6 Induced Nonstationary Transition Process

How will the nonstationarity of a medium reflect on the process of charged particle interaction with strong laser radiation?

In the current laser fields of ultrashort pulse duration and relativistic intensities any medium turns instantaneously (on a time span much smaller than one wave cycle) into a plasma, that is, abrupt change of the medium properties, particularly the dielectric permittivity, occurs in time.

On the other hand, with the abrupt change in time of the dielectric permittivity of a medium, charged particle radiation occurs similar to transition radiation on the boundary of two media with different dielectric permittivity.

In the presence of an external EM radiation field this nonstationary transition process acquires induced character and the inverse process of radiation absorption by a charged particle is actualized, particularly in plasmas where in the stationary states the radiation or absorption of quanta of a transversal EM radiation field (monochromatic radiation such as a laser one) by a free particle cannot proceed.

With the abrupt change in time of the medium dielectric permittivity the production of hard quanta of relativistic energies from the laser radiation is possible and, consequently, electron–positron pair creation in nonstationary plasma of common densities is available. Meanwhile, for electron–positron pair production in a stationary plasma (a medium should be plasmalike for this process) by a γ*-quantum a superdense plasma with electron densities greater than* $10^{34}\,cm^{-3}$ *is necessary. Such superdense matter exists in astrophysical objects (in the core of neutron stars — pulsars), leading to special interest in the processes of electron–positron pair production and annihilation in superdense plasma. On the other hand, the matter in the astrophysical objects may also be in a strongly nonstationary state.*

Hence, it is important to study the induced nonstationary transition process in the strong EM radiation field in a medium with an arbitrary dielectric permittivity changing abruptly in time.

6.1 Effect of Abrupt Temporal Variation of Dielectric Permittivity of a Medium

In the investigation of charged particle interaction with strong EM radiation in a medium, overall it was supposed that the electromagnetic properties of the latter, i.e., the dielectric (ε_0) and magnetic (μ_0) permittivities and, consequently, refractive index n_0, are not changed in the field and the medium being initially in the stationary state maintains its electromagnetic characteristics $n_0 = \sqrt{\varepsilon_0 \mu_0} = \text{const}$.

Consider now how the nonstationarity of a medium will reflect on the process of charged particle interaction with strong EM radiation. From the physical point of view it is clear that the effects that arise here because of the nonstationarity of a medium will be essential at the abrupt temporal change of the dielectric permittivity (as it is generally assumed the magnetic permittivity of the medium will be taken as $\mu_0 = 1$). Under the abrupt change of ε here we mean its change at the time $\Delta t << 2\pi/\omega$, where ω is the characteristic frequency because of the nonstationarity of a medium (then radiation frequency by a charged particle in this process). Such abrupt change of the dielectric permittivity occurs with the propagation of ultrashort laser pulses of relativistic intensities in a medium when the tunneling ionization of atoms on a time span smaller than a few femtoseconds/attoseconds occurs and the medium instantaneously becomes a plasma.

Let a charged particle with constant initial velocity $\mathbf{v}_0$ move in a spatially homogeneous and isotropic medium whose dielectric permittivity ε changes abruptly at the time from a value ε_1 to ε_2

$$\varepsilon = \begin{cases} \varepsilon_1, & t < 0, \\ \varepsilon_2, & t > 0, \end{cases} \tag{6.1}$$

and let a strong EM wave propagate in this medium. To determine the electromagnetic field in that type of nonstationary medium one should solve the macroscopic Maxwell equations

$$\text{rot}\mathbf{H}(\mathbf{r},t) = \frac{1}{c}\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} + \frac{4\pi}{c}\mathbf{J}(\mathbf{r},t), \tag{6.2}$$

$$\text{rot}\mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} \tag{6.3}$$

for $t < 0$ and for $t > 0$, then the obtained solutions should be laced at the instant of time $t = 0$. At the discontinuity of the dielectric permittivity (in general, properties of the medium) only the derivatives of the physical quantities can have large values. Hence, the conditions of the lacing can be obtained by the integration of the Maxwell equations (6.2) and (6.3) over t in the arbitrary small region including the instant of time $t = 0$ at which the stepwise discontinuity of the dielectric permittivity (6.1) occurs. The latter means that the integration should be made between the moments $t_1 = -\Delta t$ and $t_2 = \Delta t$ and then one should take the limit $\Delta t \to 0$. Taking into account that the quantities rot**H**, rot**E**, and **J** are finite, after this procedure we obtain

$$\mathbf{D}\left(\mathbf{r},t\right)\big|_{t=-0} = \mathbf{D}\left(\mathbf{r},t\right)\big|_{t=+0},$$

$$\mathbf{B}\left(\mathbf{r},t\right)\big|_{t=-0} = \mathbf{B}\left(\mathbf{r},t\right)\big|_{t=+0}.$$

These equations can be written in terms of electric and magnetic field strengths with the help of the constitutive equations

$$\mathbf{D}\left(\mathbf{r},t\right) = \varepsilon\left(t\right)\mathbf{E}\left(\mathbf{r},t\right); \qquad \mathbf{B}\left(\mathbf{r},t\right) = \mathbf{H}\left(\mathbf{r},t\right),$$

which yield to "boundary conditions"

$$\varepsilon_1\,\mathbf{E}\left(\mathbf{r},t\right)\big|_{t=-0} = \varepsilon_2\,\mathbf{E}\left(\mathbf{r},t\right)\big|_{t=+0}, \tag{6.4}$$

$$\mathbf{H}\left(\mathbf{r},t\right)\big|_{t=-0} = \mathbf{H}\left(\mathbf{r},t\right)\big|_{t=+0}. \tag{6.5}$$

Under the conditions (6.4) and (6.5) the charged particle radiation will occur in the nonstationary medium similar to transition radiation on the boundary of two media with different dielectric permittivity. This spontaneous radiation field can be obtained from the Maxwell equations (6.2), (6.3) with the corresponding current density of a charged particle $\mathbf{J}\left(\mathbf{r},t\right)$ under the conditions (6.4) and (6.5). However, we will not describe here the spontaneous nonstationary transition radiation effect and refer the reader interested in this process to the original work presented in the bibliography of this chapter. We will consider the induced nonstationary transition process in the external EM wave field. For the latter one needs also to clear up the question of how the change of the dielectric permittivity (6.1) of the medium affects the external monochromatic wave.

If a plane monochromatic wave of frequency ω_0, wave vector $\mathbf{k}_0$, and electric field amplitude $\mathbf{E}_0$ propagates in a medium with the mentioned properties, then at $t < 0$ when $\varepsilon = \varepsilon_1$

$$\mathbf{E}\left(\mathbf{r},t\right) = \mathbf{E}_0 e^{i(\omega_0 t - \mathbf{k}_0\mathbf{r})} + \text{c.c.}; \quad t < 0 \tag{6.6}$$

and at $t > 0$ when $\varepsilon = \varepsilon_2$ there are two waves — transmitted and reflected:

$$\mathbf{E}\left(\mathbf{r},t\right) = \mathbf{E}_1 e^{i(\omega_1 t - \mathbf{k}_1\mathbf{r})} + \mathbf{E}_2 e^{i(-\omega_2 t - \mathbf{k}_2\mathbf{r})} + \text{c.c.}; \quad t > 0. \tag{6.7}$$

Here ω_1, $\mathbf{k}_1$, $\mathbf{E}_1$ and ω_2, $\mathbf{k}_2$, $\mathbf{E}_2$ are the frequencies, wave vectors, and amplitudes of the electric fields of the transmitted and reflected waves, respectively. Since the medium is assumed to be spatially homogeneous, for the wave vectors the condition takes place:

$$\mathbf{k}_0 = \mathbf{k}_1 = \mathbf{k}_2 = \text{const}, \tag{6.8}$$

and the nonstationarity of the medium leads to a change of frequency. From the condition for the wave vectors (6.8) follows the relations between the frequencies of the incident, transmitted, and reflected waves:

$$\omega_0\sqrt{\varepsilon_1} = \omega_1\sqrt{\varepsilon_2} = \omega_2\sqrt{\varepsilon_2}. \tag{6.9}$$

Let the wave propagate along the axis OX with the vector of electric field amplitude $\mathbf{E}_0$ directed along the OY axis. Then using conditions (6.4), (6.5) and Maxwell equations (6.2), (6.3) for the field (6.6), (6.7) in the case of the wave linear polarization, for the amplitudes of the electric field of the transmitted and reflected waves we obtain

$$E_1 = \frac{\sqrt{\varepsilon_1}(\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2})}{2\varepsilon_2} E_0, \tag{6.10}$$

$$E_2 = \frac{\sqrt{\varepsilon_1}(\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2})}{2\varepsilon_2} E_0. \tag{6.11}$$

Equations (6.10), (6.11) with the analogous equations for the magnetic strengths, and Eqs. (6.8), (6.9) determine the electromagnetic fields of the transmitted and reflected waves at the propagation of a plane monochromatic EM wave in a medium the dielectric permittivity of which changes abruptly at the time.

6.2 Classical Description of Induced Nonstationary Transition Process

As was mentioned above in the presence of an external EM radiation field the nonstationary transition process acquires induced character and the interaction of a charged particle with the incident plane monochromatic wave in a medium will proceed with the actual energy change and the acceleration of the particles or induced coherent radiation will take place. It is of special interest, in particular, in plasmas where for the stationary states the real energy change between a charged particle and a transversal EM wave cannot proceed because of the violation of the conservation law of energy-momentum for the absorption/emission of quanta in the field of a plane monochromatic wave by a free charged particle. Hence, we will study the classical and quantum dynamics of the induced nonstationary transition process in the external wave field on the basis of relativistic equations of motion for a charged particle.

Consider first the classical dynamics of the particle–wave interaction in a medium with the abrupt temporal change of the dielectric permittivity. Then, the initial monochromatic wave is transformed into a continuous wave spectrum (in general, finite since the change of ε actually occurs in finite time). This spectrum of frequencies (ω) depends on the time during which the electromagnetic properties of the medium are changed. If the characteristic time $\tau << 2\pi/\omega$, then the abrupt temporal change of the dielectric permittivity can be described by the stepwise function ε (6.1).

With the stepwise discontinuity of the dielectric permittivity (6.1) the initial monochromatic wave (of linear polarization) is transformed into a spectrum that can be found via Fourier transformation over t

$$E_y\left(x,t\right) = \int_{-\infty}^{\infty} E_y\left(x,\omega\right) e^{i\omega t} d\omega. \tag{6.12}$$

Then for the field (6.6), (6.7) the Fourier transform $E_y\left(x,\omega\right)$ may be presented in the form

$$E_y\left(x,\omega\right) = \frac{e^{-ik_0x}}{2\pi}\left\{E_0\int_{-\infty}^{0} e^{\epsilon t}e^{i(\omega_0-\omega)t}dt + E_1\int_{0}^{\infty} e^{-\epsilon t}e^{i(\omega_1-\omega)t}dt\right.$$

$$\left. + E_2\int_{0}^{\infty} e^{-\epsilon t}e^{-i(\omega_1+\omega)t}dt\right\} + \frac{e^{ik_0x}}{2\pi}\left\{E_0\int_{-\infty}^{0} e^{\epsilon t}e^{-i(\omega_0+\omega)t}dt\right.$$

$$\left. + E_1\int_{0}^{\infty} e^{-\epsilon t}e^{-i(\omega_1+\omega)t}dt + E_2\int_{0}^{\infty} e^{-\epsilon t}e^{i(\omega_1-\omega)t}dt\right\}, \tag{6.13}$$

where we have introduced an arbitrarily small damping factor $\epsilon \to 0$ to switch on/off adiabatically the wave at $t = \mp\infty$. After the integration in Eq. (6.13) for the Fourier transform of the field we obtain

$$E_y\left(x,\omega\right) = \frac{e^{-ik_0x}}{2\pi i}\left\{\frac{E_2}{\omega+\omega_1-i\epsilon} + \frac{E_1}{\omega-\omega_1-i\epsilon} - \frac{E_0}{\omega-\omega_0+i\epsilon}\right\}$$

$$+\frac{e^{ik_0x}}{2\pi i}\left\{\frac{E_2}{\omega-\omega_1-i\epsilon} + \frac{E_1}{\omega+\omega_1-i\epsilon} - \frac{E_0}{\omega+\omega_0+i\epsilon}\right\}. \tag{6.14}$$

The infinitesimal quantity $i\epsilon$ in the poles of Eq. (6.14) indicates the path that should be chosen at the integration over ω (at the inverse Fourier transformation as well). Taking into account Eqs. (6.9), (6.10), and (6.11) for the

$E_y\,(x,\omega)$ we will have

$$E_y\,(x,\omega) = E(\omega)e^{-ik_0x} - E(-\omega)e^{ik_0x}, \tag{6.15}$$

where

$$E(\omega) = \frac{E_0}{2\pi i}\left(\frac{\varepsilon_1}{\varepsilon_2} - 1\right)\frac{\omega^2}{(\omega - \omega_0)\left(\omega^2 - \omega_0^2\frac{\varepsilon_1}{\varepsilon_2}\right)}. \tag{6.16}$$

Here we have omitted the infinitesimal $i\epsilon$ bearing in mind the role of the poles bypass.

