mathrm{v}}_z \mathcal{E}_{||}} \right) \frac{\epsilon\overline{\mathrm{v}}_z}{\Omega}$$

$$\left(\cos\alpha\sin\beta\cos\varphi\cos\varphi_1 - (-1)^{s''} \sin\alpha\sin\varphi\cos\varphi_1 \right) \Bigg]$$

$$+ J_s'(\varkappa) \Bigg[a\cos\beta\sin\varphi\cos\varphi_1 + (-1)^{s'} b\sin\alpha\sin\beta\sin\varphi\sin\varphi_1$$

$$+ (-1)^{s'+s''} b\cos\alpha\cos\varphi\sin\varphi_1 - \left(1 + \frac{2c\left|\mathcal{E}_\perp\right|}{\overline{\mathrm{v}}_z \mathcal{E}_{||}} \right) \frac{\epsilon\overline{\mathrm{v}}_z}{\Omega}$$

$$\times \left(\cos\alpha\sin\beta\sin\varphi\sin\varphi_1 + (-1)^{s''} \sin\alpha\sin\varphi_1\cos\varphi \right) \Bigg] \Bigg\}, \qquad (7.45)$$

where the parameters $\varkappa$, φ_1, and φ are

$$\varkappa = \sqrt{\varkappa_1^2 + \varkappa_2^2},$$

$$\varphi_1 = \frac{\varkappa_1}{|\varkappa_1|} \arcsin \frac{\varkappa_2}{\varkappa}, \tag{7.46}$$

$$\varphi = \omega_0 t_0 - n_0 \frac{\omega_0}{c} z_0 \cos\alpha \cos\beta + a\epsilon n_0 \frac{\omega_0}{c} \sin\beta - s\varphi_1 .$$

The physical analysis of Eq. (7.45) is the same as was made for the positron planar channeling. So, we will not repeat the analogous analysis, noting only that the condition of resonance at the axial channeling for coherent energy exchange (7.45) is given by Eq. (7.29), where the frequency of transversal oscillations Ω of the electron is determined by Eq. (7.39).

Equation (7.46) corresponding to general geometry of the electron axial channeling in the arbitrary propagation and polarization directions of the wave is very bulky. It is rather simplified if the wave propagates along the direction of the electron motion in the channel (axis z) with the components of the electric field strength directed along the axes x and y, as well as the electron energy should not exceed the value $m^2c^4/\mathcal{E}_\perp$. For the induced energy exchange we have the following ultimate equation:

$$\Delta\mathcal{E} = -eE_0 \Omega \Delta t \left\{ aJ_s'(\varkappa) + b(-1)^{s'+s''} \frac{s}{\varkappa} J_s(\varkappa) \right\}$$

$$\times \sin\left(\omega_0 t_0 - n_0 \frac{\omega_0}{c} z_0\right) . \tag{7.47}$$

The existence of diverse harmonics in Eq. (7.47) is related to the anharmonic character of the electron transversal oscillations in the field (7.35) (in contrast to Eq. (7.34) for the planar channeling, at which the positron is a harmonic oscillator in the channel).

In addition, note that Eqs. (7.45) and (7.47) due to their coherent dependence on the interaction phase lead to the electron beam classical modulation and bunching after the interaction with the stimulating wave at the axial channeling analogously to the positron beam bunching at the planar channeling.

7.3 Quantum Description of the Induced Planar Channeling Effect

Consider the interaction of the particles channeled in a crystal and a plane monochromatic EM wave in the scope of the quantum theory. First we will study the case of a weak wave when the one-photon absorption and emission processes dominate and the induced channeling effect may be described within the quantum perturbation theory by the particle wave function in the linear over the field approximation with respect to the initial state in the potential field of the crystal channel. It means that the latter should be described exactly.

We will start from the Dirac equation which in the case of the planar channeling of a positron in the field of an external EM wave is written as

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(\widehat{H}_0 + \widehat{V}\right) \Psi, \tag{7.48}$$

$$\widehat{H}_0 = c\widehat{\alpha}\widehat{\mathbf{p}} + \widehat{\beta} mc^2 + U(x); \quad \widehat{V} = -e\widehat{\alpha}\mathbf{A}, \tag{7.49}$$

where $\widehat{\alpha}$, $\widehat{\beta}$ are the Dirac matrices in the standard representation (3.2). According to perturbation theory we seek the solution of Eq. (7.49) in the form

$$\Psi = \Psi_0 + \Psi_1 + \cdots; \quad |\Psi_1| \ll |\Psi_0|, ...,$$

where Ψ_0 satisfies the following equation for the positron in the electrostatic field of the crystal channel:

$$i\hbar \frac{\partial \Psi_0}{\partial t} = \left[c\widehat{\alpha}\widehat{\mathbf{p}} + \widehat{\beta} mc^2 + U(x)\right] \Psi_0 \tag{7.50}$$

with the effective potential $U(x)$ (7.3). The particular solution of Eq. (7.50) may be presented in the form

$$\Psi_0(\mathbf{r}, t) = b \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-\frac{i}{\hbar}\mathcal{E}t}, \tag{7.51}$$

where φ and χ are spinor functions, $\mathcal{E}$ is the total energy of the positron in the potential field of the channel, and b is the normalization coefficient. From Eq. (7.50) for the spinor functions φ and χ we obtain the following set of equations:

$$\mathcal{E}\varphi = c(\sigma\widehat{\mathbf{p}})\chi + mc^2\varphi + U(x)\varphi,$$

$$\mathcal{E}\chi = c(\sigma\widehat{\mathbf{p}})\varphi - mc^2\chi + U(x)\chi, \tag{7.52}$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices (1.79). Eliminating χ from the first equation (7.52):

$$\chi = \frac{c\sigma\widehat{\mathbf{p}}}{\mathcal{E} + mc^2 - U(x)}\varphi, \tag{7.53}$$

for the spinor function φ we obtain a differential equation of the second order:

$$\Delta\varphi + \frac{1}{\hbar^2 c^2}\left([\mathcal{E} - U(x)]^2 - m^2c^4\right)\varphi + \frac{\sigma\nabla U(x)}{\mathcal{E} + mc^2 - U(x)}(\sigma\nabla)\varphi = 0. \quad (7.54)$$

The solution of Eq. (7.54) is sought in the form

$$\varphi = w\psi(x)\, e^{\frac{i}{\hbar}\mathbf{p}_{\shortparallel}\mathbf{r}} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \psi(x)\, e^{\frac{i}{\hbar}\mathbf{p}_{\shortparallel}\mathbf{r}}, \quad (7.55)$$

where $\psi(x)$ is the positron wave function corresponding to the transversal motion in the potential well of the channel, and w is a constant spinor which should be defined from the wave function normalization condition

$$w^{\dagger}w = w_1^* w_1 + w_2^* w_2 = 1.$$

Neglecting the small terms of the order $U_{\max}/\mathcal{E} << 1$ (or $\mathcal{E}_{\perp}/\mathcal{E} << 1$) in Eq. (7.54), for the positron wave function describing the transversal motion in the crystal channel we obtain a one-dimensional Schrödinger equation in the potential field $U(x)$

