1. Energy Transfer Between Radiation and Atomic Transitions

In this introductory chapter we shall outline the basic ideas underlying the operation of solid-state lasers. In-depth treatments of laser physics can be found in a number of excellent textbooks [1.1, 2].

1.1 Optical Amplification

To understand the operation of a laser we have to know some of the principles governing the interaction of radiation with matter.

Atomic systems such as atoms, ions, and molecules can exist only in discrete energy states. A change from one energy state to another, called a transition, is associated with either the emission or the absorption of a photon. The wavelength of the absorbed or emitted radiation is given by Bohr's frequency relation

$$E_2 - E_1 = h\nu_{21}, \tag{1.1}$$

where E_2 and E_1 are two discrete energy levels, ν_{21} is the frequency, and h is the Planck's constant. An electromagnetic wave whose frequency ν_{21} corresponds to an energy gap of such an atomic system can interact with it. To the approximation required in this context, a solid-state material can be considered an ensemble of very many identical atomic systems. At thermal equilibrium, the lower energy states in the material are more heavily populated than the higher energy states. A wave interacting with the substance will raise the atoms or molecules from lower to higher energy levels and thereby experience absorption.

The operation of a laser requires that the energy equilibrium of a laser material be changed such that energy is stored in the atoms, ions, or molecules of this material. This is achieved by an external pump source which transfers electrons from a lower energy level to a higher one. The pump radiation thereby causes a "population inversion." An electromagnetic wave of appropriate frequency, incident on the "inverted" laser material, will be amplified because the incident photons cause the atoms in the higher level to drop to a lower level and thereby emit additional photons. As a result, energy is extracted from the atomic system and supplied to the radiation field. The release of the stored energy by interaction with an electromagnetic wave is based on stimulated or induced emission.

Stated very briefly, when a material is excited in such a way as to provide more atoms (or molecules) in a higher energy level than in some lower level, the material will be capable of amplifying radiation at the frequency corresponding to the energy

level difference. The acronym "laser" derives its name from this process: "Light Amplification by Stimulated Emission of Radiation."

A quantum mechanical treatment of the interaction between radiation and matter demonstrates that the stimulated emission is, in fact, completely indistinguishable from the stimulating radiation field. This means that the stimulated radiation has the same directional properties, same polarization, same phase, and same spectral characteristics as the stimulating emission. These facts are responsible for the extremely high degree of coherence, which characterizes the emission from lasers. The fundamental nature of the induced or stimulated emission process has already been described by A. Einstein and M. Planck.

In solid-state lasers, the energy levels and the associated transition frequencies result from the different quantum energy levels or allowed quantum states of the electrons orbiting about the nuclei of atoms. In addition to the electronic transitions, multiatom molecules in gases exhibit energy levels that arise from the vibrational and rotational motions of the molecule as a whole.

1.2 Interaction of Radiation with Matter

Many of the properties of a laser may be readily discussed in terms of the absorption and emission processes which take place when an atomic system interacts with a radiation field. In the first decade of this century Planck described the spectral distribution of thermal radiation, and in the second decade Einstein, by combining Planck's law and Boltzmann statistics, formulated the concept of stimulated emission. Einstein's discovery of stimulated emission provided essentially all of the theory necessary to describe the physical principle of the laser.

1.2.1 Blackbody Radiation

When the electromagnetic radiation in an isothermal enclosure, or cavity, is in thermal equilibrium at temperature T, the distribution of radiation density $\varrho(\nu)\,d\nu$, contained in a bandwidth $d\nu$, is given by Planck's law

$$\varrho(\nu)\,d\nu = \frac{8\pi\nu^2\,d\nu}{c^3}\frac{h\nu}{e^{h\nu/kT}-1}, \tag{1.2}$$

where $\varrho(\nu)$ is the radiation energy density per unit frequency ((J s)/cm^3), k is Boltzmann's constant, and c is the velocity of light. The spectral distribution of thermal radiation energy vanishes at $\nu = 0$ and $\nu \to \infty$, and has a peak, which depends on the temperature.

The factor

$$\frac{8\pi\nu^2}{c^3} = p_n \tag{1.3}$$

in (1.2) gives the density of radiation modes per unit volume and unit frequency interval. The factor p_n can also be interpreted as the number of degrees of freedom associated with a radiation field, per unit volume, per unit frequency interval. The expression for the mode density p_n ((modes s)/cm^3) plays an important role in connecting the spontaneous and the induced transition probabilities.

For a uniform, isotropic radiation field, the following relationship is valid:

$$W = \frac{\varrho(\nu)c}{4}, \tag{1.4}$$

where W is the blackbody radiation (W/cm^2), which will be emitted from an opening in the cavity of the blackbody. Many solids radiate like a blackbody. Therefore, the radiation emitted from the surface of a solid can be calculated from (1.4).

According to the Stefan–Boltzmann equation, the total black body radiation is

$$W = \sigma T^4, \tag{1.5}$$

where $\sigma = 5.68 \times 10^{-12}$ W/(cm^2 K^4). The emitted radiation W has a maximum which is obtained from Wien's displacement law

$$\frac{\lambda_{\max}}{\mu\mathrm{m}} = \frac{2893}{T/\mathrm{K}}. \tag{1.6}$$

For example, a blackbody at a temperature of 5200 K has its radiation peak at 5564 Å, which is about the center of the visible spectrum.

A good introduction to the fundamentals of radiation and its interaction with matter can be found in [1.2].

1.2.2 Boltzmann's Statistics

According to a basic principle of statistical mechanics, when a large collection of similar atoms is in thermal equilibrium at temperature T, the relative populations of any two energy levels E_1 and E_2, such as the ones shown in Fig. 1.1, must be related by the Boltzmann ratio

$$\frac{N_2}{N_1} = \exp\left(\frac{-(E_2 - E_1)}{kT}\right), \tag{1.7}$$

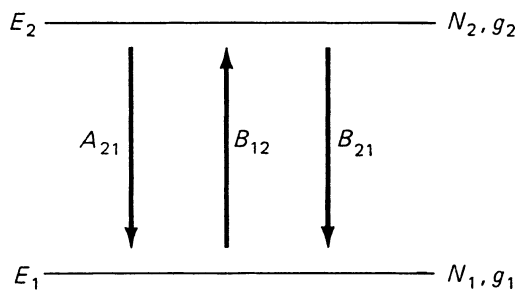

Fig. 1.1. Two energy levels with population N_1, N_2 and degeneracies g_1, g_2, respectively

where N_1 and N_2 are the number of atoms in the energy levels E_1 and E_2, respectively. For energy gaps large enough that $E_2 - E_1 = h\nu_{21} \gg kT$, the ratio is close to zero, and there will be very few atoms in the upper energy level at thermal equilibrium. The thermal energy kT at room temperature ($T \approx 300$ K) corresponds to an energy gap $h\nu$ with $\nu \approx 6 \times 10^{12}$ Hz, which is equivalent in wavelength to $\lambda \approx 50\,\mu$m. Therefore, for any energy gap whose transition frequency ν_{21} lies in the near-infrared or visible regions, the Boltzmann exponent will be very small at normal temperatures. The number of atoms in any upper level will then be very small compared to the lower levels. For example, in ruby the ground level E_1 and the upper laser level E_2 are separated by an energy gap corresponding to a wavelength of $\lambda \approx 0.69\,\mu$m. Since $h = 6.6 \times 10^{-34}$ W s^2, then $E_2 - E_1 = hc/\lambda = 2.86 \times 10^{-19}$ W s. With $k = 1.38 \times 10^{-23}$ (W s)/K and $T = 300$ K, it follows that $N_2/N_1 \approx \exp(-69)$. Therefore at thermal equilibrium virtually all the atoms will be in the ground level.

Equation (1.7) is valid for atomic systems having only nondegenerate levels. If there are g_i different states of the atom corresponding to the energy E_i, then g_i is defined as the degeneracy of the ith energy level.

We recall that atomic systems, such as atoms, ions, molecules, can exist only in certain stationary states, each of which corresponds to a definite value of energy and thus specifies an energy level. When two or more states have the same energy, the respective level is called degenerate, and the number of states with the same energy is the multiplicity of the level. All states of the same energy level will be equally populated therefore the number of atoms in levels 1 and 2 is $N_1 = g_1 N_1'$ and $N_2 = g_2 N_2'$, where N_1' and N_2' refer to the population of any of the states in levels 1 and 2, respectively. It follows then from (1.7) that the populations of the energy levels 1 and 2 are related by the formula

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}\frac{N_2'}{N_1'} = \frac{g_2}{g_1}\exp\left(\frac{-(E_2 - E_1)}{kT}\right). \tag{1.8}$$

At absolute zero temperature, Boltzmann's statistics predicts that all atoms will be in the ground state. Thermal equilibrium at any temperature requires that a state with a lower energy be more densely populated than a state with a higher energy. Therefore, N_2/N_1 is always less than unity for $E_2 > E_1$ and $T > 0$. This means that optical amplification is not possible in thermal equilibrium.

