Shape Optimization of Transfer Functions*

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Summary. We show how to optimize the shape of the transfer function of a linear time invariant (LTI) single-input-single-output (SISO) system. Since any transfer function is rational, this can be formulated as an optimization problem for the coefficients of polynomials. After characterizing the cone of polynomials which are nonnegative on intervals, we formulate this problem using semidefinite programming (SDP), which can be solved efficiently. This work extends prior results for discrete LTI SISO systems to continuous LTI SISO systems.

Key words: Linear system, transfer function, shape optimization, nonnegative polynomials, convex cone, semidefinite programming.

1 Introduction

Consider the following linear time invariant (LTI) single-input-single-output (SISO) system

$$\dot{x} = Ax + bu \tag{1}$$

$$y = c^T x + du \tag{2}$$

where $A \in \mathbb{R}^{n\times n}, b, c \in \mathbb{R}^n, d \in \mathbb{R}$. The transfer function is $H(s) = d + c^T(sI - A)^{-1}b$, which can also be written as the rational function

$$\frac{\sum_{k=0}^{n} \alpha_k s^k}{\sum_{k=0}^{n} \beta_k s^k} \equiv \frac{q_1(s)}{q_2(s)} \quad .$$

Note that $\deg(q_1) \leq \deg(q_2) \leq n$.

Conversely any function $H(s)$ of this kind is the transfer function of some LTI system. Any such a LTI system is called a realization of $H(s)$. There are many such (algebraically equivalent) LTI systems [CD91, chap. 9].

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In many engineering applications, we want the transfer function to have certain attractive properties. For example, we may want the Bode plot (the graph of $|H(s)|$ along the pure imaginary axis $s = j \cdot \omega$) to have a certain shape corresponding to some kind of filtering. In this paper we study the shape optimization problem of choosing the coefficients of the rational function $H(s)$ so its Bode plot has some desired shape.

Now consider a *discrete* LTI system, i.e. the governing differential equation (1)-(2) is replaced by the difference equation $\sum_{k=1}^{p} \alpha_k z(n-k) = \sum_{\ell=1}^{q} \beta_\ell v(n-\ell)$ where $\{v(k)\}_{k=1}^{\infty}$ is a sequence of discrete inputs and $\{z(k)\}_{k=1}^{\infty}$ is the sequence of state variables. In this case there are several nice papers [AV02, GHN00, WBV97] that show how to formulate the filter design problem as the solution of the feasibility problem for certain convex sets. The main idea is to apply the spectral factorization of trigonometric polynomials, a characterization of nonnegative univariate polynomials, and semi-infinite programming. This approach can be used to design the transfer function to be a bandpass filter, piecewise constant or polynomial, or even have an arbitrary shape.

Our contribution is to extend these results to continuous time LTI SISO systems (1)-(2). In this case the transfer function is not a trigonometric polynomial and hence we cannot directly apply spectral factorization. Fortunately our transfer function is a univariate rational function, which lets us apply certain characterizations of nonnegative univariate polynomials over the whole axis $(-\infty, \infty)$, semi-axis $(0, \infty)$, or some finite interval $[a, b]$; see section 2. Using these characterizations, we show how to solve the shape optimization problem for the following shapes:

1. standard bandpass filter design;
2. arbitrary piecewise constant shape;
3. arbitrary piecewise polynomial shape;
4. general nonnegative function.

We will show that the first three shape optimization problems can be solved by testing the feasibility of certain convex sets, which are the intersections of certain hyperplanes and the cone of semidefinite matrices. This feasibility testing can be done efficiently using semidefinite programming (SDP) [VB96]. The fourth shape optimization problem can be solved by semi-infinite programming (SIP) [Pol97, WBV97].

We introduce some notation. For any $m \in \mathbb{N}$, denote by S^m the vector space of $m - by - m$ symmetric matrices, and let S_+^m be the intersection of S^m and the positive semidefinite matrices. $A \succ B (A \succeq B\ resp.)$ means that $A - B$ is positive definite(semidefinite resp.). $\lfloor r \rfloor$ denotes the largest integer no greater than r. $deg(p)$ is the degree of the polynomial $p(\cdot)$. Given a cone $K \subseteq \mathbb{R}^N$, $y \succ_K 0$ means that $y \in \text{int}K$, the interior of K. K^* denotes the dual cone of K, i.e., $K^* = \{u \in \mathbb{R}^N : u^T y \geq 0,\ \forall y \in K\}$.