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m_{eff}}{\hbar^2}[\mathcal{E}_{\perp} - U(x)]\psi(x) = 0, \quad (7.56)$$

with the effective mass m_{eff} corresponding to the energy $\mathcal{E}_{\shortparallel}$ of relativistic longitudinal motion

$$m_{eff} = \frac{\mathcal{E}_{\shortparallel}}{c^2} = \sqrt{\frac{\mathbf{p}_{\shortparallel}^2}{c^2} + m^2}. \quad (7.57)$$

In Eq. (7.56) $\mathcal{E}_{\perp} = \mathcal{E} - \mathcal{E}_{\shortparallel}$ is the energy of transversal motion, which parametrically depends on the energy of longitudinal motion $\mathcal{E}_{\perp} = \mathcal{E}_{\perp}(\mathcal{E}_{\shortparallel})$. In the case of planar channeling of positrons with the harmonic potential (7.3), Eq. (7.56) describes the quantum harmonic oscillator the solution of which is given by

$$\psi_n(x) = \left(\frac{\mathcal{E}_{\shortparallel}\Omega}{\pi\hbar c^2}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} e^{-\frac{\mathcal{E}_{\shortparallel}\Omega}{2\hbar c^2}x^2} \mathcal{H}_n\left(\sqrt{\frac{\mathcal{E}_{\shortparallel}\Omega}{\hbar c^2}}x\right), \quad (7.58)$$

where

$$\mathcal{H}_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n e^{-\xi^2}}{d\xi^n} \quad (7.59)$$

are the Hermit polynomials, and the quantization law for the positron transversal energy is

$$\mathcal{E}_{\perp}\left(n, \mathcal{E}_{\shortparallel}\right)=\left(n+\frac{1}{2}\right) \hbar \Omega, \tag{7.60}$$

where Ω is given by Eq. (7.15).

Finally, with the help of Eqs. (7.55) and (7.51) the solution of Eq. (7.48) for the positron wave function with the longitudinal momentum $\mathbf{p}_{\shortparallel}$ in the n-th bound state of the transversal motion and spin state σ can be written as

$$\Psi_{\mathbf{p}_{\shortparallel}, n, \sigma}(\mathbf{r}, t)=\sqrt{\frac{\mathcal{E}_{\shortparallel}+m c^{2}}{2 \mathcal{E}_{\shortparallel}}}\begin{pmatrix}\varphi_{\sigma} \\ \frac{c \sigma \widehat{\mathbf{p}}}{\mathcal{E}+m c^{2}-U(x)} \varphi_{\sigma}\end{pmatrix} \psi_{n}(x) e^{\frac{i}{\hbar}\left(\mathbf{p}_{\shortparallel} \mathbf{r}-\mathcal{E} t\right)}, \tag{7.61}$$

where φ_{σ} are the spinors (3.11), and the total energy $\mathcal{E}$ is given by the relation

$$\mathcal{E}\left(\mathbf{p}_{\shortparallel}, n\right)=\sqrt{c^{2} \mathbf{p}_{\shortparallel}^{2}+m^{2} c^{4}}+\left(n+\frac{1}{2}\right) \hbar \Omega. \tag{7.62}$$

Now we can evaluate the wave function of the channeled positron at the induced interaction with an external EM wave in the first approximation of perturbation theory (Ψ_1) on the basis of Eqs. (7.61), (7.62) for unperturbed (by the wave) state in the crystal channel (Ψ_0).

Before the interaction with a plane monochromatic EM wave assume that a positron with an initial longitudinal momentum $\mathbf{p}_{\shortparallel}=(0, p_y, p_z)$ is situated in the bound state of the crystal channel characterized by the quantum numbers n, σ, that is, the initial state is described by the wave function

$$\Psi_{0}(\mathbf{r}, t)=\Psi_{\mathbf{p}_{\shortparallel}, n, \sigma}(\mathbf{r}, t). \tag{7.63}$$

The positron wave function Ψ_1 perturbed by the EM wave will be expanded in terms of the full basis of the eigenstates (7.63) with Eqs. (7.61), (7.62):

$$\Psi_{1}(\mathbf{r}, t)=\sum_{\mathbf{p}_{\shortparallel}^{\prime}, n^{\prime}, \sigma^{\prime}} a_{\mathbf{p}_{\shortparallel}^{\prime}, n^{\prime}, \sigma^{\prime}}(t) \Psi_{\mathbf{p}_{\shortparallel}^{\prime}, n^{\prime}, \sigma^{\prime}}(\mathbf{r}, t), \tag{7.64}$$

where $a_{\mathbf{p}_{\shortparallel}^{\prime}, n^{\prime}, \sigma^{\prime}}(t)$ are unknown functions, and the summation is made over all possible states of the positron transversal motion in the potential well corresponding to planar channeling. Substituting the wave function $\Psi=\Psi_0+\Psi_1$ with Eqs. (7.63) and (7.64) in the Dirac equation (7.48) and neglecting the small terms of the second order by the quantity $\sim e\widehat{\alpha}\mathbf{A}\Psi_1$ (in accordance with the perturbation theory) we obtain the following differential equation for the expansion coefficients $a_{\mathbf{p}_{\shortparallel}^{\prime}, n^{\prime}, \sigma^{\prime}}$:

$$\sum_{\mathbf{p}'_{\shortparallel},n',\sigma''} \hbar \frac{\partial a_{\mathbf{p}'_{\shortparallel},n',\sigma'}}{\partial t} \Psi_{\mathbf{p}'_{\shortparallel},n',\sigma'}(\mathbf{r},t) = ie\widehat{\alpha}\mathbf{A}(\mathbf{r},t)\Psi_{\mathbf{p}_{\shortparallel},n,\sigma}(\mathbf{r},t) . \tag{7.65}$$

