Network Equilibrium Models for Analyzing Toll Highways

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Summary. The construction of toll highways by concessions awarded to private companies leads to the need of forecasting their usage in order to estimate the future stream of revenues. Two main modeling approaches for this problem that result in variants of multiclass network equilibrium models, are presented and commented upon.

Key words: Traffic equilibrium, congestion pricing, transportation.

1 Introduction

The construction of new highways, in both developed and developing countries, is often assigned to private companies which operate these new facilities as concessions. The users are charged tolls according to the extent that they travel on the new facilities. The derived revenues finance the construction and operation of the highway for a certain period of time, after which the highway becomes property of the state government that awarded the concession.

Economic theory is not respected by such toll highway enterprises. If one were to follow the dictates of the economic literature on tolling congested facilities, then a toll would have to be imposed on some or all of the links of the congested network. In 1952, William Vickrey, a Nobel Prize winner in Economics and the father of Congestion Pricing, suggested that fares for New York City subways should be increased in peak times and in high-traffic sections and be lowered in others. Later, he made a similar proposal for road pricing. Vickrey considered time-of-day pricing as a classic application of market forces to balance supply and demand. Those who are able can shift their schedules to cheaper hours, reducing congestion, air pollution and energy use – and increasing use of roads or other utilities. According to Vickrey, "you're not reducing traffic flow, you're increasing it, because traffic is spread more evenly over time." He also claimed that "even some proponents of congestion pricing don't understand that."

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Despite the sound economic theory that supports it, the public in general opposes tolling. Because of this, elected government officials are reluctant to impose tolls on roads and highways, a resource often thought of as "free" good. When the committee chaired by Professor Reuben Smead of University College in the U.K. supported the proposition that charging road tolls would increase economic welfare, Sir Alec D. Home, the prime minister at the time, lamented that "if we are re-elected we will never again set up a study like this one." Notwithstanding the public's opposition, the mayor of London (Mr. Ken Livingstone) recently implemented a flat congestion toll of £5 for access to the city center. In doing so, London joins Singapore and Oslo as one of a few cities around the world to impose systematic congestion tolls.

The study of tolls within the context of network equilibrium models was advanced recently by the contributions of Hearn and Ramana [HR98] and Hearn and Yildirim [HY02]. This line of research, initiated by Don Hearn, led to the understanding of a variety of toll schemes that all render a "user optimal" route choice to a "system optimal" route choice. The latter minimizes travel time for all the travelers on the network.

However, the models described in this paper correspond to the actual analyses carried out in many countries for the construction of toll highways as stand alone enterprises. There is no value judgment implied by the statement of these models; rather, they are a testimony to the flexibility and adaptability of network equilibrium models to a variety of different situations and circumstances. The purpose of this paper is to identify and analyse the various approaches that have been used to predict the usage of tolled facilities among different classes of users. Essentially, the new facilities (e.g., new highways) provide shorter travel times and, given the value of time of different classes of users, one must determine the trade-off between increased travel cost and reduced travel time in order to predict their usage.

The paper is organized as follows. The next section introduces notation and definitions. Section 3 deals with deterministic models and Section 4 described a demand function based approach to this problem. In Section 5 a small numerical example is given. In Section 6, some large scale applications of these models are described and Section 7 offers some conclusions.

2 Notation and Definitions

A road network $R = (N, A)$ consists of nodes n, $n \in N$ and directed arcs a, $a \in A$ which may carry vehicular traffic. The demand for travel is subdivided into classes c, $c \in C$ which may correspond to different vehicle types or different socio-economic characteristics. The demand for travel of class c for origin-destination (O-D) pairs i, $i \in I \subset N \times N$ is denoted g_i^c. These demands use paths k, $k \in K_i^c$ where K_i^c is the set of paths used by class c for travel between O-D pair i. In its simplest form, the travel cost function for class c on arc a is the sum of the travel time function denoted as $s_a(\cdot)$ and a toll, t_a^c, that is converted into time units by the factor θ^c:

$$s_a^c(v_a) = s_a(v_a) + \theta^c t_c^a, \quad \forall a \in A. \tag{1}$$

In the above equation, v_a^c denotes the number of class c vehicles on arc a and $v_a = \sum_{c \in C} v_a^c$.