The rest of this paper is organized as follows. In Section 2 we give a characterization of the cone of polynomials which are nonnegative on certain intervals. In Section 3, we reformulate the shape optimization problem for transfer functions to be convex optimization, and also discuss related work. In Section 4 we show how to recover the transfer function from its absolute value. Section 5 draws conclusions.

2 Cone of nonnegative polynomials on intervals

We characterize univariate polynomials which are nonnegative on certain intervals. For a survey paper see [PR00].

First, we characterize the nonnegative polynomials on the positive semi-axis $[0, \infty)$. The following result is due to Markov and Lukacs about one century ago.

Theorem 1 (Markov, Lukacs [Luk18, Mar48, PS76]). *Let $q(t) \in \mathbb{R}[t]$ be a real polynomial of degree n. Let $n_1 = \lfloor \frac{n}{2} \rfloor$ and $n_2 = \lfloor \frac{n-1}{2} \rfloor$. If $q(t) \geq 0$ for all $t \geq 0$, then $q(t) = q_1(t)^2 + tq_2(t)^2$ where $deg(q_1) \leq n_1$ and $deg(q_2) \leq n_2$.*

Now we apply this theorem to characterize the transfer function, which is similar to the spectral factorization for trigonometric polynomials. Observe that

$$
\begin{aligned}
|H(j\omega)|^2 = \frac{|q_1(j\omega)|^2}{|q_2(j\omega)|^2} &= \frac{|q_{1,even}(j\omega) + q_{1,odd}(j\omega)|^2}{|q_{2,even}(j\omega) + q_{2,odd}(j\omega)|^2} \\
&= \frac{q_{11}(\omega^2)^2 + \omega^2 q_{12}(\omega^2)^2}{q_{21}(\omega^2)^2 + \omega^2 q_{22}(\omega^2)^2} \\
&\equiv \frac{p_1(w)}{p_2(w)} \quad \text{where } w = \omega^2
\end{aligned}
$$

Here $q_{i,even}$ and $q_{i,odd}$ denotes the even and odd parts of the polynomial q_i, and $q_{ij}, i, j = 1, 2$ are defined accordingly. Note that $p_1(w)$ and $p_2(w)$ are nonnegative polynomials on $w \in [0, \infty)$. Conversely, by Theorem 1, given any such nonnegative $p_1(w)$ and $p_2(w)$, it is possible to reconstruct the $q_{ij}(w)$, and so $q_i(j\omega)$ and $H(j\omega)$. In other words, $p_1(w)$ and $p_2(w)$ with $deg(p_1) \leq deg(p_2)$ satisfy $|H(j\omega)|^2 = p_1(w)/p_2(w)$ where $w = \omega^2$ for some transfer function $H(j\omega)$ if and only if they are nonnegative on $[0, \infty)$.

The characterization of polynomials nonnegative on some finite interval $[a, b]$ is analogous:

Theorem 2 (Markov, Lukacs [Luk18, Mar48, PS76]). *Let $q(t) \in \mathbb{R}[t]$ be a real polynomial. Suppose $q(t) \geq 0$ for all $t \in [a, b]$, then one of the following holds.*

1. *If $deg(q) = n = 2m$ is even, then $q(t) = q_1(t)^2 + (t-a)(b-t)q_2(t)^2$ where $deg(q_1) \leq m$ and $deg(q_2) \leq m - 1$.*
2. *If $deg(q) = n = 2m + 1$ is odd, then $q(t) = (t-a)q_1(t)^2 + (b-t)q_2(t)^2$ where $deg(q_1) \leq m$ and $deg(q_2) \leq m$.*

In our algorithms we will need to compute the polynomials $q_i(j\omega)$ from $p_i(w)$, i.e. we need computationally effective versions of Theorems 1 and 2. These are given in section 4.