Multiplying Eq. (7.65) on the left-hand side by $\Psi^{\dagger}_{\mathbf{p}'_{\shortparallel},n',\sigma'}(\mathbf{r},t)$ and integrating over $d\mathbf{r}dt$ one can present the solution of Eq. (7.65) in the form

$$a_{\mathbf{p}'_{\shortparallel},n',\sigma'} = i\frac{eA_0}{4}\sqrt{\frac{2\hbar\Omega}{\mathcal{E}_{\shortparallel}}}\delta_{\sigma'\sigma}\left[\sqrt{n}\delta_{n'+1,n} - \sqrt{n+1}\delta_{n'-1,n}\right]$$

$$\times\left[\delta_{\mathbf{p}'_{\shortparallel},\mathbf{p}_{\shortparallel}+\hbar\mathbf{k}_0}\frac{e^{-\frac{i}{\hbar}\left(\mathcal{E}(\mathbf{p}_{\shortparallel},n)-\mathcal{E}\left(\mathbf{p}'_{\shortparallel},n'\right)+\hbar\omega_0\right)t}}{\mathcal{E}(\mathbf{p}_{\shortparallel},n)-\mathcal{E}\left(\mathbf{p}'_{\shortparallel},n'\right)+\hbar\omega_0}\right.$$

$$\left.+\delta_{\mathbf{p}'_{\shortparallel},\mathbf{p}_{\shortparallel}-\hbar\mathbf{k}_0}\frac{e^{-\frac{i}{\hbar}\left(\mathcal{E}(\mathbf{p}_{\shortparallel},n)-\mathcal{E}\left(\mathbf{p}'_{\shortparallel},n'\right)-\hbar\omega_0\right)t}}{\mathcal{E}(\mathbf{p}_{\shortparallel},n)-\mathcal{E}\left(\mathbf{p}'_{\shortparallel},n'\right)-\hbar\omega_0}\right] . \tag{7.66}$$

In Eq. (7.66) it was assumed that the wave propagates in the plane yz with the vector potential directed along the axis x:

$$A_x = A_0\cos\left(\omega_0 t - \mathbf{k}_0\mathbf{r}\right) ,$$

and was taken into account that for actual cases $\hbar\omega_0/\mathcal{E}_{\shortparallel} << 1$ and the positron energies $\mathcal{E} < m^2c^4/U_0$ as well.

As is seen from Eq. (7.66) only the following expansion coefficients differ from zero

$$a_{\mathbf{p}_{\shortparallel}+\hbar\mathbf{k}_0,n-1,\sigma}(t) = \mathcal{D}\sqrt{n}\frac{e^{-i(\omega+\Omega)t}}{\omega+\Omega},$$

$$a_{\mathbf{p}_{\shortparallel}+\hbar\mathbf{k}_0,n+1,\sigma}(t) = -\mathcal{D}\sqrt{n+1}\frac{e^{-i(\omega-\Omega)t}}{\omega-\Omega},$$

$$a_{\mathbf{p}_{\shortparallel}-\hbar\mathbf{k}_0,n-1,\sigma}(t) = -\mathcal{D}\sqrt{n}\frac{e^{i(\omega-\Omega)t}}{\omega-\Omega}, \tag{7.67}$$

$$a_{\mathbf{p}_{\shortparallel}-\hbar\mathbf{k}_0,n+1,\sigma}(t) = \mathcal{D}\sqrt{n+1}\frac{e^{i(\omega+\Omega)t}}{\omega+\Omega},$$

where the quantity $\mathcal{D}$ is

$$\mathcal{D} = i\frac{eA_0}{2\hbar}\sqrt{\frac{\hbar\Omega}{2\mathcal{E}_{\shortparallel}}}, \tag{7.68}$$

and the Doppler-shifted wave frequency ω is

$$\omega = \omega_0 - \mathbf{k}_0 \mathbf{v}_{\shortparallel}; \quad \mathbf{v}_{\shortparallel} = \frac{c^2 \mathbf{p}_{\shortparallel}}{\mathcal{E}_{\shortparallel}}. \tag{7.69}$$

The expressions in Eq. (7.67) show that the second and third coefficients have a resonance character due to which the induced channeling effect occurs — resonance absorption of the wave photons by a channeled particle and coherent emission of the photons into the wave. Hence, neglecting in Eq. (7.64) the small terms with nonresonant expansion coefficients (first and fourth ones in Eq. (7.67)) of the perturbed wave function for the probability density of the positron at the planar channeling we will have

$$W(\mathbf{r}, t) = \varphi_n^2(x) + \frac{eA_0}{\hbar(\omega - \Omega)} \sqrt{\frac{\hbar\Omega}{2\mathcal{E}}} \varphi_n(x)$$

$$\times \left[\sqrt{n+1}\varphi_{n+1}(x) - \sqrt{n}\varphi_{n-1}(x)\right] \sin(\mathbf{k}_0 \mathbf{r} - \omega_0 t). \tag{7.70}$$

In the case of the exact resonance ($\omega = \Omega$) Eq. (7.70) is not applicable. In this case the solution of Eq. (7.65) for the probability density of the positron gives

$$W(\mathbf{r}, t) = \varphi_n^2(x) + \frac{eA_0}{\hbar} \sqrt{\frac{\hbar\Omega}{2\mathcal{E}}} \varphi_n(x)$$

$$\times \left[\sqrt{n}\varphi_{n-1}(x) - \sqrt{n+1}\varphi_{n+1}(x)\right] \Delta t \cos(\mathbf{k}_0 \mathbf{r} - \omega_0 t), \tag{7.71}$$

where Δt is the period of channeled positron interaction with EM wave.

As is seen from the Eqs. (7.70) and (7.71) the probability density of the positron due to the induced channeling effect is modulated at the stimulating wave frequency (in the one-photon approximation; in the next orders of perturbation theory we will obtain modulation at the harmonics of the wave fundamental frequency).

The condition of validity of the perturbation theory at which the obtained formulas are applicable we can obtain from Eq. (7.71):

$$\frac{eE_0 \mathrm{v}_{xm} \Delta t}{\hbar\omega_0} \ll 1, \tag{7.72}$$

where v_{xm} is the maximal velocity of transversal motion of the positron in the channel of the crystal (see Eq. (7.18)):

$$\mathrm{v}_{xm} = c\sqrt{\frac{2n\hbar\Omega}{\mathcal{E}_{\shortparallel}}} = c\sqrt{\frac{2\mathcal{E}_{\perp}}{\mathcal{E}_{\shortparallel}}}. \tag{7.73}$$

7.4 Quantum Description of the Induced Axial Channeling Effect

At the axial channeling the state of the electron is characterized by the projection of the momentum p_z on the crystal axis z, and due to the axial symmetry of the effective electrostatic potential of an atomic chain within the channel the projection of the orbital moment of the electron on the same axis is conserved.

