Exact penalty functions for generalized Nash problems

Francisco Facchinei[1] and Jong-Shi Pang[2]

[1] Università di Roma "La Sapienza", Dipartimento di Informatica e Sistemistica, Via Buonarroti 12, 00185 Roma, Italy (facchinei@dis.uniroma1.it)
[2] Rensselaer Polytechnic Institute, Department of Mathematical Sciences and Department of Decision Sciences and Engineering Systems, Troy, New York, 12180-3590 U.S.A. (pangj@rpi.edu)

Summary. We propose the use exact penalty functions for the solution of generalized Nash equilibrium problems (GNEPs). We show that by this approach it is possible to reduce the solution of a GNEP to that of a usual Nash problem. This paves the way to the development of numerical methods for the solution of GNEPs. We also introduce the notion of generalized stationary point of a GNEP and argue that convergence to generalized stationary points is an appropriate aim for solution algorithms.

Key words: (Generalized) Nash equilibrium problem, Penalty function.

1 Introduction

In this paper we consider the following Generalized Nash Equilibrium Problem (GNEP) with two players:

$$\begin{aligned} &\text{minimize}_x \ \theta_{\rm I}(x,y) & & & &\text{minimize}_y \ \theta_{\rm II}(x,y) \\ &\text{subject to } h^{\rm I}(x) \quad \leq 0 & &\text{and} & &\text{subject to } h^{\rm II}(y) \quad \leq 0 \\ &\qquad\qquad g^{\rm I}(x,y) \leq 0 & & & &\qquad\qquad g^{\rm II}(x,y) \leq 0 \end{aligned} \tag{1}$$

where

- $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$;
- $\theta_{\rm I}(x,y)$ ($\theta_{\rm II}(x,y)$) is a continuously differentiable function from $\mathbb{R}^{n_1+n_2}$ to $\mathbb{R}$, such that, for every fixed y (x), $\theta_{\rm I}(\cdot,y)$ ($\theta_{\rm II}(x,\cdot)$) is convex;
- $h^{\rm I}(x)$ ($h^{\rm II}(y)$) is a continuously differentiable convex function from $\mathbb{R}^{n_1}$ ($\mathbb{R}^{n_2}$) to $\mathbb{R}^{m_1}$ ($\mathbb{R}^{m_2}$);
- $g^{\rm I}(x,y)$ ($g^{\rm II}(x,y)$) is a continuously differentiable function from $\mathbb{R}^{n_1+n_2}$ to $\mathbb{R}^{m_1}$ ($\mathbb{R}^{m_2}$) such that, for every fixed y (x), $g_{\rm I}(\cdot,y)$ ($g_{\rm II}(x,\cdot)$) is convex.

The extension of all the results of the paper to the case of a finite number of players is trivial and we do not discuss this more general case just for the sake of notational simplicity.

For every fixed y, denote by $\mathcal{S}(y)$ the (possibly empty) solution set of the first player and, similarly, for every fixed x, $\mathcal{S}(x)$ is the (possibly empty) solution set of the second player. We further denote by $\mathcal{F} \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ the feasible set of the GNEP, i.e. the set of points (x, y) that are feasible for the first and second player at the same time. Note that, by the assumptions made, once y is fixed, the first player's problem is convex in x and analogously we have that the second player's optimization problem is convex in y for every fixed x. A point $(\bar{x}, \bar{y})$ is a solution of the GNEP if $\bar{x} \in \mathcal{S}(\bar{y})$ and $\bar{y} \in \mathcal{S}(\bar{x})$. Solutions of the GNEP are also often called *equilibria*.

