8 ALLOCATION MODELS

8.1 INTRODUCTION

The preceding chapters focused on the technical-physical aspects of production for use in situations where unit price and unit cost information are not available, or where their uses are limited because of variability in the prices and costs that might need to be considered. This chapter turns to the topic of "allocative efficiency" in order to show how DEA can be used to identify types of inefficiency which can emerge for treatment when information on prices and costs are known exactly. Technology and cost are the wheels that drive modern enterprises; some enterprises have advantages in terms of technology and others in cost. Hence, the management is eager to know how and to what extent their resources are being effectively and efficiently utilized, compared to other similar enterprises in the same or a similar field.

Regarding this subject, there are two different situations: one with common unit prices and costs for all DMUs and the other with different prices and costs from DMU to DMU. Section 2 of this chapter deals with the former case. However, the common price and cost assumption is not always valid in actual business and it is demonstrated that efficiency measures based on this assumption can be misleading. So we introduce a new cost-efficiency related model along with new revenue and profit efficiency models in Section 3. Section 4 develops a new formula for decomposition of the observed actual cost based on the new cost efficiency model. Using this formula, we can decompose actual cost as the sum of the minimum cost and the losses due to technical, price and allocative inefficiencies.

8.2 OVERALL EFFICIENCY WITH COMMON PRICES AND COSTS

8.2.1 Cost Efficiency

Figure 8.1 introduces concepts dealing with "allocative efficiency" that can be traced back to M.J. Farrell (1957) and G. Debreu (1951)[1] who originated many of the ideas underlying DEA. Färe, Grosskopf and Lovell (1985)[2] developed linear programming formulations of these concepts.

The solid lines in this figure are segments of an isoquant that represents all possible combinations of the input amounts (x_1, x_2) that are needed to produce the same amount of a single output. P is a point in the interior of the production possibility set representing the activity of a DMU which produces this same amount of output but with greater amounts of both inputs.[3]

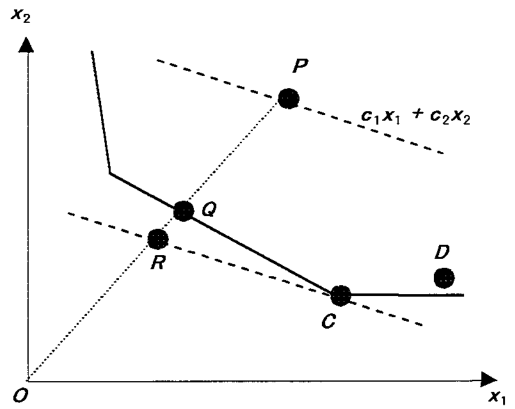

Figure 8.1. Technical, Allocative and Overall Efficiency

To evaluate the performance of P we can use the customary Farrell measure of radial efficiency. Reverting to the notation of Section 1.4, Chapter 1, we can represent this measure in ratio form as

$$0 \leq \frac{d(O,Q)}{d(O,P)} \leq 1,$$

and interpret this as the distance from O to Q relative to the distance from O to P. The result is the measure of technical efficiency that we have customarily represented as θ_o^*.

The components of this ratio lie on the dotted line from the origin through Q to P. To bring price-cost and, hence, "allocative efficiency" considerations into the picture we turn to the broken line passing through P for which the budget (or cost) line is associated with $c_1x_1 + c_2x_2 = k_1$. However, this cost can be reduced by moving this line in parallel fashion until it intersects the isoquant at C. The coordinates of C then give $c_1x_1^* + c_2x_2^* = k_0$ where $k_0 < k_1$ shows the amount by which total cost can be reduced. Further parallel movement in a downward direction is associated with reduced output so the position of the broken line passing through C is minimal at the prescribed output level. This optimal point C is obtained as the optimal solution $\boldsymbol{x}^*$ of the following LP (Farrell (1957)):

$$
\begin{aligned}
\text{[Cost]} \quad & \boldsymbol{cx}^* = \min_{\boldsymbol{x},\boldsymbol{\lambda}} \boldsymbol{cx} \\
\text{subject to} \quad & \boldsymbol{x} \geq X\boldsymbol{\lambda} \\
& \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} \\
& \boldsymbol{\lambda} \geq \boldsymbol{0},
\end{aligned}
\tag{8.1}
$$

where $\boldsymbol{c} = (c_1, \ldots, c_m)$ is the common unit input-price or unit-cost vector.

Now we note that we can similarly determine the relative distances of R and Q to obtain the following ratio,

$$0 \leq \frac{d(O,R)}{d(O,Q)} \leq 1.$$

Farrell refers to this as a measure of "price efficiency" but the more commonly used term is "allocative efficiency." In either case it provides a measure of the extent to which the technically efficient point, Q, falls short of achieving minimal cost because of failure to make the substitutions (or reallocations) involved in moving from Q to C along the efficiency frontier.

There is one further measure that is commonly referred to as "overall efficiency," or "cost efficiency." We can represent this by means of the following ratio,

$$0 \leq \frac{d(O,R)}{d(O,P)} = \frac{\boldsymbol{cx}^*}{\boldsymbol{cx}_o} \leq 1. \tag{8.2}$$

This is a measure of the extent to which the originally observed values at P, represented in the denominator, have fallen short of achieving the minimum cost represented in the numerator.

To put this in a way that relates all three of these efficiency concepts to each other, note that

$$\frac{d(O,R)}{d(O,Q)} \cdot \frac{d(O,Q)}{d(O,P)} = \frac{d(O,R)}{d(O,P)}. \tag{8.3}$$

In sum, "overall (cost) efficiency" (on the right) is equal to the product of "allocative" times "technical efficiency," (on the left).

Furthermore, the technical efficiency can be decomposed into the pure technical efficiency and the scale efficiency as defined in Section 4.5 of Chapter 4. Thus, we have the following decomposition:

$$\text{Overall Eff.} = \text{Allocative Eff.} \times \text{Pure Technical Eff.} \times \text{Scale Eff.}. \tag{8.4}$$

Or,

$$\text{OE} = \text{AE} \times \text{TE} = \text{AE} \times \text{PTE} \times \text{SE}. \tag{8.5}$$

8.2.2 Revenue Efficiency

Given the common unit price vector $\boldsymbol{p} = (p_1, \ldots, p_s)$ for the output $\boldsymbol{y}$, we evaluate the revenue efficiency of DMU$_o$ as follows:

$$\begin{aligned} &[\text{Revenue}] \quad && \boldsymbol{p}\boldsymbol{y}^* = \max_{\boldsymbol{y},\boldsymbol{\lambda}} \boldsymbol{p}\boldsymbol{y} \\ &\text{subject to} && \boldsymbol{x}_o \geq X\boldsymbol{\lambda} \\ &&& \boldsymbol{y} \leq Y\boldsymbol{\lambda} \\ &&& L \leq \boldsymbol{e}\boldsymbol{\lambda} \leq U \\ &&& \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned} \tag{8.6}$$

We inserted an additional constraint on scale ($L \leq \boldsymbol{e}\boldsymbol{\lambda} \leq U$) in order to cope with various returns-to-scale assumptions.

This model allows substitutions in outputs. Let the optimal solution be ($\boldsymbol{y}^*$, $\boldsymbol{\lambda}^*$). Then, the *revenue efficiency* is defined in ratio form as:

$$E_R \text{ (Revenue Efficiency)} = \frac{\boldsymbol{p}\boldsymbol{y}_o}{\boldsymbol{p}\boldsymbol{y}^*}. \tag{8.7}$$

We have $0 \leq E_R \leq 1$ and DMU($\boldsymbol{x}_o$, $\boldsymbol{y}_o$) is *revenue efficient* if and only if $E_R = 1$.

8.2.3 Profit Efficiency

To express the profit of DMU$_o$ we use the common unit price vector $\boldsymbol{p}$ and unit cost vector $\boldsymbol{c}$, to obtain the following LP problem:

$$\begin{aligned} &[\text{Profit}] \quad && \boldsymbol{p}\boldsymbol{y}^* - \boldsymbol{c}\boldsymbol{x}^* = \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda}} \boldsymbol{p}\boldsymbol{y} - \boldsymbol{c}\boldsymbol{x} \\ &\text{subject to} && \boldsymbol{x} = X\boldsymbol{\lambda} \leq \boldsymbol{x}_o \\ &&& \boldsymbol{y} = Y\boldsymbol{\lambda} \geq \boldsymbol{y}_o \\ &&& L \leq \boldsymbol{e}\boldsymbol{\lambda} \leq U \\ &&& \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned} \tag{8.8}$$

This formulation extends (8.1) with an additional constraint on scale ($L \leq \boldsymbol{e}\boldsymbol{\lambda} \leq U$). So, substitutions in inputs or outputs are not allowed in this case. Here, the purpose is to find a profit-maximization mix in the production possibility set $P = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq X\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, L \leq \boldsymbol{e}\boldsymbol{\lambda} \leq U, \boldsymbol{\lambda} \geq \mathbf{0}\}$. Based on an optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$, the *profit efficiency* can be defined in ratio form by

$$E_P \text{(Profit Efficiency)} = \frac{\boldsymbol{p}\boldsymbol{y}_o - \boldsymbol{c}\boldsymbol{x}_o}{\boldsymbol{p}\boldsymbol{y}^* - \boldsymbol{c}\boldsymbol{x}^*}. \tag{8.9}$$

where $\boldsymbol{y}^*$, $\boldsymbol{x}^*$ are optimal for (8.8) and $\boldsymbol{y}_o, \boldsymbol{x}_o$ are the vectors of observed values for DMU$_o$.