To determine the choices that travelers make between "toll" and "non-toll" alternatives, stated preference analyses are carried out. Usually, the result of a stated preference analysis is a set of logit functions of the form

$$p(\text{using toll facility}) = \frac{1}{1 + \exp(\alpha^c \Delta\text{cost} + \beta^c \Delta\text{time})}, \quad \forall c \in C, \tag{2}$$

where α^c and β^c are nonnegative parameters, Δcost is the difference in the cost of the trip (usually positive if a toll facility is used) and Δtime is the difference in the trip time (usually negative if a toll facility is used).

The perceived value of time for each class c of travellers is determined as the ratio $\theta^c = \beta^c/\alpha^c$. The cost of a path is denoted $s_k^c(v)$ and is simply

$$s_k^c(v) = \sum_{a \in A} \delta_{ak} s_a^c(v_a) = \sum_{a \in A} \delta_{ak} \left(s_a(v_a) + \theta^c t_a^c\right), \quad \forall k \in K_i^c, i \in I, c \in C, \tag{3}$$

where $\delta_k^a = 1$ if arc a belongs to path k and zero otherwise. Later, it is useful to write the cost of a path as

$$s_k^c(v) = \sum_{a \in A} \delta_{ak} s_a(v_a) + t_k^c, \quad \forall k \in K_i^c, i \in I, c \in C, \tag{4}$$

where $t_k^c = \sum_{a \in A} \delta_{ak} \theta^c t_a^c$ may be viewed as the toll cost of path k. The link fixed costs t_a^cmay be used to model toll plazas or tolls which vary with the distance traveled on the toll facility. It suffices to define t_a^c proportional to the length of the arc.

3 Models Based on Generalized Cost Path Choice

In such models, the demand for each class, g_i^c, $c \in C$, $i \in I$ is fixed and known and users are assumed to make their choice of a toll based only on the generalized cost differences between paths that include tolled facilities and those that do not. The usage of the tolled facilities may then be deduced from the flows on links a, $a \in A$ with positive tolls, i.e., $t_a^c > 0$. The resulting model is the classical multiclass (or multi-user) network equilibrium models which satisfies the user equilibrium condition of [Wa52]

$$\left.\begin{array}{l} s_k^c(v) = u_i^c \text{ if } h_k > 0 \\ s_k^c(v) \geq u_i^c \text{ if } h_k = 0 \end{array}\right\} k \in K_i^c,\ i \in I,\ c \in C, \tag{5}$$

where u_i^c are the shortest travel times for O-D pairs i, $i \in I$ and classes c, $c \in C$ and subject to conservation of flow and nonnegativity constraints. It is well-known (see [Da73], [Va76], [Sp95] that this network equilibrium problem is equivalent to solving the convex cost minimisation problem

$$\min \sum_{a \in A} \int_0^{v_a} s_a(x)dx + \sum_{c \in C} \sum_{a \in A} v_a^c \theta^c t_a^c \tag{6}$$

$$s.t. \sum_{k \in K_i^c} h_k = g_i^c,\ i \in I,\ c \in C \tag{7}$$

$$h_k \geq 0,\ k \in K_i^c,\ i \in I \tag{8}$$

$$(v_a^c = \sum_{k \in K_i^c} \delta_{ak} h_k,\ a \in A,\ c \in C). \tag{9}$$

The numerical solution of this model by the linear approximation method is well-known and will not be repeated again here. It is perhaps worthwhile to point out that the flows by class, v_a^c, are not unique, nor are the path flows h_k, but the arc flows v_a are indeed unique.

This model has been used extensively in many toll facility studies since most popular transportation planning software packages offer, as a standard model, a generalized cost multi-class network equilibrium model. The only published references known to the author are [Me95] and [Me95]. These articles describe the models used for the analysis of Highway 407, a toll facility which bypasses the city of Toronto, Canada.

In this formulation the link cost functions, $s_a(v_a)$, $a \in A$, are relatively simple, since they do not model asymmetric costs due to different vehicle types. If more complex functions were used, the resulting multiclass model would be considerably more complex and would require the solution of a variational inequality model (see [FH95]).