The presence of the "coupling" constraints $g^{\mathrm{I}}(x, y)$ and $g^{\mathrm{II}}(x, y)$ which make the feasible set of one player depend on the variables of the other player is what distinguish the GNEP from the standard Nash Equilibrium Problem NEP and actually makes the GNEP an extremely difficult problem. To date there have been very few attempts to define algorithms for the calculation of an equilibrium of a GNEP. One possibility is to write the optimality conditions of the two players using the minimum principle thus obtaining a Quasi Variational Inequality (QVI). This has been recognized by Bensoussan [1] as early as 1974 (see also [11] for the finite-dimensional case). However there is also a lack of proposals for the solution of QVIs, so this reduction is not very appealing from the algorithmic point of view. There are a few other scattered proposals in the literature, based either on fixed point approaches or the use of the Nikaido-Isoda function: [2, 3, 14, 16, 19] (see [15] for the definition of the Nikaido-Isoda function). These kind of algorithms, however, require quite stringent assumptions that cannot be expected to be satisfied in general. On the other hand there is certainly a wealth of interesting applications that call for the solution of GNEPs: see, as a way of example, [4, 10, 13, 17, 18]

Recently, Fukushima and Pang [10] proposed a promising *sequential* penalty approach whereby a solution is sought by solving a sequence of smooth NEPs problems for values of a penalty parameter increasing to infinity. The advantage of this idea is that the the penalized NEPs can be reformulated as Variational Inequalities (VI) to which, in principle, well understood solution methods can be applied, see [9]. In this work we propose a solution framework whereby, by using *exact* penalization techniques, the GNEP is reduced to the solution of a single NEP. The advantage is that we only deal with a single NEP with a finite value of the penalty parameter. The disadvantage is that the players in this NEP have nonsmooth objective functions.

Before describing our approach more in detail we think it is important to set the goal. The GNEP has a structure that exhibits many "convexities" and so one could think that a reasonable goal for a numerical method is to find a solution (or to determine that the problem has no solutions): this would parallel what happens for a convex optimization problem. However the GNEP is really a "non convex" problem. For example, under our assumptions, the feasible set $\mathcal{F}$ is non convex and even finding a feasible point, let alone a solution, is a difficult task. Therefore we cannot expect that we can solve GNEP unless some (more) stringent assumptions are made.

In nonlinear programming a research line has emerged that attempts to analyze algorithms under minimal assumptions; an algorithm is considered successful if it can be shown to find a "generalized stationary point" under these minimal assumptions. Roughly speaking, a "generalized stationary point" of a nonlinear program is a either a Karush-Kuhn-Tucker (KKT) point or Fritz-John (FJ) point or an (unfeasible) stationary point of some measure of violation of the constraints. After this analysis has been carried out, it is investigated under which additional assumptions one can guarantee that, indeed, the algorithm converges to a KKT point, thus ruling out other undesirable possibilities. This point of view seems very sensible and enriches the usual approach in that when one applies an algorithm to the solution of a nonlinear optimization problem, one does not usually know in advance that the problem satisfies the regularity assumptions required by algorithm (i.e. linear independence of active constrains, positive linear independence of violated constraints and so on). It is then of interest to show that, in any case, the algorithm behaves in a reasonable way, locating a generalized stationary point, and to show that, if in addition some more stringent regularity conditions are satisfied, convergence actually occurs to a KKT point of the problem.

In this paper we parallel these kind of developments and show that the penalization technique we propose can only converge to a "Nash generalized stationary point". Then we give additional conditions to guarantee that convergence occurs to a solution. Our first task on the agenda is then to give a suitable definition of Nash generalized stationary point. Our definition is inspired by similar definitions in the case of nonlinear optimization problems and also takes into account the convexities that are present in the GNEP.

Definition 1. A point $(x, y) \in \mathbb{R}^{n_1+n_2}$ is a Nash generalized stationary point if

1. x is either a KKT or a FJ point of

$$\begin{aligned} &\text{minimize}_x \ \theta_{\mathrm{I}}(x,y) \\ &\text{subject to } h^{\mathrm{I}}(x) \quad \le 0 \\ &\qquad\qquad\quad g^{\mathrm{I}}(x,y) \le 0 \end{aligned} \tag{2}$$

or a global minimizer of $\|h^{\mathrm{I}}(\cdot)_+, g^{\mathrm{I}}(\cdot,y)_+\|_2$ with $\|h^{\mathrm{I}}(x)_+, g^{\mathrm{I}}(x,y)_+\|_2 > 0$;

2. y is either a KKT or a FJ point of

$$\begin{aligned} &\text{minimize}_y \ \theta_{\mathrm{II}}(x,y) \\ &\text{subject to } h^{\mathrm{II}}(y) \quad \le 0 \\ &\qquad\qquad\quad g^{\mathrm{II}}(x,y) \le 0 \end{aligned} \tag{3}$$

or a global minimizer of $\|h^{\mathrm{II}}(\cdot)_+, g^{\mathrm{II}}(x,\cdot)_+\|_2$ with $\|h^{\mathrm{I}}(y)_+, g^{\mathrm{I}}(x,y)_+\|_2 > 0$.

