Chapter 3-2

PROPOSAL FOR A FRAMEWORK TO ANALYSE MATHEMATICS EDUCATION IN EASTERN AND WESTERN TRADITIONS

Looking at England, France, Germany and Japan

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1. INTRODUCTION

Empirical surveys within the framework of large international comparison studies, such as the TIMSS accompanying "Survey of Mathematics and Science Opportunities" (SMSO), are indicating a strong cultural determination of lessons, even in subjects like mathematics and science. Thus the authors of the SMSO state:

> "Mathematics and science, unlike culturally embedded subjects such as history and language, are often thought to be a-cultural. For example, many believe 'numeration is numeration' – the concept is the same across all contexts. ... One can argue that if there is something universal about mathematics and science content, there should be something universal about the way this content is presented to students. Our results, of course, suggest that this second assumption needs re-evaluating ... Countries have developed their own ways of engaging students in the substance of mathematics and science. There appear to be strong cultural components, even national ideologies, in the teaching of these subjects." (Schmidt et al. 1996, p.132)

Following their analyses French teaching of mathematics emphasises formal knowledge, because French mathematics teachers identify themselves strongly with their disciplinary counterparts at universities. US educators appear bent on prolonging childhood as long as possible, at least as evidenced by the tendency to exposure to more basic and early-introduced topics in mathematics and science well into lower secondary school level.

Pepin (1997) reveals in her analyses also a strong dependence on cultural traditions for mathematics teaching in England, Germany and France. She states that of course it is interesting and exciting to read descriptions of mathematics teaching in other countries, but the aim of comparative education must be to find explanations for the observed differences and similarities in order to benefit from comparing teaching and learning mathematics in different countries. Thus she concludes that comparative education in mathematics should answer the following questions:

> "How can we understand teachers' practices in the light of what we see and experience. How can we understand teachers' practices in the light of what we know about the different countries? If we believe that the teaching and learning of mathematics is 'culturally embedded', what are the cultural and intellectual underpinnings that influence the teaching and learning of mathematics? Where do the cultural and educational traditions stem from, and how do they feed into the classroom? These and more questions have to be posed and answered, if we want to benefit from comparing teaching and learning mathematics in different countries." (Pepin 2002, p. 246)

These questions have not been answered yet, neither in older nor in recent comparative studies, but seem to be an unalterable step for the development of an educational theory on comparative education. They show the direction in which theory development in comparative education should move in order to be successful in explaining cultural differences in education and proposing effective measures for change.

In the following we will describe a proposal that permits a description or even a classification of mathematics teaching in different countries, in Eastern and Western countries as well, within a broader framework. By means of this proposal for a classification system questions about the reasons for the origins and philosophical bases of the differences in mathematics teaching in various countries will not yet be answered. However, it helps to recognise differences and similarities in mathematics teaching of different countries. Based on further studies this classification system might help to clarify how far these differences are influenced by educational philosophies or external economical and social aspects. Within the framework of a comparative study on German and English mathematics teaching we have set forth exemplarily such an approach (see Kaiser, 2002). The study describes the educational conceptions and philosophies that have been developed during the last centuries in England and Germany and shows their impact on the national approaches to teaching mathematics. Such classificatory attempts for educational systems have a long tradition in comparative education and seem suitable to be continued as a reference for the above mentioned developments (for an overview of such classifications see Holmes 1981).

The framework proposed in this paper focuses on central orientations of mathematics teaching on which a comparison of mathematics teaching can be based. This means concretely that the suggested classification system characterises mathematics teaching according to its epistemic orientation – i.e. the position of the subject's structure and the role of mathematical theory – and the arising aspects that contribute to the construction of an understanding of mathematics. These aspects are the lessons' structures and the approaches to mathematics, the position of formulae and algorithms or calculations, the relevance of the introduction of new concepts, the role of proof, the importance of mathematical language, the consideration of real world aspects and the central teaching-and-learning styles.

The classification system developed here is based on empirical research of the comparative study of mathematics teaching in England and Germany (see Kaiser 1999). For this comparison, mathematics teaching of both countries has been described as idealised types which were developed from observed empirical phenomena but do not exist in this pure way in reality (for details concerning this 'ideal typus' approach see section 2). Furthermore, mathematics teaching of both countries has been described through contrasting characterisations. In another study dealing with mathematics teaching in France and Germany characteristics of mathematics teaching are developed through focusing on the teaching of proofs (see Knipping 2003). A comparison of the results of these two studies makes clear that the polarisation of German and English mathematics teaching is not suitable anymore if regarded from a new perspective: concerning the understanding of mathematics, English and French mathematics must be regarded much more as poles, while position of German mathematics teaching and its orientations is positioned "between" the two countries.

If the perspective of this empirically grounded theory is extended, we must inevitably ask whether the comparative framework is restricted to these three countries or can also be applied to other countries; whether this framework is euro-centric or whether it is appropriate to be used to analyse mathematics teaching of Eastern traditions. These aspects were investigated in a subsequent study which deals with the question of how far the developed characterisations can be used to describe Japanese mathematics teaching. For logistical reasons no classroom observation could be done for this study. Thus, it is based on curricula and textbook analyses, and additional questioning of teachers and discussions with experts were performed. Because the proposed framework is a kind of 'meta-study', different types of data are acceptable, even if the developed descriptions tend to be less close to lessons taught. Further this lower proximity to teaching reality seems not to produce serious problems, because the questioning of the teachers demonstrates how strongly mathematics teaching in Japan is orientated towards textbooks.

2. METHODOLOGICAL APPROACH

The following description of the analysis framework for mathematics teaching in various countries can be described as empirical grounded theory, consisting of a meta-analysis of three studies, which have been carried out successively building on each other.

The first study comparing English and German mathematics teaching (Kaiser 1999) is an ethnographic study embedded in a qualitatively oriented paradigm of the social sciences. In the following, features of the ethnographical approach are briefly described, before details of the concrete methodical procedure are given. Ethnographical methods have been developed from the method of participant observation, which has a long research tradition in social anthropology and ethnology. Participant observation has been understood as a flexible contextualised strategy, which comprises multiple methods. Ethnographical studies are aimed at a description of a social context, in which people live and work. The main effort is to evaluate how different social realities are constructed, i.e. how the situation-related means are used by the actors, within a social situation, in order to construct social phenomena from a participating perspective.

The following three aspects can be formulated as characteristics of ethnographical research: Central is the long-term presence in the research field in order to assume an inside perspective. The process of diving into the research field can be described as process of partial enculturation (Amann & Hirschauer 1997). The second characteristic is the flexible research strategy, i.e. the researcher has to adapt his or her methodical approach to the situation and has to find a balance between the research interest and the requirements of the situation. In order to study the culture of the participants before producing explanations for their behaviour, participant observation and relatively unstructured interviews are the main ethnographical research methods. Formalisations and standardisations of the research procedures are therefore not adequate; by contrast, the methods consider that researcher and research actions are part of the cultural environment that is examined (Hammersley & Atkinson 1983). The third characteristic is ethnographical writing, which is centrally based on detailed field-notes taken during the observations. Field-notes, which are either made on the spot or written up as soon as possible after leaving the field, have to be seen as interpreted reconstructions of the observations. The question how to evaluate ethnographical data, in order to fulfil necessary scientific standards, has therefore become the focus of interest in the discussion of the last few years.