4 Models Based on Explicit Choice of Tolled Facilities

Such models are based on logit functions obtained from stated preference analyses to determine the probability (or proportion) that a user in each class will use paths that include tolled facilities. Let g_i^{ct} denote the number of users in class c who are willing to pay tolls and g_i^{cn} denote the number of those who are not. That is, $g_i^c = g_i^{ct} + g_i^{cn}$, $i \in I$, $c \in C$, and, as in the path based approach, the total demand for each class g_i^c, $i \in I$, $c \in C$ is assumed to be fixed and known. Also, let K_i^{ct} and K_i^{cn} denote the sets of paths that contain tolled facilities and those that do not, respectively. The resulting multi-class network equilibrium model with explicit choice functions may be stated as follows:

$$\left.\begin{array}{l} s_k^{ct}(v) = u_i^{ct}, \text{ if } h_k > 0 \\ s_k^{ct}(v) \geq u_i^{ct}, \text{ if } h_k = 0 \end{array}\right\} k \in K_i^{ct},\ i \in I,\ c \in C \tag{10}$$

$$\left.\begin{array}{l} s_k^{cn}(v) = u_i^{cn}, \text{ if } h_k > 0 \\ s_k^{cn}(v) \geq u_i^{cn}, \text{ if } h_k = 0 \end{array}\right\} \; k \in K_i^{cn},\; i \in I,\; c \in C \tag{11}$$

$$\sum_{k \in K_i^{ct}} h_k - g_i^{ct} = 0, \;\; i \in I, \; c \in C \tag{12}$$

$$\sum_{k \in K_i^{cn}} h_k - g_i^{cn} = 0, \;\; i \in I, \; c \in C \tag{13}$$

$$g_i^{ct} = g_i^c / \left\{1 + \exp\left(\alpha^c t_i^c + \beta^c\left(u_i^{ct} - u_i^{cn}\right)\right)\right\}, \; i \in I, \; c \in C; \; \left(g_i^{cn} = g_i^c - g_i^{ct}\right) \tag{14}$$

$$h_k \geq k \in K_i^{ct}, \;\; k \in K_i^{cn}, \; c \in C, \; i \in I \tag{15}$$

$$g_i^{cn}, \; g_i^{ct} \geq 0, \;\; i \in I, \;\; c \in C, \tag{16}$$

where t_i^{ct} is the average toll paid for all traffic of O-D pair i that uses toll roads.

Clearly there is a difficulty with this formulation, since the paths used are not known before computing the equilibrium flows. In addition, the number of possible paths is exceedingly large. On the other hand, there is an equivalent formulation in terms of p_k, the *proportion* of demand that uses path k. In particular, the path flows, h_k, may be written as

$$\left.\begin{array}{l} h_k = p_k g_i^{ct}, \; k \in K_i^{ct} \\ h_k = p_k g_i^{cn}, \; k \in K_i^{cn} \end{array}\right\} \; i \in I, \; c \in C, \tag{17}$$

and the arc flows may be expressed as

$$v_a^{ct} = \sum_{k \in K_i^{ct}} \delta_{ak} p_k g_i^{ct}, \; a \in A, \; c \in C \tag{18}$$

$$v_a^{cn} = \sum_{k \in K_i^{cn}} \delta_{ak} p_k g_i^{cn}, \; a \in A^n \tag{19}$$

$$A^n = A - \{\text{toll links}\} \tag{20}$$

$$v_a = \sum_{c \in C} \left(v_a^{ct} + v_a^{cn}\right), \; a \in A. \tag{21}$$

Then, the costs of paths containing and not containing tolled facilities are, respectively,

$$s_k^{ct}(v) = \sum_{a \in A} \delta_{ak} s_a(v_a), \quad k \in K_i^{ct}, \; i \in I \tag{22}$$

$$s_k^{cn}(v) = \sum_{a \in A^n} \delta_{ak} s_a(v_a), \quad k \in K_i^{cn}, \; i \in I \tag{23}$$

The formulation in the space of path flow proportions , p_k, consists of

1) The user equilibrium inequalities (9) – (10).

