4 ALTERNATIVE DEA MODELS

4.1 INTRODUCTION

In the preceding chapters, we discussed the CCR model, which is built on the assumption of *constant* returns to scale of activities as depicted for the production frontier in the single input-single output case shown in Figure 4.1. More generally, it is assumed that the production possibility set has the following property: If (x, y) is a feasible point, then (tx, ty) for any positive t is also feasible. This assumption can be modified to allow production possibility sets with different postulates. In fact, since the very beginning of DEA studies, various extensions of the CCR model have been proposed, among which the BCC (Banker-Charnes-Cooper) model[1] is representative. The BCC model has its production frontiers spanned by the convex hull of the existing DMUs. The frontiers have piecewise linear and concave characteristics which, as shown in Figure 4.2, leads to *variable* returns-to-scale characterizations with (a) increasing returns-to-scale occurring in the first solid line segment followed by (b) decreasing returns-to-scale in the second segment and (c) constant returns-to-scale occurring at the point where the transition from the first to the second segment is made.

In this chapter, we first introduce the BCC model in Section 2. Then, in Section 3, the "Additive Model" will be described. This model has the same production possibility set as the BCC and CCR models and their variants but treats the slacks (the input excesses and output shortfalls) directly in the objective function.

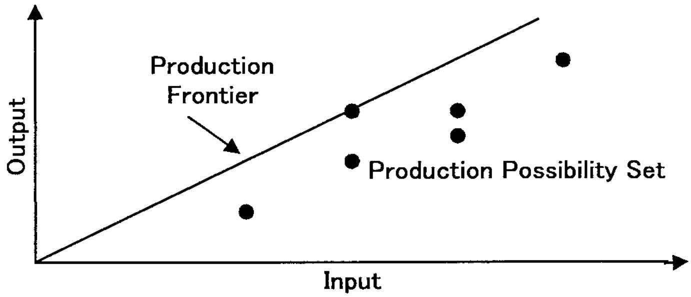

Figure 4.1. Production Frontier of the CCR Model

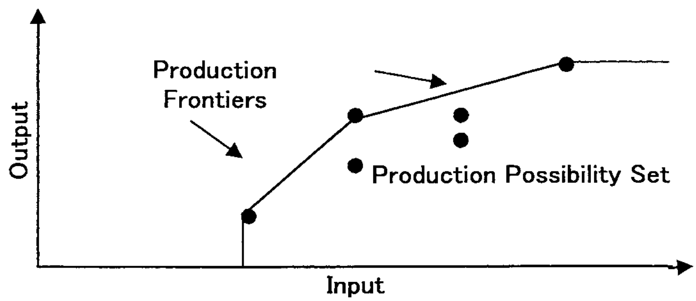

Figure 4.2. Production Frontiers of the BCC Model

CCR-type models, under weak efficiency, evaluate the radial (proportional) efficiency θ^* but do not take account of the input excesses and output shortfalls that are represented by non-zero slacks. This is a drawback because θ^* does not include the nonzero slacks. Although the Additive model deals with the input excesses and output shortfalls directly and can discriminate efficient and inefficient DMUs, it has no means to gauge the depth of inefficiency by a scalar measure similar to the θ^* in the CCR-type models. To eliminate this deficiency, we will introduce a slacks-based measure of efficiency (SBM), which was proposed by Tone (1997) [2] and is also related to the "Enhanced Russell Measure," in Section 4 in the form of a scalar with a value not greater than the corresponding CCR-type measure θ^*. This measure reflects nonzero slacks in inputs and outputs when they are present.

The BCC and CCR models differ only in that the former, but not the latter, includes the convexity condition $\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0, \forall j$ in its constraints. Thus, as might be expected, they share properties in common and exhibit differences. They also share properties with the corresponding Additive models. Thus, the Additive model without the convexity constraint will characterize a DMU as efficient if and only if the CCR model characterizes it as efficient. Similarly, the BCC model will characterize a DMU as efficient if and only if the corresponding Additive model also characterizes it as efficient.

The concept of "translation invariance" that will be introduced in this chapter deals with lateral shifts of the constraints so that negative data, for instance, may be converted to positive values that admit of treatment by our solution methods, which assume that all data are non-negative. As we will see, the Additive models which include the convexity constraint, are translation invariant but this is not true when the convexity constraint is omitted. CCR models are also not translation invariant while BCC models are translation invariant to changes in the data for only some of their constraints.

These (and other) topics treated in this chapter will provide an overview of model selection possibilities and this will be followed up in further detail in chapters that follow.

The CCR, BCC, Additive and SBM models do not exhaust the available DEA models. Hence in the Appendix to this chapter we introduce a variant, called the Free Disposal Hull (FDH) model which assumes a nonconvex (staircase) production possibility set. Finally, in the problems we ask readers to use their knowledge of the Additive model to develop yet another, known as the "multiplicative model."

4.2 THE BCC MODELS

Let us begin this section with a simple example. Figure 4.3 exhibits 4 DMUs, A, B, C and D, each with one input and one output.

The efficient frontier of the CCR model is the dotted line that passes through B from the origin. The frontiers of the BCC model consists of the bold lines connecting A, B and C. The production possibility set is the area consisting of the frontier together with observed or possible activities with an excess of input and/or a shortfall in output compared with the frontiers. A, B and C are on the frontiers and BCC-efficient. The same is true for all points on the solid lines connecting A and B, and B and C. However, only B is CCR-efficient.

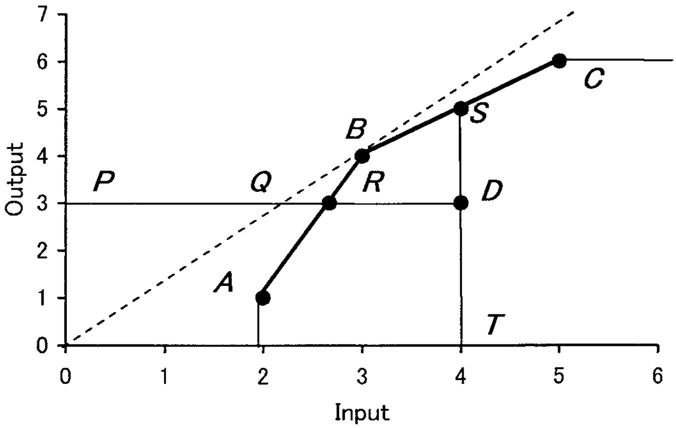

Figure 4.3. The BCC Model

Reading values from the this graph, the BCC-efficiency of D is evaluated by

$$\frac{PR}{PD} = \frac{2.6667}{4} = 0.6667,$$

while its CCR-efficiency is smaller with value

$$\frac{PQ}{PD} = \frac{2.25}{4} = 0.5625.$$

Generally, the CCR-efficiency does not exceed BCC-efficiency.

In the output-oriented BCC model, we read from the vertical axis of Figure 4.3 to find D evaluated by

$$\frac{ST}{DT} = \frac{5}{3} = 1.6667.$$

This means that achievement of efficiency would require augmenting D's output from its observed value to $1.6667 \times 3 = 5$ units. The comparable augmentation for the CCR model is obtained from the reciprocal of its input inefficiency — viz., $1/0.5625 = 1.7778$ so, as the diagram makes clear, a still greater augmentation is needed to achieve efficiency. (Note: this simple "reciprocal relation" between input and output efficiencies is not available for the BCC model.)

Banker, Charnes and Cooper (1984) published the BCC model whose production possibility set P_B is defined by:

$$P_B = \left\{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq X\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{e}\boldsymbol{\lambda} = 1, \boldsymbol{\lambda} \geq \boldsymbol{0}\right\}, \tag{4.1}$$

where $X = (\boldsymbol{x}_j) \in R^{m \times n}$ and $Y = (\boldsymbol{y}_j) \in R^{s \times n}$ are a given data set, $\boldsymbol{\lambda} \in R^n$ and $\boldsymbol{e}$ is a row vector with all elements equal to 1. The BCC model differs from the CCR model only in the adjunction of the condition $\sum_{j=1}^{n} \lambda_j = 1$ which we also write $e\boldsymbol{\lambda} = 1$ where e is a row vector with all elements unity and $\boldsymbol{\lambda}$ is a column vector with all elements non-negative. Together with the condition $\lambda_j \geq 0$, for all j, this imposes a convexity condition on allowable ways in which the observations for the n DMUs may be combined.

4.2.1 *The BCC Model*

The input-oriented BCC model evaluates the efficiency of DMU$_o$ $(o = 1, \ldots, n)$ by solving the following (envelopment form) linear program:

$$
\begin{aligned}
(BCC_o) \quad & \min_{\theta_B, \boldsymbol{\lambda}} \ \theta_B && (4.2)\\
\text{subject to} \quad & \theta_B \boldsymbol{x}_o - X\boldsymbol{\lambda} \geq \mathbf{0} && (4.3)\\
& Y\boldsymbol{\lambda} \geq \boldsymbol{y}_o && (4.4)\\
& e\boldsymbol{\lambda} = 1 && (4.5)\\
& \boldsymbol{\lambda} \geq \mathbf{0}, && (4.6)
\end{aligned}
$$

where θ_B is a scalar.

The dual multiplier form of this linear program (BCC_o) is expressed as:

$$
\begin{aligned}
& \max_{\boldsymbol{v},\boldsymbol{u},u_0} \ z = \boldsymbol{u}\boldsymbol{y}_o - u_0 && (4.7)\\
\text{subject to} \quad & \boldsymbol{v}\boldsymbol{x}_o = 1 && (4.8)\\
& -\boldsymbol{v}X + \boldsymbol{u}Y - u_0 e \leq \mathbf{0} && (4.9)\\
& \boldsymbol{v} \geq \mathbf{0}, \ \boldsymbol{u} \geq \mathbf{0}, \ \ u_0 \ \text{ free in sign}, && (4.10)
\end{aligned}
$$

where $\boldsymbol{v}$ and $\boldsymbol{u}$ are vectors and z and u_o are scalars and the latter, being "free in sign," may be positive or negative (or zero). The equivalent BCC fractional program is obtained from the dual program as:

$$
\begin{aligned}
& \max \ \frac{\boldsymbol{u}\boldsymbol{y}_o - u_0}{\boldsymbol{v}\boldsymbol{x}_o} && (4.11)\\
\text{subject to} \quad & \frac{\boldsymbol{u}\boldsymbol{y}_j - u_0}{\boldsymbol{v}\boldsymbol{x}_j} \leq 1 \ \ (j = 1, \ldots, n) && (4.12)\\
& \boldsymbol{v} \geq \mathbf{0}, \ \ \boldsymbol{u} \ \geq \ \mathbf{0}, \ \ u_0 \ \text{free}. && (4.13)
\end{aligned}
$$

Correspondences between the primal-dual constraints and variables can be represented as in Table 4.1.

It is clear that a difference between the CCR and BCC models is present in the free variable u_0, which is the dual variable associated with the constraint $e\boldsymbol{\lambda} = 1$ in the envelopment model that also does not appear in the CCR model.