Two observations are in order. Every solution is clearly a Nash generalized stationary point, but the viceversa does not hold. If x is a KKT point (by which we mean that x, together with appropriate multipliers, satisfies the KKT conditions) then x solves, for the given fixed y, the optimization problem (2) and similarly for y. If x is FJ point, instead, this reflects a "lack of constraint qualification" and in this case we cannot say whether x is or is not a solution to (2). Finally, if x is a global minimizer of the function $\|h^{\mathrm{I}}(\cdot)_+, g^{\mathrm{I}}(\cdot,y)_+\|_2$ with $\|h^{\mathrm{I}}(x)_+, g^{\mathrm{I}}(x,y)_+\|_2 > 0$, this means that, for the given y, problem (2) is unfeasible. Note that the definition of generalized stationary point extends in a natural way similar definitions valid for nonlinear optimization problems. We remark that under the assumptions we will make to analyze the algorithm the existence of a solution is not guaranteed, so it would be unreasonable to expect that any algorithm can surely find one!

2 Exact penalty functions for the GNEP

Our aim is to transform the GNEP (1) problem into a(n unconstrained), nondifferentiable Nash problem by using a penalty approach. To this end we consider the following penalization of the GNEP:

$$\begin{aligned}&\min_x \theta_{\mathrm{I}}(x,y) + \rho_{\mathrm{I}}\,\|h^{\mathrm{I}}_+(x), g^{\mathrm{I}}_+(x,y)\|_2\\ &\text{and} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4)\\ &\min_y \theta_{\mathrm{II}}(x,y) + \rho_{\mathrm{II}}\,\|h^{\mathrm{II}}_+(y), g^{\mathrm{II}}_+(x,y)\|_2,\end{aligned}$$

where ρ_{I} and ρ_{II} are positive penalty parameters. In this paper all the norms are always Euclidean norms; therefore from now on we will always write $\|\cdot\|$ instead of $\|\cdot\|_2$. By setting

$$P_{\mathrm{I}}(x,y;\rho_{\mathrm{I}}) = \theta_{\mathrm{I}}(x,y) + \rho_{\mathrm{I}}\,\|h^{\mathrm{I}}_+(x), g^{\mathrm{I}}_+(x,y)\|$$

and

$$P_{\mathrm{II}}(x,y;\rho_{\mathrm{II}}) = \theta_{\mathrm{II}}(x,y) + \rho_{\mathrm{II}}\,\|h^{\mathrm{II}}_+(y), g^{\mathrm{II}}_+(x,y)\|,$$

problem (4) can be rewritten as

$$\min_x P_{\mathrm{I}}(x,y;\rho_{\mathrm{I}}) \qquad \text{and} \qquad \min_y P_{\mathrm{II}}(x,y;\rho_{\mathrm{II}}).$$

It is also possible to penalize only some of constraints, the most natural choice being to penalize only the coupling constraints $g^{\mathrm{I}}_+(x,y) \leq 0$ and $g^{\mathrm{II}}_+(x,y) \leq 0$. This would give rise to the following penalized Nash problem

$$\min_{x\in X} \theta_{\mathrm{I}}(x,y) + \rho_{\mathrm{I}}\,\|g^{\mathrm{I}}_+(x,y)\| \qquad \text{and} \qquad \min_{y\in Y} \theta_{\mathrm{II}}(x,y) + \rho_{\mathrm{II}}\,\|g^{\mathrm{II}}_+(x,y)\|,$$

where $X = \{x : h^{\mathrm{I}}(x) \le 0\}$ and $Y = \{y : h^{\mathrm{II}}(y) \le 0\}$. Different penalizations could further be considered where, maybe, also some "difficult" constraints in the definition of X or Y are penalized while the simple ones (e.g. box constraints) are not penalized. We do not consider all this variants in this paper, but with some obvious technical changes all the developments we consider for the penalization (4) can be extended to these different penalizations.