The analogous equations can be obtained for the magnetic field strength:

$$H_z\,(x,\omega) = H(\omega)e^{-ik_0x} - H(-\omega)e^{ik_0x}, \tag{6.17}$$

$$H(\omega) = \frac{\sqrt{\varepsilon_1}\omega_0}{\omega}E(\omega).$$

Now the problem of the particle–wave interaction in a nonstationary medium with the abrupt temporal change of the dielectric permittivity reduces to the particle interaction with the EM field possesing the spectral components (6.15), (6.17). Consequently, the relativistic classical equations of motion of the particle take the form

$$\frac{dp_x}{dt} = \frac{e}{c}\mathrm{v}_y\int_{-\infty}^{\infty}\left[H(\omega)e^{-ik_0x} - H(-\omega)e^{ik_0x}\right]e^{i\omega t}d\omega, \tag{6.18}$$

$$\frac{dp_y}{dt} = e\int_{-\infty}^{\infty}\left[E(\omega)e^{-ik_0x} - E(-\omega)e^{ik_0x}\right]e^{i\omega t}d\omega$$

$$-\frac{e}{c}\mathrm{v}_x\int_{-\infty}^{\infty}\left[H(\omega)e^{-ik_0x} - H(-\omega)e^{ik_0x}\right]e^{i\omega t}d\omega, \tag{6.19}$$

$$\frac{dp_z}{dt} = 0. \tag{6.20}$$

The energy change of the particle is given by the equation

$$\frac{d\mathcal{E}}{dt} = e\mathrm{v}_y\int_{-\infty}^{\infty}\left[E(\omega)e^{-ik_0x} - E(-\omega)e^{ik_0x}\right]e^{i\omega t}d\omega. \tag{6.21}$$

The equations of motion (6.18)–(6.20) can be presented in the form

$$\frac{dp_x}{dt} = -i\frac{e}{c}k_0 \int_{-\infty}^{\infty} \mathrm{v}_y F(\omega, x, t) d\omega, \tag{6.22}$$

$$\frac{dp_y}{dt} = i\frac{e}{c} \int_{-\infty}^{\infty} (k_0 \mathrm{v}_x - \omega) F(\omega, x, t) d\omega \ , \tag{6.23}$$

$$\frac{dp_z}{dt} = 0, \tag{6.24}$$

where the kernel in the integrals (6.22), (6.23)

$$F(\omega, x, t) = A(\omega) \exp[i(\omega t - k_0 x)] - A^*(\omega) \exp[-i(\omega t - k_0 x)],$$

and

$$A(\omega) = \frac{cE_0}{2\pi} \left(\frac{\varepsilon_1}{\varepsilon_2} - 1\right) \frac{\omega}{(\omega - \omega_0)\left(\omega^2 - \omega_0^2 \frac{\varepsilon_1}{\varepsilon_2}\right)} \tag{6.25}$$

is the spectral amplitude of the vector potential of the field (6.12).

We shall solve the set of equations (6.22)–(6.24) in the approximation of the perturbation theory by the field. The parameter of the perturbation theory is $\xi_0 = eE_0/mc\omega_0 << 1$. As long as the particle motion along the z axis remains free we can choose the initial velocity of the particle in the xy plane: $\mathbf{v}_0 = \{\mathrm{v}_0 \cos\theta, \mathrm{v}_0 \sin\theta, 0\}$. According to perturbation theory

$$\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}; \quad |\Delta\mathbf{p}| << |\mathbf{p}_0|,$$

and from the Eqs. (6.22), (6.23) in first-order approximation by ξ_0 (keeping only the uniform part of motion $x(t) = x_0 + \mathrm{v}_{0x} t$ on the right-hand side of the equations) for the changes of the particle momentum in the field $\Delta\mathbf{p}$ we will obtain the following equations:

$$\frac{d\Delta p_x}{dt} = -i\frac{e}{c}k_0 \int_{-\infty}^{\infty} \mathrm{v}_{0y} F(\omega, x_0 + \mathrm{v}_{0x} t, t) d\omega, \tag{6.26}$$

$$\frac{d\Delta p_y}{dt} = i\frac{e}{c} \int_{-\infty}^{\infty} (k_0 \mathrm{v}_{0x} - \omega) F(\omega, x_0 + \mathrm{v}_{0x} t, t) d\omega \ . \tag{6.27}$$

Integrating Eqs. (6.26) and (6.27) over t from $-\infty$ to $+\infty$ we obtain in first-order approximation by ξ_0 the following expressions for the particle momentum change after the interaction:

$$\Delta p_x = -i\frac{2\pi e k_0}{c} \mathrm{v}_{0y} \int_{-\infty}^{\infty} \left[A(\omega) e^{-ik_0 x_0}\right.$$

$$-A^{*}(\omega)\,e^{ik_0x_0}\big]\,\delta\left(\omega-k_0\mathrm{v}_{0x}\right)d\omega, \tag{6.28}$$

$$\Delta p_y=i\frac{2\pi e}{c}\int_{-\infty}^{\infty}\left(k_0\mathrm{v}_{0x}-\omega\right)\big[A(\omega)\,e^{-ik_0x_0}$$

$$-A^{*}(\omega)\,e^{ik_0x_0}\big]\,\delta\left(\omega-k_0\mathrm{v}_{0x}\right)d\omega. \tag{6.29}$$

The δ-function in these expressions defines the condition of induced radiation/absorption by a free charged particle in the field of a transversal monochromatic EM wave under the nonstationary transition process:

$$\omega-\mathbf{k}_0\mathbf{v}_0=0. \tag{6.30}$$

Integrating in the same way Eqs. (6.21) and taking into account Eq. (6.30) for the particle momentum and energy changes after the interaction we obtain the following ultimate formulas:

$$\triangle p_y=\triangle p_z=0,\qquad \triangle p_x=\frac{\triangle\mathcal{E}}{\mathrm{v}_0\cos\theta}, \tag{6.31}$$

$$\triangle\mathcal{E}=2mc^2\xi_0\frac{\mathrm{v}_0^3}{c^3}\left(\varepsilon_1-\varepsilon_2\right)\frac{\sin\theta\cos^2\theta}{\left(1-\sqrt{\varepsilon_1}\frac{\mathrm{v}_0}{c}\cos\theta\right)\left(1-\varepsilon_2\frac{\mathrm{v}_0^2}{c^2}\cos^2\theta\right)}$$

$$\times\sin\left(\omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_0\cos\theta}{c}t_0\right). \tag{6.32}$$

Here t_0 is the instant of time corresponding to the initial phase of the particle in the external EM wave. Note that Eq. (6.32) besides the induced nonstationary transition process describes generally the induced Cherenkov effect as well (see the denominator) if a medium initially (at $t<0$) was dielectriclike (in principle, it includes also the Cherenkov effect at $t>0$ if $\varepsilon_2>1$, but for actual physical cases we assume that the stepwise discontinuity of ε (6.1) may be realistic at the abrupt transformation of a dielectriclike medium into a plasma for which $\varepsilon_2<1$ and the induced Cherenkov effect is excluded).

As is seen from Eq. (6.32) depending on the initial phase

$$\Phi_0=\omega_0t_0\sqrt{\varepsilon_1}\left(\mathrm{v}_0/c\right)\cos\theta$$

the particle is either accelerated after the interaction or is decelerated radiating coherently into the wave. This real energy exchange is due to the direct and inverse induced nonstationary transition effect. In the case of a particle beam, various particles situated initially in the diverse phases Φ_0 will acquire or lose different energies in the field and the particles' free drift after the interaction will result in bunching of an initially homogeneous particle beam.

6.3 Quantum Description of Multiphoton Interaction

Consider now the quantum dynamics of the induced nonstationary transition process. Quantitative analysis of Eqs. (6.31) and (6.32) shows that the classical energy exchange of a particle with strong EM radiation in a nonstationary medium as a result of the induced nonstationary transition effect corresponds to absorption and emission of a large number of photons. On the basis of the quantum theory such multiphoton process can be described by the quasiclassical-type wave function neglecting, in fact, the quantum recoil at the absorption/emission of photons by the particle. The latter corresponds to a slowly varying wave function for which the derivatives of the second order of the particle wave function can be neglected with respect to the first order ones that have been made in the consideration of the multiphoton processes in the previous chapters. The role of the particle spin is inessential here, hence by neglecting the spin interaction the Dirac equation in quadratic form is written as the Klein–Gordon equation (3.30) for the particle in the specified EM field. Assuming the same geometry as in Section 6.1 the latter takes the form

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} = \left[-\hbar^2 c^2 \nabla^2 + 2iec\hbar \nabla_y A_y(x,t) + e^2 A_y^2(x,t) + m^2 c^4\right] \Psi, \quad (6.33)$$

where

$$A_y(x,t) = \int_{-\infty}^{\infty} \left[A(\omega) e^{-ik_0 x} + A(-\omega) e^{ik_0 x}\right] e^{i\omega t} d\omega \quad (6.34)$$

is the vector potential of the field (6.12) expressed via the spectral amplitude $A(\omega)$ (6.25).

Equation (6.33) will be solved in the mentioned approximation by the particle wave function

$$\Psi(\mathbf{r},t) = \sqrt{\frac{N_0}{2\mathcal{E}_0}} f(x,t) \exp\left[\frac{i}{\hbar}(\mathbf{p}_0 \mathbf{r} - \mathcal{E}_0 t)\right], \quad (6.35)$$

where $f(x,t)$ is a slowly varying function with respect to the free-particle wave function (see Section 3.5). Taking into account the conditions (3.92) and Eq. (6.35) from Eq. (6.33) for $f(x,t)$ we will obtain the differential equation of the first order:

$$\frac{\partial f}{\partial t} + \mathrm{v}_{0x} \frac{\partial f}{\partial x} = \frac{i}{2\hbar \mathcal{E}_0} \left[2ecp_{0y} A_y(x,t) + e^2 A^2(x,t)\right] f(x,t). \quad (6.36)$$

The conditions (3.92) correspond to a small change of the momentum and energy of the electron in the field compared with the initial values $\triangle p << p_0$ and $\triangle \mathcal{E} << \mathcal{E}_0$, that is, the approximation made in the classical consideration, where the intensity of the EM wave is restricted by the condition $\xi_0 << 1$. Then for actual values of parameters $p_{0y}/mc >> \xi_0$ and the last term $\sim A^2$ in Eq. (6.36) will be neglected.