The Dirac equation for an electron at the axial channeling is written in the form (7.48) with the Hamiltonian

$$\widehat{H}_0 = c\widehat{\alpha}\widehat{\mathbf{p}} + \widehat{\beta}mc^2 + U(\rho), \tag{7.74}$$

where $U(\rho)$ is given by Eq. (7.35). The interaction of the electron with the external EM wave will again be taken into account by perturbation theory (in the one-photon approximation):

$$\Psi = \Psi_0 + \Psi_1; \quad |\Psi_1| \ll |\Psi_0|,$$

where Ψ_0 is the electron wave function in a crystal at the axial channeling, which satisfies the equation

$$i\hbar\frac{\partial \Psi_0}{\partial t} = \left[c\widehat{\alpha}\widehat{\mathbf{p}} + \widehat{\beta}mc^2 + U(\rho)\right]\Psi_0. \tag{7.75}$$

The solution of Eq. (7.75) may be presented in the form

$$\Psi_0(\mathbf{r}, t) = b\begin{pmatrix}\Phi \\ \chi\end{pmatrix} e^{\frac{i}{\hbar}(p_z z - \mathcal{E}t)}, \tag{7.76}$$

where $\mathcal{E}$ is the total energy of the electron and b is the normalization coefficient. The bispinors Φ and χ are connected by the relation

$$\chi = \frac{cp_z\sigma_z + c\widehat{\mathbf{p}}\sigma}{\mathcal{E} + mc^2 - U(\rho)}\Phi. \tag{7.77}$$

From Eq. (7.75) for the wave function of the electron transversal motion in the channel with the accuracy of a small term $\sim U_0/\mathcal{E}$ we obtain the equation

$$\Delta_{\rho,\varphi}\Phi(\rho,\varphi) + \frac{2\mathcal{E}_{\shortparallel}}{\hbar^2c^2}\left[\mathcal{E}_{\perp} - U(\rho)\right]\Phi(\rho,\varphi) = 0, \tag{7.78}$$

where

$$\Delta_{\rho,\varphi} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$$

is the two-dimensional Laplacian,

$$\mathcal{E}_{\shortparallel} = \sqrt{c^2p_z^2 + m^2c^4}$$

is the energy of the electron longitudinal motion, and $\mathcal{E}_\perp = \mathcal{E} - \mathcal{E}_{\shortparallel}$ is the transversal one.

As is seen from Eq. (7.78) for wave function $\Phi(\rho,\varphi)$ the variables are separated and the eigenvalue of the operator

$$\widehat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$$

— the projection of the orbital moment of the electron on the z axis is conserved. Then the wave function $\Phi(\rho,\varphi)$ can be represented in the form

$$\Phi(\rho,\varphi) = \Phi(\rho)\, e^{i\mathfrak{m}\varphi}; \quad \mathfrak{m} = 0, \pm1, \pm2, ..., \tag{7.79}$$

where $\mathfrak{m}$ is the azimuthal quantum number, and from Eq. (7.78) for the function

$$R(\rho) = \frac{\Phi(\rho)}{\sqrt{\rho}} \tag{7.80}$$

we obtain the equation

$$R'' + \frac{2}{\rho}R' + \left[\frac{2\mathcal{E}_{\shortparallel}}{\hbar^2c^2}\left(\mathcal{E}_\perp + \frac{\alpha_c}{\rho}\right) - \frac{\mathfrak{m}^2 - 1/4}{\rho^2}\right]R = 0. \tag{7.81}$$

For the solution of Eq. (7.81) we pass from ρ to a new variable

$$r = \frac{2}{\hbar c}\sqrt{2\mathcal{E}_{\shortparallel}\,|\mathcal{E}_\perp|}\rho, \tag{7.82}$$

and making a notation

$$n = \frac{\alpha_c}{\hbar c}\sqrt{\frac{\mathcal{E}_{\shortparallel}}{2\,|\mathcal{E}_\perp|}}, \tag{7.83}$$

then introducing the function $R(r)$ in the form

$$R(r) = r^{|\mathfrak{m}|-1/2}e^{-r/2}w(r), \tag{7.84}$$

for the new function $w(r)$ we obtain the equation

$$rw'' + \left[2\left(|\mathfrak{m}| - \frac{1}{2}\right) + 2 - r\right] w' + \left(n - |\mathfrak{m}| - \frac{1}{2}\right) w = 0. \tag{7.85}$$

The solution of Eq. (7.85) should not diverge at infinity more quickly than a limited power r and must be confined at $r = 0$. The function satisfying the second condition is the degenerated hypergeometric function

$$w(r) = F\left(-n + |\mathfrak{m}| + \frac{1}{2}, 2|\mathfrak{m}| + 1, r\right), \tag{7.86}$$

and the solution satisfying the first condition at infinity will be obtained only at the integer negative (or equal to zero) values of the argument $-n+|\mathfrak{m}|+1/2$ when the function (7.86) turns to polynomial with the power $n - |\mathfrak{m}| - 1/2$. Otherwise it diverges at infinity as e^r. Hence, the number n must be a positive half-integer, and at the specified number $\mathfrak{m}$ it is necessary that

$$n \geq |\mathfrak{m}| + \frac{1}{2}; \quad n = |\mathfrak{m}| + \frac{1}{2} + n_\rho; \quad n_\rho = 0, 1, 2, \ldots. \tag{7.87}$$

These conditions determine the quantization law of the electron transversal motion in the potential well of the crystal at the axial channeling. Thus, from Eq. (7.83) for the spectrum of the transversal energy eigenvalues of the electron bound states in the potential field (7.35) we obtain

$$\mathcal{E}_\perp = -\frac{\alpha_c^2 \mathcal{E}_{\shortparallel}}{2\hbar^2 c^2 n^2}. \tag{7.88}$$

With the help of Eqs. (7.77), (7.79), (7.84) and (7.86) for the wave function of the channeled electron (7.76), normalized for one particle per unit volume, we will have the equation

$$\Psi_0(\mathbf{r}, t) = \Psi_{p_z, n, \mathfrak{m}, \sigma}(\mathbf{r}, t) = \sqrt{\frac{\mathcal{E}_{\shortparallel} + mc^2}{2\mathcal{E}_{\shortparallel}}} \begin{pmatrix} \varphi_\sigma \\ \frac{c\sigma\mathbf{p}}{\mathcal{E} + mc^2 - U(\rho)} \varphi_\sigma \end{pmatrix}$$

$$\times \sqrt{\frac{\rho}{2\pi}} R_{n,|\mathfrak{m}|-1/2}(\rho)\, e^{i\mathfrak{m}\varphi} e^{\frac{i}{\hbar}(p_z z - \mathcal{E}t)}, \tag{7.89}$$

where φ_σ is a constant spinor determined in Eq. (7.61), and the function $R_{n,|\mathfrak{m}|-1/2}(\rho)$ is

$$R_{n,|\mathfrak{m}|-1/2}(\rho) = \left(\frac{\mathcal{E}_{\shortparallel}\alpha_c}{\hbar^2 c^2}\right)^{3/2} \frac{4}{n^{|\mathfrak{m}|+3/2}} \sqrt{\frac{2(n + |\mathfrak{m}| - 1/2)!}{(n - |\mathfrak{m}| - 1/2)!}} \left(\frac{4\mathcal{E}_{\shortparallel}\alpha_c\rho}{\hbar^2 c^2}\right)^{|\mathfrak{m}|-1/2}$$

$$\times \exp\left\{-\frac{2\mathcal{E}_{\shortparallel}\alpha_c}{n\hbar^2c^2}\rho\right\} F\left(-n+|\mathfrak{m}|+1/2, 2|\mathfrak{m}|+1, \frac{4\mathcal{E}_{\shortparallel}\alpha_c}{n\hbar^2c^2}\rho\right). \tag{7.90}$$