1.2.3 Einstein's Coefficients

We can most conveniently introduce the concept of Einstein's A and B coefficients by loosely following Einstein's original derivation. To simplify the discussion, let us consider an idealized material with just two nondegenerate energy levels 1 and 2 having populations of N_1 and N_2, respectively. The total number of atoms in these two levels is assumed to be constant

$$N_1 + N_2 = N_{\text{tot}}. \tag{1.9}$$

Radiative transfer between the two energy levels which differ by $E_2 - E_1 = h\nu_{21}$ is allowed. The atom can transfer from state E_2 to the ground state E_1 by emitting energy; conversely, transition from state E_1 to E_2 is possible by absorbing energy. The energy removed or added to the atom appears as quanta of $h\nu_{21}$. We can identify three types of interaction between electromagnetic radiation and a simple two-level atomic system: absorption, spontaneous emission, and stimulated emission.

Absorption. If a quasimonochromatic electromagnetic wave of frequency ν_{21} passes through an atomic system with energy gap $h\nu_{21}$, then the population of the lower level will be depleted at a rate proportional to both the radiation energy density $\varrho(\nu)$ and the population N_1 of that level

$$\frac{\partial N_1}{\partial t} = -B_{12}\varrho(\nu)N_1, \tag{1.10}$$

where B_{12} is a constant of proportionality with dimensions $\mathrm{cm}^3/(\mathrm{s}^2\ \mathrm{J})$.

The product $B_{12}\varrho(\nu)$ can be interpreted as the probability per unit frequency for which transitions are induced by the effect of the field.

Spontaneous Emission. After an atom has been raised to the upper level by absorption, the population of the upper level decays spontaneously to the lower level at a rate proportional to the upper level population.

$$\frac{\partial N_2}{\partial t} = -A_{21}N_2, \tag{1.11}$$

where A_{21} is a constant of proportionality with dimensions s^{-1}. The quantity A_{21}, being a characteristic of the pair of energy levels in question, is called the spontaneous transition probability because this coefficient gives the probability that an atom in level 2 will spontaneously change to a lower level 1 within a unit of time.

Spontaneous emission is a statistical function of space and time. With a large number of spontaneously emitting atoms there is no phase relationship between the individual emission processes; the quanta emitted are incoherent. Spontaneous emission is characterized by the lifetime of the electron in the excited state, after which it will spontaneously return to the lower state and radiate away the energy. This can occur without the presence of an electromagnetic field.

Equation (1.11) has a solution

$$N_2(t) = N_2(0)\ \exp\left(\frac{-t}{\tau_{21}}\right), \tag{1.12}$$

where τ_{21} is the lifetime for spontaneous radiation of level 2. This radiation lifetime is equal to the reciprocal of the Einstein's coefficient,

$$\tau_{21} = A_{21}^{-1}. \tag{1.13}$$

In general, the reciprocal of the transition probability of a process is called its lifetime.

Stimulated Emission. Emission takes place not only spontaneously but also under stimulation by electromagnetic radiation of appropriate frequency. In this case, the atom gives up a quantum to the radiation field by "induced emission" according to

$$\frac{\partial N_2}{\partial t} = -B_{21}\varrho(\nu_{21})N_2, \tag{1.14}$$

where B_{21} again is a constant of proportionality.

Radiation emitted from an atomic system in the presence of external radiation consists of two parts. The part whose intensity is proportional to A_{21} is the spontaneous radiation; its phase is independent of that of the external radiation. The part whose intensity is proportional to $\varrho(\nu)B_{21}$ is the stimulated radiation; its phase is the same as that of the stimulating external radiation.

The probability of induced transition is proportional to the energy density of external radiation in contrast to spontaneous emission. In the case of induced transition there is a firm phase relationship between the stimulating field and the atom. The quantum which is emitted to the field by the induced emission is coherent with it.

But we shall see later that the useful parameter for laser action is the B_{21} coefficient; the A_{21} coefficient represents a loss term and introduces into the system photons that are not phase-related to the incident photon flux of electric field. Thus the spontaneous process represents a noise source in a laser.

If we combine absorption, spontaneous, and stimulated emission, as expressed by (1.10, 11, and 14), we can write for the change of the upper and lower level populations in our two-level model

$$\frac{\partial N_1}{\partial t} = -\frac{\partial N_2}{\partial t} = B_{21}\varrho(\nu)N_2 - B_{12}\varrho(\nu)N_1 + A_{21}N_2. \tag{1.15}$$

The relation

$$\frac{\partial N_1}{\partial t} = -\frac{\partial N_2}{\partial t} \tag{1.16}$$

follows from (1.9).

In thermal equilibrium, the number of transitions per unit time from E_1 to E_2 must be equal to the number of transitions from E_2 to E_1. Certainly, in thermal equilibrium

$$\frac{\partial N_1}{\partial t} = \frac{\partial N_2}{\partial t} = 0. \tag{1.17}$$

Therefore we can write

$$\begin{array}{ccccc} N_2A_{21} & + & N_2\varrho(\nu)B_{21} & = & N_1\varrho(\nu)B_{12} \\ \text{Spontaneous} & & \text{Stimulated} & & \text{Absorpion}. \\ \text{emission} & & \text{emission} & & \end{array} \tag{1.18}$$

Using the Boltzmann equation (1.8) for the ratio N_2/N_1, we then write the above expression as

$$\varrho(\nu_{21}) = \frac{(A_{21}/B_{21})}{(g_1/g_2)(B_{12}/B_{21})\exp(h\nu_{21}/kT) - 1}. \tag{1.19}$$

Comparing this expression with the blackbody radiation law (1.2), we see that

$$\frac{A_{21}}{B_{21}} = \frac{8\pi\nu^2 h\nu}{c^3} \quad \text{and} \quad B_{21} = \frac{g_1 B_{12}}{g_2}. \tag{1.20}$$

The relations between the A's and B's are known as Einstein's relations. The factor $8\pi\nu^2/c^3$ in (1.20) is the mode density p_n given by (1.3).

In solids the speed of light is $c = c_0/n$, where n is the index of refraction and c_0 is the speed of light in vacuum.

For a simple system with no degeneracy, that is, one in which $g_1 = g_2$, we see that $B_{21} = B_{12}$. Thus, the Einstein coefficients for stimulated emission and absorption are equal. If the two levels have unequal degeneracy, the probability for stimulated absorption is no longer the same as that for stimulated emission.

1.2.4 Phase Coherence of Stimulated Emission

The stimulated emission provides a phase-coherent amplification mechanism for an applied signal. The signal extracts from the atoms a response that is directly proportional to, and phase-coherent with, the electric field of the stimulating signal. Thus the amplification process is phase-preserving. The stimulated emission is, in fact, completely indistinguishable from the stimulating radiation field. This means that the stimulated emission has the same directional properties, same polarization, same phase, and same spectral characteristics as the stimulating emission. These facts are responsible for the extremely high degree of coherence, which characterizes the emission from lasers. The proof of this fact is beyond the scope of this elementary introduction, and requires a quantum mechanical treatment of the interaction between radiation and matter. However, the concept of induced transition, or the interaction between a signal and an atomic system, can be demonstrated, qualitatively, with the aid of the classical electron-oscillator model.

Electromagnetic radiation interacts with matter through the electric charges in the substance. Consider an electron which is elastically bound to a nucleus. One can think of electrons and ions held together by spring-type bonds which are capable of vibrating around equilibrium positions. An applied electric field will cause a relative displacement between electron and nucleus from their equilibrium position. They will execute an oscillatory motion about their equilibrium position. Therefore, the model exhibits an oscillatory or resonant behavior and a response to an applied field. Since the nucleus is so much heavier than the electron, we assume that only the electron moves. The most important model for understanding the interaction of light and matter is that of the harmonic oscillator. We take as our model a single electron, assumed to be bound to its equilibrium position by a linear restoring force. We may visualize the electron as a point of mass suspended by springs. Classical electromagnetic theory asserts that any oscillating electric charge will act as a miniature antenna or dipole and will continuously radiate away electromagnetic energy to its surroundings.

A detailed description of the electric dipole transition and the classical electron-oscillator model can be found in [1.3].