Due to its focus on descriptions of real life and the construction of social phenomena, the ethnographical research approach seems to be especially adequate for the evaluation of mathematics education in England and Germany, its constituents and its determining patterns. Especially bequeathed educational philosophies, which influence the actions of the participants in the educational field significantly, are well known – sometimes even unconsciously – by all actors and are therefore only seldom made explicit. The view into another culture gives us insight into our own teaching culture and the determining constituents. The method of participant observation with its detailed field-notes allows a diving into the field, which is not possible with technically more ambitious research methods. A central basis of this study is, apart from the field-notes of the classroom observations, discussions with teachers, in the staff room during lunch, or after the classroom visits, the participation in school assemblies and discussions with pupils after lesson. Only for the analyses of the teaching-and-learning process, which needed verbatim statements, were audio-tape records made.

In the following we will describe a few more technical details of the study. The study included 17 different schools in England, of which 14 were state-run comprehensive schools and 3 were private schools with selective character. Two of the state schools were grant-maintained. The 14 state schools were comprehensive schools, except for one Grammar School, and 4 of the 14 schools were single-sex schools. The schools were spread all over England. The study is limited to the English school system, as the school systems and the educational philosophies in Scotland and Ireland are quite different from the English ones. As already mentioned, the study relied heavily on classroom observations, apart from the participation into school life, especially in England. 242 lessons were observed from Year 6 to A-level, mainly restricted to Years 9 to 11. In Germany, schools from the three-tier system were included as well as comprehensive schools of different types (using streaming or setting systems). 6 of the schools were situated in the Federal state of Hessen, the others from various regions spread over Germany. 102 lessons were observed from Year 8 to Year 10. The study focussed in both countries on age groups at the end of lower secondary level. Concerning the achievement level of the pupils included, the study put its emphasis on the two higher tiers of the German school system or on the top sets in the English school system. The reason for this choice points to a major problem, well known in comparative qualitative studies: Many teachers of both countries were hesitant about opening the classroom with lower achievement students for observations by a visitor. The classroom observations were mainly carried out from 1990-95. Further research has shown that mathematics teaching in Germany has not changed in a significant manner since then, despite the TIMSS shock and political claims of the necessity for change. The English mathematics teaching has undergone a relevant change since the beginning of the nineties of the last century, especially concerning teaching-and-learning methods and the relevance of the subject structure through the introduction of the National curriculum and the accompanying key stage tests. These changes became visible already during the study and are covered by the classroom observations. Newer change is mainly focussing on primary education, which this study does not deal with.

In general, the study aims – as already stated – at generating general knowledge, based on which pedagogical phenomena might be interpreted and partly explained. Under a narrower perspective, the study aims to generate qualitative hypotheses on the differences between teaching mathematics under the educational systems in England and in Germany. Due to the use of the ethnographical method, the study cannot make any 'lawlike' statements; in contrast, the study refers to the approach of the 'ideal typus' developed by Max Weber (*Webersche Idealtypen*), and describes idealised types of mathematics teaching reconstructed from the classroom observations in England and Germany. That means that typical aspects of mathematics teaching are reconstructed on the basis of the whole qualitative studies rather than on one existing empirical case. The 'ideal typus' does not really describe empirical phenomena, it is constructed by overemphasising typical issues of single phenomena observed and by a combination of different phenomena (for details see Hempel 1971, Weber 1904).

In the second study, comparing French and German mathematics teaching, proving processes in class are described and contrasted (Knipping 2003). Comparative analyses of the processes observed in class illustrate different proving practices. These analyses reveal different functions and roles of proving in mathematics teaching. In addition, the analyses show on the one hand that mathematical theory, mathematical concepts and language have a different status in teaching and on the other hand that real-world problems are important in German teaching, while they are not important in French lessons.

French curricula apply nationally, so all classes in the *Collège* are intended to study the same material. The decision to carry out investigations in different *Collèges* in the Paris region has not resulted in a special sample, with the exception of 2 bilingual classes. In contrast, substantial differences in the topic emphasis of the German curricula can be found not only on a regional level, but also, and more so, among the different school types. While a special value is given to proofs nationally in the *Gymnasium* curricula, in the curricula of comprehensive schools proofs clearly play an inferior role. In the curricula of comprehensive schools this different valuing of proofs is usually reflected in different targets for courses of the upper and lower sets. Analyses of the curricula suggested that it would be difficult to observe proofs and proving processes outside the *Gymnasium* and perhaps the upper sets in comprehensive schools. Consultations with teachers confirmed this, and so, early in the research, the decision was made to choose German classes selectively. It was decided to examine classes in both the *Gymnasium* and the upper sets in comprehensive schools in case there were differences in their classroom proving processes.

The empirical investigation involved proving processes in 6 French and 6 German classes. The data collection was carried out at 6 *Collèges* in the Paris region and 3 *Gymnasien* and 2 comprehensive schools in Hamburg. Two of the observed classes in France are classes in a bilingual stream and are highly selective. French and German curricula, which have been analysed before the beginning of the data collection, list proofs as an explicit topic in geometry for the first time in grade 8. For this reason instructional observations were done in geometry classes at level 8/9 (13-14 year old students). The instructional units were selected according to curricular criteria and cover topics in geometry, including the Pythagorean Theorem. The observations were documented with audio-tape recordings and photos of figures and writing at the blackboard. In addition, observations were recorded in the form of process notes which were made after each session. The tape recordings were transcribed to make detailed analysis of the classroom discourses during the proving processes possible, which was in particular necessary for the reconstruction of argumentations.

In contrast to the first study, the analyses and the comparison of the data were structured by theoretical considerations based on research in the field of proof and argumentation. Analyses of the classroom processes were carried out based on historical and philosophical work (Lakatos 1976; Jahnke 1978; Rav 1999), and research in the field of argumentation (Krummheuer & Brandt 2001), in particular the functional analysis of arguments exposed in the Toulmin model (Toulmin 1958).

Based on Max Weber's methodological concept of the *ideal type,* ideal-typical characterisations of proving processes were developed by comparing processes both on the level of context analyses and on the level of argumentation analyses, with the aim of developing a typology. This involved comparative analyses of all observed episodes "*from an initial interpretation of those episodes to a later theoretical exploration of those episodes*" (Krummheuer & Brandt 2001, p. 78). Prototypical cases or *prototypes* form the basis for the construction of these ideal-typical characterisations of proving processes. A prototype is a case *"in the sense of a concrete model"* (Zerssen 1973, p. 53), not an ideal type, i.e. not an ideally formed theoretical construct. Rather it is a case that can apply to a group as representative in the sense that through it special characteristics of a group of cases become clear (Kluge 1999). Descriptive typical characteristics can be worked out by the characterisation of the prototype. Singling out prototypes forms an intermediate step in the process of constructing ideal types. The comparison of prototypes with further cases is also crucial here. In the light of other cases, typical features become clearer in contrast to individual characteristics. The ideal-typical characterisations developed in this way have a heuristic function, because *"the pure type contains a hypothesis of a possible occurrence"* (Gerhardt 1991, p. 437). The cases discussed below represent prototypes in this sense.

The comparison of prototypical cases has also been an important element of the meta-analyses presented here. These analyses made it possible to specify ideal-typical characterisations of English, French and German mathematics teaching, which are presented here as a proposal for a framework to analyse mathematics teaching. The bi-polar characterisations developed in the two studies have been re-analysed and used to characterise more precisely different poles in mathematics teaching, with respect to the status of mathematical theory, the introduction of mathematical concepts and methods, the position and function of proofs, the role of justifications and examples, the status of precise language and the role of real-world examples.