Under the assumption $\boldsymbol{p}\boldsymbol{y}_o > \boldsymbol{c}\boldsymbol{x}_o$, we have $0 < E_P \leq 1$ and DMU $(\boldsymbol{x}_o,\ \boldsymbol{y}_o)$ is *profit efficient* if and only if $E_P = 1$. The differences between $\boldsymbol{x}^*$ and $\boldsymbol{x}_o$ and between $\boldsymbol{y}^*$ and $\boldsymbol{y}_o$ may suggest directions for managerial improvement and this can be analyzed, constraint by constraint, in (8.8).

8.2.4 An Example

The data in the following table will provide examples to illustrate the use of these models. Here each of three DMUs produces a single output in the amount y, shown in the column for y under "Output," by using two inputs in the amounts x_1 and x_2 shown in the two columns headed by x_1 and x_2. The common unit costs and price are exhibited in the columns for c_1, c_2 and p, respectively.

Table 8.1. Sample Data for Allocative Efficiency

	Input				Output	
DMU	x_1	c_1	x_2	c_2	y	p
A	3	4	2	2	3	6
B	1	4	3	2	5	6
C	4	4	6	2	6	6

We solved this data set under the constant returns-to-scale assumption so the constraint $L \leq \boldsymbol{e}\boldsymbol{\lambda} \leq U$ was omitted. We then obtained the results shown in Table 8.2. DMU B is the only one that is efficient and is the best performer in all efficiency measures.

Table 8.2. Efficiencies

	Efficiency				
DMU	Technical	Cost	Allocative	Revenue	Profit
A	0.9	0.375	0.417	0.9	0.15
B	1	1	1	1	1
C	0.6	0.429	0.715	0.6	0.2

8.3 NEW COST EFFICIENCY UNDER DIFFERENT UNIT PRICES

Firstly we observe an unacceptable property of the traditional Farrell-Debreu cost efficiency models described in the preceding section which can occur when the unit prices of input are not identical among DMUs.

Suppose that DMUs A and B have the same amount of inputs and outputs, i.e., $\boldsymbol{x}_A = \boldsymbol{x}_B$ and $\boldsymbol{y}_A = \boldsymbol{y}_B$. Assume further that the unit cost of DMU A is twice that of DMU B for each input, i.e., $\boldsymbol{c}_A = 2\boldsymbol{c}_B$. Under these assumptions, we have the following theorem:

Theorem 8.1 (Tone(2002)[4]) *Using the Farrell-Debreu cost efficiency model both DMUs A and B have the same cost (overall) and allocative efficiencies even when the latter is more costly than the former.*

Proof: Since DMUs A and B have the same inputs and outputs, they have the same technical efficiency, i.e., $\theta_A^* = \theta_B^*$. The Farrell measure of cost efficiency for DMU A (or DMU B) can be obtained by solving the following LP (see (8.1)):

$$\min\ \boldsymbol{c}_A\boldsymbol{x}(= 2\boldsymbol{c}_B\boldsymbol{x}) \tag{8.10}$$

$$\text{subject to}\quad x_i \geq \sum_{j=1}^{n} x_{ij}\lambda_j\ (i = 1,\ldots,m) \tag{8.11}$$

$$y_{rA}(= y_{rB}) \leq \sum_{j=1}^{n} y_{rj}\lambda_j\ (r = 1,\ldots,s) \tag{8.12}$$

$$\lambda_j \geq 0.\ (\forall j) \tag{8.13}$$

Apparently, DMUs A and B have the same optimal solution (inputs) $\boldsymbol{x}_A^* = \boldsymbol{x}_B^*$, and hence the same cost efficiency, since we have:

$$\gamma_A^* = \boldsymbol{c}_A\boldsymbol{x}_A^*/\boldsymbol{c}_A\boldsymbol{x}_A = 2\boldsymbol{c}_B\boldsymbol{x}_B^*/2\boldsymbol{c}_B\boldsymbol{x}_B = \boldsymbol{c}_B\boldsymbol{x}_B^*/\boldsymbol{c}_B\boldsymbol{x}_B = \gamma_B^*.$$

By definition, they also have the same allocative efficiency. □

This is not acceptable, since DMUs A and B have the same cost and allocative efficiencies but the cost of DMU B is half that of DMU A.

8.3.1 A New Scheme for Evaluating Cost Efficiency

The previous example reveals a serious shortcoming in the traditional Farrell-Debreu cost and allocative efficiency measures. These shortcomings are caused by the structure of the supposed production possibility set P as defined by:

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq X\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \boldsymbol{0}\}. \tag{8.14}$$

P is defined only by using technical factors $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in R^{m\times n}$ and $Y = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n) \in R^{s\times n}$, but excludes consideration of the unit input costs $C = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n)$.

Let us define another cost-based production possibility set P_c as:

$$P_c = \left\{(\overline{\boldsymbol{x}}, \boldsymbol{y}) | \overline{\boldsymbol{x}} \geq \overline{X}\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \boldsymbol{0}\right\}, \quad (8.15)$$

where $\overline{X} = (\overline{\boldsymbol{x}}_1, \ldots, \overline{\boldsymbol{x}}_n)$ with $\overline{\boldsymbol{x}}_j = (c_{1j}x_{1j}, \ldots, c_{mj}x_{mj})^T$.

Here we assume that the matrices X and C are non-negative. We also assume that the elements of $\overline{x}_{ij} = (c_{ij}x_{ij})$ $(\forall(i,j))$ are denominated in homogeneous units, viz., dollars, so that adding up the elements of $\overline{x}_{ij}$ has a well defined meaning.

Based on this new production possibility set P_c, a new "technical efficiency" measure, $\overline{\theta}^*$, is obtained as the optimal solution of the following LP problem:

$$[\text{NTec}] \qquad \overline{\theta}^* = \min_{\overline{\theta}, \boldsymbol{\lambda}} \overline{\theta} \quad (8.16)$$

$$\text{subject to} \qquad \overline{\theta}\overline{\boldsymbol{x}}_o \geq \overline{X}\boldsymbol{\lambda} \quad (8.17)$$

$$\boldsymbol{y}_o \leq Y\boldsymbol{\lambda} \quad (8.18)$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0}. \quad (8.19)$$

The new cost efficiency $\overline{\gamma}^*$ is defined as

$$\overline{\gamma}^* = \boldsymbol{e}\overline{\boldsymbol{x}}_o^* / \boldsymbol{e}\overline{\boldsymbol{x}}_o, \quad (8.20)$$

where $\boldsymbol{e} \in R^m$ is a row vector with all elements being equal to 1, and $\overline{\boldsymbol{x}}_o^*$ is the optimal solution of the LP given below:

$$[\text{NCost}] \qquad \boldsymbol{e}\overline{\boldsymbol{x}}_o^* = \min_{\overline{\boldsymbol{x}}, \boldsymbol{\lambda}} \boldsymbol{e}\overline{\boldsymbol{x}} \quad (8.21)$$

$$\text{subject to} \qquad \overline{\boldsymbol{x}} \geq \overline{X}\boldsymbol{\lambda} \quad (8.22)$$

$$\boldsymbol{y}_o \leq Y\boldsymbol{\lambda} \quad (8.23)$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0}. \quad (8.24)$$

Theorem 8.2 *The new "cost efficiency", $\overline{\gamma}^*$ in (8.20), is not greater than the new "technical efficiency" $\overline{\theta}^*$ in (8.16).*

Proof : Let an optimal solution for (8.16)-(8.19) be $(\overline{\theta}^*, \boldsymbol{\lambda}^*)$. Then, $(\overline{\theta}^*\overline{\boldsymbol{x}}_o, \boldsymbol{\lambda}^*)$ is feasible for (8.21)-(8.24). Hence, it follows that $\boldsymbol{e}\overline{\theta}^*\overline{\boldsymbol{x}}_o \geq \boldsymbol{e}\overline{\boldsymbol{x}}_o^*$. This leads to $\overline{\theta}^* \geq \boldsymbol{e}\overline{\boldsymbol{x}}_o^*/\boldsymbol{e}\overline{\boldsymbol{x}}_o = \overline{\gamma}^*$. □

The new *allocative* efficiency $\overline{\alpha}^*$ is then defined as the ratio of $\overline{\gamma}^*$ to $\overline{\theta}^*$, i.e.,

$$\overline{\alpha}^* = \overline{\gamma}^* / \overline{\theta}^*. \quad (8.25)$$

We note that the new efficiency measures $\overline{\theta}^*$, $\overline{\gamma}^*$ and $\overline{\alpha}^*$ are all units invariant so long as $\overline{X}$ has a common unit of cost, e.g., dollars, cents or pounds.