1.3 Absorption and Optical Gain

In this section we will develop the quantitative relations that govern absorption and amplification processes in substances. This requires that we increase the realism of our mathematical model by introducing the concept of atomic lineshapes. Therefore, the important features and the physical processes which lead to different atomic lineshapes will be considered first.

1.3.1 Atomic Lineshapes

In deriving Einstein's coefficients we have assumed a monochromatic wave with frequency ν_{21} acting on a two-level system with an infinitely sharp energy gap $h\nu_{21}$. We will now consider the interaction between an atomic system having a finite transition linewidth $\Delta\nu$ and a signal with a bandwidth $d\nu$.

Before we can obtain an expression for the transition rate for this case, it is necessary to introduce the concept of the atomic lineshape function $g(\nu, \nu_0)$. The distribution $g(\nu, \nu_0)$, centered at ν_0, is the equilibrium shape of the linewidth-broadened transitions. Suppose that N_2 is the total number of ions in the upper energy level considered previously. The spectral distribution of ions per unit frequency is then

$$N(\nu) = g(\nu, \nu_0)N_2. \tag{1.21}$$

If we integrate both sides over all frequencies we have to obtain N_2 as a result:

$$\int_0^\infty N(\nu)\,d\nu = N_2 \int_0^\infty g(\nu, \nu_0)\,d\nu = N_2. \tag{1.22}$$

Therefore the lineshape function must be normalized to unity:

$$\int_0^\infty g(\nu, \nu_0)\,d\nu = 1. \tag{1.23}$$

If we know the function $g(\nu, \nu_0)$, we can calculate the number of atoms $N(\nu)\,d\nu$ in level 1 which are capable of absorbing in the frequency range ν to $\nu + d\nu$, or the number of atoms in level 2 which are capable of emitting in the same range. From (1.21) we have

$$N(\nu)\,d\nu = g(\nu, \nu_0)\,d\nu\,N_2. \tag{1.24}$$

From the foregoing it follows that $g(\nu, \nu_0)$ can be defined as the probability of emission or absorption per unit frequency. Therefore $g(\nu)\,d\nu$ is the probability that a given transition will result in an emission (or absorption) of a photon with energy between $h\nu$ and $h(\nu + d\nu)$. The probability that a transition will occur between $\nu = 0$ and $\nu = \infty$ has to be 1.

It is clear from the definition of $g(\nu, \nu_0)$ that we can, for example, rewrite (1.11) in the form

$$-\frac{\partial N_2}{\partial t} = A_{21}N_2 g(\nu, \nu_0)\,d\nu, \tag{1.25}$$

where N_2 is the total number of atoms in level 2 and $\partial N_2/\partial t$ is the number of photons spontaneously emitted per second between ν and $\nu + d\nu$.

The linewidth and lineshape of an atomic transition depends on the cause of line broadening. Optical frequency transitions in gases can be broadened by lifetime, collision, or Doppler broadening, whereas transitions in solids can be broadened by lifetime, dipolar or thermal broadening, or by random inhomogeneities. All these linewidth-broadening mechanisms lead to two distinctly different atomic lineshapes, the homogeneously and the inhomogeneously broadened line [1.4].

The Homogeneously Broadened Line

The essential feature of a homogeneously broadened atomic transition is that every atom has the same atomic lineshape and frequency response, so that a signal applied to the transition has exactly the same effect on all atoms in the collection. This means that within the linewidth of the energy level each atom has the same probability function for a transition.

Differences between homogeneously and inhomogeneously broadened transitions show up in the saturation behavior of these transitions. This has a major effect on the laser operation. The important point about a homogeneous lineshape is that the transition will saturate uniformly under the influence of a sufficiently strong signal applied anywhere within the atomic linewidth.

Mechanisms which result in a homogeneously broadened line are lifetime broadening, collision broadening, dipolar broadening, and thermal broadening.

Lifetime Broadening. This type of broadening is caused by the decay mechanisms of the atomic system. Spontaneous emission or fluorescence has a radiative lifetime. Broadening of the atomic transition due to this process is related to the fluorescence lifetime τ by $\Delta\omega_a\tau = 1$, where $\Delta\omega_a$ is the bandwidth.

Actually, physical situations in which the lineshape and linewidth are determinded by the spontaneous emission process itself are vanishingly rare. Since the natural or intrinsic linewidth of an atomic line is extremely small, it is the linewidth that would be observed from atoms at rest without interaction with one another.

Collision Broadening. Collision of radiating particles (atoms or molecules) with one another and the consequent interruption of the radiative process in a random manner leads to broadening. As an atomic collision interrupts either the emission or the absorption of radiation, the long wave train which otherwise would be present becomes truncated. The atom restarts its motion after the collision with a completely random initial phase. After the collision the process is restarted without memory of the phase of the radiation prior to the collision. The result of frequent collisions is the presence of many truncated radiative or absorptive processes.

Since the spectrum of a wave train is inversely proportional to the length of the train, the linewidth of the radiation in the presence of collision is greater than that of an individual uninterrupted process.

Collision broadening is observed in gas lasers operated at higher pressures, hence the name pressure broadening. At higher pressures collisions between gas atoms limit their radiative lifetime. Collision broadening, therefore, is quite similar to lifetime broadening, in that the collisions interrupt the initial state of the atoms.

Dipolar Broadening. Dipolar broadening arises from interactions between the magnetic or electric dipolar fields of neighboring atoms. This interaction leads to results very similar to collision broadening, including a linewidth that increases with increasing density of atoms. Since dipolar broadening represents a kind of coupling between atoms, so that excitation applied to one atom is distributed or shared with other atoms, dipolar broadening is a homogeneous broadening mechanism.

Thermal Broadening. Thermal broadening is brought about by the effect of the thermal lattice vibrations on the atomic transition. The thermal vibrations of the lattice surrounding the active ions modulate the resonance frequency of each atom at a very high frequency. This frequency modulation represents a coupling mechanism between the atoms, and therefore a homogeneous linewidth is obtained. Thermal broadening is the mechanism responsible for the linewidth of the ruby laser and Nd:YAG laser.

The lineshape of homogeneous broadening mechanisms lead to a Lorentzian lineshape for atomic response. For the normalized Lorentz distribution, the equation

$$g(\nu) = \left(\frac{\Delta\nu}{2\pi}\right)\left[(\nu - \nu_0)^2 + \left(\frac{\Delta\nu}{2}\right)^2\right]^{-1} \tag{1.26}$$

is valid. Here, ν_0 is the center frequency and $\Delta\nu$ is the width between the half-power points of the curve. The factor $\Delta\nu/2\pi$ assures normalization of the area under the curve according to (1.23). The peak value for the Lorentz curve is

$$g(\nu_0) = \frac{2}{\pi\Delta\nu}. \tag{1.27}$$

The Inhomogeneously Broadened Line

Mechanisms which cause inhomogeneous broadening tend to displace the center frequencies of individual atoms, thereby broadening the overall response of a collection without broadening the response of individual atoms. Different atoms have slightly different resonance frequencies on the same transition, for example, owing to Doppler shifts. As a result, the overall response of the collection is broadened. An applied signal at a given frequency within the overall linewidth interacts strongly only with those atoms whose shifted resonance frequencies lie close to the signal frequency. The applied signal does not have the same effect on all the atoms in an inhomogeneously broadened collection.

Since in an inhomogeneously broadened line interaction occurs only with those atoms whose resonance frequencies lie close to the applied signal frequency, a strong

signal will eventually deplete the upper laser level in a very narrow frequency interval. The signal will eventually "burn a hole" in the atomic absorption curve. Examples of inhomogeneous frequency-shifting mechanisms include Doppler broadening and broadening due to crystal inhomogeneities.

Doppler Broadening. The apparent resonance frequencies of atoms undergoing random motions in a gas are shifted randomly so that the overall frequency response of the collection of atoms is broadened. A particular atom moving with a velocity component v relative to an observer in the z direction will radiate at a frequency measured by the observer as $\nu_0(1 + v/c)$. When these velocities are averaged, the resulting lineshape is Gaussian. Doppler broadening is one form of inhomogeneous broadening, since each atom emits a different frequency rather than one atom having a probability distribution for emitting any frequency within the linewidth. In the actual physical situation, the Doppler line is best visualized as a packet of homogeneous lines of width $\Delta\nu_n$, which superimpose to give the observed Doppler shape. The He–Ne laser has a Doppler-broadened linewidth. Most visible and near-infrared gas laser transitions are inhomogeneously broadened by Doppler effects.