Passing from x, t to characteristic coordinates $\tau' = t - x/\mathrm{v}_{0x}$, $\eta' = t$ and integrating Eq. (6.36) we obtain

$$f(\tau', \eta') = \exp\left\{\frac{ie\mathrm{v}_{0y}}{\hbar c}\int_{-\infty}^{\eta'} A_y(\mathrm{v}_{0x}(\eta'' - \tau'), \eta'')d\eta''\right\}. \tag{6.37}$$

Then after the interaction ($\eta' \to +\infty$) taking into account Eq. (6.34) we obtain

$$f(\tau) = \exp\left\{\frac{i4\pi e\mathrm{v}_{0y}}{\hbar c} A\left(\omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_{0x}}{c}\right)\cos\left(\omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_{0x}}{c}\tau\right)\right\}. \tag{6.38}$$

The spectral amplitude in Eq. (6.38) is determined by Eq. (6.25):

$$A\left(\omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_{0x}}{c}\right) = \frac{E_0}{2\pi\omega_0^2}\frac{\varepsilon_1 - \varepsilon_2}{\sqrt{\varepsilon_1}}\frac{\mathrm{v}_0\cos\theta}{\left(\sqrt{\varepsilon_1}\frac{\mathrm{v}_0}{c}\cos\theta - 1\right)\left(\varepsilon_2\frac{\mathrm{v}_0^2}{c^2}\cos^2\theta - 1\right)}. \tag{6.39}$$

Returning to coordinates x, t and expanding the exponential (6.38) into a series by the Bessel functions and taking into account Eq. (6.39) for the total wave function (6.35) we will have

$$\Psi(\mathbf{r}, t) = \sqrt{\frac{N_0}{2\mathcal{E}_0}}\exp\left[\frac{i}{\hbar}p_{0y}y\right]\sum_{s=-\infty}^{+\infty} i^s J_s(\alpha)$$

$$\times\exp\left\{\frac{i}{\hbar}\left[p_{0x} - s\hbar\sqrt{\varepsilon_1}\frac{\omega_0}{c}\right]x - \frac{i}{\hbar}\left[\mathcal{E}_0 - s\hbar\omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_0}{c}\cos\theta\right]t\right\}, \tag{6.40}$$

where the argument of the Bessel function is

$$\alpha = 2\xi_0\frac{m\mathrm{v}_0^2}{\hbar\omega_0}\frac{\varepsilon_1 - \varepsilon_2}{\sqrt{\varepsilon_1}}\frac{\sin\theta\cos\theta}{\left(1 - \sqrt{\varepsilon_1}\frac{\mathrm{v}_0}{c}\cos\theta\right)\left(1 - \varepsilon_2\frac{\mathrm{v}_0^2}{c^2}\cos^2\theta\right)}. \tag{6.41}$$

As is seen from Eq. (6.40), due to the induced nonstationary transition effect the particle absorbs or emits s photons, as a result of which the momentum and energy after the interaction are changed:

$$\triangle p_x = s\hbar\frac{\omega_0}{c}\sqrt{\varepsilon_1}, \quad \triangle p_y = 0, \qquad \triangle\mathcal{E} = s\hbar\omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_0}{c}\cos\theta. \tag{6.42}$$

The probability of the induced s-photon process is

$$W_s = J_s^2\left(\frac{2\xi_0 m\mathrm{v}_0^2\left(\varepsilon_1-\varepsilon_2\right)\sin\theta\cos\theta}{\hbar\omega_0\sqrt{\varepsilon_1}\left(1-\sqrt{\varepsilon_1}\frac{\mathrm{v}_0}{c}\cos\theta\right)\left(1-\varepsilon_2\frac{\mathrm{v}_0^2}{c^2}\cos^2\theta\right)}\right). \tag{6.43}$$

The comparison of the expression for α with the amplitude of the classical change of the particle momentum $(\triangle p_x)_{\max}$ (6.31) and energy $(\triangle\mathcal{E})_{\max}$ (6.32) shows that

$$\alpha = \frac{(\triangle p_x)_{\max}}{\hbar k_0}, \tag{6.44}$$

in accordance with the correspondence principle ($s \sim \alpha >> 1$).

At the small value of α or small number of photons s when the interaction has entirely quantum character it is necessary to take into account the quantum recoil as well. It is especially important in this process, because at the abrupt temporal variation of the dielectric permittivity the hard quanta in the spectrum of the initial radiation arise. We will solve for this purpose Eq. (6.33) keeping also the derivatives of the second order of the particle wave function for a single-photon absorption or emission. Correspondingly, in first-order approximation of the perturbation theory from Eq. (6.33) we have the following equation for the particle wave function at the single-photon interaction with the field (6.35) in the nonstationary transition process:

$$\frac{\partial^2\Psi_1}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\Psi_1}{\partial t^2} - \frac{1}{\hbar^2c^2}\left(m^2c^4 + c^2p_{0y}^2\right)\Psi_1$$

$$= -2\frac{ep_{0y}}{c\hbar^2}\left[A_y(t)\,e^{-ik_0x} + A_y^*(t)\,e^{ik_0x}\right]\Psi_0, \tag{6.45}$$

where

$$\Psi_0(\mathbf{r},t) = \sqrt{\frac{N_0}{2\mathcal{E}_0}}\exp\left[\frac{i}{\hbar}\left(\mathbf{p}_0\mathbf{r} - \mathcal{E}_0t\right)\right] \tag{6.46}$$

is the initial wave function of the particle (normalized on N_0 particles per unit volume). The solution of Eq. (6.45) is sought in the form

$$\Psi_1(\mathbf{r},t) = \left[\Phi_1(t)e^{-ik_0x} + \Phi_2(t)e^{ik_0x}\right]\exp\left[\frac{i}{\hbar}\left(\mathbf{p}_0\mathbf{r} - \mathcal{E}_0t\right)\right]. \tag{6.47}$$

Substituting Eq. (6.47) in Eq. (6.45) for the functions $\Phi_1(t)$ and $\Phi_2(t)$ we obtain the equations:

$$\frac{d^2\Phi_1}{dt^2} - 2i\frac{\mathcal{E}_0}{\hbar}\frac{d\Phi_1}{dt} - c^2 k_0 \left(2\frac{p_{0x}}{\hbar} - k_0\right)\Phi_1 = 2\sqrt{\frac{N_0}{\mathcal{E}_0}}\frac{ecp_{0y}}{\hbar^2}A_y(t)\,, \qquad (6.48)$$

$$\frac{d^2\Phi_2}{dt^2} - 2i\frac{\mathcal{E}_0}{\hbar}\frac{d\Phi_2}{dt} + c^2 k_0 \left(2\frac{p_{0x}}{\hbar} + k_0\right)\Phi_2 = 2\sqrt{\frac{N_0}{\mathcal{E}_0}}\frac{ecp_{0y}}{\hbar^2}A_y^*(t)\,. \qquad (6.49)$$

The solution of Eq. (6.48) is

$$\Phi_1(t) = -2i\sqrt{\frac{N_0}{\mathcal{E}_0}}\frac{ecp_{0y}}{\hbar^2(\Omega_1 - \Omega_2)}$$

$$\times\left[e^{i\Omega_1 t}\int_{-\infty}^{t} e^{-i\Omega_1 t'}A_y(t')\,dt' - e^{i\Omega_2 t}\int_{-\infty}^{t} e^{-i\Omega_2 t'}A_y(t')\,dt'\right], \qquad (6.50)$$

where the characteristic frequencies Ω_1 and Ω_2 are given by the expressions

$$\Omega_{1,2} = \frac{\mathcal{E}_0}{\hbar} \mp \left[\left(\frac{\mathcal{E}_0}{\hbar} - \omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_{0x}}{c}\right)^2 + \omega_0^2\varepsilon_1\left(1 - \frac{\mathrm{v}_{0x}^2}{c^2}\right)\right]^{1/2} \qquad (6.51)$$

with the signs "$\mp$" correspondingly.

Passing from $A_y(t)$ to the Fourier component of the field we obtain for $\Phi_1(t)$ after the interaction ($t \to +\infty$)

$$\Phi_1(t) = -4i\sqrt{\frac{N_0}{\mathcal{E}_0}}\frac{\pi ecp_{0y}}{\hbar^2(\Omega_1 - \Omega_2)}\left[A(\Omega_1)\,e^{i\Omega_1 t} - A(\Omega_2)\,e^{i\Omega_2 t}\right], \qquad (6.52)$$

where the spectral amplitudes of the wave vector potential $A(\Omega_1)$ and $A(\Omega_2)$ are determined by Eq. (6.25).

Solving Eq. (6.49) in an analogous way for the function $\Phi_2(t)$ we obtain

$$\Phi_2(t) = -4i\sqrt{\frac{N_0}{\mathcal{E}_0}}\frac{\pi ecp_{0y}}{\hbar^2(\Omega_1' - \Omega_2')}\left[A^*(-\Omega_1')\,e^{i\Omega_1' t} - A^*(-\Omega_2')\,e^{i\Omega_2' t}\right], \qquad (6.53)$$

with the characteristic frequencies

$$\Omega_{1,2}' = \frac{\mathcal{E}_0}{\hbar} \mp \left[\left(\frac{\mathcal{E}_0}{\hbar} + \omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_{0x}}{c}\right)^2 + \omega_0^2\varepsilon_1\left(1 - \frac{\mathrm{v}_{0x}^2}{c^2}\right)\right]^{1/2}. \qquad (6.54)$$

Equations (6.51) and (6.54) correspond to the energy-momentum conservation law for a particle in the induced nonstationary transition process: the particle can emit only the photons with frequencies $\Omega_{1,2}$ and absorb photons with frequencies $\Omega'_{1,2}$. As long as $\mathcal{E}_0/\hbar >> \omega_0\sqrt{\varepsilon_1}\mathrm{v}_{0x}/c$ for the frequencies of a strong coherent radiation field we expand the square roots in Eqs. (6.51), (6.54) in a series and retain only the small terms of first order. We then obtain for the radiation frequencies:

$$\Omega_1 \simeq \omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_{0x}}{c} - \varepsilon_1\frac{\hbar\omega_0^2}{2\mathcal{E}_0}\left(1-\frac{\mathrm{v}_{0x}^2}{c^2}\right),$$

$$\Omega_2 \simeq 2\frac{\mathcal{E}_0}{\hbar} - \omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_{0x}}{c} + \varepsilon_1\frac{\hbar\omega_0^2}{2\mathcal{E}_0}\left(1-\frac{\mathrm{v}_{0x}^2}{c^2}\right) \tag{6.55}$$

and for the absorption frequencies:

$$\Omega'_1 \simeq -\omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_{0x}}{c} - \varepsilon_1\frac{\hbar\omega_0^2}{2\mathcal{E}_0}\left(1-\frac{\mathrm{v}_{0x}^2}{c^2}\right),$$

$$\Omega'_2 \simeq 2\frac{\mathcal{E}_0}{\hbar} + \omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_{0x}}{c} + \varepsilon_1\frac{\hbar\omega_0^2}{2\mathcal{E}_0}\left(1-\frac{\mathrm{v}_{0x}^2}{c^2}\right). \tag{6.56}$$

These expressions show that the emission of a photon with frequency Ω_2 and absorption with frequency Ω'_2 has a clearly quantum character, and its probability, as is seen from Eq. (6.25), depends on the change of the dielectric permittivity of the medium $\varepsilon_1-\varepsilon_2$. We therefore consider two cases: $\varepsilon_1/\varepsilon_2 \lesssim 1$ and $\varepsilon_1/\varepsilon_2 >> 1$.