After developing descriptions of mathematics teaching in England, France and Germany based on these two studies, the framework of a third study was carried out in order to clarify whether this framework would make sense for an analysis of Japanese mathematics teaching. This third study (by Hino) focussed on mathematics education in public schools at lower secondary level (Year 7–Year 9). First, a tentative description was developed based on formal documents such as the National Course of Study and on results of international comparative studies such as the TIMSS-Video-Studies, in combination with discussions with several mathematics educators and an analysis of 6 widespread, common mathematics textbooks. This description was modified and confirmed by the results of a questionnaire carried out with 51 Japanese mathematics teachers at Nara city. In the questionnaire, two aspects of mathematics teaching were studied: the teachers' dependency on textbooks in their daily teaching practice and the relevance of mathematical theory and related issues covered in the framework.

With this approach we got, as already mentioned, two different kinds of data in our analyses of the four countries: data from classroom observations in England, France and Germany and another kind of data referring to recently published descriptions of Japanese teaching. The reason for not carrying out classroom observations was mainly time and capacity restrictions. We consider the description of the Japanese mathematics teaching as highly reliable and close to classroom reality due to the following reasons: The discussions with professors for mathematics education served as expert discussions, because Japanese university professors visit classrooms and observe lessons occasionally as a process of lesson study (see Stigler & Hiebert 1999). They usually have an adequate image of the current trends of classroom teaching. The other reason is the validity of textbook analyses as a means of comparison. In Japan, the content of a textbook is strictly determined by the National Course of Study and authorised by the Ministry of Education. Moreover, there have been data of international comparison such as TIMSS (see National Institute for Educational Research 1998) and OECD/CERI (Shigematsu 1998) that show repeatedly Japanese mathematics teachers' strong dependency on the textbooks in their teaching. The results of the questionnaire confirm this aspect pointing out that the teachers usually rely on textbooks in every major occasion of their teaching: i.e. entering new textbook chapters, introducing new mathematical concepts and procedures, consolidating and summarising the learned content. Therefore, the study of 6 textbooks seemed to be an appropriate means in order to gain insight into mathematics teaching in Japan, especially concerning the relevance and status of mathematical theory and related issues.

3. ANALYSES WITH THE PROPOSED FRAMEWORK

The idealised description of English and German mathematics teaching developed in the first study consisted of polarised descriptions in order to clarify the distinctions made. In the light of the second study – the comparison of French and German mathematics teaching by Knipping (2003) – these descriptions had to be qualified. In an overall description of European educational approaches in mathematics, France and England might be seen as diametrically opposed to each other with German conceptions having an intermediate position. This holds especially with the aspect of understanding mathematical theory. Until now there exist only first attempts for the development of such a frame. The three-country-study of Pepin (1997) covering France, Germany and England limits itself to the perspective of the teacher and does therefore not provide such a frame, but may be taken as an empirical basis for further research.

In the following we will start each aspect by a short description of the characteristics of the two polar mathematics educations, usually England and France, followed by a description of the place of the German and the Japanese mathematics education.

3.1 Understanding of mathematical theory – scientific knowledge versus pragmatic understanding

Two contrasting characteristics of French and English teaching can be reconstructed as contrasting poles relating to the understanding of mathematical theory. Thus **French mathematics** teaching can be described by the ideal type characteristic of a scientific understanding of theory, this means that theoretical mathematical considerations are of great importance. Generally speaking, mathematics teaching in France is characterised by its focus on the subject structure of school mathematics ("savoir enseigné"). This means that theory is made explicit by means of concepts, theorems and formulae.

From an ideal type perspective, **English** mathematics teaching can be described by its pragmatic understanding of theory, which means a practical and purpose-depending handling of theory. Differences between the comprehensive school, the dominant kind of school, and the selective school system, which for the most part consists of private schools, could be recognised. These fundamentally different orientations of French and English mathematics teaching on a level of understanding of theory can be seen from various aspects, such as the introduction of new concepts, the meaning of proof, importance of rules or precise mathematical language, which will be described in the following.

German mathematics teaching is characterised by its focus on the subject structure of mathematics and on mathematical theory. Theoretical reflections emphasising the mathematical subject structure play a dominant role in the higher type of the tripartite school system, still prevailing in Germany, but are of less importance in the other columns of the school system. Theory is often reduced to rules and algorithms, especially in the two lower types of the tripartite school system, subject-related reflections play a more important role in the higher type of the tripartite school system, but are often restricted to remarks by the teacher or remarks in the textbooks.

In **Japan**, the National Course of Study states objectives and content of school mathematics, which emphasises the subject structure of school mathematics. Mathematical theory is made explicit by means of concepts, formulae and theorems and also by means of rules and algorithms although, in the practice of teaching, teachers treat mathematical theory in the classroom in the context of problem solving activities, i.e. teachers spend substantial time on a small number of selected problems. In the current Course of Study (valid since 2002), "mathematical activity" is considered as an important way of learning mathematics. In describing the new Course of Study, Nemoto (1999) states:

> "... for the purpose of making connections with daily life, fostering students' ability of investigating phenomena mathematically and heightening their ability of solving problems by using mathematical ways of viewing and thinking, we tried that students can engage actively in mathematical activities such as finding relationships and rules in the phenomena by means of observations, manipulation and experimentations and reflecting on and thinking of the results once they have reached." (p. 100. original in Japanese, translation by Hino)

3.2 Organisation by subject structure versus spiral-type curriculum

The characteristic of **French mathematics teaching,** a subject-based understanding of theory, leads to a curriculum whose lessons go along with the subject structure of mathematics guided by didactical considerations. In lessons mathematical concepts and methods are taught in a subject-scheduled order as prescribed in the national curriculum. Lessons start from general concepts and rules, and then proceed with special conclusions and applications. The subject-based understanding of theory is given shape also by the great importance of mathematical theorems. The great importance of theorems becomes more obvious in the topic areas of geometry, where the relevance and structure of mathematical theory often shall be demonstrated exemplarily. The units are complete in themselves, but connect subjects including geometric and algebraic issues.

The characteristic of **English mathematics teaching,** which is a pragmatic understanding of theory, is apparent from the spiral-shaped structure of mathematics lessons and curricula, which means that mathematical concepts and methods are introduced quite early but on a more elementary level. Later, in higher classes, they are picked up again. This spiral-shaped approach implies that smaller and easily comprehensible topic areas are discussed, which are not taught in a subject-oriented structure. A fast switching from one topic to another is typical for English mathematics lessons. Sometimes even several topics are dealt with at the same time. Altogether, English mathematics teaching is rarely based on a subject-based systematic. The subject structure of the National Curriculum, which has been obligatory since the beginning of the nineties of the twentieth century, did not lead to subject-structured lessons. As the curricular goals in the National

Curriculum are strongly individual based, there does not exist any obligatory canon of knowledge, to which teachers could refer to for continuing a topic as scheduled in the spiral-shaped curriculum.

This pragmatic understanding of theory, which does not put the subject structure to the foreground, corresponds with the minor emphasis on mathematical formulae, rules and theorems, because the creation of mathematical tools is regarded as being more important than structural analyses. Therefore, theorems like Pythagoras' theorem, which play a central role in a subject-structured curriculum, are called *patterns* in English teaching or they are not taught at all. Theorems and their meaning are not the focus of interest, but rather the constructive aspect of geometrical contents and the algorithmic function of algebraic contents (formulated as rules and formulae) in connection with problem solving.