2) The conservation of flow equations

$$\sum_{k\in K_i^{ct}} p_k g_i^{ct} - g_i^{ct} = 0 \quad \Rightarrow \quad g_i^{ct}\left(\sum_{k\in K_i^{ct}} p_k - 1\right) = 0,\ \ i \in I,\ \ c \in C \quad (24)$$

$$\sum_{k\in K_i^{cn}} p_k g_i^{cn} - g_i^{cn} = 0 \quad \Rightarrow \quad g_i^{cn}\left(\sum_{k\in K_i^{cn}} p_k - 1\right) = 0,\ \ i \in I,\ \ c \in C \quad (25)$$

3)

$$g_i^{ct} = g_i^c \Big/ \left\{1 + \exp\left(\alpha^c\left[\sum_{k\in K_i^{ct}} p_k t_k^c\right] + \beta^c\left(u_i^{ct} - u_i^{cn}\right)\right)\right\},$$
$$i \in I,\ \ c \in C; \left(g_i^{cn} = g_i^c - g_i^{ct}\right) \quad (26)$$

4) Nonnegativity constraints (14) – (15).

This formulation highlights the importance of the path proportions p_k and the large dimension of the problem. For example, with 9 user classes, one would have 18 flow vectors for each link. For a network of 1000 $\times$ 1000 O-D pairs, one would consider explicitly a number of paths of the order of 10^6 and one would need to keep at least 18 matrices, each of size 10^6. While it is possible to restate this model in the form of a variational inequality and search for rigorous solution algorithms, the actual solution methods used in most applications rely on heuristic algorithms that have performed well but that are not supported by convergence proofs.

In order to simplify the model, it is sometimes assumed that the vehicles of the different classes are homogeneous and that the O-D travel costs (impedances) may be simplified to

$$u_i^{ct} = u_i^t \text{ and } u_i^{cn} = u_i^n,\ \ i \in I,\ \ c \in C \quad (27)$$

that is, all the toll payers may be aggregated into one class and all the non toll payers may be aggregated into one class. This is partly justified by the implicit assumption that the toll is perceived at the demand function level, prior to the trip, and once the decision to pay or not to pay the toll is made, the path choice is no longer governed by generalized cost, but only by time. However, this assumption is *not* made in the following "heuristic" solution algorithm:

Explicit Choice Tolled Assignment Heuristic

Step 0 (Initialization) : $l = 0$, choose $g_i^{ct(0)},\ g_i^{cn(0)},\ i \in I,\ c \in C$;

Step 1 (Compute path costs and times)
Solve a two-class network equilibrium problem by the linear approximation method:

$$\min \sum_{a \in A} \int_{0}^{v_a} s_a(x)dx + \sum_{c \in C} \sum_{a \in A} v_a^c \theta^c t_a^c \tag{28}$$

$$s.t. \sum_{k \in K_i^t} h_k = g_i^{ct(l)}, \ \sum_{k \in K_i^n} h_k = g_i^{cn(l)}, \ i \in I \tag{29}$$

$$h_k \geq 0, \ \ k \in K_i^t, \ \ k \in K_i^c, \ \ i \in I, \ \ c \in C \tag{30}$$

to find $u_i^{t(l)}$ and $u_i^{n(l)}$ and, while doing so compute

$$t_i^{t(l)} = \sum_{k \in K_i^t} p_k^{(l)} t_k^{ct(l)} \tag{31}$$

which are the tolls for each class and O-D pair. The path proportions $p_k^{(l)}$, $k \in K_i^{ct(l)}$, $k \in K_i^{cn(l)}$ are computed from the step sizes of the linear approximation method at each iteration.