To make the connection to semidefinite programming, we next characterize the polynomials nonnegative on an interval (either $[0,\infty)$ or $[a,b]$) by using certain convex cones. As introduced in [Nes00], define the vector of monomials $v(t) = [1\ t\ t^2 \cdots t^n]^T$ and the two convex cones of polynomials

$$\begin{aligned} K_{0,\infty} &= \{p \in \mathbb{R}^{n+1} : p^T v(t) \geq 0\ \forall\ t \geq 0\} \\ K_{a,b} &= \{p \in \mathbb{R}^{n+1} : p^T v(t) \geq 0\ \forall\ t \in [a,b]\}. \end{aligned}$$

Let $H_{n,i} \in S^{n+1}$ be the $i-th$ Hankel matrix, i.e., $H_{n,i}^{kl} = \begin{cases} 1, & \text{if } k+l = i+1, \\ 0, & \text{otherwise.} \end{cases}$

As introduced in [Nes00], define linear operators

$$\Lambda_1 : \mathbb{R}^{2n_1+1} \to S^{n_1+1}, \quad \Lambda_2 : \mathbb{R}^{2n_2+1} \to S^{n_2+1}$$

by the following

$$\Lambda_1(v) = \sum_{i=1}^{2n_1+1} v_i H_{n_1,i}, \quad \Lambda_2(v) = \sum_{i=1}^{2n_2+1} v_{i+1} H_{n_2,i}.$$

Another two operators Λ_3 and Λ_4 are defined according to whether n is even or odd. When $n = 2m$,

$$\Lambda_3 : \mathbb{R}^{n+1} \to S^{m+1}, \quad \Lambda_4 : \mathbb{R}^{n+1} \to S^m$$

are defined as

$$\Lambda_3(v) = \sum_{i=1}^{2m+1} v_i H_{m,i}, \quad \Lambda_4(v) = \sum_{i=1}^{2m-1} [(a+b)v_{i+1} - v_{i+2} - abv_i] H_{m-1,i}.$$

When $n = 2m+1$,

$$\Lambda_3 : \mathbb{R}^{n+1} \to S^{m+1}, \quad \Lambda_4 : \mathbb{R}^{n+1} \to S^{m+1}$$

are defined as

$$\Lambda_3(v) = \sum_{i=1}^{2m+1} [v_{i+1} - av_i] H_{m,i}, \quad \Lambda_4(v) = \sum_{i=1}^{2m+1} [bv_i - v_{i+1}] H_{m,i}.$$

Let $\Lambda_1^*, \Lambda_2^*, \Lambda_3^*, \Lambda_4^*$ be their adjoint operators respectively, with respect to the inner product $< A, B >= trace(A^T B)$ for symmetric matrices of the same size. The following theorem is a compact characterization of cones $K_{0,\infty}, K_{a,b}$ and their dual cones.

Theorem 3 (Nesterov [Nes00]). *The cones $K_{0,\infty}, K_{a,b}$ can be characterized as follows*

1. $K_{0,\infty}$ *and its dual* $K^*_{0,\infty}$ *are characterized as follows:*

$$K_{0,\infty} = \{p \in \mathbb{R}^{n+1} : p = \Lambda_1^*(Y_1) + \Lambda_2^*(Y_2), Y_1 \in S_+^{n_1+1}, Y_2 \in S_+^{n_2+1}\},$$
$$K^*_{0,\infty} = \{c \in \mathbb{R}^{n+1} : \Lambda_1(c) \succeq 0, \Lambda_2(c) \succeq 0\};$$

2. *when* $n = 2m$ *is even,*

$$K_{a,b} = \{p \in \mathbb{R}^{n+1} : p = \Lambda_3^*(Y_3) + \Lambda_4^*(Y_4), Y_3 \in S_+^{m+1}, Y_4 \in S_+^{m}\},$$
$$K^*_{a,b} = \{c \in \mathbb{R}^{n+1} : \Lambda_3(c) \succeq 0, \Lambda_4(c) \succeq 0\};$$

when $n = 2m+1$ *is odd,*

$$K_{a,b} = \{p \in \mathbb{R}^{n+1} : p = \Lambda_3^*(Y_3) + \Lambda_4^*(Y_4), Y_3 \in S_+^{m+1}, Y_4 \in S_+^{m+1}\},$$
$$K^*_{a,b} = \{c \in \mathbb{R}^{n+1} : \Lambda_3(c) \succeq 0, \Lambda_4(c) \succeq 0\};$$

3. *Both* $K_{0,\infty}(K_{a,b})$ *and* $K^*_{0,\infty}(K^*_{a,b})$ *are convex, closed, and pointed cones with non-empty interiors.*