The primal problem (BCC_o) is solved using a two-phase procedure similar to the CCR case. In the first phase, we minimize θ_B and, in the second phase, we maximize the sum of the input excesses and output shortfalls, keeping $\theta_B = \theta_B^*$ (the optimal objective value obtained in Phase one). The evaluations secured

Table 4.1. Primal and Dual Correspondences in BCC Model

Linear Programming Form

Envelopment form constraints	Multiplier form variables	Multiplier form constraints	Envelopment form variables
$\theta_B \boldsymbol{x}_o - X\boldsymbol{\lambda} \geq \boldsymbol{0}$	$\boldsymbol{v} \geq \boldsymbol{0}$	$\boldsymbol{v}\boldsymbol{x}_o = 1$	θ
$Y\boldsymbol{\lambda} \geq \boldsymbol{y}_o$	$\boldsymbol{u} \geq \boldsymbol{0}$	$-\boldsymbol{v}X + \boldsymbol{u}Y - u_0\boldsymbol{e} \leq \boldsymbol{0}$	$\boldsymbol{\lambda} \geq \boldsymbol{0}$
$\boldsymbol{e}\boldsymbol{\lambda} = 1$	u_0		

from the CCR and BCC models are also related to each other as follows. An optimal solution for (BCC_o) is represented by $(\theta_B^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$, where $\boldsymbol{s}^{-*}$ and $\boldsymbol{s}^{+*}$ represent the maximal input excesses and output shortfalls, respectively. Notice that θ_B^* is not less than the optimal objective value θ^* of the CCR model, since (BCC_o) imposes one additional constraint, $\boldsymbol{e}\boldsymbol{\lambda} = 1$, so its feasible region is a subset of feasible region for the CCR model.

Definition 4.1 (BCC-Efficiency)
If an optimal solution $(\theta_B^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$ obtained in this two-phase process for (BCC_o) satisfies $\theta_B^* = 1$ and has no slack ($\boldsymbol{s}^{-*} = \boldsymbol{0}$, $\boldsymbol{s}^{+*} = \boldsymbol{0}$), then the DMU_o is called BCC-efficient, *otherwise it is* BCC-inefficient.

Definition 4.2 (Reference Set)
For a BCC-inefficient DMU_o, we define its reference set, *E_o, based on an optimal solution $\boldsymbol{\lambda}^*$ by*

$$E_o = \{\, j \mid \lambda_j^* > 0\} \ (j \in \{1, \ldots, n\}). \tag{4.14}$$

If there are multiple optimal solutions, we can choose any one to find that

$$\theta_B^* \boldsymbol{x}_o = \sum_{j \in E_o} \lambda_j^* \boldsymbol{x}_j + \boldsymbol{s}^{-*} \tag{4.15}$$

$$\boldsymbol{y}_o = \sum_{j \in E_o} \lambda_j^* \boldsymbol{y}_j - \boldsymbol{s}^{+*}. \tag{4.16}$$

Thus, we have a formula for improvement via the *BCC-projection*,

$$\widehat{\boldsymbol{x}}_o \Leftarrow \theta_B^* \boldsymbol{x}_o - \boldsymbol{s}^{-*} \tag{4.17}$$

$$\widehat{\boldsymbol{y}}_o \Leftarrow \boldsymbol{y}_o + \boldsymbol{s}^{+*}. \tag{4.18}$$

The following two theorems and the lemma can be proved in a way similar to proofs used for the CCR model in Chapter 3. Thus, analogous to Theorem 3.2 for the CCR model, we have

Theorem 4.1 *The improved activity* $(\widehat{\boldsymbol{x}}_o, \widehat{\boldsymbol{y}}_o)$ *is BCC-efficient.*

Similarly, the following adapts Lemma 3.1 to the BCC model,

Lemma 4.1 *For the improved activity* $(\widehat{\boldsymbol{x}}_o, \widehat{\boldsymbol{y}}_o)$*, there exists an optimal solution* $(\widehat{\boldsymbol{v}}_o, \widehat{\boldsymbol{u}}_o, \widehat{u}_0)$ *for its dual problem such that*

$$\widehat{\boldsymbol{v}}_o > \mathbf{0} \text{ and } \widehat{\boldsymbol{u}}_o > \mathbf{0} \tag{4.19}$$

$$\widehat{\boldsymbol{v}}_o \boldsymbol{x}_j = \widehat{\boldsymbol{u}}_o \boldsymbol{y}_j - \widehat{u}_0 \quad (j \in E_o) \tag{4.20}$$

$$\widehat{\boldsymbol{v}}_o X \geq \widehat{\boldsymbol{u}}_o Y - \widehat{u}_0 \boldsymbol{e}. \tag{4.21}$$

Theorem 4.2 *Every DMU in* E_o *associated with a* $\lambda_j^* > 0$ *as defined in* (4.14)*, is BCC-efficient.*

This is an extension of Theorem 3.3. Finally, however, the next theorem exposes a property of BCC-efficiency for the input-oriented version of the model. This property is not secured by the CCR model, so the two may be used to check whether this property is present.

Theorem 4.3 *A DMU that has a minimum input value for any input item, or a maximum output value for any output item, is BCC-efficient.*

Proof. Suppose that DMU$_o$ has a minimum input value for input 1, i.e., $x_{1o} < x_{1j}$ $(\forall j \neq o)$. Then, from (4.3) and (4.5), DMU$_o$ has the unique solution $(\theta_B^* = 1, \lambda_o^* = 1, \lambda_j^* = 0\ (\forall j \neq o))$. Hence, DMU$_o$ has $\theta_B^* = 1$ with no slacks and is BCC-efficient. The maximum output case can be proved analogously. □

4.2.2 The Output-oriented BCC Model

Turning to the output-oriented BCC model we write

$$(BCC - O_o) \qquad \max_{\eta_B, \boldsymbol{\lambda}} \ \eta_B \tag{4.22}$$

$$\text{subject to} \qquad X\boldsymbol{\lambda} \leq \boldsymbol{x}_o \tag{4.23}$$

$$\eta_B \boldsymbol{y}_o - Y\boldsymbol{\lambda} \leq \mathbf{0} \tag{4.24}$$

$$\boldsymbol{e}\boldsymbol{\lambda} = 1 \tag{4.25}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}. \tag{4.26}$$

This is the envelopment form of the output-oriented BCC model. The dual (multiplier) form associated with the above linear program $(BCC - O_o)$ is expressed as:

$$\min_{\boldsymbol{v}, \boldsymbol{u}, v_0} \ z = \boldsymbol{v}\boldsymbol{x}_o - v_0 \tag{4.27}$$

$$\text{subject to} \qquad \boldsymbol{u}\boldsymbol{y}_o = 1 \tag{4.28}$$

$$\boldsymbol{v}X - \boldsymbol{u}Y - v_0 \boldsymbol{e} \geq \mathbf{0} \tag{4.29}$$

$$\boldsymbol{v} \geq \mathbf{0}, \ \boldsymbol{u} \geq \mathbf{0}, \ \ v_0 \text{ free in sign,} \tag{4.30}$$

where v_0 is the scalar associated with $\boldsymbol{e\lambda} = 1$ in the envelopment model. Finally, we have the equivalent (BCC) fractional programming formulation for the latter (multiplier) model:

$$\min \frac{\boldsymbol{v}\boldsymbol{x}_o - v_0}{\boldsymbol{u}\boldsymbol{y}_o} \tag{4.31}$$

$$\text{subject to} \quad \frac{\boldsymbol{v}\boldsymbol{x}_j - v_0}{\boldsymbol{u}\boldsymbol{y}_j} \geq 1 \quad (j = 1, \ldots, n) \tag{4.32}$$

$$\boldsymbol{v} \geq \boldsymbol{0}, \quad \boldsymbol{u} \geq \boldsymbol{0}, \quad v_0 \text{ free in sign.} \tag{4.33}$$

4.3 THE ADDITIVE MODEL

The preceding models required us to distinguish between input-oriented and output-oriented models. Now, however, we combine both orientations in a single model, called the *Additive model.*

4.3.1 The Basic Additive Model

There are several types of Additive models, from which we select following:

$$(ADD_o) \quad \max_{\boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+} \quad z = \boldsymbol{e}\boldsymbol{s}^- + \boldsymbol{e}\boldsymbol{s}^+ \tag{4.34}$$

$$\text{subject to} \quad X\boldsymbol{\lambda} + \boldsymbol{s}^- = \boldsymbol{x}_o \tag{4.35}$$

$$Y\boldsymbol{\lambda} - \boldsymbol{s}^+ = \boldsymbol{y}_o \tag{4.36}$$

$$\boldsymbol{e}\boldsymbol{\lambda} = 1 \tag{4.37}$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0}, \quad \boldsymbol{s}^- \geq \boldsymbol{0}, \quad \boldsymbol{s}^+ \geq \boldsymbol{0}. \tag{4.38}$$

A variant, which we do not explore here, is an Additive model[3] which omits the condition $\boldsymbol{e\lambda} = 1$.

The dual problem to the above can be expressed as follows:

$$\min_{\boldsymbol{v}, \boldsymbol{u}, u_0} \quad w = \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}\boldsymbol{y}_o + u_0 \tag{4.39}$$

$$\text{subject to} \quad \boldsymbol{v}X - \boldsymbol{u}Y + u_0\boldsymbol{e} \geq \boldsymbol{0} \tag{4.40}$$

$$\boldsymbol{v} \geq \boldsymbol{e} \tag{4.41}$$

$$\boldsymbol{u} \geq \boldsymbol{e} \tag{4.42}$$

$$u_0 \text{ free.} \tag{4.43}$$

To explain this model we use Figure 4.4, where four DMUs A, B, C and D, each with one input and one output, are depicted. Since, by (4.35)– (4.38), the model (ADD_o) has the same production possibility set as the BCC model, the efficient frontier, which is continuous, consists of the line segments $\overline{AB}$ and $\overline{BC}$. Now consider how DMU D might be evaluated. A feasible replacement of D with s^- and s^+ is denoted by the arrows s^- and s^+ in the figure. As shown by the dotted line in the figure, the maximal value of $s^- + s^+$ is attained at B. It is clear that this model considers the input excess and the output shortfall simultaneously in arriving at a point on the efficient frontier which is most distant from D.[4]

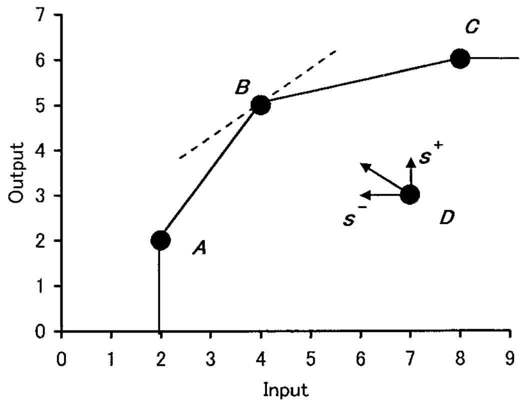

Figure 4.4. The Additive Model

Taking these considerations into account we can obtain a definition of efficiency as follows for the Additive model.

Let the optimal solutions be $(\boldsymbol{\lambda}^*,\ \boldsymbol{s}^{-*},\ \boldsymbol{s}^{+*})$. The definition of efficiency for an efficient DMU in the Additive model is then given by:

Definition 4.3 (ADD-efficient DMU)
DMU_o is ADD-efficient *if and only if* $\boldsymbol{s}^{-*} = \mathbf{0}$ *and* $\boldsymbol{s}^{+*} = \mathbf{0}$.