The focus on theory when teaching mathematics in **Germany** implies a lesson structure that goes along with the subject structure of mathematics. Mathematical theorems, rules and formulae are therefore of high importance. That varies though, with the different kinds of school of the tripartite school system. Bigger coherent topic areas are taught (lasting sometimes months), e.g. fractions, percentages, Pythagoras' theorem and others. These big thematic fields are taught independently of relations to other topic areas, and are later on not referred to again.

In **Japanese** mathematics teaching the mathematics content is classified into three areas, "number and algebraic expressions" "geometrical figures" and "mathematical relations". These areas are located in each grade and taught alternatively. The Course of Study states objectives at each grade level. Not only content but also thinking and interest are considered important. For the objectives of thinking, four levels are distinguished: knowing, understanding, processing, utilising. New trials such as problem situation learning, election of special topics of mathematics, or integrated learning are carried out. Current controversies are focused on what the power and ability is that students should acquire through school learning so as to be useful for their future lives, and how to foster such power and ability. The teaching units are somehow between the French and the English approach covering about 3-5 weeks.

3.3 Introduction of new mathematical concepts and methods

Concerning this aspect, the classification of the various educational systems is different from above, e.g. that the German and the English approach form the poles of the description.

The subject-based understanding of theory of **German mathematics teaching** leads to the high importance of the introduction of new mathematical concepts and the deduction of new methods. Normally, this is planned carefully by the teachers or they refer to detailed introductions from the text book. Often mathematical concepts and methods are illustrated by real-world examples, although the real-world examples depend on their purpose, they often appear artificial and far from real life. Partly, the introduction of concepts refers to basic understandings as representatives of the mathematical "nucleus" of a concept. The introduction of new mathematical concepts is usually done by class discussion, in which the whole learning group participates under the guidance of the teacher. There exist various kinds of teacher guidance. A characteristic of the course of a lesson is that the newly introduced concepts or methods are formulated in detail by phrases or notes on the blackboard, which then is followed by exercises.

Below we will give an example that demonstrates the high value of the introduction of concepts and the connection with exclusively formally meant basic imaginations and the application of introductory real-world examples. This sequence, about the introduction of the concept function, was observed by Kaiser in a *Realschule* of Year eight:

> It starts with a graph about the development of temperature over 24 hours on a worksheet distributed by the teacher to the pupils. The teacher writes on the blackboard "function". Then he asks what the task on the worksheet means and gives the answer by himself by the fact that he writes:
> The connection between time and temperature.
> time→ temperature
> At first various temperatures for given times are determined together by the pupils in a discussion, then the times are noted in a table. After it has been clarified at what time there is the highest and the lowest temperature, the teacher asks: How many temperatures could we declare for one time? A girl answers: one. The teacher comments on this: This we keep in mind. He writes on the blackboard:
> At each time there is only one declaration of temperature.
> He states that for this there is a specific name in mathematics and writes:
> The assignment time→ temperature is defined exactly.
> He continues that this is to be completed by the name which is already written as a headline on the blackboard, and then he writes:
> Exactly defined relations are called functions.
> Then various contrasting examples are discussed in detail. The teacher introduces the notation form of a function and the word equation of a function; then further examples are discussed with all pupils.

The pragmatic understanding of theory of the **English mathematics teaching** influences the importance given to the introduction of new mathematical concepts and methods: Thus English mathematics teaching is characterised by a low importance of the introduction of concepts and methods. This is generally done pragmatically, and often the concepts or methods are given by the teacher just as information or in the style of a recipe. Content-related information is replaced by referring to calculators or mnemonics. Especially new mathematical methods often are explained and demonstrated through experiments, by drawing and measuring. New mathematical concepts normally are not introduced explicitly but implicitly, in connection with exercise sequences. This corresponds with the fact that the introduction of concepts and methods is done in short class discussion sequences or individually with the textbook.

What follows is a description of a sequence of a lesson from English mathematics teaching observed in Year 9 of the top set at a comprehensive school, which deals with the introduction of sine and cosine in right angled triangles. The lesson demonstrates the usage of calculators instead of a content-based understanding:

> During the first lesson the tangent is introduced as follows: Pupils draw individually various right angled triangles with the same angle X, then the length of the two short sides of a rectangular triangle and their ratio are determined from the drawing. Then, in a discussion guided by the teacher, it is clarified that the ratio of the two short sides of each right triangle with equal angle is always stable. The teacher instructs the pupils to enter the angle X into their calculator, then to press the tan key. The pupils recognise that the already calculated ratio is the same as the X value of tan. In this way the tangent of an angle is defined as the ratio of the opposite and adjacent sides. Subsequently the pupils worked individually on further examples.
>
> The next lesson continued with a methodically similar introduction of sine and cosine done by individual work. At the end of the lesson a wide variety among the pupils becomes obvious: Some of them are still busy with the exercises about the tangent, others have already started with further exercises about sine and cosine.

In **France**, as in Germany, both the introduction of new concepts and the deduction of new mathematical methods are of high importance for the teaching of mathematics. They are usually either well prepared by the teacher or follow detailed introductions given in the textbooks. In contrast to German teaching, new mathematical concepts and methods are not motivated by real-world examples, but sometimes prepared by problem situations, with the aim to revise former knowledge. The introduction of new mathematical concepts and methods usually takes place during short periods of class discussion, followed by sophisticated exercises that are supposed to deepen the understanding. The whole teaching is centred towards exercises and individual work of students, while mathematical concepts and methods, which are formulated in the form of definitions or theorems and written down on the blackboard, are equally important.

Within proving processes mathematical concepts and methods are revised and made more precise. Thus proving in class gives value and importance to mathematical concepts and methods. The following example of a French lesson illustrates this.

> In this lesson an arithmetical proof for the Pythagorean Theorem is developed. In class the students and the teacher give detailed reasons why the inner quadrilateral in the proof figure is a square, and finally they note these reasons at the blackboard. As a first step they justify that the inner figure is a rhombus. Based on this conclusion they look for a right angle in the figure and for reasons why BCD is a right angle. In this way they finally conclude that the inner figure is a square and write down together the following proof.

• As ABCD has four sides of the same length, it is a rhombus	• Comme ABCD a quatre côtés de même longueur, c'est un losange.
• The acute angles of a right triangle are complementary.	• Les angles aigus d'un triangle rectangle sont complémentaires.
• The right triangles DHC and BGC are superposable, their corresponding angles are equal.	• Les triangles rectangles DHC et BGC sont superposables, leurs angles sont deux à deux égaux.
• We can deduce that the angles $\angle$ BCG and $\angle$ DCH are complementary.	• On en déduit les angles $\angle$ BCG et $\angle$ DCH sont complémentaires.
• $\angle$ HCG=180° so angle $\angle$ BCD=180 – 90= 90°.	• $\angle$ HCG=180° d'ou l'angle $\angle$ BCD=180 – 90= 90°.
• ABCD is a square.	• ABCD est un carré.

Writing down proofs and solutions of problems underlines the importance of a precise use of terminology and reasoning in class. Explicit and accurate proofs are highly valued in French teaching. In this way the mathematically correct application of concepts and terminology is fostered.

In **Japan**, introduction of new mathematical concepts and methods are of high importance for the teaching of mathematics. They are usually well prepared by the teacher or follow detailed introductions given in the textbooks. New mathematical concepts and methods are often motivated by real-world examples. In textbooks, each chapter often has opening pages that illustrate real-world examples, which is to initiate students to the basic ideas.