Now suppose we have L subintervals of $[0,\infty)$: $\{[a_i, b_i]\}_{i=1}^L$. Let $K = K_{0,\infty} \times K_{a_1,b_1} \times \cdots K_{a_L,b_L}$. Then its dual $K^* = K^*_{0,\infty} \times K^*_{a_1,b_1} \times \cdots K^*_{a_L,b_L}$. Given a matrix A of $(L+1)(n+1)$ rows and $2(n+1)$ columns, consider the following problem:

$$\text{find a vector (if it exists) } p \in \mathbb{R}^{2(n+1)} \ s.t. \ Ap \in K.$$

This can be done by solving a SDP feasibility problem by Theorem 3, say, using the SDP solver in [Stu99]. However it will introduce $2(L+1)$ symmetric matrices of size $[n/2]$ or $[n/2]+1$. In order to use interior-point methods to solve it, the complexity of one iteration will be at least $O(2(L+1)n^3)$ arithmetic operations. Fortunately, the dual cone $K^*_{a,b}$ does not involve two symmetric matrices. A natural barrier function [NN94] for K^* is given by

$$F(c) = -\ln\det\Lambda_1(c_0) - \ln\det\Lambda_2(c_0) - \sum_{i=1}^{L}(\ln\det\Lambda_3^{(i)}(c_i) + \ln\det\Lambda_4^{(i)}(c_i)),$$

where $\Lambda_3^{(i)}(\Lambda_4^{(i)})$ is the operator $\Lambda_3(\Lambda_4)$ corresponding to K_{a_i,b_i} in Theorem 3. Here the vector $c = (c_0, \cdots, c_L) \in \underbrace{\mathbb{R}^{n+1} \times \cdots \mathbb{R}^{n+1}}_{L+1 \text{ times}}$. Now solve the following analytic center problem:

$$\min F(c) \tag{3}$$
$$s.t. \ A^T c = 0, c \in \text{int}K^*. \tag{4}$$

The barrier function $F(c)$ will tend to infinity as c approaches ∂K^*. Hence the minimum will be attained in the interior of K^*, which is not empty as guaranteed by Theorem 3. The optimality condition is that

$$\begin{aligned} \nabla F(c) &= A\lambda, \ c \in \mathrm{int}K^*; \\ A^T c &= 0. \end{aligned}$$

The optimal solution c^* and its Lagrange multiplier λ^* can be found very efficiently using Newton's method. For any $c \in \mathrm{int}K^*$, it can be shown [NN94] that $\nabla F(c) \prec_K 0$. Therefore a strictly feasible point $p^* = -\lambda^*$ with $Ap^* \succ_K 0$ is obtained immediately. In Newton's method, we need to evaluate the first and second derivatives of $F(c)$, which takes $O(Ln\ln^2 n + L^2 n)$ arithmetic operations by using the displacement structure of Hankel matrices [GHN00, KS95]. The interior-point methods that solve this analytic center problem take $O(\sqrt{n}\ln\frac{1}{\epsilon})$ steps [NN94] to achieve relative accuracy ϵ. Therefore, the total complexity is $O(Ln^{1.5}(\ln^2 n + L)\ln\frac{1}{\epsilon})$.

3 Shape optimization

In this section, we will show how to design the transfer function of a LTI SISO system so that it has a desired Bode plot. Four kinds of shapes will be discussed: standard bandpass filter, piecewise constant, piecewise polynomial, and general shapes.

3.1 Bandpass filter design

The goal is to design a transfer function $|H(j\omega)|^2 = \frac{p_1(w)}{p_2(w)}$ which is close to one on some squared frequency ($w = \omega^2$) interval $[w^\ell, w^r]$ and tiny in a neighborhood just outside this interval. The design rules can be formulated as

$$\begin{aligned} p_1(w), p_2(w) &\geq 0, \ \forall\, w \geq 0 \\ 1 - \alpha \leq \frac{p_1(w)}{p_2(w)} &\leq 1 + \beta, \ \forall\, w \in [w^\ell, w^r] \\ \frac{p_1(w)}{p_2(w)} &\leq \delta, \ \forall\, w \in [w_1^\ell, w_2^\ell] \cup [w_1^r, w_2^r] \end{aligned}$$

where the interval $[w_1^\ell, w_2^\ell]$ is to the left of $[w^\ell, w^r]$, and $[w_1^r, w_2^r]$ is to the right. Here α and β are tiny tolerance parameters (say around .05). Let p_1 and p_2 be the vectors of coefficients of $p_1(w)$ and $p_2(w)$ respectively. Then the constraints above can we restated as

$$\begin{aligned} p_1, p_2 &\in K_{0,\infty} \\ p_1 - (1-\alpha)p_2 &\in K_{w^\ell, w^r} \\ (1+\beta)p_2 - p_1 &\in K_{w^\ell, w^r} \\ \delta p_2 - p_1 &\in K_{w_1^\ell, w_2^\ell} \cap K_{w_1^r, w_2^r} \end{aligned}$$