Theorem 4.4 *DMU_o is ADD-efficient if and only if it is BCC-efficient.*

A proof of this theorem may be found in Ahn *et al.*[5] Here, however, it suffices to note that the efficiency score θ^* is not measured explicitly but is implicitly present in the slacks $\boldsymbol{s}^{-*}$ and $\boldsymbol{s}^{+*}$. Moreover, whereas θ^* reflects only Farrell (=weak) efficiency, the objective in (ADD_o) reflects all inefficiencies that the model can identify in *both* inputs and outputs.

Theorem 4.5 *Let us define* $\widehat{\boldsymbol{x}}_o = \boldsymbol{x}_o - \boldsymbol{s}^{-*}$ *and* $\widehat{\boldsymbol{y}}_o = \boldsymbol{y}_o + \boldsymbol{s}^{+*}$. *Then,* $(\widehat{\boldsymbol{x}}_o,\ \widehat{\boldsymbol{y}}_o)$ *is ADD-efficient.*

By this theorem, improvement to an efficient activity is attained by the following formulae (Projection for the Additive model):

$$\widehat{\boldsymbol{x}}_o \Leftarrow \boldsymbol{x}_o - \boldsymbol{s}^{-*} \quad (4.44)$$

$$\widehat{\boldsymbol{y}}_o \Leftarrow \boldsymbol{y}_o + \boldsymbol{s}^{+*}, \quad (4.45)$$

with $(\widehat{\boldsymbol{x}}_o,\ \widehat{\boldsymbol{y}}_o)$ serving as the coordinates for the point on the efficient frontier used to evaluate DMU_o.

Example 4.1
We clarify the above with Table 4.2 which shows 8 DMUs with one input and one output. The solution of the Additive model and that of the BCC model are both exhibited. The reference set is determined as the set of DMUs which are in the optimal basic set of the LP problem. It is thus of interest to note that B, C and E are all fully efficient under both the BCC and (ADD_o) model, as asserted in Theorem 4.4. Furthermore, the nonzero slack under s^{+*} for A shows that it is not fully efficient for the BCC model (as well as (ADD_o)) even though it is weakly efficient with a value of $\theta^* = 1$. This means that A is on a part of the frontier that is not efficient, of course, while all of the other values with $\theta^* < 1$ mean that they fail even to achieve the frontier and, as can be observed, they also have nonzero slacks in their optimum (2^{nd}-stage) solutions. For the latter observations, therefore, a projection to $\widehat{\boldsymbol{x}}_o = \theta^* \boldsymbol{x}_o$ need not achieve efficiency. It could merely bring the thus adjusted performance onto a portion of the frontier which is not efficient — as was the case for A.

Table 4.2. Data and Results of Example 4.1

	Input	Output	BCC	Additive Model		
DMU	x	y	θ^*	s^{-*}	s^{+*}	Ref.
A	2	1	1	0	1	C
B	3	3	1	0	0	B
C	2	2	1	0	0	C
D	4	3	0.75	1	0	B
E	6	5	1	0	0	E
F	5	2	0.40	3	0	C
G	6	3	0.50	3	0	B
H	8	5	0.75	2	0	E

The following definition brings up another important distinction between the Additive model and BCC (or CCR) models;

Definition 4.4 (Mix) *We define "Mix" as proportions in which inputs are used or in which outputs are produced.*

Returning to the BCC (or CCR) models, it can be observed that $(1 - \theta^*)$ represents reductions which can be achieved without altering the input mix utilized and $(\eta^* - 1)$ would play a similar role for output expansions which do not alter the output mix.

In the literature of DEA, as well as economics, this proportion is referred to as "technical inefficiency" as contrasted with "scale inefficiencies," "allocative (or price) inefficiencies" and other type of inefficiencies that will be discussed in subsequent chapters.

With all such technical inefficiencies accounted for in the BCC and CCR models, the second stage optimization is directed to maximize the slacks in order to see whether further inefficiencies are present. Altering any nonzero slack obtained in the second stage optimization must then necessarily alter the mix.

As is now apparent, the CCR and BCC models distinguish between technical and mix inefficiencies and this is a source of trouble in trying to arrange for a single measure of efficiency. The allocative model makes no such distinction, however, and so in Section 4.4 we will use the opportunity provided by these models to develop a measure that comprehends *all* of the inefficiencies that the model can identify.

As noted above, there is a distinction to be made between the technical and mix inefficiencies identified by the BCC and CCR models. The DEA literature (much of it focused on weak efficiency) does not make this distinction and we will follow the literature in its rather loose usage of "technical efficiency" to cover both types of inefficiency. When it is important to do so, we should refine this terminology by using "purely technical inefficiency" for θ^* and η^* and distinguish this from the mix inefficiencies given in Definition 4.4.

4.3.2 Translation Invariance of the Additive Model

In many applications it may be necessary (or convenient) to be able to handle negative data in some of the inputs or outputs. For instance, in order to determine whether mutual insurance companies are more (or less) efficient than their stock-ownership counterparts, it was necessary to be able to go beyond the assumption of semipositive data (as defined at the start of Chapter 3) in order to handle losses as well as profits treated as output.[6] This was dealt with by employing a property of (ADD_o) known as "translation invariance" which we now develop from the following definition.

Definition 4.5 (Translation Invariance)
Given any problem, a DEA model is said to be translation invariant *if translating the original input and/or output data values results in a new problem that has the same optimal solution for the envelopment form as the old one.*

First we examine the input-oriented BCC model. In Figure 4.5, DMU D has the BCC-efficiency PR/PD. This ratio is invariant even if we shift the output value by changing the origin from O to O'. Thus, the BCC model is translation invariant with respect to *outputs* (but not *inputs*). Similar reasoning shows that the output oriented BCC model is invariant under the translation of *inputs* (but not *outputs*). (See Problem 4.3.)

Turning to the Additive model, Figure 4.6 shows that this model is translation invariant in both *inputs* and *outputs*, since the efficiency evaluation does not depend on the origin of the coordinate system when this model is used.

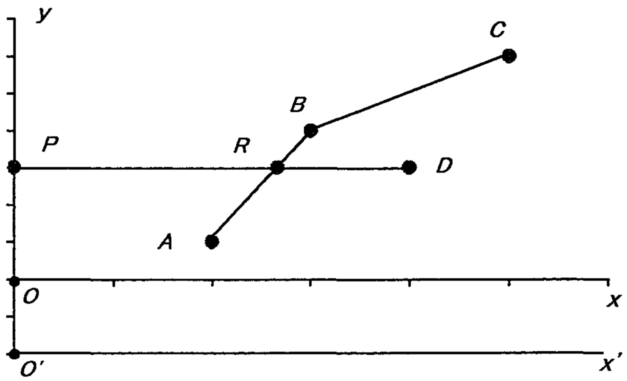

Figure 4.5. Translation in the BCC Model

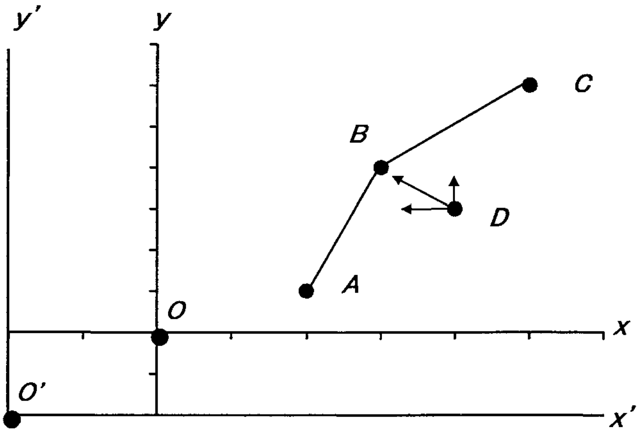

Figure 4.6. Translation in the Additive Model

We now develop this property of the Additive model in detail.

Let us translate the data set $(X,\ Y)$ by introducing arbitrary constants $(\alpha_i:\ i=1,\ldots,m)$ and $(\beta_r:\ r=1,\ldots,s)$ to obtain new data

$$x'_{ij} = x_{ij} + \alpha_i \quad (i=1,\ldots,m:\ j=1,\ldots,n) \tag{4.46}$$

$$y'_{rj} = y_{rj} + \beta_r, \quad (r=1,\ldots,s:\ j=1,\ldots,n) \tag{4.47}$$

To show that this model is invariant under this arbitrary translation we observe that the x values (4.35) become

$$\sum_{j=1}^{n}(x'_{ij} - \alpha_i)\lambda_j + s_i^- = \sum_{j=1}^{n} x'_{ij}\lambda_j + s_i^- - \alpha_i = x'_{io} - \alpha_i$$

so that

$$\sum_{j=1}^{n} x'_{ij}\lambda_j + s_i^- = x'_{io} \quad (i = 1, \ldots, m)$$

which are the same λ_j, s_i^- that satisfy

$$\sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{io}. \quad (i = 1, \ldots, m)$$

Similarly, the same λ_j, s_r^+ that satisfy

$$\sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{ro}. \quad (r = 1, \ldots, s)$$

will also satisfy

$$\sum_{j=1}^{n} y'_{rj}\lambda_j - s_r^+ = y'_{ro}. \quad (r = 1, \ldots, s)$$

Notice that the convexity condition $\boldsymbol{e\lambda} = 1$ is a key factor in deriving the above relations. The above equalities show that if $(\boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$ is an optimal solution of the original problem, then it is also optimal for the translated problem and *vice versa*. Finally, we also have

$$\begin{aligned} s_i^- &= \sum_{j=1}^{n} x_{ij}\lambda_j - x_{io} = \sum_{j=1}^{n} x'_{ij}\lambda_j - x'_{io} \\ s_r^+ &= \sum_{j=1}^{n} y_{rj}\lambda_j - y_{ro} = \sum_{j=1}^{n} y'_{rj}\lambda_j - y'_{ro} \end{aligned}$$

so the value of the objective is also not affected and we therefore have the following theorem for the Additive model (4.34)-(4.38) from Ali and Seiford (1990).[7]

Theorem 4.6 (Ali and Seiford (1990)) *The Additive model given by (4.34)-(4.38) is translation invariant.*

4.4 A SLACKS-BASED MEASURE OF EFFICIENCY (SBM)

We now augment the Additive models by introducing a measure that makes its efficiency evaluation, as effected in the objective, invariant to the units of measure used for the different inputs and outputs. That is, we would like this summary measure to assume the form of a scalar that yields the same efficiency value when distances are measured in either kilometers or miles. More generally, we want this measure to be the same when x_{io} and x_{ij} are replaced by $k_i x_{io} = \widehat{x}_{io}$, $k_i x_{ij} = \widehat{x}_{ij}$ and y_{ro} and y_{rj} are replaced by $c_r y_{ro} = \widehat{y}_{ro}$, $c_r y_{rj} = \widehat{y}_{rj}$ where the k_i and c_r are arbitrary positive constants, $i = 1, \ldots, m$; $r = 1, \ldots, s$.