In introducing concepts and methods, real-world examples (which are quite often rather artificial) are used to foster students' interest and motivation toward thinking. In class, introduction of new mathematical concepts and methods takes place in a problem-solving situation. It is also the case that the teacher explains the new concepts and methods. Class discussion, during which newly introduced concepts and methods are formulated, is desired but not easily realised. Introduction is followed by exercises. The number of exercises in the textbook is not large. Teachers often allocate students additional workbooks to do more exercises.

When looking at the 6 textbooks in the chapter of linear function (Year 8), all of them started with at least a few situations that contain one or several linear (non-linear) relationships. Rather than just giving detailed instructions, students are asked to do some work along with key questions that initiate them to the basic idea, that is to look at the phenomenon from the perspective of "changing quantities."

> One of the textbooks (Chuugaku suugaku, 2 [Mathematics in lower secondary school 2]. p. 41, published by Osaka-shoseki, 2002) contained the following situation: Water is poured with the speed of 5 cm height per minute in a box-shaped tank with the height of 50 cm that already contains water up to the height of 10 cm. The description is accompanied by the following table:

TIME (MIN)	0	1	2	3	4	5	...
HEIGHT (CM)	10						

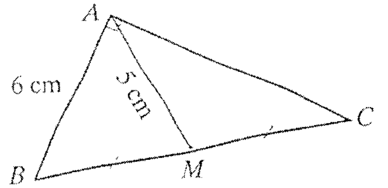
> Further, the task contains an illustration with three students talking with each other. In bubbles is written: "Let's look at the relationship between the time spent for pouring water and the height," "I wonder if the relationship would be somewhat different from proportion" and "When will the tank be filled up with water?" Then there is a goal statement "Let's learn the relationship between two quantities that vary together with a fixed rate."

In the questionnaire, 51% of the teachers confirmed that they sometimes ask students to do investigative work with real-world examples in order to initiate them to the basic ideas of linear function (33% said that they do so occasionally). 86% of the teachers reported that they make connections between new mathematical concepts (e.g. linear function) with the concepts the students have already learned (e.g. direct proportions). This result shows the relevance of establishing connections between older and newer concepts and methods within the teaching process.

3.4 The position and function of proofs

The understanding of theory in **French** mathematics teaching can be seen from the strong emphasis put on proof. Proofs are considered important for introducing students into a theoretical understanding of mathematics and for developing their skills in mathematical argumentation. Especially in the context of geometry, proofs are studied and students have to carry out proof problems by themselves. The public justification of mathematical relations and facts is the main function of mathematical proving processes in class. The knowledge of the class, i.e. the concepts, theorems and methods already studied in class that build public accepted knowledge, is extended by new knowledge that is first justified before it becomes part of the accepted public knowledge of the class. Justifying a new theorem means going back to theorems, definitions and methods that are already accepted as public knowledge in class. Justifications, in the form of discussions in class and written texts at the blackboard, that form a discursive culture in class characterise this type of proving process. In problem solving not the solution itself, but the justification of the solution by tracking it back to publicly accepted knowledge, is of primary importance. Written proofs are also a model for justifying solutions of problems. The following example of a French lesson illustrates this.

> In an exercise students have to calculate two sides of a right angled triangle, given one side of the triangle and the length of the median. In order to solve the problem the Pythagorean Theorem has to be applied as well as the circumcircle theorem. The new theorem is explicitly connected with knowledge that was already studied in class and so is inscribed into the knowledge of the class. Writings at the blackboard foster this inscription.



Calculer les longueurs BC et AC.

Hachette, Cinq sur Cinq 1998, p. 161, n° 29

• As ABCD has four sides of the same length, it is a rhombus • The acute angles of a right triangle are complementary. • The right triangles DHC and BGC are superposable, their corresponding angles are equal. • We can deduce that the angles ∠ BCG and ∠ DCH are complementary. • ∠ HCG=180° so angle ∠ BCD=180 – 90= 90°. • ABCD is a square. • As ABCD has four sides of the same length, it is a rhombus	• Comme ABCD a quatre côtés de même longueur, c'est un losange. • Les angles aigus d'un triangle rectangle sont complémentaires. • Les triangles rectangles DHC et BGC sont superposables, leurs angles sont deux à deux égaux. • On en déduit les angles ∠ BCG et ∠ DCH sont complémentaires. • ∠ HCG=180° d'ou l'angle ∠ BCD=180 – 90= 90°. • ABCD est un carré. • Comme ABCD a quatre côtés de même longueur, c'est un losange.

Formal proof is of low importance **in English mathematics teaching**, both in the selective and in the non-selective school sector. Theorems are often developed experimentally with some examples. Teachers do not make clear that example-related explanations are not sufficient as proof, or that such considerations are not a proof in a formal sense. Very often teachers use the term "proof" for example-related explanations. Consequently, both pupils and teachers do not make a difference between proof and example-related checking. This leads to the fact that many pupils in their own mathematical investigations end their work with example-based checking of formulae or solutions they found, without trying to find out a general explanation. The low importance of proofs corresponds with the fact that mathematical theorems and methods are quite often just announced by the teachers without any attempt to give reasons for them. The following sequence of a lesson observed in Year 10 at the Top Set of a comprehensive school shows exemplarily how proof and example-related checking are not distinguished.

> In the lesson before this one, various theorems on the size of angles of triangles inscribed in circles have been discovered by the pupils themselves through individual work.
>
> At the beginning of the next lesson, after a review, the teacher asks the pupils to start with the practical check of the size of the angle at the circumference of a triangle inscribed in a circle (so-called Umfangswinkelsatz).
>
> "Draw three diagrams with triangles in it, measure the angle and show that it is right, what we said yesterday." He points out that it is important to draw accurately. While the pupils are working individually and the

teacher is walking around and helping some pupils he formulated several times:

"Yesterday we delivered the theory, today we will prove it."

After a couple of minutes most of the pupils have finished three drawings and recognised that the angles in each triangle over a chord are nearly the same. However, as a part of the drawings were done inaccurately, there occurred quite big differences. The teacher asks the pupils what might have been the reason for asking them to do three drawings. Then the following discussion started:

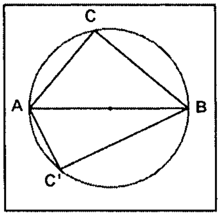
Teacher:	What do you think, why I have asked you to do three drawings?
Pupil:	In order to have three checks.
Teacher:	Do you think I would be satisfied if I would have a theory and just check it with three examples?
Several pupils:	No, no.
Teacher:	So, what do you think, how high is the probability that you all take the same triangle?
Some pupils:	Unlikely.
Teacher:	So we actually have 60 checks. We would be rather sure that if our theory has worked for 60 cases, it will work on the whole.

After this, exercises from the text are done which are to be finished as homework. In this way the efforts for a "proof" ended.

The subject-based understanding of theory in **German mathematics teaching** influences the emphasis put on proof in German mathematics teaching, limited to the *Gymnasium*. A formula-related understanding of proof prevails, while content-related proofs are carried out quite seldom. There are great differences between the various school types, as proof is done less in the *Realschule* than at the *Gymnasium,* and at the *Hauptschule* they almost do not exist at all. Especially in geometry teaching at the *Gymnasium* great meaning is granted to the carrying out of proof. In this connection the importance of proof within the framework of the structures of mathematical theorems is made clear. For this the need of proof for theorems is of high relevance. Thus its meaning is to explain that experimental, practical proofs are not sufficient for the control of the validity of general statements and therefore formal proofs are necessary. These characteristics – the great importance of explaining the need of proof for theorems – is demonstrated by the following example from Year 8 of a *Gymnasium*, which deals with the so-called theorem of Thales.