Line Broadening Due to Crystal Inhomogeneities. Solid-state lasers may be inhomogeneously broadened by crystalline defects. This happens only at low temperatures where the lattice vibrations are small. Random variations of dislocations, lattice strains, etc., may cause small shifts in the exact energy level spacings and transition frequencies from ion to ion. Like Doppler broadening, these variations do not broaden the response on an individual atom, but they do cause the exact resonance frequencies of different atoms to be slightly different. Thus random crystal imperfection can be a source of inhomogeneous broadening in a solid-state laser crystal.

A good example of an inhomogeneously broadened line occurs in the fluorescence of neodymium-doped glass. As a result of the so-called glassy state, there are variations, from rare earth site to rare earth site, in the relative atomic positions occupied by the surrounding lattice ions. This gives rise to a random distribution of static crystalline fields acting on the rare earth ions. Since the line shifts corresponding to such crystal-field variations are larger, generally speaking, than the width contributed by other factors associated with the transition, an inhomogeneous line results.

The inhomogeneous-broadened linewidth can be represented by a Gaussian frequency distribution. For the normalized distribution, the equation

$$g(\nu) = \frac{2}{\Delta\nu}\left(\frac{\ln 2}{\pi}\right)^{1/2} \exp\left[-\left(\frac{\nu - \nu_0}{\Delta\nu/2}\right)^2 \ln 2\right] \tag{1.28}$$

is valid, where ν_0 is the frequency at the center of the line and $\Delta\nu$ is the linewidth at which the amplitude falls to one-half. The peak value of the normalized Gaussian curve is

$$g(\nu_0) = \frac{2}{\Delta\nu}\left(\frac{\ln 2}{\pi}\right)^{1/2}. \tag{1.29}$$

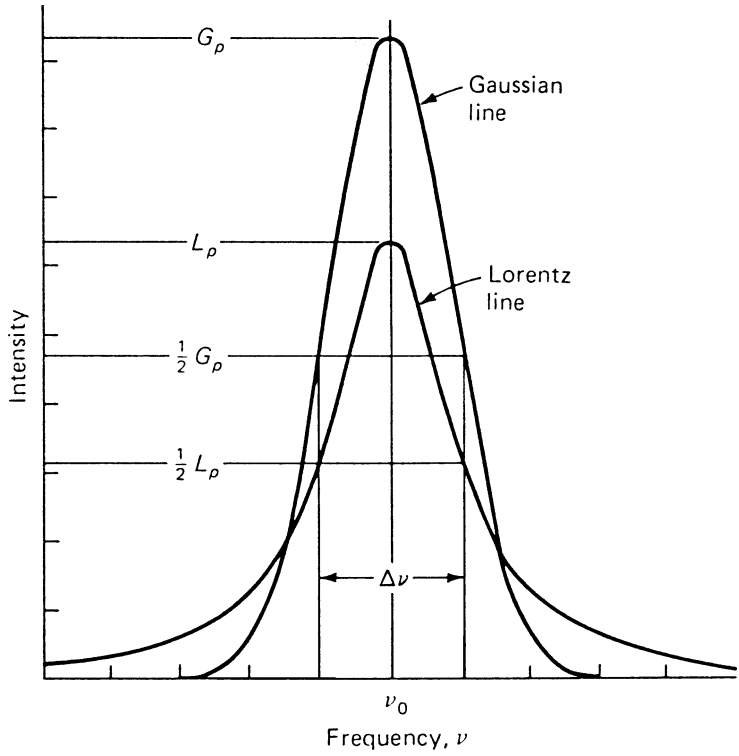

Fig. 1.2. Gaussian and Lorentz lines of common linewidth (G_p and L_p are the peak intensities)

In Fig. 1.2 the normalized Gaussian and Lorentz lines are plotted for a common linewidth.

1.3.2 Absorption by Stimulated Transitions

We assume a quasicollimated beam of energy density per frequency $\varrho(\nu)$ incident on a thin absorbing sample of thickness dx; as before, we consider the case of an optical system that operates only between two energy levels as illustrated schematically in Fig. 1.1. The populations of the two levels are N_1 and N_2, respectively. Level 1 is the ground level and level 2 is the excited level. We consider absorption of radiation in the material and emission from the stimulated processes but neglect the spontaneous emission. From (1.15 and 1.20) we obtain

$$-\frac{\partial N_1}{\partial t} = \varrho(\nu) B_{21} \left(\frac{g_2}{g_1} N_1 - N_2 \right). \tag{1.30}$$

As we recall, this relation was obtained by considering infinitely sharp energy levels separated by $h\nu_{21}$ and a monochromatic wave of frequency ν_{21}.

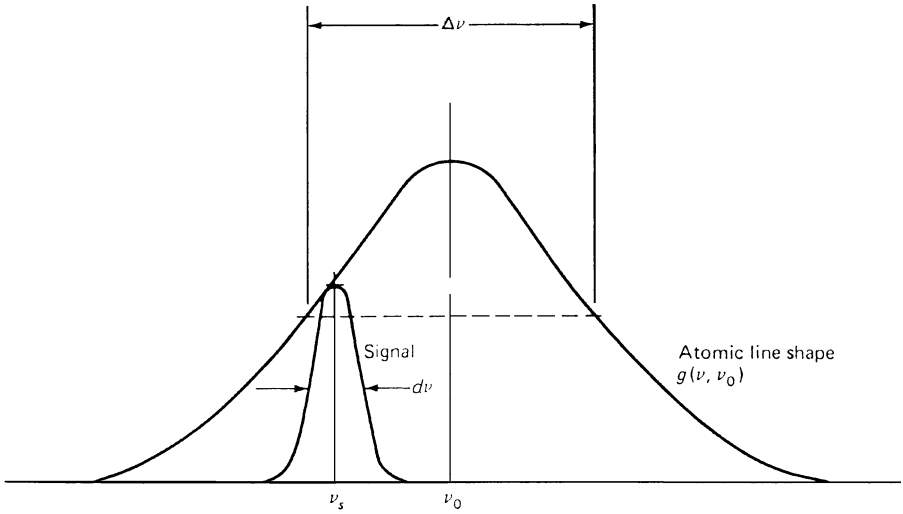

Fig. 1.3. Linewidth-broadened atomic transition line centered at ν_0 and narrow band signal centered at ν_s

We will now consider the interaction between two linewidth-broadened energy levels with an energy separation centered at ν_0, and a half-width of $\Delta\nu$ characterized by $g(\nu, \nu_0)$ and a signal with center frequency ν_s and bandwidth $d\nu$. The situation is shown schematically in Fig. 1.3. The spectral width of the signal is narrow, as compared to the linewidth-broadened transition. If N_1 and N_2 are the total number of atoms in level 1 and level 2, then the number of atoms capable of interacting with a radiation of frequency ν_s and bandwidth $d\nu$ are

$$\left(\frac{g_2}{g_1}N_1 - N_2\right) g(\nu_s, \nu_0)\, d\nu. \tag{1.31}$$

The net change of atoms in energy level 1 can be expressed in terms of energy density $\varrho(\nu)\, d\nu$ by multiplying both sides of (1.30) with photon energy $h\nu$ and dividing by the volume V. We will further express the populations N_1 and N_2 as population densities n_1 and n_2.

Equation (1.30) now becomes

$$-\frac{\partial}{\partial t}[\varrho(\nu_s)\, d\nu] = \varrho(\nu_s)\, d\nu\, B_{21} h\nu g(\nu_s, \nu_0)\left(\frac{g_2}{g_1}n_1 - n_2\right). \tag{1.32}$$

This equation gives the net rate of absorbed energy in the frequency interval $d\nu$ centered around ν_s. In an actual laser system the wavelength of the emitted radiation, corresponding to the signal bandwidth $d\nu$ in our model, is very narrow as compared to the natural linewidth of the material. Ruby, for example, has a fluorescent linewidth of 5Å, whereas the linewidth of the laser output is typically 0.1–0.01Å. The operation of a laser, therefore, can be fairly accurately characterized as the interaction of linewidth-broadened energy levels with a monochromatic wave. The photon

density of a monochromatic radiation of frequency ν_0 can then be represented by a delta function $\delta(\nu - \nu_0)$. After integrating (1.32) in the interval $d\nu$, we obtain, for a monochromatic signal of frequency ν_s and a linewidth-broadened transition,

$$-\frac{\partial \varrho(\nu_s)}{\partial t} = \varrho(\nu_s) B_{21} h\nu_s g(\nu_s, \nu_0) \left(\frac{g_2}{g_1} n_1 - n_2 \right). \tag{1.33}$$