The total energy $\mathcal{E}$ in Eq. (7.89) is given by the relation

$$\mathcal{E}(p_z, n) = \sqrt{c^2p_z^2 + m^2c^4} - \frac{2\alpha_c^2\mathcal{E}_{\shortparallel}}{\hbar^2c^2n^2}. \tag{7.91}$$

To determine the electron wave function Ψ_1 perturbed by the EM wave in the next approximation of perturbation theory one needs the concrete form of the wave vector potential. Let it have the form

$$A_x = A_0 \cos(\omega_0 t - k_0 z),$$

$$A_y = A_0 \sin(\omega_0 t - k_0 z). \tag{7.92}$$

Expanding Ψ_1 in terms of the full basis of the eigenstates (7.89)

$$\Psi_1(\mathbf{r}, t) = \sum_{p_z', n', m', \sigma'} c_{p_z', n', \mathfrak{m}', \sigma'}(t)\, \Psi_{p_z', n', \mathfrak{m}'\sigma'}(\mathbf{r}, t), \tag{7.93}$$

and substituting the wave function in the first approximation of perturbation theory $\Psi_0 + \Psi_1$ into Eq. (7.48) with Eqs. (7.89)–(7.92), then after the solution of the obtained equation for unknown expansion coefficients $c_{p_z', n', \mathfrak{m}'}(t)$ we will have

$$c_{p_z', n', \mathfrak{m}', \sigma'} = -i\frac{eA_0}{2c}\Omega_{n'n}\mathcal{D}_{n'n}^{\mathfrak{m}'\mathfrak{m}}\delta_{\sigma\sigma'}\left\{\frac{e^{-\frac{i}{\hbar}\left(\mathcal{E}(p_z,n) - \mathcal{E}\left(p_z',n'\right) + \hbar\omega_0\right)t}}{\mathcal{E}(p_z, n) - \mathcal{E}(p_z', n') + \hbar\omega_0}\delta_{\mathfrak{m}', \mathfrak{m}+1}\right.$$

$$\left.\times\delta_{p_z', p_z + \hbar k_0} + \frac{e^{-\frac{i}{\hbar}\left(\mathcal{E}(p_z,n) - \mathcal{E}\left(p_z',n'\right) - \hbar\omega_0\right)t}}{\mathcal{E}(p_z, n) - \mathcal{E}(p_z', n') - \hbar\omega_0}\delta_{\mathfrak{m}', \mathfrak{m}-1}\delta_{p_z, p_z - \hbar k_0}\right\}, \tag{7.94}$$

where

$$\mathcal{D}_{n'n}^{\mathfrak{m}'\mathfrak{m}} = \int_0^\infty \rho^3 R_{n', |\mathfrak{m}'|-1/2}(\rho)\, R_{n, |\mathfrak{m}|-1/2}(\rho)\, d\rho, \tag{7.95}$$

and

$$\Omega_{n'n} = \frac{\mathcal{E}_{\perp n'} - \mathcal{E}_{\perp n}}{\hbar} = -\frac{2\mathcal{E}_{\shortparallel}\alpha_c^2}{\hbar^3c^2n'^2n^2}(n' + n)(n' - n) \tag{7.96}$$

is the transition frequency between the initial and excited states of the transversal motion of the electron in the crystal channel.

Equations (7.93) and (7.94) determine the wave function of the one-photon induced axial channeling effect. With the help of the latter the probability density ($\Psi^{+}\Psi$) of the electron after the interaction can be presented in the form

$$W = \frac{\rho}{2\pi} R^2_{n,|\mathfrak{m}|-1/2}(\rho) + \frac{eA_0\rho}{2\pi\hbar} R_{n,|\mathfrak{m}|-1/2}(\rho)$$

$$\times \left\{ \sum_{n' \geqslant |\mathfrak{m}+1|+1/2} \Omega_{n'n} \frac{R_{n,|\mathfrak{m}+1|-1/2}(\rho)}{\omega - \Omega_{n'n}} \mathcal{D}^{\mathfrak{m}+1\mathfrak{m}}_{n'n} \right.$$

$$\left. + \sum_{n' \geqslant |\mathfrak{m}-1|+1/2} \Omega_{n'n} \frac{R_{n',|\mathfrak{m}-1|-1/2}(\rho)}{\omega + \Omega_{n'n}} \mathcal{D}^{\mathfrak{m}-1\mathfrak{m}}_{n'n} \right\} \sin(k_0 z - \omega_0 t + \varphi), \quad (7.97)$$

where the Doppler-shifted wave frequency ω is

$$\omega = \omega_0 \left(1 - n_0 \frac{c p_z}{\mathcal{E}_{\shortparallel}}\right). \quad (7.98)$$

As in the case of the planar channeling the electron probability density is modulated at the wave frequency. Consequently, the electric current density in the case of an electron beam will be modulated at the stimulating wave frequency and its harmonics (corresponding equations for the modulation at the harmonics can be found in the next approximation of perturbation theory). Equation (7.97) is complicated enough for general forms of the functions $R_{n,\mathfrak{m}}(\rho)$ and $\mathcal{D}^{\mathfrak{m}'\mathfrak{m}}_{n'n}$. It is rather simplified for resonant transitions of the electron from the initial bound state of transversal motion to the neighbor ones. Thus, from Eqs. (7.88), (7.95), and (7.96) we obtain that in the expression of the modulation depth quantity $\Omega_{n'n}\mathcal{D}^{\mathfrak{m}'\mathfrak{m}}_{n'n} \sim \sqrt{\mathcal{E}_{\perp}/\mathcal{E}_{\shortparallel}}$. The latter is the amplitude of the velocity of the electron transversal motion in the channel $\mathrm{v}_{\perp m}$. Besides, the resonant denominators in Eq. (7.97) define the period of coherent interaction of the electron with the EM wave in the channel: $(\omega - \Omega_{n'n})^{-1} \to \Delta t$. Hence, the modulation depth $\sim eE_0 \mathrm{v}_{\perp m} \Delta t/\omega << 1$ in accordance with the perturbation theory.