Note that for every fixed y, the first player subproblem in (4) is convex, although nondifferentiable, and the same holds for the second player. In this section we show that under appropriate conditions and for sufficiently large (but finite) values of the penalty parameters, the solutions sets of the GNEP (1) and that of the Penalized GNEP (PGNEP) (4) are strongly related. In the next section we will give a general scheme that allows us to iteratively update the penalty parameters in an appropriate way supposing that a minimization algorithm is available for the solution of the PGNEP (4) for fixed values of the penalty parameters.

The first result we discuss is basically known (see [12] for example), however we report it here with a proof for sake of completeness and also because we could not find a reference with the precise statement below that we need.

Proposition 1. Consider the minimization problem

$$\begin{aligned} &\text{minimize}_z\ f(z) \\ &\text{subject to } v(z) \le 0, \end{aligned} \tag{5}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $v : \mathbb{R}^n \to \mathbb{R}^m$ are C^1 and (componentwise) convex. Let $\bar{z}$ be a solution of this problem and assume that set M of KKT multipliers is non-empty. Let λ be any multiplier in M. Then, for every $\rho > \|\lambda\|$, the solution sets of (5) and

$$\text{minimize}_z\ f(z) + \rho\, \|v_+(z)\|$$

coincide.

Proof. Let $\tilde{z}$ be a solution of (5) and suppose that $\rho > \|\lambda\|$. We show that $\tilde{z}$ is a minimum point of the penalty function $P(z;\rho) \equiv f(z) + \rho\, \|v_+(z)\|$ (which is also convex). We recall that $\tilde{z}$ is a minimum point of the Lagrangian of the constrained problem (5), that is

$$f(z) + \lambda^T g(z) \ge f(\tilde{z}) + \lambda^T g(\tilde{z}), \qquad \forall z \in \mathbb{R}^n$$

(λ is a fixed KKT multiplier; recall that the set M of multipliers of problem (5) does not depend on the solution considered). Therefore we get, for any $z \in \mathbb{R}^n$,

$$
\begin{aligned}
f(z) + \rho \|v_+(z)\| &> f(z) + \|\lambda\| \, \|v_+(z)\| \\
&\geq f(z) + \lambda^T v(z)_+ \\
&\geq f(z) + \lambda^T v(z) \\
&\geq f(\bar{z}) + \lambda^T v(\bar{z}) \\
&= f(\bar{z}) \\
&= f(\bar{z}) + \rho \|v_+(\bar{z})\|,
\end{aligned}
$$

where we have used the facts that $\lambda \geq 0$, $\lambda^T v(\bar{z}) = 0$ and $v_+(\bar{z}) = 0$. Suppose now that $\tilde{z}$ is a (global) minimum point of the penalty function $P(z, \rho)$. Since the penalty function P and the objective function f coincide on the feasible set of problem (5), in order to show that $\tilde{z}$ is a solution of problem (5) it is sufficient to verify that $\tilde{z}$ is feasible to problem (5). If this were not the case

$$
\begin{aligned}
P(\tilde{z}; \rho) &> P(\tilde{z}; \|\lambda\|) \\
&\geq f(\tilde{z}) + \lambda^T v(\tilde{z})_+ \\
&\geq f(\tilde{z}) + \lambda^T v(\tilde{z}) \\
&\geq f(\bar{z}) + \lambda^T g(\bar{z}) \\
&= f(\bar{z}) \\
&= P(\bar{z}; \rho),
\end{aligned}
$$

(where the fourth inequality derives again from the fact that $\bar{z}$ is a minimum point of the Lagrangian of the constrained problem (5)). This would show that $\tilde{z}$ is not a global minimum of the penalty function thus giving a contradiction and therefore $\tilde{z}$ must be feasible. △

Consider now the first player's problem in (1). We can see it as a collection of convex optimization problems, one for each possible y. The same holds for the second player's problem, with the role of "parameter" taken here by x. The previous theorem suggests that if we want to penalize (1) and obtain some kind of equivalence to the penalized problem for finite values of the penalty parameters, we should require the boundedness of the multipliers of the player's problems for each value of the other player's variables. This lead us to the following assumption.