Using Theorem 3, we see that the above cone constraints can be expressed as $Ap \in K$ where

$$A = \begin{bmatrix} I_{n+1} & 0 \\ 0 & I_{n+1} \\ I_{n+1} & (\alpha-1)I_{n+1} \\ -I_{n+1} & (1+\beta)I_{n+1} \\ -I_{n+1} & \delta I_{n+1} \\ -I_{n+1} & \delta I_{n+1} \end{bmatrix}, \ p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},$$

and $K = K_{0,\infty} \times K_{0,\infty} \times K_{w^\ell,w^r} \times K_{w^\ell,w^r} \times K_{w_1^\ell,w_2^\ell} \times K_{w_1^r,w_2^r}$. Given (α, β, δ), solve the analytic center problem (3)-(4) and then recover the coefficients p.

As introduced in [GHN00] for the discrete case, we can also consider the following objectives:

- minimize $\alpha + \beta$ for fixed δ and n
- minimize δ for fixed α, β, and n
- minimize the degree n of p_1 and p_2 for fixed α, β, and δ.

These optimization problems with objectives are no longer convex, but quasi-convex. This means that we can use bisection to find the solution by solving a sequence of analytic center problems.

A design example is given in figure 1. For the simplicity of programming, we used SeDuMi[Stu99] to solve the primal feasibility problem.

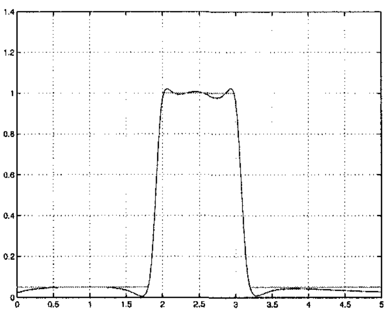

Fig. 1. The design filter shape for $[w^l \ w^r] = [2\ 3]$, $[w_1^l \ w_2^l] = [0\ 1.8]$, $[w_1^r \ w_2^r] = [3.2\ 5]$, $\alpha = \beta = 0.05$, $\delta = 0.05$, $n = 10$.

3.2 Piecewise constant shape design

Here we extend the shape design technique from the last section to piecewise constant shapes. In other words we want the transfer function to be close to given constant values $c_1, ..., c_m$ in a set of m disjoint intervals $\omega^2 = w \in [a_k, b_k]$, where $a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m$. More precisely we want the transfer function to lie in the interval $[(1-\alpha)c_k, (1+\beta)c_k]$ for $w \in [a_k, b_k]$. By picking enough intervals (picking m large enough) we can approximate any continuous function as closely as we like.

These constraints may be written

$$p_1(w), p_2(w) \geq 0, \; \forall \; w \geq 0$$
$$(1-\alpha)c_k \leq \frac{p_1(w)}{p_2(w)} \leq (1+\beta)c_k, \; \forall \; w \in [a_k, b_k], \; k = 1, \cdots, m.$$

Using Theorem 3 as before, these constraints can be rewritten as the cone constraints

$$p_1(w), p_2(w) \in K_{0,\infty}$$
$$p_1 - (1-\alpha)c_k p_2, \; (1+\beta)c_k p_2 - p_1 \in K_{a_k, b_k}, k = 1, \cdots, m.$$

As before, find vector p such that $Ap \in K$ where

$$A = \begin{bmatrix} I_{n+1} & 0 \\ 0 & I_{n+1} \\ I_{n+1} & (\alpha-1)c_1 I_{n+1} \\ (1+\beta)c_1 I_{n+1} & -I_{n+1} \\ \vdots & \vdots \\ I_{n+1} & (\alpha-1)c_m I_{n+1} \\ (1+\beta)c_m I_{n+1} & -I_{n+1} \end{bmatrix}, \; p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},$$

and $K = K^2_{0,\infty} \times K^2_{a_1,b_1} \times \cdots \times K^2_{a_m,b_m}$. Solve the analytic center problem (3)-(4) again and recover the vector p.