On monotonicity of the new measures with respect to unit cost, we have the following theorem.

Theorem 8.3 (Tone(2002)) *If* $\boldsymbol{x}_A = \boldsymbol{x}_B$, $\boldsymbol{y}_A = \boldsymbol{y}_B$ *and* $\boldsymbol{c}_A \geq \boldsymbol{c}_B$, *then we have the following inequalities:* $\overline{\theta}_A^* \leq \overline{\theta}_B^*$ *and* $\overline{\gamma}_A^* \leq \overline{\gamma}_B^*$. *Furthermore, strict inequalities hold if* $\boldsymbol{c}_A > \boldsymbol{c}_B$.

Proof. Since $\overline{\boldsymbol{x}}_A \geq \overline{\boldsymbol{x}}_B$ and $\boldsymbol{y}_A = \boldsymbol{y}_B$, the new technical measure $\overline{\theta}_A^*$ is less than or equal to $\overline{\theta}_B^*$ and a strict inequality holds if $\boldsymbol{c}_A > \boldsymbol{c}_B$. Regarding the new cost efficiency, we note that the optimal solution of [NCost] depends only on $\boldsymbol{y}_o$. Hence, DMUs $(\overline{\boldsymbol{x}}_A, \boldsymbol{y}_A)$ and $(\overline{\boldsymbol{x}}_B, \boldsymbol{y}_B)$ with $\boldsymbol{y}_A = \boldsymbol{y}_B$ have a common optimal solution $\overline{\boldsymbol{x}}^*$. Therefore, we have $\overline{\gamma}_A^* = \boldsymbol{e}\overline{\boldsymbol{x}}^*/\boldsymbol{e}\overline{\boldsymbol{x}}_A \leq \boldsymbol{e}\overline{\boldsymbol{x}}^*/\boldsymbol{e}\overline{\boldsymbol{x}}_B = \overline{\gamma}_B^*$ and strict inequality holds if $\boldsymbol{c}_A > \boldsymbol{c}_B$. □

Thus, the new measure eliminates the possible occurrence of the phenomenon observed at the beginning of this section.

8.3.2 *Differences Between the Two Models*

We now comment on the differences existing between the traditional "Farrell-Debreu" and the new models. In the traditional model, keeping the unit cost of DMU$_o$ fixed at c_o, the optimal input mix $\boldsymbol{x}^*$ that produces the output $\boldsymbol{y}_o$ is found. In the new model, we search for the optimal input mix $\overline{\boldsymbol{x}}^*$ for producing $\boldsymbol{y}_o$ (or more). More concretely, the optimal mix is described as:

$$\overline{x}_i^* = \sum_{j=1}^{n} c_{ij} x_{ij} \lambda_j^*. \ (i = 1, \ldots, m) \tag{8.26}$$

Hence, it is assumed that, for a given output $\boldsymbol{y}_o$, the optimal input mix can be found (and realized) independently of the current unit cost $\boldsymbol{c}_o$ of DMU$_o$.

These are fundamental differences between the two models. Using the traditional "Farrell-Debreu" model we can fail to recognize the existence of other cheaper input mixes, as we have demonstrated earlier. We demonstrate this with a simple example involving three DMUs A, B and C with each using two inputs (x_1, x_2) to produce one output (y) along with input costs (c_1, c_2). The data and the resulting measures are exhibited in Table 8.3.

For DMUs A and B, the traditional model gives the same technical (θ^*), cost (γ^*) and allocative (α^*) efficiency scores — as expected from Theorem 1. DMU C is found to be the only efficient performer in this framework.

The new scheme devised as in Tone (2002) — see footnote 4 — distinguishes DMU A from DMU B by according them different technical and cost efficiency scores. (See New Scheme in Table 8.3). This is due to the difference in their unit costs. Moreover, DMU B is judged as technically, cost and allocatively efficient with improvement in cost efficiency score from $0.35(\gamma_B^*)$ to $1(\overline{\gamma}_B^*)$ as exhibited in these two tables. As shown in Table 8.3, this cost difference produces a drop in DMU A's cost efficiency score from $0.35(\gamma_A^*)$ to $0.1(\overline{\gamma}_A^*)$. This drop in DMU A's performance is explained by its higher cost structure. Lastly, DMU C is no longer efficient in any of its technical, cost or allocative efficiency performances.

Table 8.3. Comparison of Traditional and New Scheme

						Traditional Efficiency		
						Tech.	Cost	Alloc.
	x_1	c_1	x_2	c_2	y	θ^*	γ^*	α^*
A	10	10	10	10	1	0.5	0.35	0.7
B	10	1	10	1	1	0.5	0.35	0.7
C	5	3	2	6	1	1	1	1
						New Scheme Efficiency		
						Tech.	Cost	Alloc.
	$\overline{x}_1$	e_1	$\overline{x}_2$	e_2	y	$\overline{\theta}^*$	$\overline{\gamma}^*$	$\overline{\alpha}^*$
A	100	1	100	1	1	0.1	0.1	1
B	10	1	10	1	1	1	1	1
C	15	1	12	1	1	0.8333	0.7407	0.8889

8.3.3 An Empirical Example

In this section, we apply our new method to a set of hospital data. Table 8.4 records the performances of 12 hospitals in terms of two inputs, number of doctors and nurses, and two outputs identified as number of outpatients and inpatients (each in units of 100 persons/month). Relative unit costs of doctors and nurses for each hospital are also recorded in columns 4 and 6.

Multiplying the number of doctors and nurses by their respective unit costs we obtain the new data set $(\overline{X}, Y)$ exhibited in Table 8.5. The results of efficiency scores: CCR(θ^*), New technical ($\overline{\theta}^*$), New cost ($\overline{\gamma}^*$) and New allocative ($\overline{\alpha}^*$), are also recorded.

From the results, it is seen that the best performer is Hospital B with all its efficiency scores being equal to one. Regarding the cost-based measures, Hospitals E and L received full efficiency marks even though they fell short in their CCR efficiency score. Conversely, although E has the worst CCR score (0.763), its lower unit costs are sufficient to move its cost-based performance to the top rank. This information obtained from $\theta^* = 0.763$ shows that this hospital still has room for input reductions compared with other technically efficient hospitals. Hospital L, on the other hand, may be regarded as positioned in the best performer group. These two DMUs show that the usual assumption does not hold and thus technical efficiency (CCR $\theta^* = 1$) being achieved is not a necessary condition for the new cost and allocative efficiencies. This is caused by the difference between the two production possibility sets, i.e., the technology-based (X, Y) and the cost-based $(\overline{X}, Y)$. On the other hand, Hospital D is rated worst with respect to cost-based measures, although it receives full efficiency marks in terms of its CCR score. This gap is due to its

Table 8.4. Data for 12 Hospitals

		Inputs				Outputs	
		Doctor		Nurse		Outpat.	Inpat.
No.	DMU	Number	Cost	Number	Cost	Number	Number
1	A	20	500	151	100	100	90
2	B	19	350	131	80	150	50
3	C	25	450	160	90	160	55
4	D	27	600	168	120	180	72
5	E	22	300	158	70	94	66
6	F	55	450	255	80	230	90
7	G	33	500	235	100	220	88
8	H	31	450	206	85	152	80
9	I	30	380	244	76	190	100
10	J	50	410	268	75	250	100
11	K	53	440	306	80	260	147
12	L	38	400	284	70	250	120
	Average	33.6	435.8	213.8	85.5	186.3	88.2

Table 8.5. New Data Set and Efficiencies

		Data				Efficiency			
		$\overline{X}$		Y		CCR	Tech.	Cost	Alloc.
No.	DM	Doctor	Nurse	Inp.	Outp.	θ^*	$\overline{\theta}^*$	$\overline{\gamma}^*$	$\overline{\alpha}^*$
1	A	10000	15100	100	90	1	.994	.959	.965
2	B	6650	10480	150	50	1	1	1	1
3	C	11250	14400	160	55	.883	.784	.724	.923
4	D	16200	20160	180	72	1	.663	.624	.941
5	E	6600	11060	94	66	.763	1	1	1
6	F	24750	20400	230	90	.835	.831	.634	.764
7	G	16500	23500	220	88	.902	.695	.693	.997
8	H	13950	17510	152	80	.796	.757	.726	.959
9	I	11400	18544	190	100	.960	.968	.953	.984
10	J	20500	20100	250	100	.871	.924	.776	.841
11	K	23320	24480	260	147	.955	.995	.863	.867
12	L	15200	19880	250	120	.958	1	1	1

high cost structure. Hospital D needs reductions in its unit costs to attain good cost-based scores.