If $\varepsilon_1/\varepsilon_2 \lesssim 1$ we get from Eq. (6.25)

$$A(\Omega_2) \simeq A\left(2\frac{\mathcal{E}_0}{\hbar}\right) << A(\Omega_1) \simeq A\left(\omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_{0x}}{c}\right), \tag{6.57}$$

so that in this case we can neglect in Eqs. (6.52) and (6.53) the pure quantum process of emission and absorption of hard quanta $\Omega_2 \simeq 2\mathcal{E}_0/\hbar$. Then for the amplitudes of the particle wave function $\Phi_1(t)$ and $\Phi_2(t)$ we will have correspondingly

$$\Phi_{1,2}(t) = i\sqrt{\frac{N_0}{\mathcal{E}_0}}\frac{e\mathrm{v}_0^2E_0}{\hbar\omega_0^2c}\frac{\varepsilon_1-\varepsilon_2}{\sqrt{\varepsilon_1}}\frac{\sin\theta\cos\theta}{\left(1-\sqrt{\varepsilon_1}\frac{\mathrm{v}_0}{c}\cos\theta\right)\left(1-\varepsilon_2\frac{\mathrm{v}_0^2}{c^2}\cos^2\theta\right)}$$

$$\times\exp\left\{i\omega_0\left[\pm\sqrt{\varepsilon_1}\frac{\mathrm{v}_0}{c}\cos\theta - \frac{\varepsilon_1\hbar\omega_0}{2\mathcal{E}_0}\left(1-\frac{\mathrm{v}_0^2}{c^2}\cos^2\theta\right)\right]t\right\} \tag{6.58}$$

with the signs "±" correspondingly. Equation (6.58) with Eq. (6.47) determines the particle's wave function after the single-photon interaction with the field (6.35) in the nonstationary transition process. In this case ($\varepsilon_1/\varepsilon_2 \lesssim 1$) we obtain for the current density ($\sim |\Psi_0 + \Psi_1|^2$) of the particles after the interaction

$$\mathbf{j}(x,t) = \mathbf{j}_0 \left\{ 1 + 2\alpha \sin \left[\varepsilon_1 \frac{\hbar\omega_0^2}{2\mathcal{E}_0} \left(1 - \frac{\mathrm{v}_0^2}{c^2} \cos^2\theta \right) t \right] \right.$$

$$\left. \times \cos \left[\omega_0 \sqrt{\varepsilon_1} \frac{\mathrm{v}_0 \cos\theta}{c} \left(t - \frac{x}{\mathrm{v}_0 \cos\theta} \right) \right] \right\}, \tag{6.59}$$

where $\mathbf{j}_0 = \text{const}$ is the particle's initial current density and α is defined by Eq. (6.41) or (6.44). As is seen from Eq. (6.59) as a result of the stimulated absorption and emission of the photons of frequency

$$\Omega_1 = \omega_0 \sqrt{\varepsilon_1} \frac{\mathrm{v}_0}{c} \cos\theta$$

the quantum modulation of the particle's probability density and, consequently, current density at this frequency occurs with a depth $\Gamma_1 = 2\alpha$. Also, in contrast to the effect of quantum modulation in coherent processes considered in previous chapters, the pure temporal modulation here takes place as well that is caused by the nonstationarity of the medium. The period of this temporal modulation is

$$T_1 = \frac{4\pi\mathcal{E}_0}{\hbar\omega_0^2 \varepsilon_1 \left(1 - \frac{\mathrm{v}_0^2}{c^2} \cos^2\theta \right)}.$$

If we derive the particle's wave function in the next orders of perturbation theory, then we obtain the modulation at higher harmonics of the wave frequency. The modulation depth at the s-th harmonic will be $\Gamma_s \sim \Gamma_1^s$.

For $\varepsilon_1/\varepsilon_2 >> 1$, it is necessary to also take into account in Eqs. (6.52), (6.53) the pure quantum process of emission and absorption of hard quanta $\Omega_2 \simeq 2\mathcal{E}_0/\hbar$. The spectral amplitude of the wave vector potential $A(\Omega_2)$ at such frequencies is

$$A(\Omega_2) \simeq \frac{cE_0}{8\pi} \frac{\varepsilon_1}{\varepsilon_2} \left(\frac{\mathcal{E}_0^2}{\hbar^2} - \frac{\varepsilon_1}{\varepsilon_2} \frac{\omega_0^2}{4} \right)^{-1}. \tag{6.60}$$

In an analogous way for the particles current density after the interaction we will have

$$\mathbf{j}(x,t) = \mathbf{j}_0 \left\{ 1 + \Gamma_1 \sin \left[\varepsilon_1 \frac{\hbar\omega_0^2}{2\mathcal{E}_0} \left(1 - \frac{\mathrm{v}_0^2}{c^2} \cos^2\theta \right) t \right] \right.$$

$$\times \cos\left[\omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_0\cos\theta}{c}\left(t - \frac{x}{\mathrm{v}_0\cos\theta}\right)\right]$$

$$+\mathit{\Gamma}_2 \sin\left(2\frac{\mathcal{E}_0}{\hbar}t\right)\cos\left[\omega_0\sqrt{\varepsilon_1}\frac{\mathrm{v}_0\cos\theta}{c}\left(t + \frac{x}{\mathrm{v}_0\cos\theta}\right)\right]\Bigg\}, \qquad (6.61)$$

where $\mathit{\Gamma}_1 = 2\alpha$, and the modulation depth $\mathit{\Gamma}_2$ due to the absorption-emission of hard quanta $\mathit{\Omega}_2$ is

$$\mathit{\Gamma}_2 = \xi\frac{m\mathrm{v}_0 c\hbar\omega_0}{\mathcal{E}_0^2}\frac{\varepsilon_1}{\varepsilon_2}\frac{\sin\theta}{1 - \frac{\varepsilon_1}{\varepsilon_2}\left(\frac{\hbar\omega}{2\mathcal{E}_0}\right)^2}. \qquad (6.62)$$

The period of temporal modulation in this case is $T_2 = \pi\hbar/\mathcal{E}_0$.

As the modulated particle beam radiates coherently this mechanism can be of interest in astrophysics where the radiating matter may be in a strongly nonstationary state.

6.4 Electron–Positron Pair Production by a γ-Quantum in a Medium

The formation of hard γ-quanta of frequencies $\sim \mathcal{E}_0/\hbar$ in the spectrum of a strong monochromatic EM wave propagating in a nonstationary medium, the dielectric permittivity of which abruptly changes in time, makes available the single-photon production of electron–positron (e^-, e^+) pairs from the intense light fields in a nonstationary medium.

In general, the single-photon reaction $\gamma \to e^- + e^+$ as well as the inverse reaction of the electron–positron annihilation ($e^- + e^+ \to \gamma$) can proceed in a medium that must be plasmalike (for the satisfaction of conservation laws for these reactions one needs $n(\omega) < 1$). However, as will be shown below, excessively large densities of the plasma in this case are required. Meanwhile, the single-photon production of e^-, e^+ pairs in a nonstationary plasma is possible at ordinary densities. Moreover, this process can proceed in the strong light fields in an arbitrary medium turning abruptly into a plasma (with the temporal variation law of ε (6.1)). Hence, we will consider both single-photon reactions $\gamma \rightleftarrows e^- + e^+$ in a stationary plasma and the production of e^-, e^+ pairs from the intense light beam in a nonstationary medium.

Consider first the production of electron–positron pairs by a γ-quantum and its annihilation in a stationary medium. It is easy to see from the conservation laws of the energy and momentum for the single-photon reactions $\gamma \rightleftarrows e^- + e^+$

$$\hbar\mathbf{k} = \mathbf{p}_1 + \mathbf{p}_2; \qquad \hbar\omega = \mathcal{E}_1 + \mathcal{E}_2 \qquad (6.63)$$

(ω, $\mathbf{k}$ are the γ-quantum frequency and wave vector, $|\mathbf{k}| = n(\omega)\omega/c$, $\mathbf{p}_{1,2}$ and $\mathcal{E}_{1,2}$ are the momenta and energies of the electron and positron, respectively) that the phase velocity of a γ-quantum $\mathrm{v}_{ph} = c/n(\omega)$ must be larger than c, i.e., a medium for these processes must be plasmalike: $n(\omega) < 1$. The latter restricts the energy of a γ-quantum because of the dispersive properties of a medium. Indeed, for the macroscopic meaning of the refractive index of a medium for a γ-quantum at least one particle within a distance of the order of $\lambda/2$ is required (λ is the wavelength of the γ-quantum), that is, the condition $\lambda/2 \gtrsim l$ must be satisfied, where l is the distance between the electrons in a plasma. Therefore, besides the threshold condition that follows from the conservation laws (6.63):

$$\hbar\omega > \frac{2mc^2}{\sqrt{1 - n^2(\omega)}}, \tag{6.64}$$

for the reactions $\gamma \rightleftarrows e^- + e^+$ in a medium the following requirement on the plasma density N/V for a specified frequency ω of a γ-quantum arises:

$$\omega \lesssim \pi \left(\frac{N}{V}\right)^{1/3} \equiv \omega_{\text{lim}}. \tag{6.65}$$

Hence, condition (6.65) determines the lower bound for the density of the medium or the upper bound for the energy of the γ-quantum, while threshold condition (6.64) determines the lower bound for the energy of the γ-quantum to cause the reactions $\gamma \rightleftarrows e^- + e^+$ to proceed in a medium.

From the standpoint of single-photon pair creation and annihilation in plasma, the latter must compensate the longitudinal momentum $\triangle p = [1 - n(\omega)]\hbar\omega/c$ transferred in these processes. Consequently, the characteristic length in the macroscopic description of the dispersion of the medium is the wavelength $\hbar/\triangle p$, which corresponds to the transferred momentum, and the condition necessary for this is $\hbar/\triangle p > (V/N)^{1/3}$. Since $n(\omega) < 1$, this condition is satisfied automatically when condition (6.65) is satisfied.

The plasma densities satisfying conditions (6.64) and (6.65) are at least: $N/V > 10^{33}\text{cm}^{-3}$. Such superdense matter exists only in astrophysical objects, particularly in the core of the neutron stars (pulsars). At these densities the electron component of the superdense plasma is highly degenerate (the dispersion of the transverse electromagnetic waves is determined by electrons). Actually, the degeneracy temperature of the electron component of such plasma is $T_F > 10^{10}$ K. On the other hand, because of neutrino energy losses, the physically attainable temperatures in an equilibrium system are much lower than this: $T << T_F$ and the superdense plasma is fully degenerate.

Since the Fermi energy at the densities $N/V > 10^{33}\text{cm}^{-3}$ is $\mathcal{E}_F > mc^2$ we need the dispersion law of the fully degenerate relativistic plasma. To determine the dispersion relation $n = n(\omega)$ of the latter we shall solve the self-consistent set of Maxwell–Vlasov equations for the transverse monochromatic EM wave in the relativistic collisionless plasma with the distribution function $f(\mathbf{p}, \mathbf{r}, t)$ (we will not consider the ions' motion).

The characteristic equations of $f(\mathbf{p}, \mathbf{r}, t)$ coincide with the single particle equation of motion. The latter has been solved for an arbitrary medium in Section 2.1 and in the case of plasma we have the following solutions in the wave field with the vector potential $\mathbf{A} = \{0, A_0 \cos(\omega t - n(\omega)\omega x/c), 0\}$:

$$p_x = p_{0x} - \frac{n(\omega)}{c(1 - n^2(\omega))}\Big\{\mathcal{E}_0 - n(\omega) c p_{0x}$$

$$-\sqrt{(\mathcal{E}_0 - n(\omega) c p_{0x})^2 + (1 - n^2(\omega))\left[e^2 A_y^2 - 2ecp_{0y}A_y\right]}\Big\}, \qquad (6.66)$$

$$p_y = p_{0y} - \frac{e}{c} A_y; \qquad p_z = p_{0z}, \qquad (6.67)$$

and for the energy of the particle in the field:

$$\mathcal{E} = \mathcal{E}_0 - \frac{1}{1 - n^2(\omega)}\Big\{\mathcal{E}_0 - n(\omega) c p_{0x}$$

$$-\sqrt{(\mathcal{E}_0 - n(\omega) c p_{0x})^2 + (1 - n^2(\omega))\left[e^2 A_y^2 - 2ecp_{0y}A_y\right]}\Big\}. \qquad (6.68)$$

The density of the electric current induced in the plasma can be defined by the equation

$$\mathbf{j}(\mathbf{r}, t) = e \int \mathbf{v} f(\mathbf{p}, \mathbf{r}, t)\, d\mathbf{p}, \qquad (6.69)$$

where $\mathbf{v} = c^2 \mathbf{p}/\mathcal{E}$ is the velocity of the electrons with the distribution function in the field $f(\mathbf{p}, \mathbf{r}, t)$. According to the Liouville theorem for the collisionless plasma we have

$$f(\mathbf{p}, \mathbf{r}, t) = f_0(\mathbf{p}_0, \mathbf{r}_0, t_0) = f_0(p_0), \qquad (6.70)$$

since the electrons before the interaction were distributed stationary, uniformly and isotropic.