This property is known by names such as "dimension free"[8] and "units invariant." In this section we introduce such a measure for Additive models in the form of a single scalar called "SBM," (Slacks-Based Measure) which was introduced by Tone (1997, 2001) and has the following important properties:

1. (P1) The measure is invariant with respect to the unit of measurement of each input and output item. (Units invariant)
2. (P2) The measure is monotone decreasing in each input and output slack. (Monotone)

4.4.1 *Definition of SBM*

In order to estimate the efficiency of a DMU ($\boldsymbol{x}_o$, $\boldsymbol{y}_o$), we formulate the following fractional program in $\boldsymbol{\lambda}$, $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$.

$$
\begin{aligned}
(SBM) \qquad & \min_{\boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+} \quad \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r^+/y_{ro}} \qquad (4.48)\\
\text{subject to} \qquad & \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^- \\
& \boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\
& \boldsymbol{\lambda} \geq \boldsymbol{0}, \ \boldsymbol{s}^- \geq \boldsymbol{0}, \ \boldsymbol{s}^+ \geq \boldsymbol{0}.
\end{aligned}
$$

In this model, we assume that $X \geq O$. If $x_{io} = 0$, then we delete the term s_i^-/x_{io} in the objective function. If $y_{ro} \leq 0$, then we replace it by a very small positive number so that the term s_r^+/y_{ro} plays a role of penalty.

It is readily verified that the objective function value ρ satisfies (P1) because the numerator and denominator are measured in the same units for every item in the objective of (4.48). It is also readily verified that an increase in either s_i^- or s_r^+, all else held constant, will decrease this objective value and, indeed, do so in a strictly monotone manner.

Furthermore, we have

$$0 \leq \rho \leq 1. \qquad (4.49)$$

To see that this relation holds we first observe that $s_i^- \leq x_{io}$ for every i so that $0 \leq s_i^-/x_{io} \leq 1$ $(i = 1, \ldots, m)$ with $s_i^-/x_{io} = 1$ only if the evidence shows that only a zero amount of this input was required. It follows that

$$0 \leq \frac{\sum_{i=1}^{m} s_i^-/x_{io}}{m} \leq 1.$$

This same relation does not hold for outputs since an output shortfall represented by a nonzero slack can exceed the corresponding amount of output produced. In any case, however, we have

$$0 \leq \frac{\sum_{r=1}^{s} s_r^+/y_{ro}}{s}.$$

Thus these represent ratios of average input and output mix inefficiencies with the upper limit, $\rho = 1$, reached in (4.48) only if slacks are zero in all inputs and outputs.

4.4.2 Interpretation of SBM as a Product of Input and Output Inefficiencies

The formula for ρ in (4.48) can be transformed into

$$\rho = \left(\frac{1}{m} \sum_{i=1}^{m} \frac{x_{io} - s_i^-}{x_{io}} \right) \left(\frac{1}{s} \sum_{r=1}^{s} \frac{y_{ro} + s_r^+}{y_{ro}} \right)^{-1}.$$

The ratio $(x_{io} - s_i^-)/x_{io}$ evaluates the relative reduction rate of input i and, therefore, the first term corresponds to the mean proportional reduction rate of inputs or *input mix inefficiencies.* Similarly, in the second term, the ratio $(y_{ro} + s_r^+)/y_{ro}$ evaluates the relative proportional expansion rate of output r and $(1/s)\sum(y_{ro} + s_r^+)/y_{ro}$ is the mean proportional rate of output expansion. Its inverse, the second term, measures *output mix inefficiency.* Thus, SBM ρ can be interpreted as the ratio of mean input and output mix inefficiencies. Further, we have the theorem:

Theorem 4.7 *If DMU A dominates DMU B so that* $\boldsymbol{x}_A \le \boldsymbol{x}_B$ *and* $\boldsymbol{y}_A \ge \boldsymbol{y}_B$, *then* $\rho_A^* \ge \rho_B^*$.

4.4.3 Solving SBM

(SBM) as formulated in (4.48) can be transformed into the program below by introducing a positive scalar variable t. See Problem 3.1 at the end of the preceding chapter.

$$(SBMt) \qquad \min_{t,\boldsymbol{\lambda},\boldsymbol{s}^-,\boldsymbol{s}^+} \quad \tau = t - \frac{1}{m}\sum_{i=1}^{m} ts_i^-/x_{io} \tag{4.50}$$

$$\begin{aligned} \text{subject to} \quad & 1 = t + \frac{1}{s}\sum_{r=1}^{s} ts_r^+/y_{ro} \\ & \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^- \\ & \boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\ & \boldsymbol{\lambda} \ge \mathbf{0},\ \boldsymbol{s}^- \ge \mathbf{0},\ \boldsymbol{s}^+ \ge \mathbf{0},\ t > 0. \end{aligned}$$

Now let us define

$$\boldsymbol{S}^- = t\boldsymbol{s}^-, \boldsymbol{S}^+ = t\boldsymbol{s}^+, \text{and } \boldsymbol{\Lambda} = t\boldsymbol{\lambda}.$$

Then $(SBMt)$ becomes the following linear program in t, $\boldsymbol{S}^-$, $\boldsymbol{S}^+$, and $\boldsymbol{\Lambda}$:

$$(LP) \quad \min \quad \tau = t - \frac{1}{m}\sum_{i=1}^{m} S_i^-/x_{io} \tag{4.51}$$

$$\begin{aligned} \text{subject to} \quad & 1 = t + \frac{1}{s}\sum_{r=1}^{s} S_r^+/y_{ro} \\ & t\boldsymbol{x}_o = X\boldsymbol{\Lambda} + \boldsymbol{S}^- \\ & t\boldsymbol{y}_o = Y\boldsymbol{\Lambda} - \boldsymbol{S}^+ \\ & \boldsymbol{\Lambda} \ge \mathbf{0},\ \boldsymbol{S}^- \ge \mathbf{0},\ \boldsymbol{S}^+ \ge \mathbf{0},\ t > 0. \end{aligned}$$

Note that $t > 0$ by virtue of the first constraint. This means that the transformation is reversible. Thus let an optimal solution of (LP) be

$$(\tau^*, t^*, \boldsymbol{\Lambda}^*, \boldsymbol{S}^{-*}, \boldsymbol{S}^{+*}).$$

We then have an optimal solution of (SBM) defined by,

$$\rho^* = \tau^*, \ \boldsymbol{\lambda}^* = \boldsymbol{\Lambda}^*/t^*, \ \boldsymbol{s}^{-*} = \boldsymbol{S}^{-*}/t^*, \ \boldsymbol{s}^{+*} = \boldsymbol{S}^{+*}/t^*. \tag{4.52}$$

From this optimal solution, we can decide whether a DMU is *SBM-efficient* as follows:

Definition 4.6 (SBM-efficient) *A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-efficient if and only if $\rho^* = 1$.*

This condition is equivalent to $\boldsymbol{s}^{-*} = \mathbf{0}$ *and* $\boldsymbol{s}^{+*} = \mathbf{0}$, i.e., no input excess and no output shortfall in an optimal solution.

For an SBM-inefficient DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, we have the expression:

$$\begin{aligned} \boldsymbol{x}_o &= X\boldsymbol{\lambda}^* + \boldsymbol{s}^{-*} \\ \boldsymbol{y}_o &= Y\boldsymbol{\lambda}^* - \boldsymbol{s}^{+*}. \end{aligned}$$

The DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ can be improved and becomes efficient by deleting the input excesses and augmenting the output shortfalls. This is accomplished by the following formulae (SBM-projection):

$$\widehat{\boldsymbol{x}}_o \Leftarrow \boldsymbol{x}_o - \boldsymbol{s}^{-*} \tag{4.53}$$

$$\widehat{\boldsymbol{y}}_o \Leftarrow \boldsymbol{y}_o + \boldsymbol{s}^{+*}, \tag{4.54}$$

which are the same as for the Additive model. See (4.44) and (4.45).

Based on $\boldsymbol{\lambda}^*$, we define the reference set for $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ as:

Definition 4.7 (Reference set) *The set of indices corresponding to positive λ_j^*s is called the* reference set *for $(\boldsymbol{x}_o, \boldsymbol{y}_o)$.*

For multiple optimal solutions, the reference set is not unique. We can, however, choose any one for our purposes.

Let R_o be the reference set designated by

$$R_o = \{\, j \mid \lambda_j^* > 0\} \quad (j \in \{1, \ldots, n\}). \tag{4.55}$$

Then using R_o, we can also express $(\widehat{\boldsymbol{x}}_o, \widehat{\boldsymbol{y}}_o)$ by,

$$\widehat{\boldsymbol{x}}_o = \sum_{j \in R_o} \boldsymbol{x}_j \lambda_j^* \tag{4.56}$$

$$\widehat{\boldsymbol{y}}_o = \sum_{j \in R_o} \boldsymbol{y}_j \lambda_j^*. \tag{4.57}$$

This means that $(\widehat{\boldsymbol{x}}_o, \widehat{\boldsymbol{y}}_o)$, a point on the efficient frontier, is expressed as a positive combination of the members of the reference set, R_o, each member of which is also efficient. See Definition 4.2.

4.4.4 SBM and the CCR Measure

In its weak efficiency form, the CCR model can be formulated as follows:

$$
\begin{array}{llr}
(CCR) & \min_{\theta,\boldsymbol{\mu},\boldsymbol{t}^-,\boldsymbol{t}^+} \quad \theta & \\
\text{subject to} & \theta \boldsymbol{x}_o = X\boldsymbol{\mu} + \boldsymbol{t}^- & (4.58) \\
& \boldsymbol{y}_o = Y\boldsymbol{\mu} - \boldsymbol{t}^+ & (4.59) \\
& \boldsymbol{\mu} \geq \mathbf{0},\ \boldsymbol{t}^- \geq \mathbf{0},\ \boldsymbol{t}^+ \geq \mathbf{0}. &
\end{array}
$$

Now let an optimal solution of (CCR) be $(\theta^*, \boldsymbol{\mu}^*, \boldsymbol{t}^{-*}, \boldsymbol{t}^{+*})$. From (4.58), we can derive

$$\boldsymbol{x}_o = X\boldsymbol{\mu}^* + \boldsymbol{t}^{-*} + (1-\theta^*)\boldsymbol{x}_o \tag{4.60}$$

$$\boldsymbol{y}_o = Y\boldsymbol{\mu}^* - \boldsymbol{t}^{+*}. \tag{4.61}$$

Let us define

$$\boldsymbol{\lambda} = \boldsymbol{\mu}^* \tag{4.62}$$

$$\boldsymbol{s}^- = \boldsymbol{t}^{-*} + (1-\theta^*)\boldsymbol{x}_o \tag{4.63}$$

$$\boldsymbol{s}^+ = \boldsymbol{t}^{+*}. \tag{4.64}$$

Then, $(\boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+)$ is feasible for (SBM) and, by inserting the definitions in (4.63) and (4.64), its objective value can be expressed as:

$$\rho = \frac{1 - \frac{1}{m}\{\sum_{i=1}^m t_i^{-*}/x_{io} + m(1-\theta^*)\}}{1 + \frac{1}{s}\sum_{r=1}^s t_r^{+*}/y_{ro}} = \frac{\theta^* - \frac{1}{m}\sum_{i=1}^m t_i^{-*}/x_{io}}{1 + \frac{1}{s}\sum_{r=1}^s t_r^{+*}/y_{ro}}. \tag{4.65}$$

Evidently, the last term is not greater than θ^*. Thus, we have:

Theorem 4.8 *The optimal SBM ρ^* is not greater than the optimal CCR θ^*.*

This theorem reflects the fact that SBM accounts for *all* inefficiencies whereas θ^* accounts only for "purely technical" inefficiencies. Notice that the coefficient $1/(m\, x_{io})$ of the input excesses s_i^- in ρ plays a crucial role in validating Theorem 4.8.