As in the preceding subsection, various design objectives can be considered by applying bisection. A design example for a step function with 3 steps is given in figure 2.

3.3 Piecewise polynomial shape design

Here we extend the techniques of the last section to piecewise polynomials. Thus, on each interval $[a_k, b_k]$ we ask that $p_1(w)/p_2(w)$ be close to a given polynomial $\phi_k(w)$, in particular that it lie in an interval $[(1-\alpha)\phi_k(w), (1+\beta)\phi_k(w)]$. This leads to the constraints

$$p_1(w), p_2(w) \geq 0, \; \forall \; w \geq 0$$
$$(1-\alpha)\phi_k(w) \leq \frac{p_1(w)}{p_2(w)} \leq (1+\beta)\phi_k(w), \; \forall \; w \in [a_k, b_k], k = 1, \cdots, m.$$

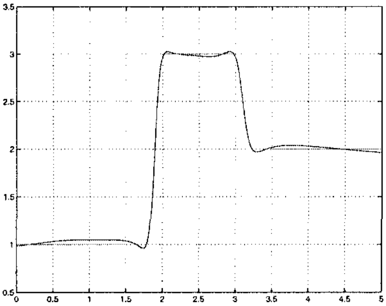

Fig. 2. The step function shape design for $[a_1\ b_1] = [0\ 1.8]$, $[a_2\ b_2] = [2\ 3]$, $[a_3\ b_3] = [3.2\ 5]$, $c_1 = 1, c_2 = 3, c_3 = 2$, $\alpha = \beta = 0.05$, $n = 10$.

Using Theorem 3 once again, we transform these to the following cone constraints

$$\begin{aligned} p_1, p_2 &\in K_{0,\infty} \\ p_1 - (1-\alpha)\phi_k(w)p_2(w) &\in K_{a_k,b_k}, k = 1, \cdots, m; \\ (1+\beta)p_2(w)\phi_k(w) - p_1 &\in K_{a_k,b_k}, k = 1, \cdots, m. \end{aligned}$$

which are again a set of linear equations and LMI's. As before, p_1 and p_2 can be obtained by solving some appropriate analytic center problem (3)-(4), and bisection can be used to achieve certain design goals.

3.4 General shape design

So far we have considered bandpass filter design, piecewise constant shape design, and piecewise polynomial shape design. Here we discuss general shape design. The goal is to design a transfer function $|H(j\omega)|^2 = \frac{p_1(w)}{p_2(w)}$ so that it behaves like some general nonnegative function $f(w)$ for $w \in [a, b]$ where $0 < a < b$. In other words we want:

$$p_1(w), p_2(w) \geq 0, \ \forall w \in [0, \infty) \tag{5}$$

$$(1-\alpha)f(w) \leq \frac{p_1(w)}{p_2(w)} \leq (1+\beta)f(w), \ \forall w \in [a, b]. \tag{6}$$

Now we can not apply Theorem 3 directly, and must instead apply approximation methods.

One obvious approach is to partition $[a, b]$ into subintervals $\{[a_k, b_k]\}_{k=1}^m$ and approximate $f(w)$ by a constant or more general polynomial in each subinterval. Then we can apply the method from the preceding sections.

Another approach is to apply semi-infinite programming(SIP), as described in [WBV97]. The idea is to choose N sample points

$$a \le w_1 < w_2 < \cdots < w_N \le b$$

and replace the semi-infinite inequality constraints (5)-(6) by N simple inequality constraints. A standard rule of thumb is to choose $N = 15n$ in practice [WBV97]. Then the approximate optimization problem is solved iteratively [Pol97].