Defining from Eqs. (6.66)–(6.68) the velocity of the electrons as a function of the $\mathbf{p}_0$, $\mathbf{r}$, and t and then passing from the integration over $\mathbf{p}$ to integration over $\mathbf{p}_0$ (taking into account Eq. (6.70)), Eq. (6.69) may be presented in the form

$$\mathbf{j}(\mathbf{r},t) = ec^2 \int \frac{\mathbf{p}(\mathbf{p}_0,\mathbf{r},t)}{\mathcal{E}(\mathbf{p}_0,\mathbf{r},t)} f_0(p_0)\, J(\mathbf{p}_0,\mathbf{r},t) d\mathbf{p}_0, \tag{6.71}$$

where

$$J(\mathbf{p}_0,\mathbf{r},t) = \frac{\partial(p_x,p_y,p_z)}{\partial(p_{0x},p_{0y},p_{0z})}$$

is the Jacobian of transformation. From Eqs. (6.66), (6.67) for the latter we have

$$J(\mathbf{p}_0,\mathbf{r},t) = 1 - \frac{n(\omega)}{1-n^2(\omega)}\left(\frac{cp_{0x}}{\mathcal{E}_0} - n(\omega)\right)$$

$$\times\left[1 - \frac{\mathcal{E}_0 - n(\omega)\, cp_{0x}}{\sqrt{\left(\mathcal{E}_0 - n(\omega)\, cp_{0x}\right)^2 + \left(1-n^2(\omega)\right)\left[e^2A_y^2 - 2ecp_{0y}A_y\right]}}\right]. \tag{6.72}$$

In the linear approximation by a weak wave field (since it will be applied for a γ-quantum) Eq. (6.72) can be written as follows:

$$J(\mathbf{p}_0,\mathbf{r},t) = 1 + \frac{n(\omega)}{\left(\mathcal{E}_0 - n(\omega)\, cp_{0x}\right)^2}\left(\frac{cp_{0x}}{\mathcal{E}_0} - n(\omega)\right) ecp_{0y}A_y. \tag{6.73}$$

The components of the electric current density (6.71) in this linear regime of interaction can be expressed in the form

$$j_y(\mathbf{r},t) = ec^2 \int \left\{\frac{p_{0y}}{\mathcal{E}_0}\left(1 + \frac{\left(1-n^2(\omega)\right) cp_{0y}eA_y}{\left(\mathcal{E}_0 - n(\omega)\, cp_{0x}\right)^2}\right) - \frac{eA_y}{\mathcal{E}_0}\right\}$$

$$\times f_0(p_0)\, d\mathbf{p}_0, \tag{6.74}$$

$$j_x = j_z = 0. \tag{6.75}$$

Then turning to spherical coordinates in Eq. (6.71)

$$p_{0x} = p_0\cos\theta;\ p_{0y} = p_0\sin\theta\cos\varphi;\ p_{0z} = p_0\sin\theta\sin\varphi,$$

and taking into account that the initial distribution of the electrons in a plasma is isotropic, after the integration in the equation

$$j_y(\mathbf{r},t) = -e^2cA_y \int \left\{1 - \frac{\left(1-n^2(\omega)\right) c^2p_{0y}^2}{\left(\mathcal{E}_0 - n(\omega)\, cp_{0x}\right)^2}\right\}$$

$$\times \frac{f_0(p_0)\, p_0^2}{\mathcal{E}_0} \sin\theta d\theta d\varphi dp_0 \tag{6.76}$$

by the angles, for the electric current density induced by a wave field in the plasma we will have

$$j_y(\mathbf{r},t) = -\frac{4\pi e^2 c A_y}{n^2(\omega)} \int \frac{f(p_0)p_0^2}{\mathcal{E}_0}$$

$$\times \left\{1 - \frac{\mathcal{E}_0\left(1-n^2(\omega)\right)}{2n(\omega)\, cp_0} \ln\left\{\frac{\mathcal{E}_0 + n(\omega)\, cp_0}{\mathcal{E}_0 - n(\omega)\, cp_0}\right\}\right\} dp_0. \tag{6.77}$$

The Maxwell equation for the vector potential

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] A_y(\mathbf{r},t) = -\frac{4\pi}{c} j_y(\mathbf{r},t) \tag{6.78}$$

with the current density (6.77) gives the following equation for the refractive index of a relativistic plasma:

$$n^2(\omega) = 1 - \frac{16\pi^2 e^2 c^2}{n^2(\omega)\,\omega^2} \int \frac{f(p_0)p_0^2}{\mathcal{E}_0}$$

$$\times \left\{1 - \frac{\mathcal{E}_0\left(1-n^2(\omega)\right)}{2n(\omega)\, cp_0} \ln\left\{\frac{\mathcal{E}_0 + n(\omega)\, cp_0}{\mathcal{E}_0 - n(\omega)\, cp_0}\right\}\right\} dp_0. \tag{6.79}$$

Equation (6.79) describes in general the dispersion law of a relativistic plasma for an arbitrary electron distribution function. In principle, it is also valid for a nondegenerate (relativistic and Maxwellian) electron plasma if an equilibrium distribution with temperature $T \gtrsim T_F$ can be realized in nature.

Now consider the production of electron–positron pairs by a γ-quantum in a stationary medium (homogeneous and isotropic) with a refractive index $n(\omega) < 1$ (6.79). As this process is a QED effect of the first order, then using the general rules for constructing the matrix element of a single-vertex $\gamma \to e^- + e^+$ diagram in a dispersive medium the probability amplitude will be written in the form

$$S_{if} = -e\sqrt{\frac{1}{2\omega a_\omega n^2(\omega)}} \int \overline{\psi}_1 \widehat{\epsilon}^{(\lambda)} e^{ikx} \psi_2 d^4x. \tag{6.80}$$

Here

$$a_\omega = 1 + \frac{\omega}{n(\omega)} \frac{dn(\omega)}{d\omega},$$

$k^i(\omega, \mathbf{k})$ is the 4-dimensional wave vector of the photon, quantization volume $V = 1$, $\epsilon^{(\lambda)}$ is the four-dimensional polarization vector of the photon ($\widehat{\epsilon}^{(\lambda)} = \epsilon_\mu^{(\lambda)} \gamma^\mu$), and

$$\psi_1 = u_1(\mathbf{p}_1) e^{i(\mathbf{p}_1\mathbf{r} - \mathcal{E}_1 t)}; \quad \psi_2 = u_2(-\mathbf{p}_2) e^{-i(\mathbf{p}_2\mathbf{r} - \mathcal{E}_2 t)} \tag{6.81}$$

are the free electron and positron wave functions. Here the units $\hbar = c = 1$ are used.

Performing integration in Eq. (6.80) with the wave functions (6.81) by the standard method for the differential probability of the $\gamma \to e^- + e^+$ process per unit time and unit space volume (in the momentum volumes $d\mathbf{p}_1/(2\pi)^3$ of the electrons and $d\mathbf{p}_2/(2\pi)^3$ of the positrons, respectively) we will have

$$dW = \frac{e^2}{8\pi^2 \omega a_\omega n^2(\omega)} \left| \overline{u}_1(\mathbf{p}_1) \widehat{\epsilon}^{(\lambda)} u_2(-\mathbf{p}_2) \right|^2 \delta(\omega - \mathcal{E}_1 - \mathcal{E}_2)$$

$$\times \delta(\mathbf{k} - \mathbf{p}_1 - \mathbf{p}_2) \, d\mathbf{p}_1 d\mathbf{p}_2. \tag{6.82}$$

We will assume that the γ-quantum is nonpolarized and perform averaging by the polarization states of the γ-quantum and summation over the electron and positron spin projections. Then the probability of the e^-, e^+ pair production per unit time is given by the expression

$$W = \frac{e^2}{8\pi^2 a_\omega \omega n^2(\omega)} \int \frac{\mathcal{E}_1 \mathcal{E}_2 + m^2 - p_1 p_2 \cos\vartheta_1 \cos\vartheta_2}{\mathcal{E}_1 \mathcal{E}_2} \delta(\omega - \mathcal{E}_1 - \mathcal{E}_2)$$

$$\times \delta(\mathbf{k} - \mathbf{p}_1 - \mathbf{p}_2) \, d\mathbf{p}_1 d\mathbf{p}_2, \tag{6.83}$$

where $\vartheta_{1,2}$ is the angle between the vectors $\mathbf{k}$ and $\mathbf{p}_{1,2}$, respectively.

Integrating Eq. (6.83) over the positron momentum $\mathbf{p}_2$ we obtain the following expression for the pair production probability:

$$W = \frac{e^2}{8\pi^2 a_\omega \omega n^2(\omega)} \int \left(1 + \frac{m^2 + p_1 \cos\vartheta_1 (p_1 \cos\vartheta_1 - k)}{\mathcal{E}_1 \sqrt{\mathcal{E}_1^2 + k^2 + k p_1 \cos\vartheta_1}} \right)$$

$$\times \delta\left(\omega - \mathcal{E}_1 - \sqrt{\mathcal{E}_1^2 + k^2 + k p_1 \cos\vartheta_1} \right) d\mathbf{p}_1. \tag{6.84}$$

For the integration over the electron momentum $\mathbf{p}_1$ note that because of azimuthal symmetry

$$d\mathbf{p}_1 = 2\pi p_1 \mathcal{E}_1 d\mathcal{E}_1 \sin\vartheta_1 d\vartheta_1$$

and the integration over ϑ_1 reduces formally to the following replacement in Eq. (6.84):

$$\delta\left(\omega - \mathcal{E}_1 - \sqrt{\mathcal{E}_1^2 + k^2 + kp_1\cos\vartheta_1}\right)\sin\vartheta_1 d\vartheta_1$$

$$\to \frac{\omega - \mathcal{E}_1}{kp_1}\left[H\left(\mathcal{E}_1 - \mathcal{E}_{\min}(\omega)\right) - H\left(\mathcal{E}_1 - \mathcal{E}_{\max}(\omega)\right)\right],$$

where $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

After the integration over ϑ_1, Eq. (6.84) becomes

$$W = \frac{e^2}{4\pi a_\omega \omega^2 n^5(\omega)} \int_{\mathcal{E}_{\min}(\omega)}^{\mathcal{E}_{\max}(\omega)} \left[\left(1 - n^2(\omega)\right)\left(\mathcal{E}_1^2 - \omega\mathcal{E}_1\right) + n^2(\omega) m^2 \right.$$
$$\left. + \frac{1 - n^4(\omega)}{4}\omega^2\right] d\mathcal{E}_1. \qquad (6.85)$$

The limits of integration over $\mathcal{E}_1 \in [\mathcal{E}_{\min}, \mathcal{E}_{\max}]$ in Eq. (6.85)

$$\mathcal{E}_{\min,\max}(\omega) = \frac{\omega}{2} \mp \frac{n(\omega)}{2}\left[\omega^2 - \frac{4m^2}{1 - n^2(\omega)}\right]^{1/2} \qquad (6.86)$$

are determined by the conservation laws for the $\gamma \rightleftarrows e^- + e^+$ processes in a medium (6.63) with the threshold value (6.64). Taking into account Eq. (6.86) after the integration over the electron energy in Eq. (6.85) we obtain the total probability for the single-photon e^-, e^+ pair production in a plasma:

$$W = \frac{e^2 m^2}{6\pi\omega^2 a_\omega n^2(\omega)}\left[\omega^2 - \frac{4m^2}{1 - n^2(\omega)}\right]^{1/2}$$
$$\times \left\{\frac{1}{2}\left(\frac{\omega}{m}\right)^2\left[1 - n^2(\omega)\right] + 1\right\}. \qquad (6.87)$$

Equation (6.86) with the dispersion law (6.79) of a relativistic plasma for an arbitrary electron distribution function determine the probability of the electron–positron pair production by a γ-quantum. As the electron component the of superdense plasma required for this process is fully degenerate the Pauli principle must also be taken into account that imposes an additional restriction on the $\gamma \to e^- + e^+$ reaction. The general picture of this process taking into account the conditions (6.64), (6.65) and the Pauli principle will be analyzed together with the electron–positron annihilation process in the next section.