The lesson starts with the construction of the circle of Thales (drawn with diameter AB) with different triangles, which each pupil does individually. One girl draws this figure on the blackboard.

The teacher asks what is special about it. Some pupils assume that these triangles are always right-angled, others express their doubts about this. The teacher then defines the circle of Thales as a special circle and formulates the theorem of Thales as follows:



If we connect the points A, B with a point C on the circle of Thales, then we get a triangle with a right angle at point C (theorem of Thales).

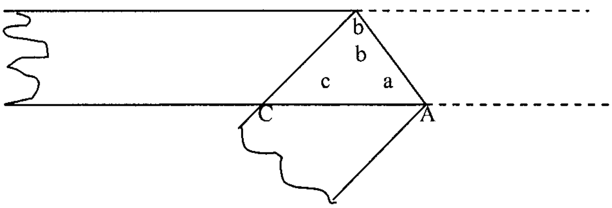
The teacher asks whether they may write down this phrase like this. A girl refers to the description of the construction. Then the teacher asks once more what the observation of Thales is and how they checked it. One girl says that she controlled the theorem with 3 or 4 examples. A boy states that it is always true with any triangle he draws. After further contributions to the discussions, the teacher summarises by stating that trying examples does not help, and he asks what to do. A girl suggests that they must argue until everybody believes.

Then, quite suddenly – and without any further inputs from the teacher – the central ideas of the proof are given by two pupils, by drawing the connecting line between the centre of the circle and point C and looking at isosceles triangles.

At the end of the lesson the teacher resumes the discussion as follows: It is important to show that triangles on the circle of Thales have a right angle. He announces the proof for the following lesson.

In **Japan**, proofs of mathematical statements play a high role when teaching mathematics. Proofs are considered important to introduce students into the theoretical frame of mathematics, especially in the context of geometry. Students also carry out proof problems by themselves. In teaching geometry in lower secondary schools, two aspects are considered as important: One is to foster understanding of concepts and properties of geometrical figures and the ability to apply them. The other is to foster a logical way of viewing and thinking, especially knowing and making use of deductive reasoning. Students are also encouraged to make conjectures for general cases about geometrical figures by means of such methods as observation, experimentation or investigation, and verifying the conjecture in a deductive way.

For Japanese students, one major difficulty is to recognise the need for proving. The textbooks often emphasise the importance of proof by repeatedly using such phrases as "explanation of the general case," "giving reason for the phenomenon" and "explanation without using manipulatives" which was the students' familiar way of explanation in elementary school. In addition, as shown below, some textbooks ask students to work with several examples and develop conjectures whether the rule they found always stands as true. Students may also question whether the manipulation in an example is sufficient for justifying the rule. Here, the need for proving is expected to emerge.

> In one of the textbooks (Suugaku, 2 [Mathematics 2], pp.94-95, published by Keirinkan, 2002), "proof" is introduced after working on a problem of folding a tape in the following way:
>
> 
>
>
> The question is "Let's measure the angles a, b and c in the triangle ABC. What do you notice?" After noticing that the measures of ∠a and ∠b are equal, the question continues, "When folding the tape differently, the measurements of the three angles change. But it seems that the equality holds whatever the case." Here students are encouraged to check with different examples. Building on these preparations, there comes a statement, "Let's think how to show that the equality always holds in whatever way the tape is folded." Then the textbook shows the deductive explanation, with a summary statement "In this way of explanation, it shows ∠a = ∠b in general, regardless of the ways of folding the tape. This way is often used when examining the properties of figures."

In the questionnaire, there were only 8% of teachers who replied 'no' when asked if they emphasise the importance of proof in class by saying, for instance, "it explains the general case" or "it gives reason for the phenol-menon."

Students also deal with proof problems by themselves. In the textbooks, aspects of language such as "assumption and conclusion" and "frame of proof" are emphasised. A frame of proof intends to help students to conduct logical proofs in a sort of formal style. In the questionnaire, 71% of the teachers confirmed that they also use "incomplete proofs with boxes" at the beginning. Here students put appropriate sentences in the boxes to complete the proof.

3.5 Focus on justifications or rules versus work with examples

Concerning this aspect there can be no polar description reconstructed:

English mathematics teaching can be characterised by its focus on work with examples, with rules and standard algorithms being of minor importance. Central algebraic theorems, such as the binomial formulae, are labelled as patterns and are not identified as general statements. This minor emphasis on rules and algorithms coincides with the minor importance of generalisation and general solving schemes in contrast to example-related explanations. Thus solutions are often formulated in connection with examples, and there often do not exist special names for general solution formulae as in German mathematics teaching.

German mathematics teaching can be characterised as rule-oriented, in which rules are manifested by algorithms. Exact execution of arithmetic and algebraic algorithms is highly important. However, there are great differences between the different school types, with a strong emphasis on algebraic algorithms in the higher type of the tripartite school system. It is important to many teachers, that pupils are able to execute the central algorithms – e.g. calculation of percentages and interest or term transformations and solving equations – with certainty by heart. Especially in the lower types of the secondary school it is regarded as important that pupils know by heart those algorithms which very often serve as a substitute for a more profound understanding of algorithms. A deeper understanding would in the opinion of many teachers mean overtaxing weak performing pupils. At the Gymnasium a competent usage of the formulae is expected, although in practice even there the transformation of terms and binomial formulae are often reduced to their calculation and practised with plenty of exercises. Besides this orientation towards execution of algorithms by heart some teachers tend to emphasise a content-related understanding of formula and the ability to develop such formulae by themselves.

In **France**, the teaching of mathematics is characterised by its focus on exercises and justification of solutions within the studied theoretical frame of mathematics. The exact and precise processing of arithmetical algorithms is considered as important and functions as a basis for solving more complex problems. Sticking to exactly prescribed procedures and following certain routines when working with algorithms is regarded as important, but for complex problems not always sufficient. This leads to high demands in students' engagement in exercises.

In exercises, extended written solutions are expected. In these solutions justifications are required at a theoretical or general level; an argument based on an example is unlikely to be accepted here. In geometry, proofs are likely to be studied based on a geometrical figure, while mathematical arguments in the proof are expected to abstract from special properties of the figure. Written proofs in class show a certain format, including explanations of applied theorems and concepts. These formats are given as models for further proofs and solutions of problems.

The exact and precise processing of arithmetical and algebraic algorithms is important in **Japanese teaching**. As for essential rules, algorithms and formulae, students have to memorise them. For many students, mathematics is a subject just concerned with memorising rules, algorithms and formulae, which is a big problem in Japanese mathematics education. Rigid and standard solution processes are considered important. In Japan, pressure to pass entrance examinations for upper schools is great. This is especially the case in lower secondary schools. This is one main reason why rigid and standard solutions are emphasised in the lessons.

Rules and formulae are often taught example-bound. For example, in the teaching of computations with square root, 79% of teachers questioned confirmed that they explain what the rule is, and why the rule is approved, by using examples. 69% of the teachers also confirmed that they explain how the rule is effective. It is also the case that many mathematics teachers use students' common mistakes in introducing the rule (84%). Textbooks often encourage students' developing conjectures whether the rule can be made. Here, the calculator is an important tool (especially recently). However, according to the questionnaire, it is still not the case that teachers use calculators in their lessons very often.