Conversely, for an optimal solution $(\rho^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$ to SBM, let us transform the constraints from (4.48) into

$$\theta \boldsymbol{x}_o = X\boldsymbol{\lambda}^* + (\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*} \tag{4.66}$$

$$\boldsymbol{y}_o = Y\boldsymbol{\lambda}^* - \boldsymbol{s}^{+*}. \tag{4.67}$$

Further, we add the constraint

$$(\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*} \geq \mathbf{0}. \tag{4.68}$$

Then, $(\theta, \boldsymbol{\lambda}^*, \boldsymbol{t}^- = (\theta-1)\boldsymbol{x}_o + \boldsymbol{s}^{-*}, \boldsymbol{t}^+ = \boldsymbol{s}^{+*})$ is feasible for (CCR).

The relationship between CCR-efficiency and SBM-efficiency is given in the following theorem:

Theorem 4.9 (Tone (1997)) *A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-efficient if and only if it is SBM-efficient.*

Proof. Suppose that $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-inefficient. Then, we have either $\theta^* < 1$ or ($\theta^* = 1$ and $(\boldsymbol{t}^{-*}, \boldsymbol{t}^{+*}) \neq (\mathbf{0}, \mathbf{0})$). From (4.65), in both cases, we have $\rho < 1$ for a feasible solution of (SBM). Therefore, $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-inefficient.

On the other hand, suppose that $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-inefficient. Then, from Definition 4.6, $(\boldsymbol{s}^{-*}, \boldsymbol{s}^{+*}) \neq (\mathbf{0}, \mathbf{0})$. By (4.66) and (4.67), $(\theta, \boldsymbol{\lambda}^*, \boldsymbol{t}^- = (\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*}, \boldsymbol{t}^+ = \boldsymbol{s}^{+*})$ is feasible for (CCR), provided $(\theta - 1)\boldsymbol{x}_o + \boldsymbol{s}^{-*} \geq \mathbf{0}$. There are two cases:

(Case 1) $\theta = 1$ and $(\boldsymbol{t}^- = \boldsymbol{s}^{-*}, \boldsymbol{t}^+ = \boldsymbol{s}^{+*}) \neq (\mathbf{0}, \mathbf{0})$. In this case, an optimal solution for (CCR) is CCR-inefficient.

(Case 2) $\theta < 1$. In this case, $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is CCR-inefficient.

Therefore, CCR-inefficiency is equivalent to SBM-inefficiency. Since the definitions of *efficient* and *inefficient* are mutually exclusive and collectively exhaustive, we have proved the theorem. □

4.4.5 The Dual Program of the SBM Model

The dual program of the problem (LP) in (4.50) can be expressed as follows, with the dual variables $\xi \in R$, $\boldsymbol{v} \in R^m$ and $\boldsymbol{u} \in R^s$:

$$(DP) \qquad \max_{\xi, \boldsymbol{v}, \boldsymbol{u}} \ \xi \tag{4.69}$$

$$\text{subject to} \qquad \xi + \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}\boldsymbol{y}_o = 1 \tag{4.70}$$

$$-\boldsymbol{v}X + \boldsymbol{u}Y \leq \mathbf{0} \tag{4.71}$$

$$\boldsymbol{v} \geq \frac{1}{m}[1/\boldsymbol{x}_o] \tag{4.72}$$

$$\boldsymbol{u} \geq \frac{\xi}{s}[1/\boldsymbol{y}_o], \tag{4.73}$$

where the notation $[1/\boldsymbol{x}_o]$ designates the row vector $(1/x_{1o}, \ldots, 1/x_{mo})$.

Using (4.70) we can eliminate ξ and we have an equivalent program:

$$(DP') \quad \max \qquad \boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o \tag{4.74}$$

$$\text{subject to} \qquad -\boldsymbol{v}X + \boldsymbol{u}Y \leq \mathbf{0} \tag{4.75}$$

$$\boldsymbol{v} \geq \frac{1}{m}[1/\boldsymbol{x}_o] \tag{4.76}$$

$$\boldsymbol{u} \geq \frac{1 - \boldsymbol{v}\boldsymbol{x}_o + \boldsymbol{u}\boldsymbol{y}_o}{s}[1/\boldsymbol{y}_o]. \tag{4.77}$$

The dual variables $\boldsymbol{v} \in R^m$ and $\boldsymbol{u} \in R^s$ can be interpreted as the "virtual" costs and prices of inputs and outputs, respectively. The dual program aims to find the optimal virtual costs and prices for the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ so that the virtual profit $\boldsymbol{u}\boldsymbol{y}_j - \boldsymbol{v}\boldsymbol{x}_j$ does not exceed zero for any DMU (including $(\boldsymbol{x}_o, \boldsymbol{y}_o)$), and maximizes the "virtual" profit $\boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o$ for the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ concerned. Apparently, the optimal profit is at best zero and hence $\xi = 1$ for the SBM efficient DMUs. Constraints (4.76) and (4.77) restrict the feasible $\boldsymbol{v}$ and $\boldsymbol{u}$ to the positive orthant.

4.4.6 Oriented SBM Models

The input (output)-oriented SBM model can be defined by neglecting the denominator (numerator) of the objective function (4.48) of (SBM). Thus, the efficiency values ρ_I^* and ρ_O^* can be obtained as follows:

[Input-oriented SBM Model]

$$(SBM - I) \qquad \rho_I^* = \min_{\boldsymbol{\lambda}, \boldsymbol{s}^-} \; 1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io} \tag{4.78}$$

$$\begin{aligned} \text{subject to} \quad & \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^- \\ & \boldsymbol{y}_o \leq Y\boldsymbol{\lambda} \\ & \boldsymbol{\lambda} \geq \mathbf{0}, \; \boldsymbol{s}^- \geq \mathbf{0}. \end{aligned}$$

[Output-oriented SBM Model]

$$(SBM - O) \qquad \rho_O^* = \min_{\boldsymbol{\lambda}, \boldsymbol{s}^+} \; \frac{1}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r^+/y_{ro}} \tag{4.79}$$

$$\begin{aligned} \text{subject to} \quad & \boldsymbol{x}_o \geq X\boldsymbol{\lambda} \\ & \boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\ & \boldsymbol{\lambda} \geq \mathbf{0}, \; \boldsymbol{s}^+ \geq \mathbf{0}. \end{aligned}$$

This leads to:

Theorem 4.10

$$\rho_I^* \geq \rho^* \quad \textit{and} \quad \rho_O^* \geq \rho^*, \tag{4.80}$$

where ρ^* is the optimal value for (4.48).

The input-oriented SBM is substantially equivalent to what is called the "Russell input measure of technical efficiency."[9] See Section 4.5, below.

4.4.7 A Weighted SBM Model

We can assign weights to inputs and outputs corresponding to the relative importance of items as follows:

$$\rho = \frac{1 - \sum_{i=1}^{m} w_i^- s_i^-/x_{io}}{1 + \sum_{r=1}^{s} w_r^+ s_r^+/y_{ro}}, \tag{4.81}$$

with

$$\sum_{i=1}^{m} w_i^- = 1 \text{ and } \sum_{r=1}^{s} w_r^+ = 1. \tag{4.82}$$

The weights should reflect the intentions of the decision-maker. If all outputs are in the same unit, e.g., dollars, a conventional scheme is that:

$$w_r^+ = \sum_{j=1}^{n} y_{rj} \Big/ \sum_{j=1}^{n}\sum_{k=1}^{s} y_{kj}. \tag{4.83}$$

This weight selection reflects the importance of the output r being proportional to its contribution to the total magnitude. The input weights can be determined analogously.

4.4.8 Numerical Example (SBM)

We illustrate SBM using an example. Table 4.3 exhibits data for eight DMUs using two inputs (x_1, x_2) to produce a single output $(y = 1)$, along with CCR, SBM scores, slacks and reference set. Although DMUs F and G have full CCR-score ($\theta^* = 1$), they have slacks compared to C and this is reflected by drops in the SBM scores to $\rho_F^* = 0.9$ and $\rho_G^* = 0.83333$. Also, the SBM scores of inefficient DMUs A, B and H dropped slightly from the CCR scores due to their slacks. Thus, the SBM measure reflects not only the weak efficiency values in θ^* but also the other (slack) inefficiencies as well.

Table 4.3. Data and Results of CCR and SBM

	Data			CCR	SBM					Mix
DMU	x_1	x_2	y	θ^*	$\rho^* = \rho_I^*$	Ref.	s_1^{-*}	s_2^{-*}	s^{+*}	Eff.[a]
A	4	3	1	0.857	0.833	D	0	1	0	0.972
B	7	3	1	0.632	0.619	D	3	1	0	0.98
C	8	1	1	1	1	C	0	0	0	1
D	4	2	1	1	1	D	0	0	0	1
E	2	4	1	1	1	E	0	0	0	1
F	10	1	1	1	0.9	C	2	0	0	0.9
G	12	1	1	1	0.833	C	4	0	0	0.833
H	10	1.5	1	0.75	0.733	C	2	0.5	0	0.978

[a]: Mix Eff. $= \rho^*/\theta^*$. See Section 5.8.2 for detail.

4.5 RUSSELL MEASURE MODELS

We now introduce a model described as the "Russell Measure Model." Actually it was introduced and named by Färe and Lovell(1978).[10] Their formulation is difficult to compute, however, so we turn to a more recent development due to Pastor, Ruiz and Sirvent.[11] This model is

$$R(\boldsymbol{x}_o, \boldsymbol{y}_o) = \min_{\theta,\phi,\lambda} \frac{\sum_{i=1}^{m} \theta_i/m}{\sum_{r=1}^{s} \phi_r/s} \tag{4.84}$$

subject to

$$\theta_i x_{io} \geq \sum_{j=1}^{n} x_{ij}\lambda_j, \; i = 1, \ldots, m$$
$$\phi_r y_{ro} \leq \sum_{j=1}^{n} y_{rj}\lambda_j, \; r = 1, \ldots, s$$
$$0 \leq \lambda_j \; \forall j$$
$$0 \leq \theta_i \leq 1; \; 1 \leq \phi_r \; \forall i, \; r.$$

Pastor *et al.* refer to this as the "Enhanced Russell Graph Measure of Efficiency" but we shall refer to it as ERM (Enhanced Russell Measure). See Färe, Grosskopf and Lovell (1985)[12] for the meaning of "graph measure." Such measures are said to be "closed" so $R(\boldsymbol{x}_o, \boldsymbol{y}_o)$ includes all inefficiencies that the model can identify. In this way we avoid limitations of the radial measures which cover only some of the input or output inefficiencies and hence measure only "weak efficiency."