6.5 Annihilation of Electron–Positron Pairs in a Medium

Now we will consider the inverse process of a single-photon annihilation of an electron–positron pair in a stationary plasma. This process is also a QED effect of the first order and the matrix element of a single-vertex $e^- + e^+ \to \gamma$ diagram is the complex conjugate to the $\gamma \to e^- + e$ diagram matrix element:

$$S'_{if} = -e\sqrt{\frac{1}{2\omega a_\omega n^2(\omega)}} \int \overline{\psi}_2 \widehat{\epsilon}^{(\lambda)} e^{-ikx} \psi_1 d^4x. \tag{6.88}$$

The differential probability of the annihilation process per unit time and unit space volume, summed by the polarization states of the created γ -quantum in the momentum volume $d\mathbf{k}/(2\pi)^3$, is given by the expression

$$dW_\gamma = \frac{\pi e^2}{2\omega a_\omega n^2(\omega)} \frac{\mathcal{E}_1\mathcal{E}_2 + m^2 - p_1 p_2 \cos\vartheta_1 \cos\vartheta_2}{\mathcal{E}_1\mathcal{E}_2}$$

$$\times \delta(\omega - \mathcal{E}_1 - \mathcal{E}_2)\, \delta(\mathbf{k} - \mathbf{p}_1 - \mathbf{p}_2)\, d\mathbf{k}. \tag{6.89}$$

Equation (6.89) determines the annihilation probability for a single e^-, e^+ pair in plasma. To obtain the total probability of annihilation of an initial positron with the plasma electrons one must define the probability of annihilation of a positron of specified energy $\mathcal{E}_2$ with the electrons of the medium in the momentum range $\mathbf{p}_1, \mathbf{p}_1 + d\mathbf{p}_1$:

$$W_\gamma = \frac{\pi e^2}{2\omega a_\omega n^2(\omega)} \int f(p_1) \frac{\mathcal{E}_1\mathcal{E}_2 + m^2 - p_1 p_2 \cos\vartheta_1 \cos\vartheta_2}{\mathcal{E}_1\mathcal{E}_2}$$

$$\times \delta(\omega - \mathcal{E}_1 - \mathcal{E}_2)\, \delta(\mathbf{k} - \mathbf{p}_1 - \mathbf{p}_2)\, d\mathbf{k} d\mathbf{p}_1, \tag{6.90}$$

where $f(p_1)$ is the distribution function of the plasma electrons. We first integrate over $\mathbf{k}$ in Eq. (6.90) and then over $\mathbf{p}_1$ taking into account that

$d\mathbf{p}_1 = 2\pi p_1 \mathcal{E}_1 d\mathcal{E}_1 \sin\vartheta d\vartheta$, where ϑ is the angle between the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$. The integration over ϑ reduces formally to the following replacement in Eq. (6.90):

$$\delta\left(\omega - \mathcal{E}_1 - \mathcal{E}_2\right)\sin\vartheta d\vartheta$$

$$\to \frac{\omega a_\omega n^2(\omega)}{p_1 p_2}\left[H\left(\mathcal{E}_1 - \mathcal{E}_{\min}(\omega)\right) - H\left(\mathcal{E}_1 - \mathcal{E}_{\max}(\omega)\right)\right],$$

where the quantities $\mathcal{E}_{\min(\max)}(\omega)$ are given by Eq. (6.86) and ω must be replaced by $\mathcal{E}_1 + \mathcal{E}_2$ according to conservation law (6.63). Then for the probability of annihilation of a positron (with an energy $\mathcal{E}_2$) with the electrons of the medium we will have

$$W_\gamma = \frac{\pi e^2}{p_2 \mathcal{E}_2}\int f\left(p_1\right)\left\{m^2 + \left(\mathcal{E}_1 + \mathcal{E}_2\right)^2 \frac{1 - n^4(\omega)}{4n^2(\omega)} - \frac{1 - n^2(\omega)}{n^2(\omega)}\mathcal{E}_1\mathcal{E}_2\right\}$$

$$\times\left[H\left(\mathcal{E}_1 - \mathcal{E}_{\min}(\omega)\right) - H\left(\mathcal{E}_1 - \mathcal{E}_{\max}(\omega)\right)\right] d\mathcal{E}_1. \tag{6.91}$$

In contrast to the pair-production process (its probability can be obtained without resorting to the explicit form of $n(\omega)$), here we must have the explicit form of the function $n = n(\omega)$ in order to be able to integrate over the electron energy $\mathcal{E}_1$ (ω is now a function of $\mathcal{E}_1$, since $\omega = \mathcal{E}_1 + \mathcal{E}_2$).

As the considered processes $\gamma \rightleftarrows e^- + e^+$ are possible in the superdense plasma where the electrons are fully degenerate, then the dispersion law of such relativistic plasma can be obtained substituting the Fermi distribution function for a fully degenerate electron gas

$$f(p_1) = \begin{cases} \frac{1}{4\pi^3}, & p_1 \le p_F \\ 0, & p_1 > p_F \end{cases} \tag{6.92}$$

in Eq. (6.79), describing in general the dispersion law of a relativistic plasma for an arbitrary distribution function of electrons $f(p_0)$. Here p_F is the boundary Fermi momentum:

$$p_F = \left(3\pi^2 \rho_e\right)^{1/3}, \tag{6.93}$$

and ρ_e is the electron density of a degenerate Fermi gas.

Integrating in Eq. (6.79) with the distribution function (6.92) over the electron momenta we obtain the following dispersion law of a relativistic degenerate plasma:

$$n^2\left(\omega\right) = 1 - \frac{2e^2}{n^2\left(\omega\right)\pi\omega^2}$$

$$\times \left\{ p_F \mathcal{E}_F - \frac{\mathcal{E}_F^2 - n^2(\omega) p_F^2}{2n(\omega)} \ln \left\{ \frac{\mathcal{E}_F + n(\omega) p_F}{\mathcal{E}_F - n(\omega) p_F} \right\} \right\}, \tag{6.94}$$

where $\mathcal{E}_F$ is the relativistic Fermi energy corresponding to boundary momentum (6.93). Inserting the dimensionless parameter

$$\beta = \frac{n(\omega) p_F}{\mathcal{E}_F}$$

Eq. (6.94) can be written in the form

$$n^2(\omega) = 1 - \frac{2e^2 p_F \mathcal{E}_F}{n^2(\omega) \pi \omega^2} \left\{ 1 - \frac{1-\beta^2}{2\beta} \ln \left\{ \frac{1+\beta}{1-\beta} \right\} \right\}, \tag{6.95}$$

or in the form more convenient for further investigation

$$n^2(\omega) = 1 - \frac{2e^2 p_F^3}{\omega^2 \pi \mathcal{E}_F} \phi(\beta), \tag{6.96}$$

where the function $\phi(\beta)$ is

$$\phi(\beta) = \frac{1}{\beta^2} \left\{ 1 - \frac{1-\beta^2}{2\beta} \ln \frac{1+\beta}{1-\beta} \right\}. \tag{6.97}$$

By analogy with the usual determination of a plasma frequency, from the equation $n(\omega_p) = 0$ we obtain the plasma frequency for a relativistic degenerate one

$$\omega_p = \sqrt{\frac{4e^2 p_F^3}{3\pi \mathcal{E}_F}}. \tag{6.98}$$

The frequency range corresponding to transverse waves that can propagate in a superdense relativistic degenerate plasma — $\omega_p \leq \omega < \infty$ — can then be obtained by varying the refractive index in the range $0 \leq n < 1$. Therefore, we present the dispersion relation (6.96) in the inverted form $\omega = \omega(n)$:

$$\omega^2 = \frac{2e^2}{\pi} \frac{p_F^3}{\mathcal{E}_F} \frac{1}{1-n^2} \phi(\beta). \tag{6.99}$$

The parameter β in Eq. (6.99) then varies in the range $0 \leqslant \beta < p_F/\mathcal{E}_F$. The analysis of the function $\phi(\beta)$, which can be expressed in the form

$$\phi(\beta) = 2 \sum_{s=1}^{\infty} \frac{\beta^{2s-2}}{4s^2 - 1},$$

shows that throughout the physically admissible range $0 \le \beta < 1$ (for superdense ultrarelativistic plasma $p_F/\mathcal{E}_F \sim 1$) the function $\phi(\beta)$ varies monotonically between the values 2/3 and 1.

The problem now reduces to the determination of the range of variation of the energies of electrons that actually participate in the annihilation process taking account of conditions (6.64), (6.65) and $\mathcal{E}_1 \le \mathcal{E}_F$ for the annihilation process. The situation may be clarified by defining this region graphically. Figure 6.1 shows the $\mathcal{E}_{\min(\max)}(\omega)$ curves and the lines corresponding to frequencies $\omega = \omega_{\lim} = (\pi/3)^{1/3} p_F$ (see Eq. (6.65)) and $\omega = \omega_{\max} = \mathcal{E}_F + \mathcal{E}_2$. The energies of the particles and γ-quantum can vary within the region $ABCA$, and the limits of integration with respect to the electron energy $\mathcal{E}_{1\min}$ and $\mathcal{E}_{1\max}$ are determined by the points at which the $\mathcal{E}_1 = \omega - \mathcal{E}_2$ line cuts the boundaries of this region.

Evaluating the integral in Eq. (6.91) with the dispersion law (6.99) we obtain a bulky expression for the total probability of the annihilation process. However, for the admissible values of $n(\omega)$ and electron density ρ_e with a great accuracy for the function $\phi(\beta)$ we have: $\phi(np_F/\mathcal{E}_F) \approx 2/3$ and the ultimate expression for the probability of the $e^- + e^+ \to \gamma$ process is rather simplified.

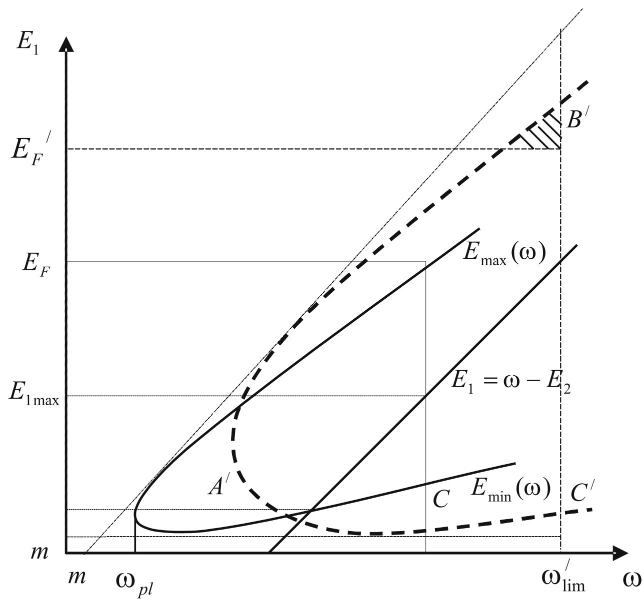

Fig. 6.1. Curves of $\mathcal{E}_{\min}(\omega)$, $\mathcal{E}_{\max}(\omega)$ and the lines corresponding to frequencies $\omega = \omega_{\lim} = (\pi/3)^{1/3} p_F$ and $\omega = \omega_{\max} = \mathcal{E}_F + \mathcal{E}_2$. The energies of the particles and γ-quantum can vary within the region $ABCA$, and the limits of integration with respect to the electron energy $\mathcal{E}_{1\min}$ and $\mathcal{E}_{1\max}$ are determined by the points at which the $\mathcal{E}_1 = \omega - \mathcal{E}_2$ line cuts the boundaries of this region.

The points of intersection of the line $\mathcal{E}_1 = \omega - \mathcal{E}_2$ and the boundaries of the region $ABCA$ then correspond to

$$\omega_1 = \frac{\omega_p}{2m^2}\left[\omega_p \mathcal{E}_2 - p_2\left(\omega_p^2 - 4m^2\right)^{1/2}\right],$$

$$\omega_2 = \begin{cases} \frac{\omega_p}{2m^2}\left[\omega_p \mathcal{E}_2 + p_2\left(\omega_p^2 - 4m^2\right)^{1/2}\right], & \mathcal{E}_2 \le \mathcal{E}_{\min}\left(\omega = \omega_{\lim}\right), \\ \omega_{\lim}, & \mathcal{E}_{\min}\left(\omega = \omega_{\lim}\right) < \mathcal{E}_2 < \mathcal{E}_{\max}\left(\omega = \omega_{\lim}\right). \end{cases} \tag{6.100}$$

Finally, the total probability of the annihilation process is

$$W_\gamma = \frac{e^2}{4\pi p_2 \mathcal{E}_2}\left[\left(m^2 + \frac{\omega_p^2}{2}\right)\left(\omega_2 - \omega_1\right) + \frac{1}{2}\omega_p\left(\mathcal{E}_2^2 + \frac{\omega_p^2}{4}\right)\right.$$

$$\left. \times \ln\frac{\left(\omega_2 - \omega_p\right)\left(\omega_1 + \omega_p\right)}{\left(\omega_2 + \omega_p\right)\left(\omega_1 - \omega_p\right)} - \frac{\mathcal{E}_2 \omega_p^2}{2}\ln\frac{\left(\omega_2 - \omega_p\right)\left(\omega_2 + \omega_p\right)}{\left(\omega_1 - \omega_p\right)\left(\omega_1 + \omega_p\right)}\right]. \tag{6.101}$$

The lower limit for the density of the medium, above which pair annihilation is possible, can be defined from the reaction threshold condition (6.64) and the dispersion law (6.96). Thus, we obtain $\omega_p > 2m$, which is equivalent to $\mathcal{E}_F > \sqrt{3\pi}m/e \approx 36m$. The electron density of the plasma corresponding to this value of $\mathcal{E}_F$ is $\rho_e > p_F^3/3\pi^2 \approx 3 \cdot 10^{34}\mathrm{cm}^{-3}$.