Assumption 2.1 (Generalized Sequential Boundedness Constraint Qualification (GSBCQ)) There exists a constant M such that for every solution $(\bar{x}, \bar{y})$ of the generalized Nash equilibrium problem there exists a corresponding couple (λ, μ) of KKT multipliers such that $\|(\lambda, \mu)\| \leq M$.

We refer to chapter 3 in [9] for more details on this constraint qualification (CQ) condition (in the case of optimization problems). Here we only stress that the GSBCQ appear to be a rather mild constraint qualification that unifies many standard CQ such as the Mangasarian-Fromovitz CQ and the constant rank one.

Under this assumption it is easy to prove the following result.

Theorem 1. Suppose that the GSBCQ is satisfied. Then there exists $\bar{\rho}_{\rm I} \geq 0$ and $\bar{\rho}_{\rm II} \geq 0$ such that for every $\rho_{\rm I} \geq \bar{\rho}_{\rm I}$ and $\rho_{\rm II} \geq \bar{\rho}_{\rm II}$ every solution of (1) is a solution of (4); viceversa every solution of (4) feasible to (1) is a solution of (1) (independent of the values of the penalty parameters).

Proof. Take $\rho_{\rm I}$ and $\rho_{\rm II}$ larger then the constant M in the definition of the GSBCQ. Suppose $(\bar{x}, \bar{y})$ is a solution of the GNEP. Then, by Proposition 1 $(\bar{x}, \bar{y})$ is also a solution of (4). Viceversa, assume that $(\bar{x}, \bar{y})$ is a solution of (4) feasible to (1). It is trivial to see then, that it is also a solution for (1). △

This result is somewhat weaker than the corresponding one in the case of constrained optimization, where in the second part there is no necessity to postulate the feasibility of the solution of (4) to conclude that the point is a solution of (1). However we do not believe that it is possible and, actually, sensible to expect such a result in the case of a GNEP. In the case of penalty functions for constrained optimization, in fact, a basic assumption is always that the optimization problem is feasible. In the case of GNEP, instead, we deal with (looking at the first player, for example) an infinite number of optimization problems, one for each possible y, and some of this problems can be expected to have no solution or even no feasible points.

Theorem 1 is certainly of interest and basically shows that a penalty approach for GNEPs has sound bases. In the next section we give a general algorithmic scheme and show, on the basis of Theorem 1, that this penalty algorithmic scheme can locate generalized stationary points.

3 Updating the penalty parameters

In general the correct value of the penalty parameters for which the solutions of the generalized Nash problem (1) and those of the Nash problem (4) coincide is not known in advance. Therefore, a strategy must be envisaged that allows to update the values of penalty parameter so that eventually the correct values are reached. In this section we show how this is possible in a broad algorithmic framework.

The aim of the penalization method is to transform the original problem into one that is easier to solve. It is clear that, in principle, (4) is easier than (1), even if the non differentiabilty of the players' objective functions is somewhat problematic, at least in practice. There exist methods to deal with nondifferentiable (4) and the equivalent VI-type reformulation. Furthermore, ad-hoc methods (such as smoothing methods, for example) could be developed to deal with the very structured nondifferentiability of the objective functions of (4). In this paper we do not go into these technical details. Our aim is, instead, to give a broad framework that is as general as possible to show the viability of the approach. To this end, we simply assume that we have a "reasonable" algorithm for the solution of the Nash problem (4). To be more precise we suppose that we have at our disposal an iterative algorithm $\mathcal{A}$ that, given a point (x^k, y^k), generates a new point $(x^{k+1}, y^{k+1}) = \mathcal{A}[(x^k, y^k)]$. We make the following absolutely natural and basic assumption on the algorithm $\mathcal{A}$.