3.6 The role of precise language

Precise language and the use of mathematically correct language are considered to be very important when teaching mathematics in **France**. Technical terms are often treated like vocabulary, which have to be learnt by heart. As a consequence, during class discussions, teachers often correct those phrases of the students that are not precise or are slightly incorrect. Concerning correct mathematical arguments neither teachers nor students use terms such as "if ... then" in a strictly logical sense. Nevertheless there is a demand for rigour in thinking and for verbalisation of reasoning in a precise way. Precise language is expected to foster mathematical reasoning, and reasoning in general. Students are asked to verbalize their thinking precisely; in particular, they have to express their reasoning in written solutions of problems in order to make their thinking precise. In school, the role of precise language is made clear to students at an early age and not only in mathematics.

In contrast in **English mathematics teaching** the development of a collectively accepted terminology with reference to the language of mathematics in the context of 'official' communication is only of minor importance. This aspect is strongly connected with the minor relevance of phases of class discussion in contrast to individual teaching-and-learning styles.

German mathematics teaching can be characterised by an intermediate position between the high importance of the usage of correct language in France and the low relevance of this issue in England: On the one hand the usage of precise mathematical language is emphasised in the context of an 'official' classroom discussion, while the grade of usage differs in the tripartite school system. As for the high meaning of results from the class discussion, this part of communication dominates other forms of communication. Therefore, teachers at the *Gymnasium* generally strive for a mathematically precise and formally correct way of speaking, which they correspondingly also demand from their pupils. This often leads to the fact that teachers interrupt the pupils' explanations and that they 'offer' correct and formally exact formulations, in order to enable them to formulate mathematically precisely. At the *Realschule*, in practice it is less uniform how important the usage of a mathematically correct language by the pupils is regarded, while at the *Hauptschule* it is generally less important. Especially at the *Gymnasium*, but at the *Realschule* too, mathematical expressions are taken as vocabulary which must be learned by heart, and sometimes this is explicitly the homework to be done.

In **Japan** precise language and the use of mathematically correct language are considered to be very important when teaching mathematics. Technical terms are often treated like vocabulary, which have to be learned by heart. During the lessons, teachers correct those phrases of the students that are not precise or are incorrect. Still, it is not likely that every teacher just urges students to memorise technical terms and notations. They recognize the importance of meaning of terms including the meaning of Chinese characters and mention conciseness and usefulness of mathematical notations, which is also stated in the textbooks.

3.7 The role of real-world examples

Concerning this aspect England and France can be described as polar approaches, with Germany and Japan in between.

English mathematics teaching can be characterised by the fairly high importance given to real-world examples, which have various educational functions. They serve to introduce and derive new mathematical concepts and methods, but also to impart abilities that enable applying mathematics to solve extra-mathematical problems. These abilities are especially supported through coursework, and by projects within the framework of statistics lessons. These projects, integrated into statistics lessons or coursework, deal with real-world examples, and normally they are realistic examples. However, besides this, 'dressed up' examples are also used for the introduction of new mathematical concepts and methods. A further characteristic of real-world contents in English mathematics teaching is that more recent mathematical topic areas, such as graph theory and network analysis from discrete mathematics with strong application references, are taught. Handling data is taught intensively and embedded into real-world contexts in English mathematics lessons. These real-world examples are often taught through an activity-oriented method, with students doing research tasks they set and then evaluate themselves, often with the aid of computers. The teaching method when dealing with applications strongly depends on the example's function: In connection with the introduction of new mathematical concepts and methods there is found both class discussions and individual work. The prevailing method with application-oriented and more extensive projects is individual work. Generally speaking, many pupils are used to formulate and to solve problems independently – if necessary with their teacher's help.

In contrast to this position real-world examples are of no importance for the teaching of mathematics in **France**. New mathematical concepts and methods are more likely to be introduced by strictly mathematical problems, if teachers decide to motivate them at all. They show students the value of new knowledge for solving problems within mathematics. Geometry, number theory and algebra are traditionally the main topics of mathematics teaching in France. New curricula put more emphasis on statistics, but in class this topic area is still given a subordinate place.

Typical for **German** mathematics teaching is a minor emphasis on real-world and modelling examples, which is nevertheless very different to the situation in France. Real-world examples mainly function as introduction of new mathematical concepts and methods or are used for exercising mathematical methods. More extensive problems, that are meant to promote extra-mathematical goals, e.g. to develop abilities to master everyday life and to solve extra-mathematical problems by means of mathematics, are rather infrequent in everyday school lessons. Normally such problems are only given within the framework of daily or weekly projects. Furthermore, it is typical for German mathematics teaching that real-world examples discussed in lessons are not authentic real-world problems, but made to illustrate mathematical contents. Therefore, these examples give a quite artificial and far from reality impression. Fairly modern mathematical areas, which widely include applications, such as graph theory, until now did not enter German mathematics lessons.

Since teaching is based on textbooks in Japan, the use of real-world examples is influenced by the textbook used. Still, in the actual teaching, the relevance of real-world examples depends on teachers. Real-world examples are often aiming at the introduction of new mathematical concepts and methods, and at exercises of mathematical methods. As described above, in many textbooks, a chapter's opening pages and front pages in textbooks contain illustrations of real-world examples. However, how they are treated and incorporated in the teaching varies according to the teacher.

3.8 Teaching and learning styles

Concerning this aspect German and English mathematics teaching form the polar approaches, with France and Japan in between.

German mathematics lessons are dominated by a teaching and learning style called class discussion - almost all mathematical concepts and methods are introduced during periods of class discussion. Individual work is of fairly low importance, and it can mostly be seen when exercises are worked on. Significant differences are apparent between different types of schools and different years. This means that, in the upper years of a Gymnasium, class discussion is almost exclusively the teaching and learning style used. In Hauptschulen, by contrast, individual work replaces class discussions and, during periods of class discussion, it is essential that students discuss with each other. Therefore, at least temporarily, students refer to each other. Furthermore, significant differences in the extent a teacher guides a class discussion can be seen, ranging from merely guiding to an authoritative directing of the class discussion. The blackboard is the essential medium of a class discussion, and the students sometimes write their solutions on it. All in all, the students shape the class discussion to an appreciable extent.

In **English** mathematics lessons, two main teaching styles are currently recognisable. The first one is more traditional, and it is focused on long periods of individual work. During these, the students work on new mathematical topics by using individual work material, or they practise known terms and methods. These periods of individual work alternate with shorter periods of class discussions, which are rigidly guided by the teacher, during which new topics are introduced or results are compared.

Besides these more traditional teaching and learning methods, another style exists which is more student focused. Its method consists of several problem-solving activities, during which the students carry out investigations and do coursework, often in the form of projects. Generally speaking, in England, class discussions are dominated by the teachers. All of the communication takes place via the teacher, and the students hardly ever refer to each other. Writing down on the blackboard is not important – if something is written down on the blackboard, this is usually done by the teacher. When teaching mathematics in England, individual differentiation often takes place. This is easily possible, since most of the learning material is designed for individual work.

In **France** periods of individual work alternate with shorter periods of class discussions, which are guided by the teacher, during which new topics, concepts, theorems and methods are introduced or results are compared. The blackboard is the essential medium of a class discussion, and the students sometimes write their solutions on it. Generally speaking, in France, class discussions are dominated by the teachers. All of the communication takes place via the teacher, and the students hardly ever refer to each other. In contrast, highly active engagement of students is asked in individual work. Students have to solve a lot of exercises in class and at home. Exercises function as the heart of mathematics teaching in France, they prepare students for new concepts and deepen their understanding of already studied concepts, theorems and formulae in class.