The signal will travel through the material of thickness dx in the time $dt = dx/c = (n/c_0)\,dx$. Then, as the wave advances from x to $x + dx$, the decrease of energy in the beam is

$$-\frac{\partial \varrho(\nu_s)}{\partial x} = h\nu_s \varrho(\nu_s) g(\nu_s, \nu_0) B_{21} \left(\frac{g_2}{g_1} n_1 - n_2 \right) \frac{1}{c}. \tag{1.34}$$

Integration of (1.34) gives

$$\frac{\varrho(\nu_s)}{\varrho_0(\nu_s)} = \exp\left[-h\nu_s g(\nu_s, \nu_0) B_{21} \left(\frac{g_2}{g_1} n_1 - n_2 \right) \frac{x}{c} \right]. \tag{1.35}$$

If we introduce an absorption coefficient $\alpha(\nu_s)$,

$$\alpha(\nu_s) = \left(\frac{g_2}{g_1} n_1 - n_2 \right) \sigma_{21}(\nu_s), \tag{1.36}$$

where

$$\sigma_{21}(\nu_s) = \frac{h\nu_s g(\nu_s, \nu_0) B_{21}}{c}. \tag{1.37}$$

Then we can write (1.35) as

$$\varrho(\nu_s) = \varrho_0(\nu_s) \exp[-\alpha(\nu_s) x]. \tag{1.38}$$

Equation (1.38) is the well-known exponential absorption equation for thermal equilibrium condition $n_1 g_2/g_1 > n_2$. The energy of the radiation decreases exponentially with the depth of penetration into the substance. The maximum possible absorption occurs when all atoms exist in the ground state n_1. For equal population of the energy states $n_1 = (g_1/g_2) n_2$, the absorption is eliminated and the material is transparent. The parameter σ_{21} is the cross section for the radiative transition $2 \rightarrow 1$. The cross section for stimulated emission σ_{21} is related to the absorption cross section σ_{12} by the ratio of the level degeneracies:

$$\frac{\sigma_{21}}{\sigma_{12}} = \frac{g_1}{g_2}. \tag{1.39}$$

The cross section is a very useful parameter to which we will refer in the following chapters. If we replace B_{21} by the Einstein relation (1.20), we obtain σ_{21} in a form which we will find most useful:

$$\sigma_{21}(\nu_s) = \frac{A_{21} \lambda_0^2}{8\pi n^2} g(\nu_s, \nu_0). \tag{1.40}$$

As we will see later, the gain for the radiation building up in a laser resonator will be highest at the center of the atomic transitions. Therefore, in lasers we are mostly dealing with stimulated transitions which occur at the center of the linewidth.

If we assume $\nu \approx \nu_s \approx \nu_0$, we obtain for the spectral stimulated emission cross section at the center of the atomic transition for a Lorentzian lineshape

$$\sigma_{21} = \frac{A_{21}\lambda_0^2}{4\pi^2 n^2 \Delta\nu}, \tag{1.41}$$

and for a Gaussian lineshape

$$\sigma_{21} = \frac{A_{21}\lambda_0^2}{4\pi n^2 \Delta\nu}\left(\frac{\ln 2}{\pi}\right)^{1/2}. \tag{1.42}$$

Here we have introduced into (1.40) the peak values of the lineshape function, as given in (1.27 and 1.29) for the Lorentzian and Gaussian curves, respectively. For example, in the case of the R_1 line of ruby, where $\lambda_0 = 6.94 \times 10^{-5}$ cm, $n = 1.76$, $\tau_{21} = (1/A_{21}) = 3$ ms, and $\Delta\nu = 330\ GHz$ one finds, according to (1.41), $\sigma_{21} = 4.0 \times 10^{-20}\,\text{cm}^2$. In comparing this value with the data provided in Table 2.2, we have to distinguish between the spectroscopic cross section and the effective stimulated emission cross section. (This will be discussed in Sect. 2.3.1 for the case of Nd:YAG). The effective stimulated emission cross section is the spectroscopic cross section times the occupancy of the upper laser level relative to the entire manifold population. In ruby, the upper laser level is split into two sublevels and therefore the effective stimulated emission cross section is about half of the value calculated from (1.41).

1.3.3 Population Inversion

According to the Boltzmann distribution (1.7), in a collection of atoms at thermal equilibrium there are always fewer atoms in a higher-lying level E_2 than in a lower level E_1. Therefore the population difference $N_1 - N_2$ is always positive, which means that the absorption coefficient $\alpha(\nu_s)$ in (1.36) is positive and the incident radiation is absorbed (Fig. 1.4).

Suppose that it were possible to achieve a temporary situation such that there are more atoms in an upper energy level than in a lower energy level. The normally positive population difference on that transition then becomes negative, and the normal stimulated absorption is correspondingly changed to stimulated emission or amplification of the applied signal. That is, the applied signal gains energy as it interacts with the atoms and hence is amplified. The energy for this signal amplification is supplied by the atoms involved in the interaction process. This situation is characterized by a negative absorption coefficient $\alpha(\nu_s)$ according to (1.36). From (1.34) it follows that $\partial\varrho(\nu)/\partial x > 0$.

The essential condition for amplification is that there are more atoms in an upper energy level than in a lower energy level, i.e., for amplification,

$$N_2 > N_1 \quad \text{if} \quad E_2 > E_1, \tag{1.43}$$

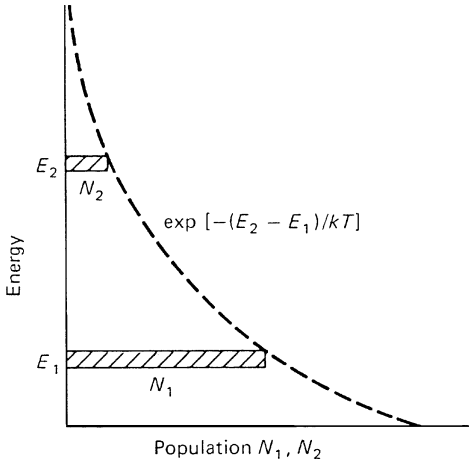

Fig. 1.4. Relative populations in two energy levels as given by the Boltzmann relation for thermal equilibrium

as illustrated in Fig. 1.5. The resulting negative sign of the population difference ($N_2 - g_2N_1/g_1$) on that transition is called a population inversion. Population inversion is clearly an abnormal situation; it is never observed at thermal equilibrium. The point at which the population of both states is equal is called the "inversion threshold."

Stimulated absorption and emission processes always occur side by side independent of the population distribution among the levels. So long as the population of the higher energy level is smaller than that of the lower energy level, the number of absorption transitions is larger than that of the emission transitions, so that there is an overall attenuation of the radiation. When the numbers of atoms in both states are equal, the number of emissions becomes equal to the number of absorptions; the material is then transparent to the incident radiation. As soon as the population of the higher level becomes larger than that of the lower level, emission processes predominate and the radiation is enhanced collectively during passage through the

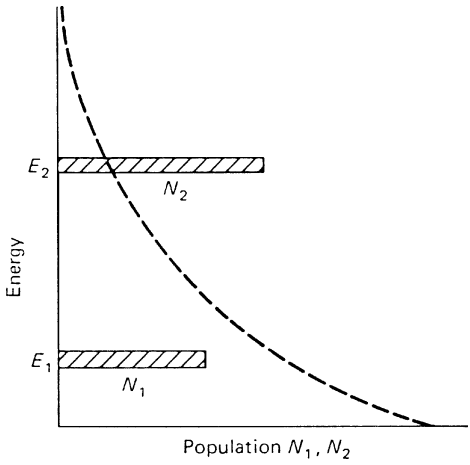

Fig. 1.5. Inverted population difference required for optical amplification

material. To produce an inversion requires a source of energy to populate a specified energy level; we call this energy the pump energy.

In Sect. 1.4 we will discuss the type of energy level structure an atomic system must possess in order to make it possible to generate an inversion. Techniques by which the atoms of a solid-state laser can be raised or pumped into upper energy levels are discussed in Sect. 6.1. Depending on the atomic system involved, an inverted population condition may be obtainable only on a transient basis, yielding pulsed laser action; or it may be possible to maintain the population inversion on a steady-state basis, yielding continuous-wave (cw) laser action.

The total amount of energy which is supplied by the atoms to the light wave is

$$E = \Delta N h\nu, \tag{1.44}$$

where ΔN is the total number of atoms which are caused to drop from the upper to the lower energy level during the time the signal is applied. If laser action is to be maintained, the pumping process must continually replenish the supply of upper-state atoms. The size of the inverted population difference is reduced not only by the amplification process but also by spontaneous emission, which always tends to return the energy level populations to their thermal equilibrium values.