Note that in general the function $\mathcal{D}^{\mathfrak{m}'\mathfrak{m}}_{n'n}$ determined by Eq. (7.95) may be presented in the form

$$\mathcal{D}^{\mathfrak{m}'\mathfrak{m}}_{n'n} = \frac{\hbar^2 c^2}{\mathcal{E}_{\shortparallel}\alpha_c} \frac{2^{|\mathfrak{m}|+|\mathfrak{m}'|}}{n^{|\mathfrak{m}|+3/2} n'^{|\mathfrak{m}'|+3/2} (2|\mathfrak{m}|)!\,(2|\mathfrak{m}'|)!}$$

$$\times \sqrt{\frac{(n+|\mathfrak{m}|-1/2)!\,(n'+|\mathfrak{m}'|-1/2)!}{(n-|\mathfrak{m}|-1/2)!\,(n'-|\mathfrak{m}'|-1/2)!}} \int_0^{\infty} z^{|\mathfrak{m}|+|\mathfrak{m}'|+2} e^{-(1/n'+1/n)z} \quad (7.99)$$

$$\times F\left(-n+|\mathfrak{m}|+\frac{1}{2},2|\mathfrak{m}|+1,\frac{2z}{n}\right)F\left(-n'+|\mathfrak{m}'|+\frac{1}{2},2|\mathfrak{m}'|+1,\frac{2z}{n'}\right)dz.$$

In Eq. (7.95) integral is known as a function

$$\mathcal{J}_{\gamma}^{sp}(\alpha,\alpha')=\int_0^{\infty}e^{-\frac{\varkappa+\varkappa'}{2}z}z^{\gamma-1+s}F(\alpha,\gamma,\varkappa z)F(\alpha',\gamma-p,\varkappa' z)\,dz,$$

which is expressed via $\mathcal{J}_{\gamma}^{00}(\alpha,\alpha')$ by the recurrent relations.

7.5 Multiphoton Induced Channeling Effect

In the quantum description of the induced channeling effect in the previous two sections the wave field was weak enough so that the interaction process had mainly one-photon character. The coherent (resonant) interaction of the channeled particles with a strong EM wave from the quantum point of view has multiphoton character. Here we will consider the induced channeling effect in the strong wave fields in the scope of quantum theory, that is, we will solve the quantum equations of motion for channeled electrons or positrons in the strong plane EM wave field.

We will assume that the wave propagates in the yz plane of a crystal and is polarized in the xy plane with the vector potential

$$\mathbf{A}=\left\{A_x\left(t-n_0\frac{z}{c}\right),A_y\left(t-n_0\frac{z}{c}\right),0\right\}, \tag{7.100}$$

where $n_0\equiv n(\omega_0)$ is the refractive index of the medium at the carrier frequency of the wave. We will consider the case when averaged potential of the crystal for a plane channeled particle is satisfactorily described by the harmonic potential

$$U(x)=\kappa\frac{x^2}{2}. \tag{7.101}$$

For the positron at the planar channeling

$$\kappa=\frac{8U_0}{d^2} \tag{7.102}$$

(see the potential (7.3)), while for the electrons the approximate potential of the channel is actually not harmonic and described by the potential

$$U(x)=-\frac{U_0}{\cosh^2\left(\frac{x}{b}\right)}. \tag{7.103}$$

Nevertheless, for the high energies it can be approximated by the harmonic potential (7.101). As we saw in previous sections, for the channeled particles the depth of the potential hole $U_0 << \mathcal{E}$, where $\mathcal{E}$ is the particle energy. The spin interaction, which is $\sim \bigtriangledown U(x)$, is again less than $\mathcal{E}$. For this reason the transverse motion of the channeled particle is described by the Schrödinger equation (7.56) with the effective mass $m_{eff} = \mathcal{E}_{\shortparallel}/c^2$. On the other hand, the spin interaction can play a role in the particle–wave interaction process at the energy of the photon comparable with the particle one: $\hbar\omega_0 \sim \mathcal{E}$. If the particle energy is not high enough, i.e., $\mathcal{E} << m^2c^4/\mathcal{E}_\perp$ (optimal cases for the channeling), then the resonant interaction of the channeled particles with an external EM wave takes place at $\hbar\omega_0 << \mathcal{E}$ and the spin effects are not essential. Hence, one may ignore the spin interaction and instead of the Dirac equation solve the Klein–Gordon equation

$$\left[i\hbar\frac{\partial}{\partial t} - U(x)\right]^2 \Psi = \left[c^2\left(\widehat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\left(t - n_0\frac{z}{c}\right)\right)^2 + m^2c^4\right]\Psi. \tag{7.104}$$

As we saw in Section 7.3 the channeled particle initial motion (before the interaction with EM wave) is separated into longitudinal (y, z) and transversal (x) degrees of freedom. For the longitudinal motion we assume an initial state with a momentum $\mathbf{p}_{\shortparallel} = \{0, p_y, p_z\}$, while for the transversal motion we assume a quantum state $\{n\}$, where by n we indicate the energy levels in the harmonic potential (7.101). As the plane wave field depends only on the retarding coordinate $\tau = t - n_0 z/c$, then using the problem symmetry the wave function of a channeled particle can be sought in the form

$$\Psi(\mathbf{r}, t) = f(x, \tau)e^{\frac{i}{\hbar}(\mathbf{p}_{\shortparallel}\mathbf{r} - \mathcal{E}t)}. \tag{7.105}$$

The multiphoton interaction of the charged particles with a strong EM wave, in general, as was shown in diverse processes is well enough described by the eikonal-type wave function corresponding to a slowly varying function $f(x, \tau)$ on the wave coordinate τ. Hence, neglecting the second derivatives of this function compared with the first-order ones in accordance with the conditions (3.92) for the function $f(x, \tau)$ we will obtain the equation

$$\left[\hbar^2\frac{\partial^2}{\partial x^2} + \frac{2\mathcal{E}_{\shortparallel}}{c^2}(\mathcal{E}_\perp - U(x)) + 2i\frac{\widetilde{p}\hbar}{c}\frac{\partial}{\partial \tau} - 2i\frac{e\hbar}{c}A_x(\tau)\frac{\partial}{\partial x}\right.$$

$$\left. +2\frac{e}{c}p_y A_y(\tau) - \frac{e^2}{c^2}\mathbf{A}^2(\tau)\right] f(x, \tau) = 0, \tag{7.106}$$

where

$$\widetilde{p} = \frac{1}{c}\left(\mathcal{E}_{\shortparallel} - n_0 c p_z\right). \tag{7.107}$$