Assumption 3.1 For every (x^0, y^0), the sequence $\{(x^k, y^k)\}$ obtained by iteratively applying the algorithm $\mathcal{A}$ is such that every of its limit points (if any) is a solution of (4).

It is clear that virtually every algorithm that can be said to "solve" (4) will satisfy this assumption. We can now consider the following algorithmic scheme. Below we denote by $\mathcal{F}(y)$ the feasible set of the first player for a given y and, similarly $\mathcal{F}(x)$ is the feasible region of the second player for a given x.

General penalty updating scheme

Algorithm. 2
Data: $(x^0, y^0) \in \mathbb{R}^{n_1+n_2}$, $\rho_{\rm I}, \rho_{\rm II} > 0$, $c_{\rm I}, c_{\rm II} \in (0,1)$. Set $k = 0$.
Step 1: If (x^k, y^k) is a solution of (1) STOP.
Step 2: If $x^k \not\in \mathcal{F}(y^k)$ and

$$\|\nabla_x \theta_{\rm I}(x^k, y^k)\| > c_{\rm I} \left[\rho_{\rm I} \left\| \nabla_x \|h_{\rm I}^+(x^k), g_{\rm I}^+(x^k, y^k)\| \right\|\right], \qquad (6)$$

then double $\rho_{\rm I}$ until (6) is not satisfied.
Step 3: If $y^k \not\in \mathcal{F}(x^k)$ and

$$\|\nabla_y \theta_{\rm II}(x^k, y^k)\| > c_{\rm II} \left[\rho_{\rm II} \left\| \nabla_y \|h_{\rm II}^+(y^k), g_{\rm II}^+(x^k, y^k)\| \right\|\right], \qquad (7)$$

then double $\rho_{\rm II}$ until (7) is not satisfied.
Step 4: Compute $(x^{k+1}, y^{k+1}) = \mathcal{A}[(x^k, y^k)]$; set $k \leftarrow k+1$ and go to step 1.

Note that if the perform the test (6) the point x^k is not feasible for the first player, so that $\|h_{\rm I}^+(x^k), g_{\rm I}^+(x^k, y^k)\| > 0$ and, since the norm is the Euclidean norm, the function $\|h_{\rm I}^+(\cdot), g_{\rm I}^+(\cdot, y^k)\|$ is continuously differentiable around x^k and the test (6) is well defined. Similar arguments can be made for the test at Step 3.

In what follows we assume that the stopping criterion at Step 1 is never satisfied and study the behavior of Algorithm 2.

Theorem 3. Let the sequence $\{(x^k, y^k)\}$ produced by the Algorithm 2 be bounded. If either $\rho_{\rm I}$ or $\rho_{\rm II}$ are updated an infinite number of times, then every limit point of the sequence $\{(x^k, y^k)\}$ is a generalized stationary point of the GNEP (1). If instead the penalty parameters $\rho_{\rm I}$ and $\rho_{\rm II}$ are updated only a finite number of times, then every limit point of the sequence $\{(x^k, y^k)\}$ is a solution of the GNEP (1).

Proof. Suppose the both penalty parameters are updated a finite number of times only. Therefore for k sufficiently large we are applying the algorithm $\mathcal{A}$

to problem (4) for fixed penalty parameters. We denote these fixed values by $\rho_{\rm I}$ and $\rho_{\rm II}$. Hence, by the assumption made on $\mathcal{A}$ we know that every limit point of the sequence $\{(x^k, y^k)\}$ is a solution of (4). Let $(\bar{x}, \bar{y})$ be such a limit point. We want to show that $(\bar{x}, \bar{y})$ is also a solution of (1). By Theorem 1 we only have to show that $(\bar{x}, \bar{y})$ is feasible for (1). Suppose this is not true and assume, without loss of generality, that the constraints of the first player are not satisfied at $(\bar{x}, \bar{y})$. Furthermore, since $(\bar{x}, \bar{y})$ is a solution of (4) and $\|h_{\rm I}^+(\bar{x}), g_{\rm I}^+(\bar{x}, \bar{y})\| > 0$, we can write

$$\nabla_x \theta_{\rm I}(\bar{x}, \bar{y}) + \rho_{\rm I} \nabla_x \|h_{\rm I}^+(\bar{x}), g_{\rm I}^+(\bar{x}, \bar{y})\| = 0,$$

from which we deduce

$$\|\nabla_x \theta_{\rm I}(\bar{x}, \bar{y})\| = \rho_{\rm I} \left\|\nabla_x \|h_{\rm I}^+(\bar{x}), g_{\rm I}^+(\bar{x}, \bar{y})\|\right\|.$$