Teaching in **Japan** is based on whole-class teaching. Still, the teacher diligently controls students' activities by shifting types of classroom interaction (Hiebert et al. 2003). Students engage in problem solving while solving a small number of main problems with the teacher in class. Problem solving activities are often carried out firstly on an individual basis. After they get their solutions, students may present their thinking on the blackboard and discuss it. Whole-class discussions are usually guided by the teacher. During individual work, teachers often walk around students' desks and give directions and suggestions. Therefore, compared with the case in Germany, it can be said that individual work plus public interaction rather than discussion among students is emphasised in Japan. Recently, fostering communication skills, including mathematical communication, has been considered important. However, in actual classroom teaching, teachers work hard in order to cover all the content in the textbooks. Time constraints together with pressure of entrance examinations hinder teachers to spend time on class discussion.

4. FURTHER PERSPECTIVES

If we try to understand the differences just described, the influence of cultural traditions on education and educational philosophies have to be considered. Already Michael Sadler, who at the beginning of the 20th century visited Germany with a British expert commission and compared the achievements of the Prussian with the British educational system, described this influence:

> "In studying foreign systems of Education we should not forget that the things outside the schools matter even more than the things inside the schools, and govern and interpret the things inside. ... A national system of Education ... has in it some of the secret workings of national life. It reflects, while it seeks to remedy, the failings of the national character. By instinct, it often lays special emphasis on those parts of training which the national character particularly needs." (Sadler 1900 (1964), p.310).

In the field of comparative education there exists a few comparative studies dealing with educational philosophies and their historical development. One of the first contributions to this was the approach of Lauwerys (1959), who distinguished the attempts of the "Liberal Education", the French "*culture générale*", the German "*Allgemeinbildung*", the American "*General education*" and the Russian "*Polytechnicalization*".

Proceeding from this, McLean (1990) developed various attempts to explain the different traditions of school knowledge, in which he distinguished several European traditions. The encyclopaedic tradition, found predominantly in the French educational system, is historically rooted in the ideals of the French Revolution. McLean characterises this attempt through several principles, from which the principles of universality and rationality are the most convincing ones. Following McLean, the principle of universality means that on the one hand teaching aims to transfer as much knowledge as possible from all important subjects to all learners. On the other hand a certain degree of standardisation and homogeneity of the transferred knowledge shall be reached. Rationality which, following McLean, is the highest objective of the encyclopaedic approach, aims to enable the learners to understand central ideas, structures, logical and ethical systems, for which the understanding of structures and systems created by reasoning, gains great importance. In particular, philosophy and mathematics originally were regarded as the subjects which suited the rational principle most closely. As a second important current of European school systems McLean (1990) defines the "humanist perspective in education" (p. 25), meaning the development of human virtues, which includes not only the development of understanding, sympathy and confession, but courage, intelligence and eloquence. He characterises it by the aim of linking thought and action in ways that would encourage human possibilities in the individual to the fullest. It focused on the individual rather than the social group. It was moral in its emphasis on the development of human virtue but this morality was extended to include aesthetic appreciation and sensibility. This approach, dominant in England and Wales, can be characterised by the principles of morality, individualism and specialism and has a strong relation to pragmatism in philosophy. The description of German educational philosophies was not convincing and we have developed our own description discriminating two different development lines (for details see Kaiser 1999). The educational philosophy dominant for mass education can be characterised as realistic education, i.e. school lessons should be more orientated towards realistic-vocational education and should incorporate concrete knowledge useful in later life. It should especially enable the students to develop social virtues through work. The education for the élite was orientated towards the development of the individual, who should receive a complete formation of humankind. Neither an early specialisation on selected subjects was allowed nor an inner differentiation within the class; all the pupils in the class should progress together at the same speed.

If we now look at the Japanese educational system a high influence of Western, especially European and US-American school traditions, can be recognised. Western schooling traditions were already introduced during the Meiji government in the late 19th century after Japan was forced to open the country to foreign influences. Around the same time, Western mathematics was introduced in Japan. These foreign influences did not come into action as they were, on the contrary they were modified and adopted to the Japanese situation focusing on teaching aspects and the situation in the class. Due to the tradition of the Wasan mathematics, including the aspect of learning elementary arithmetic by means of an abacus, people were able to adjust themselves to Western mathematics. Apart from the Western influences on the Japanese educational system there are special Japanese traditions, which have shaped Japanese teaching. One important influential factor is the tradition of the research lesson and lesson study, already going back to the time of the Meiji government. Lesson study is a collaborative and longitudinal effort (over a couple of months or even a year) of improving classroom teaching by teachers. Their focus is on lessons that they are conducting. They plan, conduct, discuss and improve lessons by studying teaching content, developing teaching materials and discussing the effect by observing each other's lessons. Here, teachers may also learn and discuss theories of teaching as a basis for developing lesson plans. However, they start from the lesson itself, instead of starting from learning theories and then trying to apply it to the classrooms, which is often the case of US American teachers (see Stigler & Hiebert 1999). The development and conduct of lesson plans, together with visits to the classrooms of colleagues, are considered to be important for the reflection of one's own teaching. Such a cultural practice has produced an image of an ideal lesson, including a joint understanding of good mathematical problems to teach certain mathematical ideas and ways the problems should be effectively dealt with in class. This image of an ideal lesson is also reflected in the textbooks, which are written by experienced mathematics educators including experienced school teachers. The structure of the textbooks mirrors the structure of class lessons, which should be like the flow of a river: It begins with cultivating students' interest and proceeds to the solution of problems on their own. This is followed by explanations of mathematical content and ends with exercises (for details see Lewis & Tsuchida 1988).

Furthermore influences of the general style of communication in Japanese culture can be detected in communication processes in teaching. Sekiguchi (2002) examined relationships between argumentation processes and mathematical proofs from a cultural perspective and argued that the teaching of mathematical proofs seems to be conceived in the communication style of a so-called "group" model, common in Japanese communication processes. The model states that cooperation rather than competition is highly valued within a community. According to Sekiguchi, the goal of proof in Japan is to reach a unanimous conclusion, which helps in establishing the harmony in the community. A proof requires one to follow the premises accepted in the community, which helps in keeping the harmony of the community members. Beyond the instruction of mathematical proof, he described that the teaching and learning styles in Japan, especially the importance of exchanging and sharing opinions in a whole-class, follow the group model. The general styles of communication in Japanese culture, in contrast to that in Western culture, seem to reflect the difference described above.

Another important tradition, which is more related to education and general pedagogical aspects, is the aversion towards differentiation of the students in the period of learning basic knowledge and skills. The spirit of giving every child equal opportunities for education since the starting of public education in the late 19^{th} century has been passed on from generation to generation. The extent to which the idea of differentiation is put into practice is one of the controversies in the topical Japanese education reform.

In total, it seems to be a general characteristic of the Japanese debate that it focusses more on the debate concerning teaching and how to improve it, than on reflections concerning educational aspects such as educational philosophies, pedagogical theories and so on.

To summarise, the reflections above show the strong influence of educational and societal philosophies on educational structures as well as on the classroom situation. The framework described in the paper might be seen as a first step to the explanation of differences observed in various studies focussing on classsroom processes. Further, the framework might enable us to see in which parts of the educational processes the different educational systems can learn from each other. Coming back to the introduction and the questions of Pepin, our framework shows the necessity to ask such questions as to the understanding of the teachers' practices, the cultural underpinnings of such practice and the sources of cultural and educational traditions and their influence on the teaching and learning of mathematics. Answers to these questions seem to be necessary in order to come to a real understanding of our own and other educational systems and to allow a reflective "transfer" of effective measures from one system to another.

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