In Eq. (7.106) the transversal and longitudinal motions are not separated. But after the definite unitarian transformation for the transformed function the variables are separated. The corresponding unitarian transformation operator is

$$\widehat{S} = e^{\frac{i}{\hbar}\{g_1(\tau)x - g_2(\tau)\widehat{p}_x\}}, \tag{7.108}$$

where the functions $g_1(\tau)$, $g_2(\tau)$ will be chosen to separate the transversal and longitudinal motions and to satisfy the initial condition. Taking into account Eq. (4.54) for transformed function

$$\Phi(x,\tau) = \widehat{S} f(x,\tau) \tag{7.109}$$

we obtain the equation

$$\Bigg[\hbar^2 \frac{\partial^2}{\partial x^2} + \frac{2\mathcal{E}_{\shortparallel}}{c^2}(\mathcal{E}_\perp - U(x)) + 2i\hbar\left(\frac{\widetilde{p}}{c}\frac{dg_2(\tau)}{d\tau} - g_1(\tau) - \frac{e}{c}A_x(\tau)\right)\frac{\partial}{\partial x}$$

$$+\frac{2}{c}\left(\widetilde{p}\frac{dg_1(\tau)}{d\tau} + \frac{\mathcal{E}_{\shortparallel}\kappa}{c}g_2(\tau)\right)x + \frac{2i\widetilde{p}\hbar}{c}\frac{\partial}{\partial \tau} + Q(\tau)\Bigg]\Phi(x,\tau) = 0, \tag{7.110}$$

where

$$Q(\tau) = \frac{\widetilde{p}}{c}\left(\frac{dg_2(\tau)}{d\tau}g_1(\tau) - \frac{dg_1(\tau)}{d\tau}g_2(\tau)\right) - g_1^2(\tau) - \frac{\mathcal{E}_{\shortparallel}\kappa}{c^2}g_2^2(\tau)$$

$$-\frac{2e}{c}A_x(\tau)\,g_1(\tau) + \frac{2e}{c}p_y A_y(\tau) - \frac{e^2}{c^2}\mathbf{A}^2(\tau)\,. \tag{7.111}$$

Let us choose $g_1(\tau)$ and $g_2(\tau)$ in such a form that the coefficients of x and $\partial/\partial x$ in Eq. (7.110) become zero. Then for the functions $g_1(\tau)$ and $g_2(\tau)$ we will obtain a classical equation of motion describing stimulated oscillations in the harmonic potential:

$$\frac{dg_1(\tau)}{d\tau} = -\frac{\mathcal{E}_{\shortparallel}\kappa}{c\widetilde{p}}g_2(\tau), \tag{7.112}$$

$$\frac{dg_2(\tau)}{d\tau} = \frac{c}{\widetilde{p}}g_1(\tau) + \frac{e}{\widetilde{p}}A_x(\tau)\,. \tag{7.113}$$

The solutions of Eqs. (7.112) and (7.113) can be written as

$$g_1(\tau) = \frac{e\Omega'}{c}\mathrm{Im}\left[e^{-i\Omega'\tau}\int\limits_{-\infty}^{\tau} A_x(\tau)\, e^{i\Omega'\tau'}d\tau'\right], \tag{7.114}$$

$$g_2(\tau) = \frac{e}{\widetilde{p}}\mathrm{Re}\left[e^{-i\Omega'\tau}\int\limits_{-\infty}^{\tau} A_x(\tau)\, e^{i\Omega'\tau'}d\tau'\right], \tag{7.115}$$

where

$$\Omega' = \frac{\Omega}{1 - n_0\frac{\mathrm{v}_z}{c}}; \quad \Omega = c\sqrt{\kappa/\mathcal{E}_{\shortparallel}}. \tag{7.116}$$

In Eqs. (7.114) and (7.115) we have taken into account the initial condition

$$g_1(-\infty) = g_2(-\infty) = 0.$$

After the unitarian transformation (7.109) for the function $\Phi(x,\tau)$ the following equation is obtained:

$$\left[\hbar^2\frac{\partial^2}{\partial x^2} + \frac{2\mathcal{E}_{\shortparallel}}{c^2}(\mathcal{E}_\perp - U(x)) + \frac{2i\widetilde{p}\hbar}{c}\frac{\partial}{\partial\tau} + Q(\tau)\right]\Phi(x,\tau) = 0. \tag{7.117}$$

Now in Eq. (7.117) the variables are separated and the solution can be written as follows:

$$\Phi(x,\tau) = N\varphi_n(x)\exp\left\{i\frac{c}{2\hbar\widetilde{p}}\int\limits_{-\infty}^{\tau} Q(\tau)\, d\tau'\right\}, \tag{7.118}$$

where $\varphi_n(x)$ coincides with the harmonic oscillator wave function (7.58) and $N = 1/\sqrt{L_y L_z}$is the normalization constant (L_y and L_z are the quantization lengths). By inverse transformation

$$f(x,\tau) = \widehat{S}^\dagger\Phi(x,\tau),$$

with the help of Eq. (4.66) we obtain the solution of the initial equation (7.104) (taking into account Eq.(7.105)):

$$\Psi(\mathbf{r},t) = N\exp\left\{\frac{i}{\hbar}(\mathbf{p}_{\shortparallel}\mathbf{r} - \mathcal{E}t)\right\}\varphi_n(x + g_2(\tau))$$

$$\times\exp\left\{\frac{i}{\hbar}\left[\frac{c}{2\widetilde{p}}\int\limits_{-\infty}^{\tau} Q(\tau)\, d\tau' - \frac{1}{2}g_1(\tau)g_2(\tau) - g_1(\tau)x\right]\right\}, \tag{7.119}$$

where the function $Q(\tau)$ can be represented in the form

$$Q(\tau) = \frac{2e}{c} p_y A_y(\tau) - \frac{e}{c} A_x(\tau) g_1(\tau) - \frac{e^2}{c^2} \mathbf{A}^2(\tau). \tag{7.120}$$

This wave function describes the multiphoton interaction of the channeled particle with the strong EM radiation field. Thus, for a monochromatic wave

$$\mathbf{A} = \{A_0 \cos(\omega_0 t - k_0 z), 0, 0\},$$

from Eqs. (7.114) and (7.115) for the functions $g_1(\tau)$ and $g_2(\tau)$ we obtain

$$g_1(\tau) = \frac{e}{c} A_0 \frac{\Omega'^2}{\Delta} \cos \omega_0 \tau,$$

$$g_2(\tau) = \frac{eA_0}{\widetilde{p}} \frac{\omega_0}{\Delta} \sin \omega_0 \tau, \tag{7.121}$$

and we will have the following wave function for the particle in the field of a strong EM wave at the planar channeling:

$$\Psi(\mathbf{r}, t) = N \exp\left\{\frac{i}{\hbar}\left(\mathbf{p}_{\shortparallel}\mathbf{r} - \mathcal{E}t - \frac{e^2 A_0^2 \omega_0^2}{4c\widetilde{p}\Delta}\tau\right)\right\} \varphi_n\left(x + \frac{eA_0\omega_0}{\widetilde{p}\Delta} \sin \omega_0 \tau\right)$$

$$\times \exp\left\{-\frac{i}{\hbar}\left[\frac{eA_0\Omega'^2}{c\Delta} x \cos \omega_0 \tau + \frac{e^2 A_0^2 \omega_0 \left(\omega_0^2 + \Omega'^2\right)}{8c\widetilde{p}\Delta^2} \sin(2\omega_0 \tau)\right]\right\}, \tag{7.122}$$

where

$$\Delta = \omega_0^2 - \Omega'^2$$

is the resonance detuning.