Hospital F has the worst allocative efficiency score, and hence needs a change in input-(cost)mix. This hospital has the current input-(cost)mix, $\overline{\boldsymbol{x}}_F = (24750, 20400)$, while the optimal mix $\overline{\boldsymbol{x}}_F^*$ is $(11697, 16947)$. So, under its current costs, F needs to reduce the number of doctors from 55 to 26 (=11697/450), and nurses from 255 to 212 (=16947/80). Otherwise, if F retains its current input numbers, it needs to reduce the unit cost of doctors from 450 to 213 (=11697/55), and that of nurses from 80 to 66 (=16947/255). Of course, there are many other adjustment plans as well. In any case our proposed new measures provide much more information than the traditional ones.

8.3.4 Extensions

We can also extend this new cost efficiency model to three other situations as follows.

1. **Revenue Efficiency**
   Given the unit price $\boldsymbol{p}_j$ for each output $\boldsymbol{y}_j$ $(j = 1, \ldots, n)$, the conventional revenue efficiency ρ_o^* of DMU$_o$ is evaluated by $\rho_o^* = \boldsymbol{p}_o\boldsymbol{y}_o/\boldsymbol{p}_o\boldsymbol{y}_o^*$. Here, $\boldsymbol{p}_o\boldsymbol{y}_o^*$ is obtained from the optimal objective value of LP problem (8.6).

   However, in the situation of different unit prices, this revenue efficiency ρ_o^* suffers from shortcomings similar to the traditional cost efficiency measure shortcomings described in the previous section. We can eliminate such shortcomings by introducing the price-based output $\overline{Y} = (\overline{\boldsymbol{y}}_1, \ldots, \overline{\boldsymbol{y}}_n)$ with $\overline{\boldsymbol{y}}_j = (p_{1j}y_{1j}, \ldots, p_{sj}y_{sj})$ into the following LP:

$$[\text{NRevenue}] \qquad \boldsymbol{e}\overline{\boldsymbol{y}}_o^* = \max_{\overline{\boldsymbol{y}}, \boldsymbol{\lambda}} \boldsymbol{e}\overline{\boldsymbol{y}} \tag{8.27}$$

$$\begin{aligned} \text{subject to} \quad & \boldsymbol{x}_o \geq X\boldsymbol{\lambda} \\ & \overline{\boldsymbol{y}} \leq \overline{Y}\boldsymbol{\lambda} \\ & L \leq \boldsymbol{e}\boldsymbol{\lambda} \leq U \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{aligned}$$

   The new revenue efficiency measure $\overline{\rho}_o$ is defined by

$$\text{New Revenue Efficiency } (\overline{\rho}_o) = \boldsymbol{e}\overline{\boldsymbol{y}}_o/\boldsymbol{e}\overline{\boldsymbol{y}}_o^*. \tag{8.28}$$

2. **Profit Efficiency**
   Using the new cost and revenue efficiency models, we can also define a new profit efficiency model as follows:

$$[\text{NProfit}] \qquad \boldsymbol{e}\overline{\boldsymbol{y}}_o^* - \boldsymbol{e}\overline{\boldsymbol{x}}_o^* = \max_{\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}, \boldsymbol{\lambda}} \boldsymbol{e}\overline{\boldsymbol{y}} - \boldsymbol{e}\overline{\boldsymbol{x}} \tag{8.29}$$

$$\begin{aligned} \text{subject to} \quad & \overline{\boldsymbol{x}} = \overline{X}\boldsymbol{\lambda} \leq \overline{\boldsymbol{x}}_o \\ & \overline{\boldsymbol{y}} = \overline{Y}\boldsymbol{\lambda} \geq \overline{\boldsymbol{y}}_o \\ & L \leq \boldsymbol{e}\boldsymbol{\lambda} \leq U \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}, \end{aligned}$$

where $\overline{X}$ and $\overline{Y}$ are defined in the new cost and revenue models, respectively. The new profit efficiency is defined as:

$$\text{New Profit Efficiency } (\overline{\pi}_o) = (e\overline{\boldsymbol{y}}_o - e\overline{\boldsymbol{x}})/(e\overline{\boldsymbol{y}}_o^* - e\overline{\boldsymbol{x}}_o^*). \tag{8.30}$$

3. **Profit Ratio Model**
We also propose a model for maximizing the revenue vs. cost ratio,

$$\frac{\text{revenue}}{\text{expenses}},$$

instead of maximizing profit (revenue − expenses), since in some situations the latter gives a negative value that is awkward to deal with. This new profit ratio model can be formulated as a problem of maximizing the *revenue/expenses* ratio to obtain the following fractional programming problem,[5]

$$\begin{aligned} \text{[Profit Ratio]} \quad & \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda}} \frac{\boldsymbol{p}_o\boldsymbol{y}}{\boldsymbol{c}_o\boldsymbol{x}} \\ \text{subject to} \quad & \boldsymbol{x} = X\boldsymbol{\lambda} \leq \boldsymbol{x}_o \\ & \boldsymbol{y} = Y\boldsymbol{\lambda} \geq \boldsymbol{y}_o \\ & L \leq e\boldsymbol{\lambda} \leq U \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{aligned} \tag{8.31}$$

We can transform this program to the linear programming problem below, by introducing a variable $t \in R$ and use the Charnes-Cooper transformation of fractional programming which sets $\widehat{\boldsymbol{x}} = t\boldsymbol{x}$, $\widehat{\boldsymbol{y}} = t\boldsymbol{y}$, $\widehat{\boldsymbol{\lambda}} = t\boldsymbol{\lambda}$. Then multiplying all terms by $t > 0$, we change (8.31) to

$$\begin{aligned} & \max_{\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}},\widehat{\boldsymbol{\lambda}},t} \boldsymbol{p}_o\widehat{\boldsymbol{y}} \\ \text{subject to} \quad & \boldsymbol{c}_o\widehat{\boldsymbol{x}} = 1 \\ & t\boldsymbol{x}_o \geq X\widehat{\boldsymbol{\lambda}} = \widehat{\boldsymbol{x}} \\ & t\boldsymbol{y}_o \leq Y\widehat{\boldsymbol{\lambda}} = \widehat{\boldsymbol{y}} \\ & Lt \leq e\widehat{\boldsymbol{\lambda}} \leq Ut \\ & \widehat{\boldsymbol{\lambda}} \geq \boldsymbol{0}. \end{aligned} \tag{8.32}$$

Let an optimal solution of this LP problem be $(t^*, \widehat{\boldsymbol{x}}^*, \widehat{\boldsymbol{y}}^*, \widehat{\boldsymbol{\lambda}}^*)$. Since $t^* > 0$ we can reverse this transformation and obtain an optimal solution to the fractional program in (8.31) from

$$\boldsymbol{x}^* = \widehat{\boldsymbol{x}}^*/t^*,\ \boldsymbol{y}^* = \widehat{\boldsymbol{y}}^*/t^*,\ \boldsymbol{\lambda}^* = \widehat{\boldsymbol{\lambda}}^*/t^*. \tag{8.33}$$

The *revenue/cost* efficiency E_{RC} of DMU$_o$ can then be related to actual revenue and costs by the following "ratio of ratios,"

$$E_{RC} = \frac{\boldsymbol{p}_o\boldsymbol{y}_o/\boldsymbol{c}_o\boldsymbol{x}_o}{\boldsymbol{p}_o\boldsymbol{y}^*/\boldsymbol{c}_o\boldsymbol{x}^*}. \tag{8.34}$$

As noted in the following remark, this efficiency index is related to profit efficiency but is applicable even when there are many deficit-DMUs, i.e. DMU_os for which $\boldsymbol{p}_o\boldsymbol{y}_o - \boldsymbol{c}_o\boldsymbol{x}_o < 0$.