But this, together with $c_{\rm I} < 1$ and some simple continuity arguments, shows that the tests at Step 2 must be satisfied eventually and $\rho_{\rm I}$ updated. This contradiction shows that $(\bar{x}, \bar{y})$ is feasible for both players.

Consider now the case in which at least one penalty parameter is updated an infinite number of times. Without loss of generality assume that it is $\rho_{\rm I}$ that is updated an infinite number of times and that the infinite subsequence of iterations where the updating occurs is K. If $\{(\bar{x}, \bar{y})\}$ is the limit of a subsequence of $\{(x^k, y^k)\}$ with $k \notin K$ we can reason as in the previous case and conclude that $\{(\bar{x}, \bar{y})\}$ is a solution of (1). Let us analyze then the case in which $\{(\bar{x}, \bar{y})\}$ is the limit of a subsequence of $\{(x^k, y^k)\}$ with $k \in K$. We have that the the sequence (x^k, y^k) is bounded by assumption and so is, by continuity, $\{\nabla_x \theta_{\rm I}(x^k, y^k)\}_K$. Therefore, since the test (6) is satisfied for every $k \in K$ and the penalty parameter goes to infinity on the subsequence K, we can conclude that

$$\left\{\left\| \nabla_x \|h_{\rm I}^+(x^k), g_{\rm I}^+(x^k, y^k)\| \right\|\right\}_K \to 0.$$

If $(\bar{x}, \bar{y})$ is infeasible we then have by continuity that $\nabla_x \|h_{\rm I}^+(\bar{x}), g_{\rm I}^+(\bar{x}, \bar{y})\| = 0$ and therefore, since $\|h_{\rm I}^+(x), g_{\rm I}^+(x, y)\|$ is convex in x (for a fixed y), this means that $\bar{x}$ is a global minimizer of the function $\|h^{\rm I}(\cdot)_+, g^{\rm I}(\cdot, y)_+\|$ with $\|h^{\rm I}(\bar{x})_+, g^{\rm I}(\bar{x}, \bar{y})_+\| > 0$.

If $(\bar{x}, \bar{y})$ is feasible, we have, taking into account that every x^k with $k \in K$ is infeasible for the first player, that

$$\nabla_x \|h_{\rm I}^+(x^k), g_{\rm I}^+(x^k, y^k)\| = \frac{\nabla_x h^{\rm I}(x^k)\, h_+^{\rm I}(x^k) + \nabla_x g^{\rm I}(x^k, y^k)\, g_+^{\rm I}(x^k, y^k)}{\sqrt{\|h_+^{\rm I}(x^k)\|^2 + \|g_+^{\rm I}(x^k, y^k)\|^2}}.$$

Passing to the limit for $k \to \infty$, $k \in K$, it is easy to check that $\bar{x}$ is a FJ point for the first player (when the second player chooses the strategy $\bar{y}$). It is now immediate to see, reasoning along similar lines, that also $\bar{y}$ must be either a solution or a FJ point for the second player or global minimizer of the function $\|h^{\mathrm{II}}(\cdot)_+, g^{\mathrm{II}}(x,\cdot)_+\|$ with $\|h^{\mathrm{I}}(\bar{x})_+, g^{\mathrm{I}}(\bar{x},\bar{y})_+\| > 0$. Hence we conclude that in any case $(\bar{x},\bar{y})$ is a generalized stationary point of the GNEP (1). $\triangle$