On the basis of the obtained wave function (7.119) consider the possibility of multiphoton excitation of transversal levels by the strong EM wave at the resonance

$$\omega_0 \simeq \frac{\Omega}{\left|1 - n_0 \frac{\mathrm{v}_z}{c}\right|}. \tag{7.123}$$

The Doppler factor $1 - n_0 \mathrm{v}_z / c$ may be positive as well as negative — anomalous Doppler effect at $n_0 > 1$. We will consider the actual case of a quasi-monochromatic EM wave with a slowly varying amplitude $A_0(\tau)$. After the interaction with the wave ($t \to +\infty$) from Eqs. (7.114) and (7.115) at the resonance condition (7.123) we have

$$g_1(\tau) = \frac{e\overline{A}_0 T \Omega'}{2c} \sin \omega_0 \tau, \tag{7.124}$$

$$g_2(\tau) = \frac{e\overline{A}_0 T}{2\widetilde{p}} \cos \omega_0 \tau, \tag{7.125}$$

where T is the coherent interaction time (for actual laser radiation T is the pulse duration) and $\overline{A}_0$ is the average value of the slowly varied envelope. Substituting Eqs. (7.124) and (7.125) into the expression for the wave function (7.119) and expanding the latter in terms of the full basis of the particle eigenstates

$$\Psi(\mathbf{r},t) = \sum_{\mathbf{p}'_{\shortparallel},n'} a_{\mathbf{p}'_{\shortparallel},n'}(t)\, \Psi_{\mathbf{p}'_{\shortparallel},n'}(\mathbf{r},t), \tag{7.126}$$

we find the probabilities of the multiphoton induced transitions between the transversal levels. To calculate the expansion coefficients

$$a_{\mathbf{p}'_{\shortparallel},n'}(t) = \int \Psi^*_{\mathbf{p}'_{\shortparallel},n'}(\mathbf{r},t)\, \Psi(\mathbf{r},t)\, d\mathbf{r}, \tag{7.127}$$

we will take into account the result of the integration (4.73). Taking into account Eqs.(7.124), (7.125), (7.119), and (7.127) we get the following expansion coefficients:

$$a_{\mathbf{p}'_{\shortparallel},n'}(t) = I_{n,n'}(\alpha)\, \delta_{p'_y,p_y} \delta_{p'_z,p_z+\mu\hbar k_0(n'-n)}$$

$$\times \exp\left\{\frac{i}{\hbar}(\mathcal{E}(\mathbf{p}'_{\shortparallel},n') - \mathcal{E}(\mathbf{p}_{\shortparallel},n) - \mu\hbar\omega_0(n'-n))t + i\phi\right\}, \tag{7.128}$$

where

$$\mu = \frac{1 - n_0 \frac{\mathrm{v}_z}{c}}{\left|1 - n_0 \frac{\mathrm{v}_z}{c}\right|},$$

and

$$\phi \equiv \frac{c}{2\hbar\widetilde{p}} \int_{-\infty}^{\infty} Q(\tau)\, d\tau'$$

is the constant phase. Here the argument of the Lagger function $I_{n,n'}(\alpha)$ is

$$\alpha = \frac{e^2 \overline{A}_0^2 T^2}{8\hbar} \frac{\Omega'}{c\widetilde{p}}. \tag{7.129}$$

According to Eq. (7.128) the transition of the particle from an initial state $\{p_y, p_z, n\}$ to a state $\{p'_y, p'_z, n'\}$ is accompanied by the emission or absorption of $|n - n'|$ number of photons. Consequently, substituting Eq. (7.128) into Eq. (7.126) we can rewrite the particle wave function in the form

$$\Psi(\mathbf{r}, t) = N \sum_{n'=0}^{\infty} I_{n,n'}(\alpha) \exp\left\{\frac{i}{\hbar}(p_y y + (p_z + \mu\hbar k_0(n' - n))z)\right\}$$

$$\times \exp\left\{-\frac{i}{\hbar}(\mathcal{E}(\mathbf{p}_{\shortparallel}, n) + \mu\hbar\omega_0(n' - n))t + i\phi\right\}\varphi_{n'}(x). \qquad (7.130)$$

Hence, the probability of the induced transitions $n \to n'$ between the energy levels of the particle transversal motion in the channel finally is defined from Eq. (7.130):

$$W_{n,n'} = I^2_{n,n'}\left(\frac{e^2\overline{A}_0^2 T^2 \Omega'}{8\hbar c\widetilde{p}}\right). \qquad (7.131)$$

Equation (7.130) shows that in the field of a strong EM wave the transversal levels are excited at the absorption of the wave quanta if $1 - n_0 \mathrm{v}_z/c > 0$ and $\mu = 1$, corresponding to the normal Doppler effect, while in the case $1 - n_0 \mathrm{v}_z/c < 0$ and $\mu = -1$ the transversal levels are excited at the emission of coherent quanta due to the anomalous Doppler effect.

Let us now estimate the average number of emitted (absorbed) photons by the particle at the resonance for the high excited levels ($n >> 1$) and for the strong EM wave. In this case the most probable number of photons in the strong wave field corresponds to the quasiclassical limit ($|n - n'| >> 1$) when multiphoton processes dominate and the nature of the interaction process is very close to the classical one. In this case the argument of the Lagger function can be represented as

$$\alpha = \frac{1}{4n}\left(\frac{\triangle\mathcal{E}_{cl}}{\hbar\omega_0}\right)^2, \qquad (7.132)$$

where

$$\triangle\mathcal{E}_{cl} = \frac{eE_0 T}{2}\frac{\overline{\mathrm{v}}_\perp}{\left|1 - n_0\frac{\mathrm{v}_z}{c}\right|}$$

is the maximal energy change of the particle according to classical perturbation theory (E_0 is the amplitude of the electric field strength of the EM wave, $\overline{\mathrm{v}}_\perp \simeq c\sqrt{2n\hbar\Omega/\mathcal{E}_{\shortparallel}}$ is the particle mean transversal velocity). Note that according to conditions (3.92) of the considered eikonal approximation $\Delta\mathcal{E} << \mathcal{E}$.

The Lagger function is maximal at $\alpha \to \alpha_0 = \left(\sqrt{n'} - \sqrt{n}\right)^2$, exponentially falling beyond α_0. Hence, for the transition $n \to n'$ and when $|n - n'| << n$ we have

$$\alpha_0 \simeq \frac{(n' - n)^2}{4n}.$$

The comparison of this expression with Eq. (7.132) shows that the most probable transitions are

$$|n - n'| \simeq \frac{\triangle \mathcal{E}_{cl}}{\hbar \omega_0},$$

in accordance with the correspondence principle.

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