[Remark] We can follow Cooper *et al.* (2005)[6] and obtain a profit-to-cost return ratio by noting that an optimal solution to (8.31) is not altered by replacing the objective with

$$\max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{\lambda}} \frac{\boldsymbol{p}_o\boldsymbol{y}}{\boldsymbol{c}_o\boldsymbol{x}} - 1.$$

This gives

$$\frac{\boldsymbol{p}_o\boldsymbol{y}^*}{\boldsymbol{c}_o\boldsymbol{x}^*} - \frac{\boldsymbol{c}_o\boldsymbol{x}^*}{\boldsymbol{c}_o\boldsymbol{x}^*} = \frac{\boldsymbol{p}_o\boldsymbol{y}^* - \boldsymbol{c}_o\boldsymbol{x}^*}{\boldsymbol{c}_o\boldsymbol{x}^*}$$

which is the commonly used profit-to-cost ratio measure of performance, after it has been adjusted to eliminate inefficiencies. Finally, if the observed profit-to-cost ratio is positive we can use the following

$$0 \le \frac{\boldsymbol{p}_o\boldsymbol{y}_o - \boldsymbol{c}_o\boldsymbol{x}_o}{\boldsymbol{c}_o\boldsymbol{x}_o} \bigg/ \frac{\boldsymbol{p}_o\boldsymbol{y}^* - \boldsymbol{c}_o\boldsymbol{x}^*}{\boldsymbol{c}_o\boldsymbol{x}^*} \le 1$$

as a measure of efficiency with unity achieved if and only if $\frac{\boldsymbol{p}_o\boldsymbol{y}_o - \boldsymbol{c}_o\boldsymbol{x}_o}{\boldsymbol{c}_o\boldsymbol{x}_o} = \frac{\boldsymbol{p}_o\boldsymbol{y}^* - \boldsymbol{c}_o\boldsymbol{x}^*}{\boldsymbol{c}_o\boldsymbol{x}^*}$.

8.4 DECOMPOSITION OF COST EFFICIENCY

Technology and cost are the wheels that drive modern enterprises. Some enterprises have advantages in terms of technology and others in cost. Hence, a management may want to know how and to what extent their resources are being effectively and efficiently utilized, compared to similar enterprises in the same or a similar field.

In an effort to address this subject, Tone and Tsutsui (2004)[7] developed a scheme for decomposing actual observed cost into the sum of the minimum cost and the loss due to input inefficiency. Furthermore, the loss due to input inefficiency can be expressed as the sum of the loss due to input technical, price and allocative inefficiencies.

8.4.1 Loss due to Technical Inefficiency

We consider n DMUs, each having m inputs for producing s outputs. We utilize the notations for denoting observed inputs ($\boldsymbol{x}_o$), outputs ($\boldsymbol{y}_o$) and input unit prices ($\boldsymbol{c}_o$). We also assume that unit input prices are not identical among DMUs. The actual (observed) input cost for DMU ($\boldsymbol{x}_o, \boldsymbol{y}_o$) can be calculated as follows:

$$C_o = \sum_{i=1}^{m} c_{io}x_{io}. \ (o = 1, \ldots, n) \tag{8.35}$$

We postulate the production possibility set P defined by

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq X\boldsymbol{\lambda},\ \boldsymbol{y} \leq Y\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq \mathbf{0}\}. \tag{8.36}$$

Let the (technically) efficient input for DMU$_o$ be $\boldsymbol{x}_o^*$, which can be obtained by solving the CCR, SBM or Hybrid models depending on the situation. The technically efficient input cost for DMU$_o$ is calculated as

$$C_o^* = \sum_{i=1}^{m} c_{io} x_{io}^*.\ (o = 1, \ldots, n) \tag{8.37}$$

Then, the loss in input cost due to technical inefficiency is expressed as follows:

$$L_o^* = C_o - C_o^* (\geq 0). \tag{8.38}$$

8.4.2 *Loss due to Input Price Inefficiency*

We now construct a cost-based production possibility set analogous to that in the preceding section as follows:

$$\overline{P}_c = \{(\overline{\boldsymbol{x}}, \boldsymbol{y}) | \overline{\boldsymbol{x}} \geq \overline{X}\boldsymbol{\lambda},\ \boldsymbol{y} \leq Y\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq \mathbf{0}\}, \tag{8.39}$$

where $\overline{X} = (\overline{\boldsymbol{x}}_1, \ldots, \overline{\boldsymbol{x}}_n) \in R^{m \times n}$, $\overline{\boldsymbol{x}}_j = (\overline{x}_{ij}, \ldots, \overline{x}_{mj})$, and $\overline{x}_{ij} = c_{ij} x_{ij}^*$. It should be noted that $\boldsymbol{x}_j^*$ represents the technically efficient input for producing $\boldsymbol{y}_j$. Hence, we utilize $c_{ij} x_{ij}^*$ instead of $c_{ij} x_{ij}$ in the preceding section in order to eliminate technical inefficiency to the maximum possible extent. Then we solve the CCR model on $\overline{P}_c$ in a manner similar to that of [NTec] in (8.16):

$$\text{[NTec-2]} \qquad \rho^* = \min_{\rho, \boldsymbol{\mu}, \boldsymbol{t}^-, \boldsymbol{t}^+} \rho \tag{8.40}$$

$$\text{subject to} \qquad \rho \overline{\boldsymbol{x}}_o = \overline{X}\boldsymbol{\mu} + \boldsymbol{t}^- \tag{8.41}$$

$$\boldsymbol{y}_o = Y\boldsymbol{\mu} - \boldsymbol{t}^+ \tag{8.42}$$

$$\boldsymbol{\mu} \geq \mathbf{0},\ \boldsymbol{t}^- \geq \mathbf{0},\ \boldsymbol{t}^+ \geq \mathbf{0}. \tag{8.43}$$

Let $(\rho^*,\ \boldsymbol{\mu}^*,\ \boldsymbol{t}^{-*},\ \boldsymbol{t}^{+*})$ be an optimal solution for [NTec-2]. Then, $\rho^* \overline{\boldsymbol{x}}_o = (\rho^* c_{1o} x_{1o}^*, \ldots, \rho^* c_{mo} x_{mo}^*)$ indicates the radially reduced input vector on the (weakly) efficient frontier of the cost-based production set $\overline{P}_c$ in (8.39). Now we define

$$\boldsymbol{c}_o^* = \rho^* \boldsymbol{c}_o = (\rho^* c_{1o}, \ldots, \rho^* c_{mo}). \tag{8.44}$$

$\boldsymbol{c}_o^*$ is the radially reduced input factor price vector for the technically efficient input $\boldsymbol{x}_o^*$ that can produce $\boldsymbol{y}_o$. The [NTec-2] projection is given by

$$\text{[NTec-2 Projection]}\ \overline{\boldsymbol{x}}_o^* = \rho^* \overline{\boldsymbol{x}}_o - \boldsymbol{t}^{-*},\ \boldsymbol{y}_o^* = \boldsymbol{y}_o + \boldsymbol{t}^{+*}. \tag{8.45}$$

We define the strongly efficient cost C_o^{**}, which is the technical and price efficient cost, and the loss L_o^{**} due to the difference of the input price as follows:

$$C_o^{**} = \sum_{i=1}^{m} \overline{x}_{io}^{*} = \sum_{i=1}^{m} (\rho^* \overline{x}_{io} - t_{io}^{-*}) \leq \rho^* \sum_{i=1}^{m} \overline{x}_{io} = \rho^* C_o^* \leq C_o^* \quad (8.46)$$

$$L_o^{**} = C_o^* - C_o^{**} (\geq 0). \quad (8.47)$$

8.4.3 *Loss due to Allocative Inefficiency*

Furthermore, we solve the [NCost] model in (8.21) for $\overline{P}_c$ as follows:

$$\text{[NCost-2]} \qquad C_o^{***} = \min_{\overline{\boldsymbol{x}}, \boldsymbol{\mu}} \ \boldsymbol{e}\overline{\boldsymbol{x}} \quad (8.48)$$

$$\text{subject to} \qquad \overline{\boldsymbol{x}} \geq \overline{X}\boldsymbol{\mu}$$

$$\boldsymbol{y}_o \leq Y\boldsymbol{\mu}$$

$$\boldsymbol{\mu} \geq \boldsymbol{0}.$$

Let $(\overline{\boldsymbol{x}}_o^{**}, \ \boldsymbol{\mu}^*)$ be an optimal solution. Then, the cost-based pair $(\overline{\boldsymbol{x}}_o^{**}, \ \boldsymbol{y}_o)$ is the minimum production cost in the assumed production possibility set $\overline{P}_c$. This set can differ substantially from P if the unit prices of the inputs vary from DMU to DMU. The (global) allocative efficiency α^* of DMU$_o$ is defined as follows:

$$\alpha^* = \frac{C_o^{***}}{C_o^{**}} (\leq 1). \quad (8.49)$$

We also define the loss L_o^{***} due to the suboptimal cost mix as

$$L_o^{***} = C_o^{**} - C_o^{***} (\geq 0). \quad (8.50)$$

8.4.4 *Decomposition of the Actual Cost*

From (8.38), (8.47) and (8.50), we can derive at the following theorem.