1.4 Creation of a Population Inversion

In this section we discuss how the necessary population inversion for laser action is obtained in solid-state lasers. We can gain considerable understanding on how laser devices are pumped and how their population densities are inverted by studying some simplified but fairly realistic models.

The discussion up to this point has been based on a hypothetical $2 \leftrightarrow 1$ transition and has not been concerned with how the levels 2 and 1 fit into the energy level scheme of the atom. This detached point of view must be abandoned when one tries to understand how laser action takes place in a solid-state medium. As already noted, the operation of the laser depends on a material with narrow energy levels between which electrons can make transitions. Usually these levels are due to impurity ions in a host crystal. The pumping and laser processes in real laser systems typically involve a very large number of energy levels, with complex excitation processes and cascaded relaxation processes among all these levels. Operation of an actual laser material is properly described only by a many-level energy diagram. The main features can be understood, however, through the familiar three-level or four-level idealizations of Figs. 1.6 and 1.7. More detailed energy level diagrams of some of the most important solid-state laser materials are presented in Chap. 2.

1.4.1 The Three-Level System

Figure 1.6 shows a diagram which can be used to explain the operation of an optically pumped three-level laser, such as ruby. Initially, all ions of the laser material are

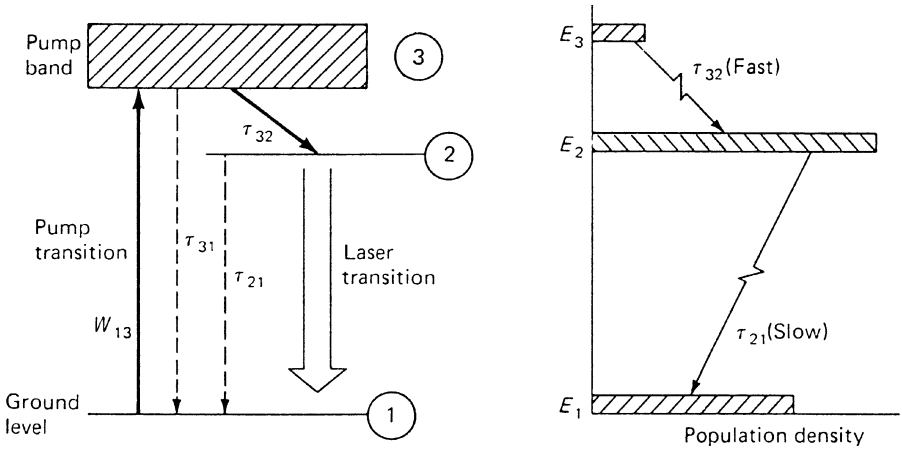

Fig. 1.6. Simplified energy level diagram of a three-level laser

in the lowest level 1. Excitation is supplied to the solid by radiation of frequencies which produce absorption into the broadband 3. Thus, the pump light raises ions from the ground state to the pump band, level 3. In general, the "pumping" band, level 3, is actually made up of a number of bands, so that the optical pumping can be accomplished over a broad spectral range. Most of the excited ions are transferred by fast radiationless transitions into the intermediate sharp level 2. In this process the energy lost by an electron is transferred to the lattice. Finally, the electron returns to the ground level by the emission of a photon. It is this last transition that is responsible for the laser action. If pumping intensity is below laser threshold, ions in level 2 predominantly return to the ground state by spontaneous emission. Ordinary fluorescence acts as a drain on the population of level 2. After the pump radiation is extinguished, level 2 is emptied by fluorescence at a rate that varies from material to material. In ruby, at room temperature, the lifetime of level 2 is 3 ms. When the pump intensity is above laser threshold, the decay from the fluorescent level consists of stimulated as well as spontaneous radiation; the stimulated radiation produces the laser output beam. Since the terminal level of the laser transition is the highly populated ground state, a very high population must be reached in the E_2 level before the $2 \rightarrow 1$ transition is inverted.

It is necessary, in general, that the rate of radiationless transfer from the uppermost level to the level at which the laser action begins be fast compared with the other spontaneous transition rates in a three-level laser. Therefore, the lifetime of the E_2 state should be large in comparison with the relaxation time of the $3 \rightarrow 2$ transition, i.e.,

$$\tau_{21} \gg \tau_{32}. \tag{1.45}$$

The number of ions N_3 in level E_3 is then negligible compared with the number of ions in the other two states, i.e., $N_3 \ll N_1, N_2$. Therefore,

$$N_1 + N_2 \approx N_{\text{tot}}. \tag{1.46}$$

A vital aspect of the three-level system is that the ions are in effect pumped directly from level 1 into the metastable level 2 with only a momentary pause as they pass through level 3. With these conditions, we can calculate as if only two levels were present. In order that an equal population is achieved between the E_2 and E_1 levels, one-half of all atoms must be excited to the E_2 level:

$$N_2 = N_1 = \frac{N_{\text{tot}}}{2}. \tag{1.47}$$

In order to maintain a specified amplification, the population of the second level must be larger than that of the first level. In most cases which are of practical importance, however, the necessary inversion $(N_2 - N_1)$ is small compared with the total number of all ions. The pump power necessary for maintaining this inversion is also small compared with the power necessary for achieving equal population of the levels.

The disadvantage of a three-level system is that more than half of the ions in the ground state must be raised to the metastable level E_2. There are thus many ions present to contribute to the spontaneous emission. Moreover, each of the ions which participate in the pump cycle transfer energy into the lattice from the $E_3 \rightarrow E_2$ transition. This transition is normally radiationless, the energy being carried into the lattice by phonons.

1.4.2 The Four-Level System

The four-level laser system, which is characteristic of the rare earth ions in glass or crystalline host materials, is illustrated in Fig. 1.7. Note that a characteristic of the three-level laser material is that the laser transition takes place between the excited laser level 2 and the final ground state 1, the lowest energy level of the system. This leads to low efficiency. The four-level system avoids this disadvantage. The pump

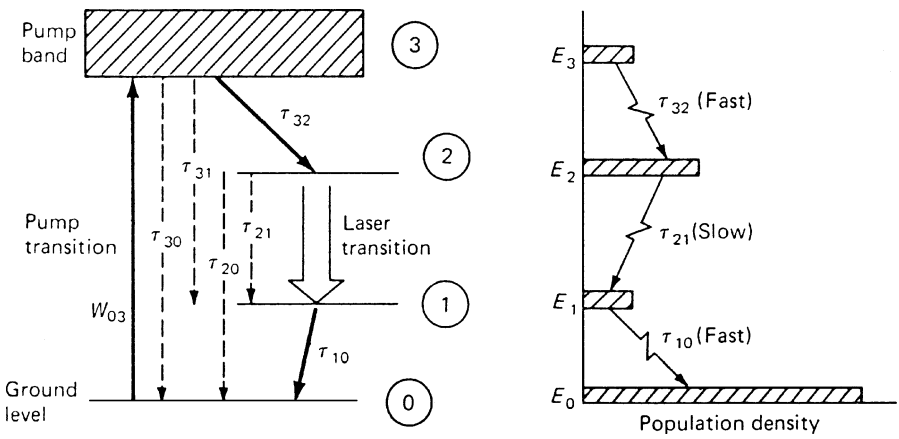

Fig. 1.7. Simplified energy level diagram of a four-level laser

transition extends again from the ground state (now level E_0) to a wide absorption band E_3. As in the case of the three-level system, the ions so excited will proceed rapidly to the sharply defined level E_2. The laser transition, however, proceeds now to a fourth, terminal level E_1, which is situated above the ground state E_0. From here the ion undergoes a rapid nonradiative transition to the ground level. In a true four-level system, the terminal laser level E_1 will be empty. To qualifiy as a four-level system a material must possess a relaxation time between the terminal laser level and the ground level, which is fast compared to the fluorescent lifetime, i.e., $\tau_{10} \ll \tau_{21}$. In addition the terminal laser level must be far above the ground state so that its thermal population is small. The equilibrium population of the terminal laser level 1 is determined by the relation

$$\frac{N_1}{N_0} = \exp\left(\frac{-\Delta E}{kT}\right), \tag{1.48}$$

where ΔE is the energy separation between level 1 and the ground state, and T is the operating temperature of the laser material. If $\Delta E \gg kT$, then $N_1/N_0 \ll 1$, and the intermediate level will always be relatively empty. In some laser materials the energy gap between the lower laser level and the ground state is relatively small and therefore they must be cooled to function as four-level lasers. In a four-level system an inversion of the $2 \to 1$ transition can occur even with vanishingly small pump power, and the high pump rate, necessary to maintain equilibrium population in the aforementioned three-level system, is no longer needed. In the most favorable case, the relaxation times of the $3 \to 2$ and $1 \to 0$ transitions in the four-level system are short compared with the spontaneous emission lifetime of the laser transition τ_{21}. Hence we can also carry out the calculations as if only the E_1 and E_2 states were populated.