3.5 Related work

There are several related papers [AV02, GHN00, WBV97] on various filter design techniques. Most of them are for discrete systems, and some techniques can be applied to continuous systems with some modifications. Note that the mapping $s = \frac{z+1}{z-1}$ maps the pure imaginary axis onto the unit circle except $(-1, 0)$. Then our transfer function $H(s)$ becomes a rational trigonometric function $R(z)$. Each interval $j[a_i, b_i]$ is mapped onto some arc $\{e^{j\omega} : \omega \in [\omega_i^1, \omega_i^2]\}$ on the unit circle. The methods described in [AV02, GHN00, WBV97] all can be applied. However, all of them will involve the constraints of the form that some trigonometric polynomial is nonnegative on some interval. The characterization of (trigonometric) polynomials nonnegative on some intervals will eventually need Theorem 3 or its equivalent form to transform to a LMI. In this paper, we transform our design problems using constraints of real polynomial nonnegativity on some intervals in the positive semi-axis. Then we may apply Theorem 3 directly to characterize these constraints using LMIs. As described at the end of Section 2, we solve an appropriate analytic center problem, instead of solving these LMIs directly. The structure of this problem can be exploited to use Newton's method to efficiently find the analytic center.

There are also several good papers [Fab02, Nes00, GHY03] on polynomials on the real axis, unit circle, pure imaginary axis, and other curves. [Nes00] is the classical paper that characterizes polynomial nonnegativity constraints by LMIs; our paper is based mostly on it. In [GHY03] the authors characterized the cone of positive pseudopolynomial matrices and discussed optimization over this cone. The authors also discussed the conditioning of such optimizations, and proposed using the basis of Chebyshev polynomials to improve conditioning. [Fab02] gives an abstract version of [Nes00, GHY03], characterizing polynomials which are nonnegative on the disjoint union of several intervals.

4 Recovery of the transfer function

In this section, we show how to use Theorems 1 and 2 effectively.

First, given polynomials $p_1(w)$ and $p_2(w)$ ($w = \omega^2$) such that $\frac{p_1}{p_2}$ has some desired shape, we need to find real polynomials q_1 and q_2 so that

$$\frac{p_1(w)}{p_2(w)} = \left| \frac{q_1(j\omega)}{q_2(j\omega)} \right|^2 .$$

To this end, given a polynomial $p(w)$ that is nonnegative on $[0,\infty)$, we will provide an algorithm to find two polynomials $q_e(w)$ and $q_o(w)$ such that $p(w) = q_e^2(w) + w \cdot q_o^2(w)$. Then q_e contains the even coefficients and q_o the odd coefficients (modulo signs) of the desired polynomial q as described in Section 2.

Lemma 1. *The following polynomial identities hold:*

1. $(f_1^2 + wg_1^2)(f_2^2 + wg_2^2) = (f_1 f_2 + w g_1 g_2)^2 + w(f_1 g_2 - f_2 g_1)^2$;
2. $(w-r)^2 + b^2 = (w - \sqrt{r^2+b^2})^2 + w \cdot 2(\sqrt{r^2+b^2} - r)$.

Proof. Verify directly. □

Lemma 2. *If a polynomial $p(w)$ is nonnegative on $[0,\infty)$, then its factorization must have the form*

$$p(w) = \alpha \left(\prod_{i=1}^{n_1} (w + c_i) \right)^2 \prod_{i=1}^{n_2} \left((w - r_i)^2 + b_i^2 \right) \prod_{i=1}^{n_3} (w + a_i)$$

where $\alpha \geq 0$, $b_i > 0$, $n_1 + 2n_2 + n_3 = n$, $0 \leq a_1 \leq \cdots \leq a_{n_3}$, $c_i < 0$.

Proof.

Write the factorization $p(w) = \alpha \prod_{k=1}^{n} (w - \rho_k)$. First consider the constant term α; it must clearly satisfy $\alpha \geq 0$ for $p(w)$ to be nonnegative over $[0,\infty)$. Next consider the three classes of roots ρ_k: real positive, complex, and real nonpositive. The positive roots must all have even multiplicity for $p(w)$ to be nonnegative, so we can write the product of all their factors $w - \rho_k$ as $\prod_{i=1}^{n_1}(w + c_i)^2$ where $c_i < 0$. Next, the complex roots come in complex conjugate pairs $\rho_k = r_k + j \cdot b_k$ and $\bar{\rho}_k = r_k - j \cdot b_k$, so we can write $(w - \rho_k)(w - \bar{\rho}_k) = (w - r_k)^2 + b_k^2$ for all n_2 complex conjugate pairs. Finally consider the n_3 nonpositive real roots $0 \geq -a_1 \geq \cdots \geq -a_{n_3}$ of $p(w)$. Their corresponding factors $w - (-a_i) = w + a_i$ are all nonnegative on $[0,\infty)$.