The "inclusive" (=closure) property is shared by SBM. In fact SBM and ERM are related as in the following theorem,

Theorem 4.11 *ERM as formulated in (4.84) and SBM as formulated in (4.48) are equivalent in that λ_j^* values that are optimal for one are also optimal for the other.*

Proof : As inspection makes clear, a necessary condition for optimality of ERM is that the constraints in (4.84) must be satisfied as equalities. Hence we can replace those constraints with

$$\theta_i = \sum_{j=1}^{n} x_{ij}\lambda_j \Big/ x_{io}, i = 1, \ldots, m \tag{4.85}$$
$$\phi_r = \sum_{j=1}^{n} y_{rj}\lambda_j \Big/ y_{ro}, r = 1, \ldots, s.$$

Following Pastor *et al.* or Bardhan *et al.*, [13] we next set

$$\theta_i = \frac{x_{io} - s_i^-}{x_{io}} = 1 - \frac{s_i^-}{x_{io}}, \; i = 1, \ldots, m \tag{4.86}$$
$$\phi_r = \frac{y_{ro} + s_r^+}{y_{ro}} = 1 + \frac{s_r^+}{y_{ro}}, \; r = 1, \ldots, s.$$

Substituting these values in (4.85) produces

$$1 = \frac{\sum_{j=1}^{n} x_{ij}\lambda_j}{x_{io}} + \frac{s_i^-}{x_{io}}, \; i = 1, \ldots, m \tag{4.87}$$
$$1 = \frac{\sum_{j=1}^{n} y_{rj}\lambda_j}{y_{ro}} + \frac{s_r^+}{y_{ro}}, \; r = 1, \ldots, s$$

or

$$x_{io} = \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^-, \; i = 1, \ldots, m \tag{4.88}$$
$$y_{ro} = \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+, \; r = 1, \ldots, s$$

which are the same as the constraints for SBM in (4.48).

Turning to the additional conditions,$0 \leq \theta_i \leq 1$ and $1 \leq \phi_r$ we again substitute from (4.86) to obtain $0 \leq s_i^- \leq x_{io}$ and $0 \leq s_r^+$. The condition $s_i^- \leq x_{io}$ is redundant since it is satisfied by the first set of inequalities in (4.84). Hence, we only have non-negativity for all slacks, the same as in (4.48).

Now turning to the objective in (4.84) we once more substitute from (4.86) to obtain

$$\frac{s}{m}\frac{\sum_{i=1}^{m}\theta_i}{\sum_{r=1}^{s}\phi_r} = \frac{s}{m}\frac{\sum_{i=1}^{m}\left(1 - \frac{s_i^-}{x_{io}}\right)}{\sum_{r=1}^{s}\left(1 + \frac{s_r^+}{y_{ro}}\right)} = \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}}} \tag{4.89}$$

which is the same as the objective for SBM. Using the relations in (4.86) we may therefore use SBM to solve ERM or *vice versa*. □

4.6 SUMMARY OF THE BASIC DEA MODELS

In Table 4.4, we summarize some important topics for consideration in choosing between basic DEA models. In this table, 'Semi-p' (=semipositive) means nonnegative with at least one positive element in the data for each DMU, and 'Free' permits negative, zero or positive data. Although we have developed some DEA models under the assumption of positive data set, this assumption can be relaxed as exhibited in the table. For example, in the BCC-I (-O) model, outputs (inputs) are free due to the translation invariance theorem. In the case of SBM, nonpositive outputs can be replaced by a very small positive number and nonpositive input terms can be neglected for consideration in the objective function. The θ^* of the output oriented model (CCR-O) is the reciprocal of $\eta^*(\geq 1)$. 'Tech. or Mix' indicates whether the model measures 'technical efficiency' or 'mix efficiency'. 'CRS' and 'VRS' mean *constant* and *variable* returns to scale, respectively. The returns to scale of ADD and SBM depends on the added convexity constraint $e\lambda = 1$.

Model selection is one of the problems to be considered in DEA up to, and including, choices of multiple models to test whether or not a result is dependent on the models (or methods) used.[14] Although other models will be developed in succeeding chapters, we will mention here some of the considerations to be taken into account regarding model selection.

1. The Shape of the Production Possibility Set.

The CCR model is based on the assumption that *constant* returns to scale

Table 4.4. Summary of Model Characteristics

Model		CCR-I	CCR-O	BCC-I	BCC-O	ADD	SBM
Data	X	Semi-p	Semi-p	Semi-p	Free	Free	Semi-p
	Y	Free	Free	Free	Semi-p	Free	Free
Trans.	X	No	No	No	Yes	Yes[a]	No
Invariance	Y	No	No	Yes	No	Yes[a]	No
Units invariance		Yes	Yes	Yes	Yes	No	Yes
θ^*		[0, 1]	[0, 1]	(0, 1]	(0, 1]	No	[0, 1]
Tech. or Mix		Tech.	Tech.	Tech.	Tech.	Mix	Mix
Returns to Scale		CRS	CRS	VRS	VRS	C(V)RS[b]	C(V)RS

[a]: The Additive model is translation invariant only when the convexity constraint is added.
[b]: C(V)RS means Constant or Variable returns to scale according to whether or not the convexity constraint is included.

prevails at the efficient frontiers, whereas the BCC and Additive models assume variable returns to scale frontiers, i.e., *increasing, constant* and *decreasing* returns to scale.[15] If preliminary surveys on the production functions identify a preferable choice by, say, such methods as linear regression analysis, (e.g., a Cobb-Douglas type) or expert opinions, then we can choose a DEA model that fits the situation. However, we should bear in mind that conventional regression-based methods deal with single output and multiple input cases, while DEA models analyze multiple outputs and multiple inputs correspondences.

2. Input or Output Oriented?

One of the main purposes of a DEA study is to project the inefficient DMUs onto the production frontiers, e.g., the CCR-projection and the BCC-projection, among others. There are three directions, one called *input-oriented* that aims at reducing the input amounts by as much as possible while keeping at least the present output levels, and the other, called *output-oriented*, maximizes output levels under at most the present input consumptions. There is a third choice, represented by the Additive and SBM models that deal with the input excesses and output shortfalls simultaneously in a way that jointly maximizes both. If achievement of efficiency, or failure to do so, is the only topic of interest, then these different models will all yield the same result insofar as technical and mix inefficiency is concerned. However, we need to note that the Additive and BCC models may give different estimates when inefficiencies are present. Moreover, as we shall see in the next chapter, the CCR and BCC models differ in that the former evaluates scale as well as technical inefficiencies simultaneously whereas the latter evaluates the two in a fashion that identifies them separately.

3. **Translation Invariance.**
As is seen from Table 4.4, we can classify the models according to whether or not they use the efficiency measure θ^*. It is to be noted that θ^* is measured in a manner that depends on the coordinate system of the data set. On the other hand, the θ^*-free models, such as the Additive model, are essentially coordinate-free and translation invariant. They evaluate the efficiency of a DMU by the l_1-metric distance from the efficient frontiers and are invariant with respect to the translation of the coordinate system. Although they supply information on the projection of the inefficient DMUs onto the production frontiers, they lack a one-dimensional efficiency measure like θ^*. The SBM model developed in Section 4.4 is designed to overcome such deficiencies in both the CCR-type and Additive models. However, SBM as represented in (4.48) is not translation invariant.

4. **Number of Input and Output Items.**
Generally speaking, if the number of DMUs (n) is less than the combined number of inputs and outputs ($m + s$), a large portion of the DMUs will be identified as efficient and efficiency discrimination among DMUs is questionable due to an inadequate number of degrees of freedom. See the opening discussion in Chapter 9. Hence, it is desirable that n exceed $m+s$ by several times. A rough rule of thumb in the envelopment model is to choose n (= the number of DMUs) equal to or greater than $\max\{m \times s, 3 \times (m + s)\}$. The selection of input and output items is crucial for successful application of DEA. We therefore generally recommend a process of selecting a small set of input and output items at the beginning and gradually enlarging the set to observe the effects of the added items. In addition, other methods, e.g., the assurance region method, the cone ratio model and others (that will be presented in the succeeding chapters) lend themselves to a sharper discrimination among DMUs.

5. **Try Different Models.**
If we cannot identify the characteristics of the production frontiers by preliminary surveys, it may be risky to rely on only one particular model. If the application has important consequences it is wise to try different models and methods, compare results and utilize expert knowledge of the problem, and possibly try other devices, too, before arriving at a definitive conclusion. See the discussion in Chapter 9 dealing with the use of statistical regressions and DEA to cross check each other.

4.7 SUMMARY OF CHAPTER 4

In this chapter, we introduced several DEA models.

1. The CCR and BCC models which are radial measures of efficiency that are either input oriented or output oriented.

2. The Additive models which identify input excesses and output shortfalls simultaneously.

3. The slacks-based measure of efficiency (SBM) which uses the Additive model and provides a scalar measure ranging from 0 to 1 that encompasses all of the inefficiencies that the model can identify.

4. We investigated the problem of model choice. Chapters 6, 7 and 9 will serve as a complement in treating this subject.

5. The translation invariance of the Additive model was introduced. This property of the Additive model (with convexity constraint) was identified by Ali and Seiford (1990) and was extended by Pastor (1996).[16] See, however, Thrall (1996)[17] for limitations involved in relying on this invariance property.

4.8 NOTES AND SELECTED BIBLIOGRAPHY

Translation invariance was first shown by Ali and Seiford (1990) and extended by Pastor *et al.* (1999). The SBM model was introduced by Tone (1997, 2001).

4.9 APPENDIX: FREE DISPOSAL HULL (FDH) MODELS

Another model which has received a considerable amount of research attention is the FDH (Free Disposal Hull) model as first formulated by Deprins, Simar and Tulkens (1984)[18] and developed and extended by Tulkens and his associates at the University of Louvain in Belgium.[19] The basic motivation is to ensure that efficiency evaluations are effected from only actually observed performances. Points like Q in Figure 3.2, for example, are not allowed because they are derived and not actually observed performances. Hence they are hypothetical.

Figure 4.7 provides an example of what is referred to as the Free Disposal Hull for five DMUs using two inputs in amounts x_1 and x_2 to produce a single output in amount $y = 1$.

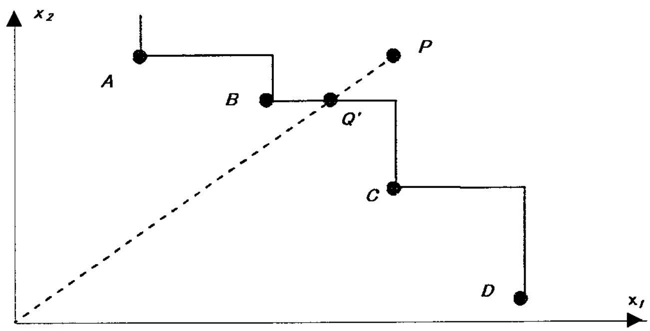

Figure 4.7. FDH Representation

The boundary of the set and its connection represents the "hull" defined as the "smallest set" that encloses all of the production possibilities that can be generated from the observations. Formally,

$$P_{FDH} = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq \boldsymbol{x}_j, \boldsymbol{y} \leq \boldsymbol{y}_j, \ \boldsymbol{x}, \ \boldsymbol{y} \geq \mathbf{0}, \ j = 1, \ldots, n\}$$

where $\boldsymbol{x}_j (\geq \mathbf{0}), \boldsymbol{y}_j (\geq \mathbf{0})$ are actually observed performances for $j = 1, \ldots, n$ DMUs. In words, a point is a member of the production possibility set if all of its input coordinates are at least as large as their corresponds in the vector of observed values $\boldsymbol{x}_j$ for any $j = 1, \ldots, n$ and if their output coordinates are no greater than their corresponds in the vectors $\boldsymbol{y}_j$ of observed values for this same j.