1.4.3 The Metastable Level

After this brief introduction to the energy level structure of solid-state lasers we can ask the question, "what energy level scheme must a solid possess to make it a useful laser?" As we have seen in the previous discussion, the existence of a metastable level is of paramount importance for laser action to occur. The relatively long lifetime of the metastable level provides a mechanism by which inverted population can be achieved. Most transitions of ions show rapid nonradiative decay because the coupling of the internal atomic oscillations to the surrounding lattice is strong. Radiative decay processes can occur readily, but most have short lifetimes and broad linewidths. Only a few transitions of selected ions in solids turn out to be decoupled from the lattice vibrations. These transitions have a radiative decay, which leads to relatively long lifetimes.

In typical laser systems with energy levels, such as illustrated by Figs. 1.6 and 7, the $3 \to 2$ transition frequencies as well as the $1 \to 0$ transition frequencies all fall within the frequency range of the vibration spectrum of the host crystal lattice. Therefore, all these transitions can relax extremely rapidly by direct nonradiative decay, i.e., by emitting a phonon to the lattice vibrations, with $\tau_{32}, \tau_{10} \approx 10^{-8}$–$10^{-11}$ s.

However, the larger $3 \to 0$, $3 \to 1$, $2 \to 0$, and $2 \to 1$ energy gaps in these atoms often correspond to transition frequencies that are higher than the highest possible vibration frequency of the crystal lattice. Such transitions cannot relax via simple single-phonon spontaneous emission, since the lattice simply cannot accept phonons at those high frequencies. These transitions must then relax either by radiative (photon) emission or by multiple-phonon processes. Since both these processes are relatively weak compared to direct single-phonon relaxation, the high-frequency transitions will have much slower relaxation rates ($\tau_{21} \approx 10^{-5}$–$10^{-3}$ s in many cases). Therefore, the various levels lumped into level 3 will all relax mostly into level 2 while level 2 itself is metastable and long-lived because there are no other levels located close below it into which it can decay directly.

The existence of metastable levels follows from quantum mechanical considerations that will not be discussed here. However, for completeness we will explain the term "forbidden transition." As we have seen in Sect. 1.2.4, the mechanism by which energy exchange takes place between an atom and the electromagnetic fields is the dipole radiation. As a consequence of quantum mechanical considerations and the ensuing selection rules, transfer between certain states cannot occur due to forbidden transitions. The term "forbidden" means that a transition among the states concerned does not take place as a result of the interaction of the electric dipole moment of the ion with the radiation field. As a result of the selection rules, an ion may get into an excited state from which it will have difficulty returning to the ground state. A state from which all dipole transitions to lower energy states are forbidden is metastable; an ion entering such a state will generally remain in that state much longer than it would in an ordinary excited state from which escape is comparatively easy.

In the absence of a metastable level, the ions which become excited by pump radiation and are transferred to a higher energy level will return either directly to the ground state by spontaneous radiation or by cascading down on intermediate levels, or they may release energy by phonon interaction with the lattice. In order for the population to increase at the metastable laser level, several other conditions have to be met. Let us consider the more general case of a four-level system illustrated in Fig. 1.7. (Note that a three-level system can be thought of as a special case of a four-level scheme where level 1 and level 0 coincide.) Pumping takes place between two levels and laser action takes place between two other levels. Energy from the pump band is transferred to the upper laser level by fast radiationless transitions. Energy is removed from the lower laser level again by fast radiationless transitions.

For electrons in the pump band at level 3 to transfer to level 2 rather than return directly to the ground state, it is required that $\tau_{30} \gg \tau_{32}$. For population to build up, relaxation out of the lower level 1 has to be fast, $\tau_{21} \gg \tau_{10}$. Thus, as a first conclusion, we may say that if the right relaxation time ratio exists between any two levels (such as 3 and 2) in an energy level system, a population inversion should be possible. If so, then obtaining a large enough inversion for successful laser operation becomes primarily a matter of the right pumping method. The optical pumping method is generally feasible only in laser materials which combine a narrow laser emission line with a broad absorption transition, so that a broadband intense light source can be used as the pump source. An exception is a solid-state laser which is pumped by

another laser, such as a diode laser. In this case the requirement for a broad absorption range for the pump band can be relaxed.

Having achieved population inversion in a material by correct combination of relaxation times and the existence of broad pump bands, the linewidth of the laser transition becomes very important. In the following chapter we will see that the optical gain for a given population inversion is inversely proportional to linewidth. Therefore, the metastable level should have a sufficiently narrow linewidth.

1.5 Laser Rate Equations

The dynamic behavior of a laser can be described with reasonable precision by a set of coupled rate equations [1.5]. In their simplest forms, a pair of simultaneous differential equations describe the population inversion and the radiation density within a spatially uniform laser medium. We will describe the system in terms of the energy-level diagrams shown in Figs. 1.6 and 1.7. As we have seen in the preceding discussions, two energy levels are of prime importance in laser action: the excited upper laser level E_2 and the lower laser level E_1. Thus for many analyses of laser action an approximation of the three- and four-level systems by a two-level representation is very useful.

The rate-equation approach used in this section involves a number of simplifying assumptions; in using a single set of rate equations we are ignoring longitudinal and radial variations of the radiation within the laser medium. In spite of these limitations, the simple rate-equation approach remains a useful tool and, if properly used, provides a great deal of insight into the behavior of real solid-state laser devices. We will derive from the rate equations the threshold condition for laser actions, and obtain a first-order approximation of the relaxation oscillations in a solid-state laser. Furthermore, in Chap. 4 we will use the rate equations to calculate the gain in a laser amplifier.

In general, the rate equations are useful in predicting the gross features of the laser output, such as average and peak power, Q-switched pulse-envelope shape, threshold condition, etc. On the other hand, many details of the nature of the laser emission are inaccessible from the point of view of a simple rate equation. These include detailed descriptions of the spectral, temporal, and spatial distributions of the laser emission. Fortunately, these details can often be accounted for independently.

In applying the rate equations to the various aspects of laser operation, we will find it more convenient to express the probability for stimulated emission $\varrho(\nu)B_{21}$ by the photon density ϕ and the stimulated emission cross section σ.

With (1.37) we can express the Einstein coefficient for stimulated emission B_{21} in terms of the stimulated emission cross section $\sigma_{21}(\nu)$,

$$B_{21} = \frac{c}{h\nu g(\nu)}\sigma_{21}(\nu), \tag{1.49}$$

where $c = c_0/n$ is the speed of light in the medium. The energy density per unit frequency $\varrho(\nu)$ is expressed in terms of the lineshape factor $g(\nu)$, the energy $h\nu$, and

the photon density ϕ (photons/cm^3) by

$$\varrho(\nu) = h\nu g(\nu)\phi. \tag{1.50}$$

From (1.49 and 50) we obtain

$$B_{21}\varrho(\nu) = c\sigma_{21}(\nu)\phi. \tag{1.51}$$

1.5.1 The Three-Level System

In order to approximate the three-level system with a two-level scheme, we assume that the transition from the pump band to the upper laser level is so fast that $N_3 \approx 0$. Therefore pumping does not affect the other processes at all except to allow a mechanism of populating the upper level and thereby obtaining population inversion ($N_2 > N_1$).

Looking at Fig. 1.6, this assumption requires that the relaxation time ratio τ_{32}/τ_{21} be very small. In solid-state lasers $\tau_{32}/\tau_{21} \approx 0$ is a good approximation. Spontaneous losses from the pump band to the ground state can be expressed by the quantum efficiency η_Q. This parameter, defined as

$$\eta_Q = \left(1 + \frac{\tau_{32}}{\tau_{31}}\right)^{-1} \leq 1, \tag{1.52}$$

specifies what fraction of the total ions excited to level 3 drop from there to level 2, thus becoming potentially useful for laser action. A small η_Q obviously requires a correspondingly larger pump power.

The changes in the electron population densities in a three-level system, based on the assumption that essentially all of the laser ions are in either level 1 or level 2, are

$$\frac{\partial n_1}{\partial t} = \left(n_2 - \frac{g_2}{g_1}n_1\right)c\phi\sigma + \frac{n_2}{\tau_{21}} - W_p n_1 \tag{1.53}$$

and

$$\frac{\partial n_2}{\partial t} = -\frac{\partial n_1}{\partial t}, \tag{1.54}$$

since

$$n_{tot} = n_1 + n_2, \tag{1.55}$$

where W_p is the pumping rate (s^{-1}).