For a nonrelativistic positron annihilation in an electron plasma we have a simple formula for the total probability:

$$W_\gamma = \frac{e^2\omega_p^3}{8\pi m^3}\left(\omega_p^2 - 4m^2\right)^{1/2}, \qquad p_2 << m. \tag{6.102}$$

Let us now analyze the results for the electron–positron pair production in a superdense relativistic degenerate plasma with the dispersion law (6.96). The Pauli principle in this case demands the satisfaction of the condition $\mathcal{E}_1 > \mathcal{E}_F$ which together with conditions (6.64) and (6.65) substantially reduces the range of parameter values for this process to proceed even in the required superdense plasma. The range of integration with respect to $\mathcal{E}_1$ in Eq. (6.85) shrinks to a point and the probability of the process $\gamma \to e^- + e^+$ tends practically to zero. With the increase of the electron density when $\mathcal{E}_F \gtrsim 150m$ ($\mathcal{E}_{\max}(\omega_{\lim}) > \mathcal{E}_F$, see Fig. 6.1), a narrow region appears and Eqs. (6.65), (6.100) show that the creation of a pair by a γ-quantum with energy $\omega_1(\mathcal{E}_2 = \mathcal{E}_F) < \omega < \omega_{\lim}$ becomes possible in this region. As a result, the lower bound of the energy of a created electron instead of $\mathcal{E}_{\min}(\omega)$ should be $\mathcal{E}_F$ and from Eq. (6.85) we obtain

$$W = \frac{e^2\left(\mathcal{E}_{\max}(\omega) - \mathcal{E}_F\right)}{4\pi a_\omega \omega^2 n^5(\omega)} \left\{ \frac{1-n^2(\omega)}{3}\left(\mathcal{E}^2_{\max}(\omega) + \mathcal{E}_F\mathcal{E}_{\max}(\omega) + \mathcal{E}_F^2\right)\right.$$

$$\left. - \frac{1-n^2(\omega)}{2}\omega\left(\mathcal{E}_{\max}(\omega) + \mathcal{E}_F\right) + n^2(\omega)m^2 + \frac{1-n^4(\omega)}{4}\omega^2 \right\}. \quad (6.103)$$

However, it is important to recall that this region $\omega \simeq \omega_{\rm lim}$ lies at the limit of validity of the macroscopic concept for a refractive index of a medium (one particle within the length $\lambda/2$).

6.6 Electron–Positron Pair Production by Strong EM Wave in Nonstationary Medium

As the probability of the single-quantum production of an electron–positron pair in a stationary plasma, as a macroscopic dispersive medium, practically equals zero (even at the required superdensities of electrons) it is reasonable to consider an exclusive possibility for a single-photon pair production in a nonstationary medium of ordinary densities by strong light fields. Namely, we assume the abrupt temporal change of the dielectric permittivity of a medium which may be described by the stepwise function ε (6.1).

In order to describe pair production in the field (6.6), (6.7) we shall employ the Dirac model (all negative-energy states of the vacuum are filled with electrons). The Dirac equation in the field (6.6), (6.7) has the form ($\hbar = c = 1$)

$$i\frac{\partial \Psi}{\partial t} = \left[\widehat{\alpha}(\mathbf{p} - e\mathbf{A}) + \widehat{\beta}m\right]\Psi, \quad (6.104)$$

where

$$\mathbf{A}(\mathbf{r},t) = \begin{cases} i\frac{\mathbf{E}_0}{\omega_0}e^{i(\omega_0 t - \mathbf{k}_0\mathbf{r})} + \text{c.c.}, & t < 0 \\ i\frac{\mathbf{E}_1}{\omega_1}e^{i\omega_1 t - \mathbf{k}_0\mathbf{r}} - i\frac{\mathbf{E}_2}{\omega_1}e^{-i\omega_1 t - \mathbf{k}_0\mathbf{r}} + \text{c.c.}, & t \geqslant 0 \end{cases} \quad (6.105)$$

is the vector potential of the EM field and $\widehat{\alpha}$, $\widehat{\beta}$ are the Dirac matrices in the standard representation (3.2).

We solve Eq. (6.104) by perturbing in the field of the wave. This method is valid if

$$\left[1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{1/2}\right]\xi_0 << 1, \qquad \xi_0 = \frac{eE_0}{m\omega_0}. \quad (6.106)$$

We expand the perturbed first-order wave function $\Psi_1(\mathbf{r},t)$ in a complete set of orthonormalized wave functions of the electrons (positrons) with momenta $\mathbf{p} - \mathbf{k}_0$ and $\mathbf{p} + \mathbf{k}_0$:

$$\Psi_1(\mathbf{r},t) = \Psi_1^{(-)}(t)e^{i(\mathbf{p}-\mathbf{k}_0)\mathbf{r}} + \Psi_1^{(+)}(t)e^{i(\mathbf{p}+\mathbf{k}_0)\mathbf{r}},$$

$$\Psi_1^{(-)}(t) = \sum_{l=1}^{4} a_l(t)u_l\left(\mathbf{p}-\mathbf{k}_0,t\right), \tag{6.107}$$

$$\Psi_1^{(+)}(t) = \sum_{j=1}^{4} b_j(t)u_j\left(\mathbf{p}+\mathbf{k}_0,t\right).$$

Here $a_l(t)$ and $b_j(t)$ are unknown functions and $u_i\left(\mathbf{p}',t\right)$ are orthonormalized bispinor functions which describe the particle states with energies $\pm\mathcal{E}' = \pm\sqrt{p'^2+m^2}$:

$$u_{1,2}\left(\mathbf{p}',t\right) = \left(\frac{\mathcal{E}'+m}{2\mathcal{E}'}\right)^{1/2} \begin{pmatrix} \varphi_{1,2} \\ \frac{\sigma\mathbf{p}'}{\mathcal{E}'+m}\varphi_{1,2} \end{pmatrix} \exp\left(-i\mathcal{E}'t\right), \tag{6.108}$$

$$u_{3,4}\left(\mathbf{p}',t\right) = \left(\frac{\mathcal{E}'+m}{2\mathcal{E}'}\right)^{1/2} \begin{pmatrix} \frac{-\sigma\mathbf{p}'}{\mathcal{E}'+m}\chi_{3,4} \\ \chi_{3,4} \end{pmatrix} \exp\left(i\mathcal{E}'t\right). \tag{6.109}$$

These functions are normalized to one particle per unit volume: $u_i^+u_j = \delta_{ij}$; the constant spinors $\varphi_{1,2}$ and $\chi_{3,4}$ are

$$\varphi_1 = \chi_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \varphi_2 = \chi_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Under the transformations (6.107)–(6.109) the Dirac equation for the perturbed wave function $\Psi = \Psi_0 + \Psi_1 + \cdots$, $(|\Psi_1| << |\Psi_0|)$:

$$\left(i\frac{\partial}{\partial t} - \widehat{\alpha}\mathbf{p} - \widehat{\beta}m\right)\Psi_1 = -e\widehat{\alpha}\mathbf{A}\Psi_0 \tag{6.110}$$

transforms into a system of 16 equations for the unknown functions $a_l(t)$ and $b_j(t)$:

$$\left(i\frac{\partial}{\partial t} - \widehat{\alpha}\mathbf{p} - \widehat{\beta}m\right)\left[\sum_{l=1}^{4} a_l(t)u_l\left(\mathbf{p}-\mathbf{k}_0,t\right)e^{i(\mathbf{p}-\mathbf{k}_0)\mathbf{r}}\right.$$

$$\left. + \sum_{j=1}^{4} b_j(t)u_j\left(\mathbf{p}+\mathbf{k}_0,t\right)e^{i(\mathbf{p}+\mathbf{k}_0)\mathbf{r}}\right]$$

$$= -e\widehat{\alpha}\left[\mathbf{A}_{(-)}(t)e^{-i\mathbf{k}_0\mathbf{r}} + \mathbf{A}_{(+)}(t)e^{i\mathbf{k}_0\mathbf{r}}\right] u_s\left(\mathbf{p}, t\right) e^{i\mathbf{pr}}, \tag{6.111}$$

where $s = 3, 4$ and

$$\mathbf{A}_{(-)}(t) = \begin{cases} i\frac{\mathbf{E}_0}{\omega_0}e^{i\omega_0 t}, & t < 0, \\ i\frac{\mathbf{E}_1}{\omega_1}e^{i\omega_1 t} - i\frac{\mathbf{E}_2}{\omega_1}e^{-i\omega_1 t}, & t \geqslant 0, \end{cases} \qquad \mathbf{A}_{(+)}(t) = \mathbf{A}^*_{(-)}(t). \tag{6.112}$$

The bispinor functions $u_s\left(\mathbf{p}, t\right)$ in Eq. (6.111) correspond to the unperturbed states of the Dirac vacuum (they are determined by Eq. (6.109) with $s = 3$ and $s = 4$, where $\mathbf{p}' = \mathbf{p}$ and $\mathcal{E}' = \mathcal{E}$ are the momenta and energies of the free vacuum electrons). According to this model, a pair is produced because of the interaction of the external field with the electrons of negative energies of the Dirac vacuum. In the first-order perturbation theory in the field this leads to electron states in the region of positive energies with the values

$$\mathcal{E}_{(-)} = \sqrt{\left(\mathbf{p} - \mathbf{k}_0\right)^2 + m^2}, \qquad \mathcal{E}_{(+)} = \sqrt{\left(\mathbf{p} + \mathbf{k}_0\right)^2 + m^2}.$$

The probabilities of these transitions are determined by the amplitudes $a_{1,2}$ and $b_{1,2}$, respectively (the indices 1 and 2 correspond to two different spin states). Therefore the problem reduces to determining the functions $a_{1,2}(t)$ and $b_{1,2}(t)$ by integrating the set of Eqs. (6.111). From the latter we obtain the following set of equations:

$$\sum_{l=1}^{4} i\frac{da_l}{dt}u_l\left(\mathbf{p} - \mathbf{k}_0, t\right) = -e\widehat{\alpha}\mathbf{A}_{(-)}(t)u_s\left(\mathbf{p}, t\right), \tag{6.113}$$

$$\sum_{j=1}^{4} i\frac{db_j}{dt}u_j\left(\mathbf{p} + \mathbf{k}_0, t\right) = -e\widehat{\alpha}\mathbf{A}_{(+)}(t)u_s\left(\mathbf{p}, t\right). \tag{6.114}$$

Multiplying Eq. (6.113) on the left by $u_l^\dagger\left(\mathbf{p} - \mathbf{k}_0, t\right)$ and Eq. (6.114) by $u_j^\dagger\left(\mathbf{p} + \mathbf{k}_0, t\right)$ and taking into account that the bispinors are orthonormal ($u_l^\dagger u_m = \delta_{lm}$) we obtain eight equations for the transitions amplitudes $a_l(t)$ and $b_j(t)$ for a given spinor state s of a vacuum electron ($s = 3$ or $s = 4$) :

$$\frac{da_l(t)}{dt} = ieu_l^\dagger\left(\mathbf{p} - \mathbf{k}_0, t\right)\widehat{\alpha}\mathbf{A}_{(-)}(t)u_s\left(\mathbf{p}, t\right), \qquad l = 1, ..., 4, \tag{6.115}$$

$$\frac{db_j(t)}{dt} = ieu_j^\dagger\left(\mathbf{p} + \mathbf{k}_0, t\right)\widehat{\alpha}\mathbf{A}_{(+)}(t)u_s\left(\mathbf{p}, t\right), \qquad j = 1, ..., 4. \tag{6.116}$$

Orienting the z axis parallel to the electric field $\mathbf{E}_0$ of the wave and the x axis parallel to the wave vector $\mathbf{k}_0$, we obtain for the amplitudes $a_{1,2}$ and $b_{1,2}$

$$a_{1,2}(t) = ieu^{\dagger}_{1,2}\left(\mathbf{p}-\mathbf{k}_0\right)\alpha_z u_s\left(\mathbf{p}\right)\int_{-\infty}^{t} A_{(-)}(t')e^{i\left(\mathcal{E}+\mathcal{E}_{(-)}\right)t'}dt', \qquad (6.117)$$

$$b_{1,2}(t) = ieu^{\dagger}_{1,2}\left(\mathbf{p}+\mathbf{k}_0\right)\alpha_z u_s\left(\mathbf{p}\right)\int_{-\infty}^{t} A_{(+)}(t')e^{i\left(\mathcal{E}+\mathcal{E}_{(+)}\right)t'}dt', \qquad (6.118)$$

where $u^{\dagger}_{1,2}\left(\mathbf{p}\mp\mathbf{k}_0\right)$ and $u_s\left(\mathbf{p}\right)$ are constant bispinors determined by Eqs. (6.108) and (6.109) (preexponential factors in Eqs. (6.108), (6.109)) .