Step 2 (Modify demand): $l = l + 1$;
$\tilde{g}_i^{ct}$, $\tilde{g}_i^{cn}$are recomputed by using the logit functions for each class c:

$$\tilde{g}_i^{ct} = g_i^c / \left\{ 1 + \exp\left(\alpha^c t_i^{ct(l)} + \beta^c \left(u_i^{t(l)} - u_i^{n(l)} \right) \right) \right\}, \ c = 1, 2, ..., c \tag{32}$$

and

$$\left. \begin{aligned} g_i^{ct(l)} &= \left(1 - \lambda^{(l)}\right) g_i^{ct(l-1)} + \lambda^{(l)} \tilde{g}_i^{ct} \\ g_i^{cn(l)} &= g_i^c - g_i^{ct(l)} \end{aligned} \right\} \ i \in I, \ \ c \in C \tag{33}$$

$$0 \leq \lambda^{(l)} \leq 1 \tag{34}$$

Step 3 (Convergence test)
If $\max_{i,c} \left\| g_i^{ct(l)} - g_i^{ct(l-1)} \right\| \leq \varepsilon$, STOP ;
otherwise, return to Step 1.

The step sizes $\lambda^{(l)}$ may be chosen to implement the method of successive averages (MSA) or any other reasonable sequence of step sizes. The algorithm still requires at least $2|C|$ O-D matrices, and there are $2|C|$ link flow vectors, v_a^{ct} and v_a^{cn}, $a \in A$.

No convergence proof is given in this paper, however in numerous applications in practice, the algorithm has demonstrated good empirical convergence. It is evident that, if the algorithm terminates, the resulting demands and flows satisfy approximately the model formulation.

A block diagram representation of this heuristic algorithm is in Figure 1.

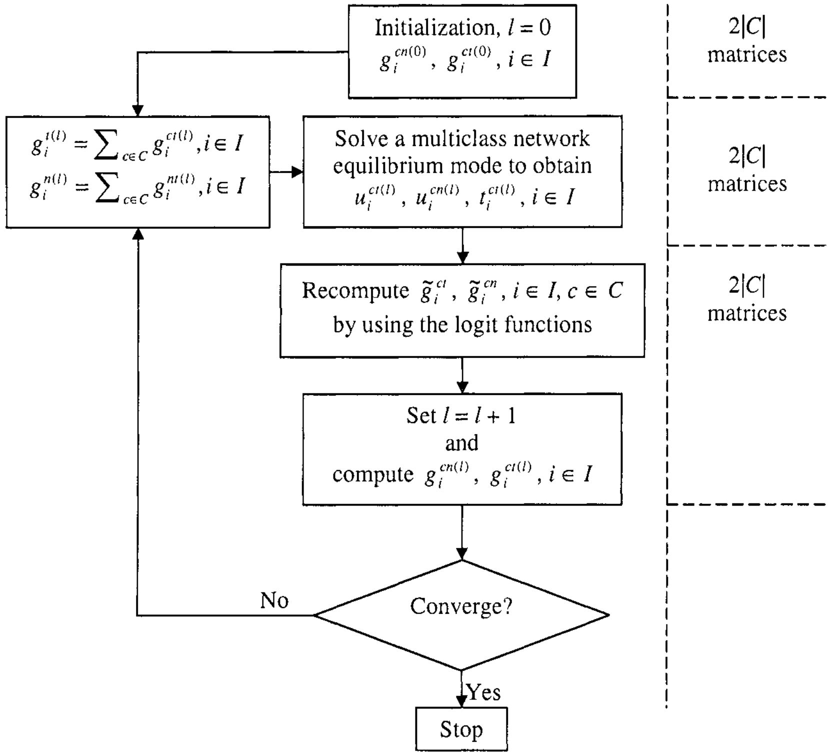

Fig. 1. Block diagram of heuristic algorithm

5 A Small Numerical Example

A small network of three links and one origin-destination pair is used to illustrate the difference between the two approaches for predicting the usage of toll highways. The network is given in Figure 2.

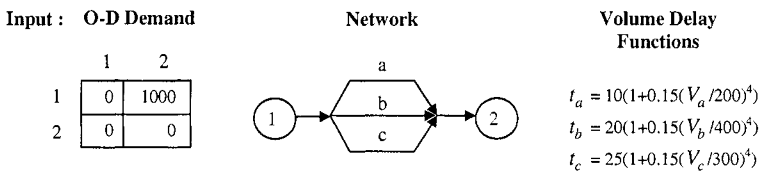

Fig. 2. The network and the demand

Figures 3,4,5 show successively the equilibrium flows without tolls, with a toll of 2 units on the middle link and a value of time of 1.2 and with the application of a demand function where the probability of using a toll facility