This gives rise to the staircase (or step) function which we have portrayed by the solid line in the simple two-input one-output example of Figure 4.7. No point below this solid line has the property prescribed for P_{FDH}. Moreover this boundary generates the smallest set with these properties. For instance, connecting points B and C in the manner of Figure 3.2 in Chapter 3 would generate the boundary of a bigger production possibility set. Tulkens and his associates use an algorithm that eliminates all dominated points as candidates for use in generating the FDH. This algorithm proceeds in pairwise comparison fashion as follows: Let DMU$_k$ with coordinate $\boldsymbol{x}_k$, $\boldsymbol{y}_k$ be a candidate. If for any DMU$_j$ we have $\boldsymbol{x}_j \leq \boldsymbol{x}_k$ or $\boldsymbol{y}_j \geq \boldsymbol{y}_k$ with either $\boldsymbol{x}_j \neq \boldsymbol{x}_k$ or $\boldsymbol{y}_j \neq \boldsymbol{y}_k$ then DMU$_k$ is dominated (strictly) and removed from candidacy. Actually this can be accomplished more simply by using the following mixed integer programming formulation,

$$\begin{aligned} \min \quad & \theta \\ \text{subject to} \quad & \theta \boldsymbol{x}_o - X\boldsymbol{\lambda} \geq \mathbf{0} \\ & \boldsymbol{y}_o - Y\boldsymbol{\lambda} \leq \mathbf{0} \\ & \boldsymbol{e}\boldsymbol{\lambda} = 1, \quad \lambda_j \in \{0, \ 1\} \end{aligned} \tag{4.90}$$

where X and Y contain the given input and output matrices and $\lambda_j \in \{0, \ 1\}$ means that the components of $\boldsymbol{\lambda}$ are constrained to be bivalent. That is, they must all have values of zero or unity so that together with the condition $\boldsymbol{e}\boldsymbol{\lambda} = 1$ one and only one of the performances actually observed can be chosen. This approach was first suggested in Bowlin *et al.*(1984)[20] where it was coupled with an additive model to ensure that the "most dominant" of the non-dominated DMUs was chosen for making the indicated efficiency evaluation. In the case of Figure 4.7, the designation would thus have been A rather than B or C in order to maximize the sum of the slacks s_1^- and s_2^- when evaluating P. However, this raises an issue because the results may depend on the units of measure employed. One way to avoid this difficulty is to use the radial measure represented by $\min \theta = \theta^*$. This course, as elected by Tulkens and associates, would yield the point Q' shown in Figure 4.7. This, however, leaves the slack in going from Q' to B unattended, which brings into play the assumption of "Free Disposal" that is incorporated in the name "Free Disposal Hull." As noted earlier this means that nonzero slacks are ignored or, equivalently, weak efficiency suffices because the slacks do not appear in the objective — or, equivalently, they are present in the objective with zero coefficients assigned to them.

One way to resolve all of these issues is to return to our slacks-based measure (SBM) as given in (4.48) or its linear programming (LP) equivalent in (4.51) with the conditions $\lambda_j \in \{0,\ 1\}$ and $e\boldsymbol{\lambda} = 1$ adjoined. This retains the advantage of the additive model which (a) selects an "actually observed performance" and (b) provides a measure that incorporates all of the thus designated inefficiencies. Finally, it provides a measure which is units invariant as well. See R.M. Thrall (1999)[21] for further discussions of FDH and its limitations.

4.10 RELATED DEA-SOLVER MODELS FOR CHAPTER 4

BCC-I (The input-oriented Banker-Charnes-Cooper model).

This code solves the BCC model expressed by (4.2)-(4.6). The data format is the same as for the CCR model. The "Weight" sheet includes the optimal value of the multiplier u_0 corresponding to the constraint $\sum_{j=1}^{n} \lambda_j = 1$, as well as $\boldsymbol{v}^*$ and $\boldsymbol{u}^*$. In addition to the results which are similar to the CCR case, this model finds the returns-to-scale characteristics of DMUs in the "RTS" sheet. For inefficient DMUs, we identify returns to scale with the projected point on the efficient frontier. BCC-I uses the projection formula (4.17)-(4.18).

BCC-O (The output-oriented BCC model).

This code solves the output-oriented BCC model expressed in (4.22)-(4.26). The optimal expansion rate η_B^* is displayed in "Score" by its inverse in order to facilitate comparisons with the input-oriented case. Usually, the efficiency score differs in both cases for inefficient DMUs. This model also finds the returns-to-scale characteristics of DMUs in the "RTS" sheet. For inefficient DMUs, we identify returns to scale with the projected point on the efficient frontier. BCC-O uses the BCC version of the formula (3.74)-(3.15). The returns-to-scale characteristics of inefficient DMUs may also change from the input-oriented case.

SBM-C(V or GRS) (The slacks-based measure of efficiency under the constant (variable or general) returns-to-scale assumption.)

This code solves the SBM model. The format of the input data and the output results is the same as for the CCR case. In the GRS (general returns-to-scale) case, L (the lower bound) and U (the upper bound) of the sum of the intensity vector $\boldsymbol{\lambda}$ must be supplied through keyboard. The defaults are $L = 0.8$ and $U = 1.2$.

SBM-I-C(V or GRS) (The input-oriented slacks-based measure of efficiency under the constant (variable or general) returns-to-scale assumption).

This code solves the SBM model in input-orientation. Therefore, output slacks (shortfalls) are not accounted for in this efficiency measure.

SBM-O-C(V or GRS) (The output-oriented SBM model under constant (variable or general) returns-to-scale assumption).

This is the output-oriented version of the SBM model, which puts emphasis on the output shortfalls. Therefore, input slacks (excesses) are not accounted for in this efficiency measure.

FDH (The free disposal hull model).

This code solves the FDH model introduced in Section 4.8 and produces the worksheets "Score," "Projection," "Graph1" and "Graph2." The data format is the same as for the CCR model.

4.11 PROBLEM SUPPLEMENT FOR CHAPTER 4

Problem 4.1

It is suggested on occasion that the weak (or Farrell) efficiency value, θ^*, be used to rank DMU performances as determined from CCR or BCC models.

Assignment : Discuss possible shortcomings in this approach.

Suggested Response : There are two main shortcomings that θ^* possesses for the proposed use. First, the value of θ^* is not a complete measure and, instead, the nonzero slacks may far outweigh the value of $(1-\theta^*)$. Second, the θ^* values will generally be determined relative to different reference groups. Note, therefore, that a value of $\theta^* = 0.9$ for DMU A means that its purely technical efficiency is 90% of the efficiency generated by members of its (efficient) reference group. Similarly a value of $\theta^* = 0.8$ for DMU B refers to its performance relative to a different reference group. This does *not* imply that DMU B is less efficient than DMU A.

Ranking can be useful, of course, so when this is wanted recourse should be had to an explicitly stated principle of ranking. This was done by the Texas Public Utility Commission, for example, which is required by law to conduct "efficiency audits" of the 75 electric cooperatives which are subject to its jurisdiction in Texas. See Dennis Lee Thomas "Auditing the Efficiency of Regulated Companies through the Use of Data Envelopment Analysis: An Application to Electric Cooperatives," Ph.D. Thesis, Austin, TX: University of Texas at Austin, Graduate School of Business, 1985.[22] As discussed in Thomas (subsequently Chairman of the Texas PUC), the ranking principle was based on dollarizing the inefficiencies. This included assigning dollar values to the slacks as well as to the purely technical inefficiencies. For instance, line losses in KWH were multiplied by the revenue charges at prices that could otherwise be earned from sales of delivered power. Fuel excesses, however, were costed at the purchase price per unit. Thomas then also remarks that the reference groups designated by DEA were used by PUC auditors to help in their evaluation by supplying comparison DMUs for the cooperative being audited. See also A. Charnes, W.W. Cooper, D. Divine, T.W. Ruefli and D. Thomas "Comparisons of DEA and Existing Ratio and Regression Systems for Effecting Efficiency Evaluations of Regulated Electric Cooperatives in Texas," *Research in Government and Nonprofit Accounting* 5, 1989, pp.187-210.

Problem 4.2

Show that the CCR model is not translation invariant.
Suggested Answer : Let us examine the CCR model via a simple one input and one output case, as depicted in Figure 4.8. The efficient frontier is the line connecting the origin O and the point B. The CCR-efficiency of A is evaluated by PQ/PA. If we translate the output value by a given amount, say 2, in the manner shown in Figure 4.8, the origin will move to O' and the efficient frontier will shift to the dotted line extending beyond B from O'. Thus, the CCR-efficiency of A is evaluated, under the new coordinate system, by PR/PA, which differs from the old value PQ/PA. A similar observation of the translation of the input item shows that the CCR model is *not* invariant for translation of both *input* and *output*. From this we conclude that the condition $\sum_{j=1}^{n} \lambda_j = 1$ plays a crucial role which is exhibited in limited fashion in the BCC model and more generally in Additive models.

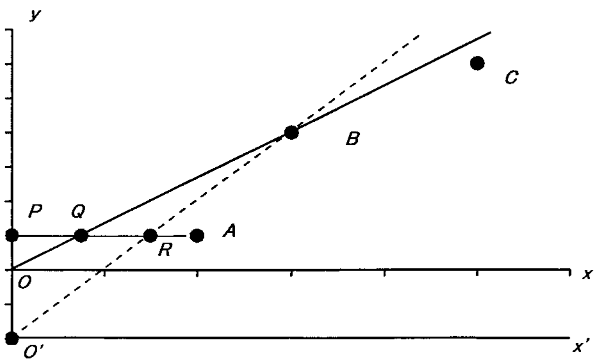

Figure 4.8. Translation in the CCR Model

Problem 4.3

As noted in the text, the critical condition for translation invariance is $\boldsymbol{e\lambda} = 1$. This suggests that parts of the BCC model may be translation invariant. Therefore,

Assignment : Prove that the input-oriented (output-oriented) BCC model is translation invariant with respect to the coordinates of outputs (inputs).

Answer : The input-oriented BCC model is:

$$\begin{aligned}
\min \quad & \theta_B \\
\text{subject to} \quad & \theta_B \boldsymbol{x}_o - X\boldsymbol{\lambda} - \boldsymbol{s}^- = \boldsymbol{0} \\
& Y\boldsymbol{\lambda} - \boldsymbol{s}^+ = \boldsymbol{y}_o \\
& \boldsymbol{e\lambda} = 1 \\
& \boldsymbol{\lambda} \geq \boldsymbol{0}, \quad \boldsymbol{s}^- \geq \boldsymbol{0}, \quad \boldsymbol{s}^+ \geq \boldsymbol{0}.
\end{aligned} \tag{4.91}$$

If we translate the output data Y by some constants $(\beta_r : r = 1, \ldots, s)$ to obtain

$$y'_{rj} = y_{rj} + \beta_r, \quad (r = 1, \ldots, s : j = 1, \ldots, n) \tag{4.92}$$

then (4.91) becomes, in terms of the components of Y',

$$\sum_{j=1}^{n}(y'_{rj} - \beta_r)\lambda_j - s_r^+ = \sum_{j=1}^{n} y'_{rj}\lambda_j - \beta_r - s_r^+ = y'_{ro} - \beta_r, \quad (r = 1, \ldots, s) \tag{4.93}$$

so subtracting β_r from both sides gives

$$\sum_{j=1}^{n} y'_{rj}\lambda_j - s_r^+ = y'_{ro}, \quad (r = 1, \ldots, s) \tag{4.94}$$

which is the same as the original expression for this output constraint. Notice that the convexity condition $\boldsymbol{e\lambda} = 1$ is used to derive the above relation. Notice also that this result extends to inputs *only* in the special case of $\theta_B^* = 1$.