The probability of electron production from a definite vacuum state $\mathbf{p}$, s is determined by the quantity $|a_1(t)|^2+|a_2(t)|^2+|b_1(t)|^2+|b_2(t)|^2$ (the probability of the production of a positron with a momentum $\mathbf{p}$ in a definite spinor state s). The differential probability of pair production, summed over the initial spin states of the Dirac vacuum, in an element of the phase volume $d\mathbf{p}/(2\pi)^3$ (the spatial normalization volume $V=1$), is

$$dW = 2\left[|a_1(t)|^2+|a_2(t)|^2+|b_1(t)|^2+|b_2(t)|^2\right]|_{t\to+\infty}\,\frac{d\mathbf{p}}{(2\pi)^3}. \qquad (6.119)$$

Integrating Eqs. (6.117), (6.118) over time with Eq. (6.112) and assuming that the EM wave is switched on and switched off adiabatically: $\mathbf{E}_0(t=-\infty)=\mathbf{E}_1(t=+\infty)=\mathbf{E}_2(t=+\infty)=0$ (the amplitudes of the incident, transmitted, and reflected waves are assumed to be slowly varying functions of time), we obtain the following expressions for the amplitudes $a_{1,2}$ and $b_{1,2}$ after the wave interaction with the Dirac vacuum:

$$a_{1,2}(t=+\infty) = \frac{ieE_0\left(\varepsilon_1-\varepsilon_2\right)\left(\mathcal{E}+\mathcal{E}_{(-)}\right)}{\varepsilon_2\left(\mathcal{E}+\mathcal{E}_{(-)}+\omega_0\right)\left(\left(\mathcal{E}+\mathcal{E}_{(-)}\right)^2-\omega_0^2\frac{\varepsilon_1}{\varepsilon_2}\right)} \times\left[u^{\dagger}_{1,2}\left(\mathbf{p}-\mathbf{k}_0\right)\alpha_z u_s\left(\mathbf{p}\right)\right], \qquad (6.120)$$

$$b_{1,2}(t=+\infty) = \frac{ieE_0\left(\varepsilon_1-\varepsilon_2\right)\left(\mathcal{E}+\mathcal{E}_{(+)}\right)}{\varepsilon_2\left(\mathcal{E}+\mathcal{E}_{(+)}-\omega_0\right)\left(\left(\mathcal{E}+\mathcal{E}_{(+)}\right)^2-\omega_0^2\frac{\varepsilon_1}{\varepsilon_2}\right)} \times\left[u^{\dagger}_{1,2}\left(\mathbf{p}+\mathbf{k}_0\right)\alpha_z u_s\left(\mathbf{p}\right)\right]. \qquad (6.121)$$

Evaluating the transition matrix elements in Eqs. (6.120), (6.121), we obtain with the help of Eq. (6.119) the differential probability of pair production by a strong EM wave in a nonstationary medium:

$$dW = \frac{e^2}{(2\pi)^3}\frac{E_0^2}{\mathcal{E}}\left(\frac{\varepsilon_1}{\varepsilon_2}-1\right)^2$$

$$\times\left\{\frac{\left(\mathcal{E}+\mathcal{E}_{(-)}\right)^2\left[\mathcal{E}\mathcal{E}_{(-)}+m^2+p_x(p_x-k_0)+p_y^2-p_z^2\right]}{\mathcal{E}_{(-)}\left(\mathcal{E}+\mathcal{E}_{(-)}+\omega_0\right)^2\left[\left(\mathcal{E}+\mathcal{E}_{(-)}\right)^2-\omega_0^2\frac{\varepsilon_1}{\varepsilon_2}\right]^2}\right.$$

$$\left.+\frac{\left(\mathcal{E}+\mathcal{E}_{(+)}\right)^2\left[\mathcal{E}\mathcal{E}_{(+)}+m^2+p_x(p_x+k_0)+p_y^2-p_z^2\right]}{\mathcal{E}_{(+)}\left(\mathcal{E}+\mathcal{E}_{(+)}-\omega_0\right)^2\left[\left(\mathcal{E}+\mathcal{E}_{(+)}\right)^2-\omega_0^2\frac{\varepsilon_1}{\varepsilon_2}\right]^2}\right\}d\mathbf{p}. \qquad (6.122)$$

As one can see from Eq. (6.122), the process exhibits azimuthal asymmetry with respect to the direction of propagation of the wave. Orienting the polar axis in this direction ($d\mathbf{p} = p\mathcal{E}d\mathcal{E}\sin\theta d\theta d\varphi$, where θ is the angle between the vectors $\mathbf{p}$ and $\mathbf{k}_0$ and φ is the azimuthal angle relative to the direction of polarization of the wave) and integrating over the energy, we obtain the angular distribution of the produced electrons (positrons). As the case of physical interest is an EM wave of frequencies $\omega << m$, Eq. (6.122) simplifies greatly and takes the form

$$dW = \frac{e^2E_0^2}{2\pi^3}\left(\frac{\varepsilon_1}{\varepsilon_2}-1\right)^2\frac{\sqrt{\mathcal{E}^2-m^2}}{\mathcal{E}}$$

$$\times\frac{m^2\sin^2\theta\cos^2\varphi+\mathcal{E}^2\left(1-\sin^2\theta\cos^2\varphi\right)}{\left(4\mathcal{E}^2-\omega_0^2\frac{\varepsilon_1}{\varepsilon_2}\right)^2}\sin\theta d\theta d\varphi d\mathcal{E}. \qquad (6.123)$$

Integrating Eq. (6.123) over the energy we obtain the number of pairs produced in the element of solid angle $do = \sin\theta d\theta d\varphi$:

$$dW(\theta,\varphi) = \frac{e^2E_0^2}{128\pi^2 m}\left(\frac{\varepsilon_1}{\varepsilon_2}-1\right)^2\left[F\left(2;\frac{1}{2};2;\frac{\omega_0^2\varepsilon_1}{4m^2\varepsilon_2}\right)\right.$$

$$\left.\times\left(1-\sin^2\theta\cos^2\varphi\right)+\frac{1}{4}F\left(2;\frac{3}{2};3;\frac{\omega_0^2\varepsilon_1}{4m^2\varepsilon_2}\right)\sin^2\theta\cos^2\varphi\right]do, \qquad (6.124)$$

where $F(\nu;\mu;\lambda;z)$ is the hypergeometric function.

For the energy distribution of the produced electrons (positrons) we have

$$dW(\mathcal{E}) = \frac{2e^2E_0^2}{3\pi^2}\left(\frac{\varepsilon_1}{\varepsilon_2}-1\right)^2 \frac{\sqrt{\mathcal{E}^2-m^2}\left(2\mathcal{E}^2+m^2\right)}{\left(4\mathcal{E}^2-\omega_0^2\frac{\varepsilon_1}{\varepsilon_2}\right)^2}d\mathcal{E}. \tag{6.125}$$

Integrating Eq. (6.124) over the angles θ and φ (or Eq. (6.125) over the energy) we obtain the total number of electron–positron pairs produced by a strong EM wave in a nonstationary medium:

$$W = \frac{2e^2E_0^2}{48\pi m}\left(\frac{\varepsilon_1}{\varepsilon_2}-1\right)^2 \left[F\left(2;\frac{1}{2};2;\frac{\omega_0^2\varepsilon_1}{4m^2\varepsilon_2}\right)\right.$$

$$\left.+\frac{1}{8}F\left(2;\frac{3}{2};3;\frac{\omega_0^2\varepsilon_1}{4m^2\varepsilon_2}\right)\right]. \tag{6.126}$$

Note that in Eqs. (6.123) and (6.125) the denominators become zero for $\omega_0\sqrt{\varepsilon_1/\varepsilon_2} = 2\mathcal{E}$. This is the conservation law for the single-photon pair production by a wave of the frequency $\omega_1 = \omega_0\sqrt{\varepsilon_1/\varepsilon_2}$ (by the transmitted and reflected waves) in a medium with the index of refraction $n_2 = \sqrt{\varepsilon_2} < 1$ (plasma). Since Eqs. (6.123)–(6.126) correspond to the case $\omega << m$, the pole in Eq. (6.123) can be reached, i.e., the conservation laws of energy and momentum for the process $\gamma \to e^- + e^+$ can be satisfied only if $\varepsilon_1/\varepsilon_2 >> 1$. Actually this is possible if $\varepsilon_2 << 1$, in agreement with the fact that pair production by a photon field requires a plasmalike medium. It is obvious from Eq. (6.126) that the total probability of the process diverges when $\omega_0^2\varepsilon_1/4m^2\varepsilon_2 = 1$. The latter is associated with the fact that these probabilities were determined for an infinitely long interaction time. In perturbation theory probabilities are proportional to the interaction time (under stationary conditions) and diverge as $t \to \infty$. Thus, this divergence is not associated with the process studied here, which is governed by the time dependence of the medium, and it can be eliminated by assuming $\omega_0^2\varepsilon_1/\varepsilon_2 < 4m^2$. Moreover, for laser frequencies and realistic values of the dielectric permittivities $\omega_0\sqrt{\varepsilon_1/\varepsilon_2} << 2\mathcal{E}$ and from Eq. (6.126) we obtain the following expression for the total number of e^-, e^+ pairs produced in the volume V due only to the medium nonstationary properties:

$$W = \frac{3e^2E_0^2V}{128\pi m}\left(1-\frac{\varepsilon_1}{\varepsilon_2}\right)^2. \tag{6.127}$$

In the general case, for arbitrary frequency of EM wave and temporal variation of the dielectric permittivity of the medium $\varepsilon_1/\varepsilon_2$ from Eq. (6.122) the following formula for the pair's probability distribution over the total energy $\mathcal{E}_t = \mathcal{E}_{e^-} + \mathcal{E}_{e^+}$ of the produced particles can be derived:

$$\frac{dW}{d\mathcal{E}_t} = \frac{e^2 E_0^2}{6\pi^2}\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right)^2 \left(1 - \frac{4m^2}{\mathcal{E}_t^2 - k_0^2}\right)^{1/2}$$

$$\times \frac{\mathcal{E}_t^2 \left(\mathcal{E}_t^2 + \omega_0^2\right)\left(\mathcal{E}_t^2 + 2m^2 - k_0^2\right)}{\left(\mathcal{E}_t^2 - \omega_0^2\right)\left(\mathcal{E}_t^2 - \omega_0^2 \frac{\varepsilon_1}{\varepsilon_2}\right)^2}. \tag{6.128}$$

Bibliography

G.S. Sahakyan, Zh. Éksp. Teor. Fiz. **38**, 843 (1960)
G.S. Sahakyan, Zh. Éksp. Teor. Fiz. **38**, 1593 (1960)
V.L. Ginzburg, Izv. VUZov, Radiofizika **16**, 512 (1973) [in Russian]
V.L. Ginzburg, V.N. Tsitovich, Zh. Éksp. Teor. Fiz. **65**, 132 (1973)
H.K. Avetissian, A.K. Avetissian, R.G. Petrossian, Zh. Éksp. Teor. Fiz. **75**, 382 (1978)
V.L. Ginzburg: Theoretical Physics and Astrophysics (Pergamon Press, Oxford 1979)
H.K. Avetissian, A.K. Avetissian, Kh.V. Sedrakian, Zh. Éksp. Teor. Fiz. **94**, 21 (1988)
H.K. Avetissian, A.K. Avetissian, Kh.V. Sedrakian, Zh. Éksp. Teor. Fiz. **100**, 82 (1991)