It should be clear that there is no need to perform the tests at steps 2 and 3 for every k. It is enough that they are performed an infinite number of times. Also, if the updating test, say that at Step 2, is passed, it is sufficient to take the new ρ_{I} larger than

$$2\frac{\|\nabla_x \theta_{\mathrm{I}}(\bar{x},\bar{y})\|}{\|\nabla_x\|h_{\mathrm{I}}^+(\bar{x}), g_{\mathrm{I}}^+(\bar{x},\bar{y})\|\|} + 1.$$

Actually any updating rule for the penalty parameter ρ_{I} will be acceptable as far as it is guaranteed that if ρ_{I} is updated an infinite number of times then it grows to infinity. We leave the details to the reader and discuss instead the more interesting issue of whether it is possible to guarantee that every limit point is a solution and not just a generalized stationary point of the GNEP (1). Note that in Theorem 3 we did not make any regularity assumption on the problem; correspondingly, and quite naturally, we could prove convergence to generalized stationary points and not to solutions. However, Theorem 3 makes clear that convergence to generalized stationary points that are not solutions can only occur if a(t least one) penalty parameter goes to infinity. In turn, the proof of Theorem 3 shows that if this occurs, then we can find a sequence $\{x^k, y^k\}$ of infeasible points such that either $\{\nabla_x\|h_{\mathrm{I}}^+(x^k), g_{\mathrm{I}}^+(x^k,y^k)\|\}$ or $\{\nabla_y\|h_{\mathrm{II}}^+(y^k), g_{\mathrm{II}}^+(x^k,y^k)\|\}$ tend to zero. The following corollary then easily follows from the above considerations.

Corollary 1. Let the sequence $\{(x^k,y^k)\}$ produced by the algorithm 2 belong to a bounded set $\mathcal{B}$. Suppose that there exists a positive constant σ such that, for every infeasible point $(x,y) \in \mathcal{B}$,

$$\|\nabla_x\|h_{\mathrm{I}}^+(x^k), g_{\mathrm{I}}^+(x^k,y^k)\|\| \geq \sigma, \qquad \|\nabla_x\|h_{\mathrm{I}}^+(x^k), g_{\mathrm{I}}^+(x^k,y^k)\|\| \geq \sigma. \quad (8)$$

Then ρ_{I} or ρ_{II} are updated an finite number of times and every limit point of the sequence $\{(x^k,y^k)\}$ is a solution of the GNEP (1). $\triangle$

In the case of one player (i.e. in the case of optimization problems) the condition (8) has been used and analyzed in detail, see [5–8]. Basically this condition can be viewed as a sort of generalization of the Mangasarian-Fromovitz CQ. Its practical meaning is rather obvious: the functions $\|h_{\mathrm{I}}^+(x), g_{\mathrm{I}}^+(x,y)\|$ and $\|h_{\mathrm{II}}^+(y), g_{\mathrm{II}}^+(x,y)\|$ which represent the violation of the constraints of the first and second player respectively, must not have stationary points outside the feasible set. This condition seems very natural and says that the "feasibility" problem which is a "part" of the generalized Nash equilibrium problem is easy (in the sense that the only stationary points of the functions representing the violation of the constraints are the global minima). From this condition we could derive several sets of sufficient conditions for Corollary 1 to hold, along the lines developed in [5–8]. We leave this for future research.

4 Conclusions

In this paper we proposed the notion of generalized stationary point for the Generalized Nash Equilibrium Problem and argued that this is an appropriate and realistic target for any numerical solution method. Furthermore we introduced an exact penalization method for the solution of the GNEP. We gave a broad algorithmic scheme and showed that this scheme is able to generate sequences converging to a generalized stationary point under a mere boundedness assumption. Finally we also discussed an additional regularity condition that guarantees convergence to solutions (as opposed to generalized stationarity points). There are certainly still many issues that deserve more study, prominent among these an effective solution procedure for the nondifferentiable (unconstrained) problem arising from the application of the exact penalty approach. It certainly was not our intention to investigate all the issues connected to a penalization approach to the solution of a GNEP. However, we remark that, given the lack of results in the study of GNEPs, we believe that the approach proposed in this paper could not only be useful from the numerical point of view, but also lead to new sensitivity and stability results.

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