□

Using the above two lemmas, we get the following algorithm.

Algorithm 4.1 *This algorithm will find $q_e(w)$ and $q_o(w)$ such that $p(w) = q_e^2(w) + w \cdot q_o^2(w)$ if $p(w)$ is nonnegative on $[0,\infty)$. Let $q_e = 1, q_o = 0$.*

Step 1 Find the factorization of
$p(w) = \alpha \left(\prod_{i=1}^{n_1}(w + c_i)\right)^2 \prod_{i=1}^{n_2}(w^2 + b_i^2) \prod_{i=1}^{n_3}(w + a_i)$
where $b_i > 0, n_1 + 2n_2 + n_3 = n, 0 \leq a_1 \leq \cdots \leq a_{n_3}, 0 > c_i \in \mathbb{R}$.

Step 2 Find the $(\cdot)^2 + w(\cdot)^2$ form of $\prod_{i=1}^{n_3}(w + a_i)$
for $k = 1 : n_3$
$\hat{q}_e = \sqrt{a_i} q_e + w \cdot q_o$
$\hat{q}_o = \sqrt{a_i} q_o - q_e$
$q_e := \hat{q}_e, q_o := \hat{q}_o$.
end

Step 3 Find the $(\cdot)^2 + w(\cdot)^2$ *form of* $\prod_{i=1}^{n_2}(w^2 + b_i^2)$
for $k = 1 : n_2$

$$\hat{q}_e = q_e(w - \sqrt{r_i^2 + b_i^2}) + w \cdot q_o\sqrt{2(\sqrt{r_i^2 + b_i^2} - r_i)}$$
$$\hat{q}_o = q_e\sqrt{2(\sqrt{r_i^2 + b_i^2} - r_i)} - q_o(w - \sqrt{r_i^2 + b_i^2})$$
$$q_e := \hat{q}_e, q_o := \hat{q}_o.$$

end

Step 4 $q_e := \alpha q_e \prod_{i=1}^{n_1}(w + c_i), \; q_o := \alpha q_o \prod_{i=1}^{n_1}(w + c_i).$

Now apply algorithm 4.1 to $p_i(w)(i = 1, 2)$ respectively, i.e., find $q_{ij}(w)(i, j = 1, 2)$ such that $p_i(w) = q_{i,1}^2(w) + w \cdot q_{i,2}^2(w)$ for $i = 1, 2$. Then we obtain the desired transfer function

$$H(s) = \frac{q_{1,1}(-s^2) + sq_{1,2}(-s^2)}{q_{2,1}(-s^2) + sq_{2,2}(-s^2)}.$$

Remark: If a polynomial $p(w)$ is nonnegative on a finite interval $[-1, 1]$ (another finite interval can be changed to this one by a linear transformation), then we can also apply the above algorithm to find two polynomials p_1 and p_2 such that $p(w) = p_1^2(w) + (1 - w)(w + 1)p_2^2(w)$. Actually, we only need to do the *Goursat transform* (see [PR00]) for $p(w)$, i.e.,

$$\tilde{p}(w) = (w + 1)^n p(\frac{1 - w}{w + 1}),$$

and then apply the above algorithm to find q_e, q_o such that

$$\tilde{p}(w) = q_e^2(w) + w \cdot q_o^2(w),$$

and then apply the inverse Goursat transform to get back $p(w)$:

$$p(w) = 2^{-n}(w + 1)^{deg(\tilde{p})}\tilde{p}(\frac{1 - w}{1 + w}).$$

5 Conclusions and discussion

This paper discusses shape optimization for a transfer function for a LTI SISO system by formulating it via semidefinite programming. Given the shape (absolute value) of the transfer function, we show how to extract the transfer function itself. Since the optimization process uses semidefinite programming, it may be done efficiently.

We do not consider any constraints on the components A, b, c, d of the LTI system. However in practice, these components may not be arbitrary, but instead have special structure and depend on certain design parameters. Thus an interesting question is finding those parameters to optimize the shape of the transfer function as we did in Section 3. This is in general not a convex problem, and can be very hard to solve. But it is still a feasibility/optimization problem about polynomials, if (A, b, c, d) are polynomials in those parameters. Therefore, we may formulate them using polynomial optimization, and then solve them by techniques such as the sum of squares and positivstellensatz (see [Par01]). But this is more difficult, and future work.

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