The terms on the right-hand side of (1.53) express the net stimulated emission, the spontaneous emission, and the optical pumping.

The time variation of the population in both levels due to absorption, spontaneous, and stimulated emission is obtained from (1.15). Note that the populations N_1 and N_2 are now expressed in terms of population densities n_1 and n_2. To take into account the effect of pumping, we have added the term $W_p n_1$, which can be thought of as the rate of supply of atoms to the metastable level 2. More precisely, $W_p n_1$ is the number

of ions transferred from the ground level to the upper laser level per unit time per unit volume. The pump rate W_p is related to the pump parameter W_{13} in Fig. 1.6 by

$$W_p = \eta_Q W_{13}. \tag{1.56}$$

The negative sign in front of $W_p n_1$ in (1.53) indicates that the pump mechanism removes atoms from the ground level 1 and increases the population of level 2.

If we now define the inversion population density by

$$n = n_2 - \frac{g_2 n_1}{g_1} \tag{1.57}$$

we can combine (1.53, 54, and 57) to obtain

$$\frac{\partial n}{\partial t} = -\gamma n \phi \sigma c - \frac{n + n_{tot}(\gamma - 1)}{\tau_f} + W_p(n_{tot} - n), \tag{1.58}$$

where

$$\gamma = 1 + \frac{g_2}{g_1} \quad \text{and} \quad \tau_f = \tau_{21}. \tag{1.59}$$

In obtaining (1.58) we have used the relations

$$n_1 = \frac{n_{tot} - n}{1 + g_2/g_1} \quad \text{and} \quad n_2 = \frac{n + (g_2/g_1)n_{tot}}{1 + g_2/g_1}. \tag{1.60}$$

Another equation, usually regarded together with (1.58), describes the rate of change of the photon density within the laser resonator,

$$\frac{\partial \phi}{\partial t} = c\phi\sigma n - \frac{\phi}{\tau_c} + S, \tag{1.61}$$

where τ_c is the decay time for photons in the optical resonator and S is the rate at which spontaneous emission is added to the laser emission.

If we consider for the moment only the first term on the right, which is the increase of the photon density by stimulated emission, then (1.61) is identical to (1.33). However, for the time variation of the photon density in the laser resonator we must also take into account the decrease of radiation due to losses in the system and the increase of radiation due to a small amount of spontaneous emission which is added to the laser emission. Although very small, this term must be included because it provides the source of radiation which initiates laser emission.

An important consideration for initiation of laser oscillation is the total number p of resonant modes possible in the laser resonator volume V_R, since in general only a few of these modes are initiated into oscillations. This number is given by the familiar expression (1.3),

$$p = 8\pi\nu^2 \frac{\Delta\nu V_R}{c^3}, \tag{1.62}$$

where ν is the laser optical frequency and $\Delta\nu$ is the bandwidth of spontaneous emission. Let p_L be the number of modes of the laser output. Then S can be expressed as the rate at which spontaneous emission contributes to stimulated emission,

namely,

$$S = \frac{p_L n_2}{p\tau_{21}}. \tag{1.63}$$

Refer to Chap. 3 for a more detailed description of the factor τ_c which appears in (1.61). For now we only need to know that τ_c represents all the losses in an optical resonator of a laser oscillator. Since τ_c has the dimension of time, the losses are expressed in terms of a relaxation time. The decay of the photon population in the cavity results from transmission and absorption at the end mirrors, "spillover" diffraction loss due to the finite apertures of the mirrors, scattering and absorptive losses in the laser material itself, etc. In the absence of the amplifying mechanism, (1.61) becomes

$$\frac{\partial \phi}{\partial t} = -\frac{\phi}{\tau_c}, \tag{1.64}$$

the solution of which is $\phi(\tau) = \phi_0 \exp(-t/\tau_c)$.

The importance of (1.61) should be emphasized by noting that the right-hand side of this equation describes the net gain per transit of an electromagnetic wave passing through a laser material.

1.5.2 The Four-Level System

We will assume again that the transition from the pump band into the upper laser level occurs very rapidly. Therefore, the population of the pump band is negligible, i.e., $n_3 \approx 0$. With this assumption the rate of change of the two laser levels in a four-level system is

$$\frac{dn_2}{dt} = W_p n_0 - \left(n_2 - \frac{g_2}{g_1} n_1\right)\sigma\phi c - \left(\frac{n_2}{\tau_{21}} + \frac{n_2}{\tau_{20}}\right), \tag{1.65}$$

$$\frac{dn_1}{dt} = \left(n_2 - \frac{g_2}{g_1} n_1\right)\sigma\phi c + \frac{n_2}{\tau_{21}} - \frac{n_1}{\tau_{10}}, \tag{1.66}$$

$$n_{tot} = n_0 + n_1 + n_2. \tag{1.67}$$

From (1.65) it follows that the upper laser level population in a four-level system increases due to pumping and decreases due to stimulated emission and spontaneous emissions into level 1 and level 0. The lower level population increases due to stimulated and spontaneous emission and decreases by a radiationless relaxation process into the ground level. This process is characterized by the time constant τ_{10}. In an ideal four-level system the terminal level empties infinitely fast to the ground level. If we let $\tau_{10} \approx 0$, then it follows from (1.66) that $n_1 = 0$. In this case the entire population is divided between the ground level 0 and the upper level of the laser transition. The system appears to be pumping from a large source that is independent of the lower laser level. With $\tau_{10} = 0$ and $n_1 = 0$, we obtain the following rate equation for the ideal four-level system:

$$n = n_2 \tag{1.68}$$

and

$$n_{\text{tot}} = n_0 + n_2 \approx n_0 \quad \text{since} \quad n_2 \ll n_0. \tag{1.69}$$

Therefore, instead of (1.58), we have

$$\frac{\partial n_2}{\partial t} = -n_2 \sigma \phi c - \frac{n_2}{\tau_{\text{f}}} + W_{\text{p}} n_0. \tag{1.70}$$

The fluorescence decay time τ_{f} of the upper laser level is given by

$$\frac{1}{\tau_{\text{f}}} = \frac{1}{\tau_{21}} + \frac{1}{\tau_{20}}, \tag{1.71}$$

where $\tau_{21} = A_{21}^{-1}$ is the effective radiative lifetime associated with the laser line. In the equation for the rate of change of the upper laser level we have again taken into account the fact that not all ions pumped to level 3 will end up at the upper laser level. It is

$$W_{\text{p}} = \eta_{\text{Q}} W_{03}, \tag{1.72}$$

where the quantum efficiency η_{Q} depends on the branching ratios, which are the relative relaxation rates for the ions along the various possible downward paths,

$$\eta_{\text{Q}} = \left(1 + \frac{\tau_{32}}{\tau_{31}} + \frac{\tau_{32}}{\tau_{30}}\right)^{-1} \leq 1. \tag{1.73}$$

As already indicated in the case of a three-level system the quantum efficiency is the probability of an absorbed pump photon producing an active atom in the upper laser level. Some of the absorbed pump photons will not produce an active ion in the upper laser level. Some, for example, may decay to manifolds other than the manifold containing the upper laser level while others may decay to the ground level by radiationless transitions. The equation which describes the rate of change of the photon density within the laser resonator is the same as in the case of the three-level system.

1.5.3 Comparison of Three- and Four-Level Lasers

The rate equation applicable to three- and four-level systems can be expressed by a single pair of equations, namely, (1.58 and 61), where $\gamma = 1 + g_2/g_1$ for a three-level system and $\gamma = 1$ for a four-level system. The factor γ can be thought of as an "inversion reduction factor" since it corresponds to the net reduction in the population inversion after the emission of a single photon. In a four-level system, see (1.70), we have $\gamma = 1$ since the population inversion density is reduced only by one for each photon emitted. For a three-level system, see (1.58), we have $\gamma = 2$ if we assume no degeneracy, i.e., $g_2/g_1 = 1$. This reflects the fact that in this case the population inversion is reduced by two for each stimulated emission of a photon because the

emitting photon is not only lost to the upper laser level, but also increases the lower laser level by one. The parameters τ_f and W_p are defined by (1.56, 59, 71, and 72) for the three- and four-level systems. The factor S in (1.61), which represents the initial noise level of ϕ due to spontaneous emission at the laser frequency, is small and needs to be considered only for initial starting of the laser action. It will be dropped from this point on.

A more detailed analysis of the laser rate equations can be found in [1.1, 3].