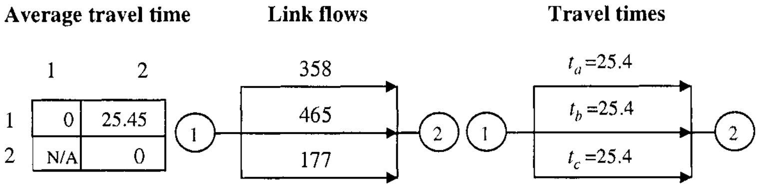

Fig. 3. The equilibrium flows without tolls

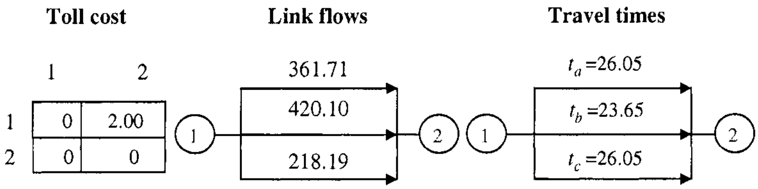

Fig. 4. The equilibrium flows with a toll of 2 units and value of time of 1.2

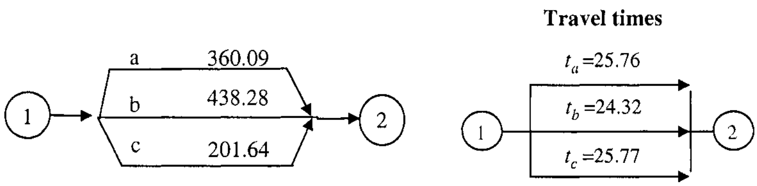

Fig. 5. The equilibrium flows obtained by using the demand function

is given by the function

$$P_r(\text{toll}) = 1/(1 + \exp(.2556(\text{time difference}) + .3067 * \text{toll})$$[1].

For this model convergence was reached after 4 iterations. The toll facility, which is the middle link, carries 420.10 trips in the simple model compared to 438.28 trips when the demand function is used. For this solution the proportion of toll trips given by the logit function is .438972

6 Some Large-Scale Applications

The algorithm described in Section 4 has been used in numerous applications in Europe, North America and Asia. Most of these applications are confidential and the results may not be reported in an academic paper. However, a pilot application of very large scale, carried out on the network used for transportation planning in Southern California may be reported in this paper. The network consists of 2,450 zones, 46,000 arcs. The demand for travel is subdivided into High Occupancy Vehicles (HOV) and Low Occupancy Vehicles

[1] The constants in this function were chosen so that their ratio is exactly 1.2.

(LOV). Tolls were envisioned on some of the regional highways. The logit function

$$P_r(\text{using toll}) = 1/(1 + \exp(0.5647(u_i^t - u_i^n) + 0.4199(t_i^t))$$

was used to determine the probability of using the toll facility. The model described in the previous section was adapted to handle the HOV and LOV demand. A two-class (HOV, LOV) network equilibrium model was used to find the initial travel times and toll costs. The logit function was used to obtain four matrices corresponding to the demand for HOV_{toll}, HOV_{notoll}, LOV_{toll}, and LOV_{notoll}, and a four-class network equilibrium assignment was carried out in Step 1 of the heuristic algorithm. The convergence criterion for an $\varepsilon = 1$ (1 trip) was satisfied after four iterations of the algorithm. The computations were carried out with the EMME/2 (INRO, 1996) software package.

Both these models were applied in Mexico City for the evaluation of a 26 km section of an urban autoroute (Chamapa Highway). They produced different results, which is not surprising. The explicit choice model was used in the final analysis. The generalized cost path based approach was used in several applications in North America, Europe, Asia and Australia.

7 CONCLUSIONS

The intuitive heuristic solution algorithm for the explicit choice function approach was used successfully in practice in numerous applications. It is an example of the compromises that one must make in order to solve large-scale non-standard network equilibrium models. The results obtained are quite sensitive to the coding of the network and the quality of the stated preference model calibrations. The costs of building toll highways are so large that they justify careful use of travel demand and network models to predict the potential ridership.

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