Theorem 8.4 (Tone and Tsutsui (2004))

$$C_o \geq C_o^* \geq C_o^{**} \geq C_o^{***}. \quad (8.51)$$

Furthermore, we can obtain the relationship among the optimal cost and losses, and the actual cost (C_o) can be decomposed into three losses and the minimum cost (C_o^{***}):

$$L_o^* = C_o - C_o^* (\geq 0) \text{ Loss due to Technical Inefficiency} \quad (8.52)$$

$$L_o^{**} = C_o^* - C_o^{**} (\geq 0) \text{ Loss due to Price Inefficiency} \quad (8.53)$$

$$L_o^{***} = C_o^{**} - C_o^{***} (\geq 0) \text{ Loss due to Allocative Inefficiency} \quad (8.54)$$

$$C_o = L_o^* + L_o^{**} + L_o^{***} + C_o^{***}. \quad (8.55)$$

For further developments of this scheme, see Problems 8.1-8.4 at the end of this chapter.

8.4.5 An Example of Decomposition of Actual Cost

We applied the above procedure to the data set exhibited in Table 8.4 and obtained the results listed in Table 8.6. We then utilized the input-oriented CCR model and the projection formulas in Chapter 3 for finding the technical efficient inputs (x_o^*).

Table 8.6. Decomposition of Actual Cost

DMU	Actual C	Cost Minimum C^{***}	C^{***}/C	Loss Tech. L^*	Loss Price L^{**}	Loss Alloc. L^{***}
A	25100	18386	0.73	0	5959	754
B	17130	17130	1	0	0	0
C	25650	18404	0.72	3557	2658	1032
D	36360	21507	0.59	0	13470	1383
E	17660	13483	0.76	4177	0	0
F	45150	27323	0.61	12911	1767	3149
G	40000	26287	0.66	4256	8931	526
H	31460	19684	0.63	6407	4230	1139
I	29944	24605	0.82	3254	1617	468
J	40600	29871	0.74	7725	0	3004
K	47800	34476	0.72	5367	5030	2927
L	35080	31457	0.90	2348	0	1275
Total	391934	282615	0.72	50002	43661	15656

As can be seen from the results, Hospital B is again the most efficient DMU in the sense that it has no loss due to technical, price or allocative inefficiencies, while Hospital D has the worst ratio $C^{***}/C(= 0.59)$ caused by losses due to price and allocative inefficiencies. Hospital A has inefficiency in cost-based aspects, i.e., price and allocation but not in technical-physical aspects, whereas E has inefficiency due to technical-physical aspects but not in cost-based aspects. Figure 8.2 exhibits the decomposition graphically.

8.5 SUMMARY OF CHAPTER 8

This chapter has covered approaches which have been studied in DEA for evaluations of efficiencies such as "allocative" and "overall" efficiencies. Problems in the standard approaches were identified with cases in which different prices or different unit costs may be associated with the performance of different firms producing the same outputs and utilizing the same inputs. Types of price-cost efficiencies were therefore identified and related to extensions of the customary production possibility sets that reflected the unit price and cost differences.

Standard approaches were also extended in models that permit substitutions so that worsening of some inputs or outputs may be made in order to improve

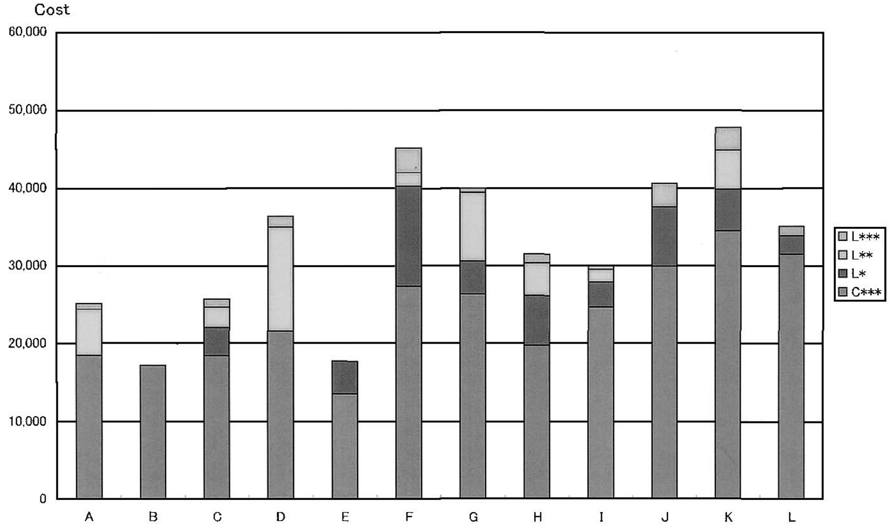

Figure 8.2. Decomposition of Actual Cost

other inputs or outputs. We also extended the traditional Farrell-Debreu cost efficiency measures and introduced new ones that can deal with non-identical cost and price situations. Furthermore, we provided a decomposition of the actual cost into the minimum cost and into losses due to other inefficiencies.

Problems in the use of these concepts may be encountered because many companies are unwilling to disclose their unit costs. As noted by Farrell, unit prices may also be subject to large fluctuations. One may, of course, use averages or other summaries of such prices and also deduce or infer which unit costs are applicable. However, this may not be satisfactory because in many cases accurate costs and prices may not really reflect criteria that are being used. Cases in point include attempts to evaluate public programs such as education, health, welfare, or military and police activities.

Earlier in this book a variety of approaches were suggested that can be applied to these types of problems. This includes the case of assurance regions and like concepts, as treated in Chapter 6, which can replace exact knowledge of prices and costs with corresponding bounds on their values. When this is done, however, the precise relations between allocative, overall and technical efficiencies may become blurred.

8.6 NOTES AND SELECTED BIBLIOGRAPHY

The concepts of cost efficiency related subjects were introduced by M.J. Farrell (1957) and G. Debreu (1951) and developed into implementable form by Färe, Grosskopf and Lovell (1985) using linear programming technologies. Cooper, Park and Pastor (1999)[8] extended these treatments to the Additive models with a new "translation invariant" measure named "RAM."

Inadequacies in attempts to move from "technical" to price based or cost based efficiencies were identified by Tone (2002). In response a new approach to cost efficiency was developed by Tone (2002) and further extended to decompositions of cost efficiency by Tone and Tsutsui (2004) in a form they applied to Japan-US electric utilities comparisons. Tone and Sahoo (2005)[9] applied the new cost efficiency model to examine the performance of Life Insurance Corporation (LIC) of India and found a significant heterogeneity in the cost efficiency scores over the course of 19 years. See also Tone and Sahoo (2005)[10] in which the issues of cost elasticity are extensively discussed based on the new cost efficiency model. Fukuyama and Weber (2004)[11] developed a variant of the new cost efficiency model using "directional distance functions" introduced in Chambers, Chung and Färe (1996)[12] to measure inefficiency. See Färe, Grosskopf and Whittaker (2004)[13] for an updated survey. See, however, Ray (2004), [14] p.95, who identified a deficiency in the failure of "directional distance functions" to account for nonzero slacks in the measures of efficiency.

8.7 RELATED DEA-SOLVER MODELS FOR CHAPTER 8

(New-)Cost-C(V) Cost-C(V) code evaluates the *cost efficiency* of each DMU as follows. First we solve the LP problem below:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} c_i x_i \\ \text{subject to} \quad & x_i \geq \sum_{j=1}^{n} x_{ij}\lambda_j \quad (i = 1, \ldots, m) \\ & y_{ro} \leq \sum_{i=1}^{n} y_{rj}\lambda_j \quad (r = 1, \ldots, s) \\ & L \leq \sum_{j=1}^{n} \lambda_j \leq U \\ & \lambda_j \geq 0 \ \ \forall j, \end{aligned} \tag{8.56}$$

where c_i is the unit cost of the input i. This model allows substitutions in inputs. Based on an optimal solution $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ of the above LP, the cost efficiency of DMU$_o$ is defined as

$$E_C = \frac{\boldsymbol{c}\boldsymbol{x}^*}{\boldsymbol{c}\boldsymbol{x}_o}. \tag{8.57}$$

The code "Cost-C" solves the case $L = 0$. $U = \infty$ (the case of constant returns to scale) and "Cost-V" for the case $L = U = 1$ (the variable returns to scale case).

The data set (X, Y, C) should be prepared in an Excel Workbook under an appropriate Worksheet name, e.g., ".Data", prior to execution of this code. See the sample format displayed in Figure B.6 in Section B.5 of Appendix B and refer to explanations above the figure.

The results will be obtained in the Worksheets of the selected Workbook: "Score", "Projection" (projection onto the efficient frontier), "Graph1", "Graph2" and "Summary."

New-Cost-C(V) solves the model described in [NCost] (8.21). Data format is the same with the Cost-C(V) model.

(New-)Revenue-C(V) Revenue-C(V) code solves the following revenue maximization program for each DMU:

$$\begin{aligned} \max \quad & \boldsymbol{py} \\ \text{subject to} \quad & \boldsymbol{x}_o \geq X\boldsymbol{\lambda} \\ & \boldsymbol{y} \leq Y\boldsymbol{\lambda} \\ & L \leq \boldsymbol{e\lambda} \leq U \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}, \end{aligned}$$

where the vector $\boldsymbol{p} = (p_1, \ldots, p_s)$ expresses the unit prices of the output. This model allows substitutions in outputs.

The code "Revenue-C" solves the case $L = 0$. $U = \infty$ (the case of constant returns to scale) and "Revenue-V", the case $L = U = 1$ (the variable returns to scale case).

Based on an optimal solution $\boldsymbol{y}^*$ of this program, the *revenue efficiency* is defined as

$$E_R = \frac{\boldsymbol{py}_o}{\boldsymbol{py}^*}. \tag{8.58}$$

E_R satisfies $0 < E_R \leq 1$, provided $\boldsymbol{py}_o > 0$. See the sample data format displayed in Figure B.7 in Section B.5 of Appendix B and refer to the explanations above the figure.

The results will be obtained in the Worksheets of the selected Workbook: "Score", "Projection" (projection onto the efficient frontier), "Graph1," "Graph2," and "Summary."

New-Revenue-C(V) solves the model described in [NRevenue] (8.27). Data format is the same with Revenue-C(V).

(New-)Profit-C(V) Profit-C(V) code solves the LP problem defined in (8.8) for each DMU. Based on an optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$, the *profit efficiency* is defined as

$$E_P = \frac{\boldsymbol{py}_o - \boldsymbol{cx}_o}{\boldsymbol{py}^* - \boldsymbol{cx}^*}. \tag{8.59}$$

Under the assumption $\boldsymbol{py}_o > \boldsymbol{cx}_o$, we have $0 < E_P \leq 1$ and DMU$_o$ is *profit efficient* if $E_P = 1$.

The data format is a combination of *Cost* and *Revenue* models. The cost columns are headed by (C) for input names and the price column are headed by (P) for output names.

The results will be obtained in the Worksheets of the selected Workbook: "Score", "Projection" (projection onto the efficient frontier), "Graph1," "Graph2," and "Summary."

New-Profit-C(V) solves the model described in [NProfit] (8.29). Data format is the same with Profit-C(V).

Ratio-C(V) This code solves the LP problem defined in (8.31). Based on the optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$, the *ratio* (revenue/cost) *efficiency* is defined as

$$E_{RC} = \frac{\boldsymbol{p}_o\boldsymbol{y}_o/\boldsymbol{c}_o\boldsymbol{x}_o}{\boldsymbol{p}_o\boldsymbol{y}^*/\boldsymbol{c}_o\boldsymbol{x}^*},$$

which satisfies $0 < E_{RC} \leq 1$ and DMU$_o$ is *ratio efficient* if $E_{RC} = 1$.

The data format is a combination of *Cost* and *Revenue* models. The cost columns are headed by (C) for input names and the price column are headed by (P) for output names.

The results will be obtained in the Worksheets of the selected Workbook: "Score", "Projection" (projection onto the efficient frontier), "Graph1," "Graph2," and "Summary."

8.8 PROBLEM SUPPLEMENT FOR CHAPTER 8

Problem 8.1

In Section 8.4, the actual cost is decomposed into the *sum* of the minimum cost and losses due to technical, price and allocative inefficiencies.

Can you decompose it in the *productive* form (not in the *sum* form)?

Suggested Response : Define the following efficiency measures:

- C^{***}/C = cost efficiency (CE)
- C^{*}/C = technical efficiency (TE)
- C^{**}/C^{*} = price efficiency (PE)
- C^{***}/C^{**} = allocative efficiency (AE)

Then, we have:

$$\text{CE} = \text{TE} \times \text{PE} \times \text{AE}.$$

Problem 8.2

Write out the decomposition of the actual profit in the same vein as described in Section 8.4.

Suggested Response : The actual profit of DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is calculated as:

$$E_o = \boldsymbol{p}_o\boldsymbol{y}_o - \boldsymbol{c}_o\boldsymbol{x}_o. \tag{8.60}$$

Using radial or non-radial technical efficiency models, e.g., the CCR and SBM, we project the DMU onto the efficient frontier and obtain the technically efficient $(\boldsymbol{x}_o^*, \boldsymbol{y}_o^*)$ with profit given by

$$E_o^* = \boldsymbol{p}_o\boldsymbol{y}_o^* - \boldsymbol{c}_o\boldsymbol{x}_o^* \ (\geq \boldsymbol{p}_o\boldsymbol{y}_o - \boldsymbol{c}_o\boldsymbol{x}_o = E_o). \tag{8.61}$$

Thus, the *loss due to technical inefficiency* is evaluated as

$$L_o^* = E_o^* - E_o. \tag{8.62}$$

We formulate the new cost-price based production possibility set as

$$P_{cp} = \left\{(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) |\ \overline{\boldsymbol{x}} \geq \overline{X}\boldsymbol{\lambda}, \overline{\boldsymbol{y}} \leq \overline{Y}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \boldsymbol{0}\right\}, \tag{8.63}$$

where $\overline{X} = (\overline{\boldsymbol{x}}_1, \ldots, \overline{\boldsymbol{x}}_n)$ with $\overline{\boldsymbol{x}}_j = (c_{1j}x_{1j}^*, \ldots, c_{mj}x_{mj}^*)$ and $\overline{Y} = (\overline{\boldsymbol{y}}_1, \ldots, \overline{\boldsymbol{y}}_n)$ with $\overline{\boldsymbol{y}}_j = (p_{1j}y_{1j}^*, \ldots, p_{sj}y_{sj}^*)$. On this PPS we form a technical efficiency model similar to [NTec-2] (8.40) as follows:

$$\text{[NTec-3]} \qquad \rho^* = \min_{\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}, \rho, \boldsymbol{\mu}} \rho \tag{8.64}$$

$$\text{subject to} \qquad \rho\overline{\boldsymbol{x}}_o \geq \overline{X}\boldsymbol{\mu} = \overline{\boldsymbol{x}}$$

$$\boldsymbol{y}_o \leq \overline{Y}\boldsymbol{\mu} = \overline{\boldsymbol{y}}$$

$$\boldsymbol{\mu} \geq \boldsymbol{0}.$$

Let the optimal solution of [NTec-3] be $(\overline{\boldsymbol{x}}_o^*, \overline{\boldsymbol{y}}_o^*, \boldsymbol{\mu}^*, \rho^*)$. (Note that, instead of [NTec-3], we can apply the non-radial and non-oriented SBM for obtaining $(\overline{\boldsymbol{x}}_o^*, \overline{\boldsymbol{y}}_o^*)$.)

We then have the technical and cost-price efficient profit given by

$$E_o^{**} = \boldsymbol{e}\overline{\boldsymbol{y}}_o^* - \boldsymbol{e}\overline{\boldsymbol{x}}_o^* \ (\geq E_o^*). \tag{8.65}$$

The loss due to cost-price inefficiency is estimated by

$$L_o^{**} = E_o^{**} - E_o^* \ (\geq 0). \tag{8.66}$$

Lastly, we solve the following max profit model on P_{cp}.

$$\text{[NProfit-2]} \qquad \boldsymbol{e}\overline{\boldsymbol{y}}_o^{**} - \boldsymbol{e}\overline{\boldsymbol{x}}_o^{**} = \max_{\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}, \boldsymbol{\lambda}} \boldsymbol{e}\overline{\boldsymbol{y}} - \boldsymbol{e}\overline{\boldsymbol{x}} \tag{8.67}$$

$$\text{subject to} \qquad \overline{\boldsymbol{x}}_o \geq \overline{X}\boldsymbol{\lambda} = \overline{\boldsymbol{x}}$$

$$\boldsymbol{y}_o \leq \overline{Y}\boldsymbol{\lambda} = \overline{\boldsymbol{y}}$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0}.$$

Let the optimal solution be $(\overline{\boldsymbol{x}}_o^{**}, \overline{\boldsymbol{y}}_o^{**})$. The allocative efficient profit is given by

$$E_o^{***} = \boldsymbol{e}\overline{\boldsymbol{y}}_o^{**} - \boldsymbol{e}\overline{\boldsymbol{x}}_o^{**}. \tag{8.68}$$

The loss due to allocative inefficiency is evaluated by

$$L_o^{***} = E_o^{***} - E_o^{**} \ (\geq 0). \tag{8.69}$$

Summing up, we have the decomposition of the actual profit into the maximum profit and the losses due to technical, cost-price and allocative inefficiencies as follows:

$$E_o = E_o^{***} - L_o^{*} - L_o^{**} - L_o^{***}. \tag{8.70}$$

Problem 8.3

Most of the models in this Chapter are developed under the constant returns-to-scale (CRS) assumption. Can you develop them under the variable returns-to-scale assumption?

Suggested Response : This can be done by adding the convex constraint on the intensity variable $\boldsymbol{\lambda}$ as follows:

$$\sum_{j=1}^{n} \lambda_j = 1.$$

Problem 8.4

In connection with the preceding problem, can you incorporate the scale inefficiency effect, i.e., loss due to scale inefficiency, in the model?

Suggested Response : In the cost efficiency case, we first solve the input-oriented technical efficiency model under the VRS assumption. Let the optimal solution be $(\boldsymbol{x}_o^{VRS}, \boldsymbol{y}_o^{VRS})$. The cost and loss due to this projection are, respectively:

$$C_o^{VRS} = \boldsymbol{c}_o \boldsymbol{x}_o^{VRS},\ L_o^{VRS} = C_o - C_o^{VRS}, \tag{8.71}$$

where C_o is the actual observed cost of DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$. Then we construct the data set (X^{VRS}, Y^{VRS}) consisting of $(\boldsymbol{x}_j^{VRS}, \boldsymbol{y}_j^{VRS})$ $j = 1, \ldots, n$. We next evaluate the technical efficiency of $(\boldsymbol{x}_o^{VRS}, \boldsymbol{y}_o^{VRS})$ with respect to (X^{VRS}, Y^{VRS}) under the constant returns-to-scale (CRS) assumption. Let the optimal solution be $(\boldsymbol{x}_o^{*}, \boldsymbol{y}_o^{*})$, with its cost $C_o^{*} = \boldsymbol{c}_o \boldsymbol{x}_o^{*}$ $(\leq C_o^{VRS})$. Thus, we obtain the loss due to scale inefficiency as follows:

$$L_o^{Scale} = C_o^{VRS} - C_o^{*}. \tag{8.72}$$

Referring to (8.52)-(8.55), we can decompose the actual cost into four losses with minimum cost as follows:

$$L_o^{VRS} = C_o - C_o^{VRS} (\geq 0) \text{ Loss due to Pure Tech. Inefficiency} \tag{8.73}$$
$$L_o^{Scale} = C_o^{VRS} - C_o^{*} (\geq 0) \text{ Loss due to Scale Inefficiency} \tag{8.74}$$
$$L_o^{**} = C_o^{*} - C_o^{**} (\geq 0) \text{ Loss due to Price Inefficiency} \tag{8.75}$$
$$L_o^{***} = C_o^{**} - C_o^{***} (\geq 0) \text{ Loss due to Allocative Inefficiency} \tag{8.76}$$
$$C_o = L_o^{VRS} + L_o^{Scale} + L_o^{**} + L_o^{***} + C_o^{***}. \tag{8.77}$$

Problem 8.5

The concluding part of Section 5.5, Chapter 5, quoted Dr. Harold Wein, a steel industry consultant, who believed that the concept of returns to scale, as formulated in economics, is useless because increases in plant size are generally accompanied by mix changes in outputs or inputs — or both.

Assignment : Formulate the responses an economist might make to this criticism.

Suggested Response : Under the assumption of profit maximization, as employed in economics, both scale and mix are determined simultaneously. This is consistent with Dr. Wein's observation from his steel industry experience.

One response to Dr. Wein's criticism of the scale concept as employed in economics (which holds mix constant) is to note that economists are interested in being able to distinguish between scale and mix changes when treating empirical data *after* the decisions have been made. Economists like Farrell[15] and Debreu[16] contributed concepts and methods for doing this, as cited in this chapter (and elsewhere in this text), and these have been further extended by economists like Färe, Grosskopf and Lovell whose works have also been cited at numerous points in this text.[17]

The above response is directed to *ex post* analyses. Another response is directed to whether economics can contribute to these decisions in an *ex ante* fashion. The answer is that there is a long-standing literature which provides guidance to scale decisions by relating marginal (incremental) costs to marginal (incremental) receipts. Generally speaking, returns to scale will be increasing as long as marginal costs are below average (unit) costs. The reverse situation applies when returns to scale is decreasing. Equating marginal costs to marginal receipts will determine the best (most profitable) scale size.

This will generally move matters into the region of decreasing returns to scale. Reasons for this involve issues of stability of solutions which we cannot treat here. Marginal cost lying below average unit cost in regions of increasing returns to scale means that average unit cost can be decreased by incrementing outputs. Hence if marginal receipts equal or exceed average unit cost it is possible to increase total profit by incrementing production.

The above case refers to single output situations. Modifications are needed to allow for complementary and substitution interactions when multiple outputs are involved. This has been accomplished in ways that have long been available which show that the above rule continues to supply general guidance. Indeed, recent years have seen this extended to ways for determining "economics of scope" in order to decide whether to add or delete product lines while simultaneously treating mix and scale decisions. See W.S. Baumol, J.C. Panzar and R.D. Willig (1982) *Contestable Markets and the Theory of Industry Structure* (New York: Harcourt Brace Jovanovich).

Much remains to be done in giving the above concepts implementable form — especially when technical inefficiencies are involved and errors and uncertainties are also involved. A start has been made in the form of what are referred to as "stochastic frontier" approaches to statistical estimation. However, these approaches are, by and large, confined to the use of single output regressions.

Comment : As already noted, profit maximization requires a simultaneous determination of the best (i.e., most profitable) combination of scale, scope, technical and mix efficiencies. The models and methods described in earlier chapters can then be used to determine what was done and whether and where any inefficiencies occurred.

Problem 8.6

Prove that the *profit ratio* model in (8.31) does not suffer from the inadequacies pointed out in Section 8.3.

Suggested Response : Since the profit ratio efficiency is defined as ratio of ratios (between revenue and cost), it is invariant when we double both the unit cost and price.

Notice that the traditional profit efficiency model [Profit] (8.8) gives the same efficiency value when we double both the unit cost and price. This is unacceptable.

Notes

1. M.J. Farrell (1957), "The Measurement of Productive Efficiency," *Journal of the Royal Statistical Society* Series A, 120, III, pp.253-281. G. Debreu (1951), "The Coefficient of Resource Utilization," *Econometrica* 19, pp.273-292.

2. R. Färe, S. Grosskopf and C.A.K. Lovell, *Measurement of Efficiency of Production* (Boston: Kluwer-Nijhoff Publishing Co., Inc., 1985).

3. This means that P lies below the production possibility surface and hence is inefficient.

4. K. Tone (2002), "A Strange Case of the Cost and Allocative Efficiencies in DEA," *Journal of the Operational Research Society* 53, pp.1225-1231.

5. This ratio model was introduced in K. Tone (1993), *Data Envelopment Analysis* (in Japanese)(Tokyo: JUSE Press, Ltd.).

6. W.W. Cooper, Z. Huang, S. Li and J.T. Pastor (2005) "Aggregation with Enhanced Russell Measure in DEA," *European Journal of Operational Research* (forthcoming).

7. K. Tone and M. Tsutsui (2004), "Decomposition of Cost Efficiency and its Application to Japan-US Electric Utility Comparisons," Research Report Series I-2004-0004, GRIPS (National Graduate Institute for Policy Studies), also forthcoming in *Socio-Economic Planning Sciences.*

8. W.W. Cooper, K.S. Park and J.T. Pastor (1999), "RAM: A Range Adjusted Measure of Inefficiency for Use with Additive Models and Relations to Other Models and Measures in DEA," *Journal of Productivity Analysis* 11, pp.5-42.

9. K. Tone and B.K. Sahoo (2005), "Evaluating Cost Efficiency and Returns to Scale in the Life Insurance Corporation of India Using Data Envelopment Analysis," *Socio-Economic Planning Sciences* 39, pp.261-285.

10. K. Tone and B.K. Sahoo (2005), "Cost-Elasticity: A Re-Examination in DEA," *Annals of Operations Research* (forthcoming).

11. H. Fukuyama and W.L. Weber (2004), "Economic Inefficiency Measurement of Input Spending When Decision-making Units Face Different Input Prices," *Journal of the Operational Research Society* 55, pp.1102-1110.

12. R.G. Chambers, Y. Chung and R. Färe (1996) "Benefit and Distance Functions," *Journal of Economic Theory* 70, pp.407-418.

13. R. Färe, S. Grosskopf and G. Whittaker (2004) "Distance Functions," Chapter 5 in W.W. Cooper, L.M. Seiford and J. Zhu, eds., *Handbook on Data Envelopment Analysis* (Norwell Mass., Kluwer Academic Publishers).

14. S. Ray (2004) *Data Envelopment Analysis: Theory and Techniques for Economics and Operations Research* (Cambridge University Press).

15. See Note 1. See also M.J. Farrell and M. Fieldhouse (1962), "Estimating Efficient Production Functions Under Increasing Returns to Scale," *Journal of the Royal Statistical Society* Series A, 125, Part 2, pp.252-267.

16. See the Note 1 reference.

17. See the Note 2 above. See also R. Färe, S. Grosskopf and C.A.K. Lovell (1994), *Production Frontiers* (Cambridge: Cambridge University Press).