It should now be readily apparent that if $(\theta_B^*, \ \lambda^*, s^{-*}, s^{+*})$ is an optimal solution of the original BCC model, then it is also optimal for the translated problem.

By the same reasoning used for the input-oriented case, we conclude that the output-oriented BCC model is translation invariant with respect to the input data X but *not* to the output data.

Comment and References : The property of translation invariance for Additive models was first demonstrated in Ali and Seiford (1990), as noted in the text. It has been carried further in J.T Pastor (1996).[23] See also R.M. Thrall (1996)[24] who shows that this property does not carry over to the dual of Additive models.

Problem 4.4

Statement : Consider the following version of an Additive model

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\ \text{subject to} \quad & \sum_{j=1}^{n} \widehat{x}_{ij}\lambda_j + s_i^- = \widehat{x}_{io}, \ i = 1, \ldots, m \\ & \sum_{j=1}^{n} \widehat{y}_{rj}\lambda_j - s_r^+ = \widehat{y}_{ro}, \ r = 1, \ldots, s \end{aligned} \tag{4.95}$$

where the variables are also constrained to be nonnegative and the symbol "$\widehat{\ }$" means that the data are stated in natural logarithmic units. (Unlike (ADD_o), as given in (4.34)-(4.38), we have omitted the condition $\boldsymbol{e\lambda} = 1$ because this restricts results to the case of constant returns to scale in the models we are now considering.

Assignment : Part I: Use anti-logs (base e) to derive yet another DEA model called the "multiplicative model," and relate the resulting formulation to the Cobb-Douglas forms of production frontiers that have played prominent roles in many econometric and statistical studies.
Part II: Relate the conditions for full (100%) efficiency to Definition 5.1 for the Additive model.

Suggested Response: : Part I: Let an optimal solution be represented by $\lambda_j^*, s_i^{-*}, s_r^{+*}$. Then, taking anti-logs of (4.95) we obtain

$$x_{io} = \prod_{j=1}^{n} x_{ij}^{\lambda_j^*} e^{s_i^{-*}} = a_i^* \prod_{j=1}^{n} x_{ij}^{\lambda_j^*},\ i = 1, \ldots, m \tag{4.96}$$
$$y_{ro} = \prod_{j=1}^{n} y_{rj}^{\lambda_j^*} e^{-s_r^{+*}} = b_r^* \prod_{j=1}^{n} y_{rj}^{\lambda_j^*},\ r = 1, \ldots, s$$

where $a_i^* = e^{s_i^{-*}}$, $b_r^* = e^{-s_r^{+*}}$. This shows that each x_{io} and y_{ro} is to be considered as having been generated by Cobb-Douglas (=log-linear) processes with estimated parameters indicated by the starred values of the variables.

Part II: To relate these results to ratio forms we proceed in a manner analogous to what was done for (4.7)-(4.10) and turn to the dual (multiplier) form of (4.95),

$$\begin{aligned}
\min \quad & \sum_{i=1}^{m} v_i \widehat{x}_{io} - \sum_{r=1}^{s} u_r \widehat{y}_{ro} \\
\text{subject to} \quad & \sum_{i=1}^{m} v_i \widehat{x}_{ij} - \sum_{r=1}^{s} u_r \widehat{y}_{rj} \geq 0,\ j = 1, \ldots, n \\
& v_i \geq 1,\ i = 1, \ldots, m \\
& u_r \geq 1,\ r = 1, \ldots, s.
\end{aligned} \tag{4.97}$$

This transformation to logarithm values gives a result that is the same as in the ordinary Additive model. Hence, no new software is required to solve this multiplicative model. To put this into an easily recognized efficiency evaluation form we apply anti-logs to (4.97). We change the objective in (4.97) and reverse the constraints to obtain:

$$\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r \widehat{y}_{ro} - \sum_{i=1}^{m} v_i \widehat{x}_{io} \\
\text{subject to} \quad & \sum_{r=1}^{s} u_r \widehat{y}_{rj} - \sum_{i=1}^{m} v_i \widehat{x}_{ij} \leq 0,\ j = 1, \ldots, n \\
& v_i \geq 1,\ i = 1, \ldots, m \\
& u_r \geq 1,\ r = 1, \ldots, s.
\end{aligned}$$

We then obtain:

$$\begin{aligned} \max \quad & \prod_{r=1}^{s} y_{ro}^{u_r} \Big/ \prod_{i=1}^{m} x_{io}^{v_i} \\ \text{subject to} \quad & \prod_{r=1}^{s} y_{rj}^{u_r} \Big/ \prod_{i=1}^{m} x_{ij}^{v_i} \leq 1, \ j = 1, \ldots, n \\ & v_i, u_r \geq 1, \forall i, \ r. \end{aligned} \tag{4.98}$$

To obtain conditions for efficiency we apply antilogs to (4.95) and use the constraints in (4.97) to obtain

$$\max \frac{\prod_{r=1}^{s} e^{s_r^+}}{\prod_{i=1}^{m} e^{-s_i^-}} = \frac{\prod_{r=1}^{s} \prod_{j=1}^{n} y_{rj}^{\lambda_j^*} \Big/ y_{ro}}{\prod_{i=1}^{m} \prod_{j=1}^{n} x_{ij}^{\lambda_j^*} \Big/ x_{io}} \geq 1. \tag{4.99}$$

The lower bound represented by the unity value on the right is obtainable if and only if all slacks are zero. Thus, the conditions for efficiency of the multiplicative model are the same as for the Additive model where the latter has been stated in logarithmic units. See Definition 5.1. We might also note that the expression on the left in (4.99) is simpler and easier to interpret and the computation via (4.95) is straightforward with the $\widehat{x}_{ij}$ and $\widehat{y}_{rj}$ stated in logarithmic units.

In conclusion we might note that (4.99) is not units invariant unless $\sum_{j=1}^{n} \lambda_j^* = 1$. This "constant-returns-to-scale condition" can, however, be imposed on (4.95) if desired. See also A. Charnes, W.W. Cooper, L.M. Seiford and J. Stutz "Invariant Multiplicative Efficiency and Piecewise Cobb-Douglas Envelopments" *Operations Research Letters* 2, 1983, pp.101-103.
Comments : The class of multiplicative models has not seen much used in DEA applications. It does have potential for uses in extending returns-to-scale treatments to situations that are not restricted to concave cap functions like those which are assumed to hold in the next chapter. See R.D. Banker and A. Maindiratta "Piecewise Loglinear Estimation of Efficient Production Surfaces" *Management Science* 32, 1986, pp.126-135. Its properties suggest possible uses not only in its own right but also in combination with other models and, as will be seen in the next chapter, it can be used to obtain scale elasticities that are not obtainable with other DEA models.

Problem 4.5

Work out the dual version of the weighted SBM model in Section 4.4.8 and interpret the results.

Suggested Response:

$$\max \quad \boldsymbol{u}\boldsymbol{y}_o - \boldsymbol{v}\boldsymbol{x}_o \tag{4.100}$$

$$\text{subject to} \quad -\boldsymbol{v}X + \boldsymbol{u}Y \leq \mathbf{0} \tag{4.101}$$

$$v_i x_{io} \geq \frac{1}{m} w_i^- \ (i = 1, \ldots, m) \tag{4.102}$$

$$u_r y_{ro} \geq \frac{1 - \boldsymbol{v}\boldsymbol{x}_o + \boldsymbol{u}\boldsymbol{y}_o}{s} w_r^+ . \ (r = 1, \ldots, s) \tag{4.103}$$

Thus, each $v_i x_{io}$ is bounded below by a value proportional to $1/m$ times the selected weight w_i^- for input and the value of unity increment for the $\boldsymbol{y}_o$ and decrement for the $\boldsymbol{x}_o$ combined in the numerator on s, the number of outputs in the denominator.

Problem 4.6

Assignment: Suppose the objective for ERM in (4.84) is changed from its fractional form to the following

$$\max \frac{\sum_{r=1}^{s} \phi_r}{s} - \frac{\sum_{i=1}^{m} \theta_i}{m}. \tag{4.104}$$

Using the relations in (4.86) transform this objective to the objective for the corresponding Additive model and discuss the properties of the two measures.

Suggested Response: Transforming (4.104) in the suggested manner produces the following objective for the corresponding Additive model.

$$\max z = \frac{\sum_{r=1}^{s}\left(1 + \frac{s_r^+}{y_{ro}}\right)}{s} - \frac{\sum_{i=1}^{m}\left(1 - \frac{s_i^-}{x_{io}}\right)}{m} = \sum_{r=1}^{s} \frac{s_r^+}{y_{ro}} + \sum_{i=1}^{m} \frac{s_i^-}{x_{io}}. \tag{4.105}$$

The ERM constraints are similarly transformed to the constraints of an Additive model as described in the proof of Theorem 4.15.

Discussion: The objective in (4.104) jointly maximizes the "inefficiencies" measured by the ϕ_r and minimizes the "efficiencies" measured by the θ_i. This is analogous to the joint minimization represented by the objective in (4.84) for ERM but its z^* values, while non-negative, may exceed unity. The objective stated on the right in (4.105) jointly maximizes the "inefficiencies" in both the s_r^+/y_{ro} and s_i^-/x_{io}. This is analogous to the joint maximization of both of these measures of inefficiency in SBM except that, again, its value may exceed unity.

Notes

1. R. D. Banker, A. Charnes and W.W. Cooper (1984), "Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis," *Management Science* 30, pp.1078-1092

2. SBM was first proposed by K. Tone (1997), "A Slacks-based Measure of Efficiency in Data Envelopment Analysis," Research Reports, Graduate School of Policy Science, Saitama University and subsequently published in *European Journal of Operational Research* 130 (2001), pp.498-509.

3. For a discussion of this constant-returns-to-scale Additive model see A.I. Ali and L.M. Seiford, "The Mathematical Programming Approach to Efficiency Measurement," in *The Measurement of Productive Efficiency: Techniques and Applications*, H. Fried, C. A. Knox Lovell, and S. Schmidt (editors), Oxford University Press, London, (1993).

4. This distance is measured in what is called an l_1-metric for the slack vectors s^- and s^+. Also called the "city-block" metric, the objective in (4.34) is directed to maximizing the distance as measured by the sum of the input plus output slacks.

5. T. Ahn, A. Charnes and W.W. Cooper (1988), "Efficiency Characterizations in Different DEA Models," *Socio-Economic Planning Sciences*, 22, pp.253-257.

6. P.L. Brockett, W.W. Cooper, L.L. Golden, J.J. Rousseau and Y. Wang, "DEA Evaluation of Organizational Forms and Distribution System in the U.S. Property and Liability Insurance Industry," *International Journal of Systems Science*, 1998.

7. A.I. Ali and L.M. Seiford (1990), "Translation Invariance in Data Envelopment Analysis," *Operations Research Letters* 9, pp.403-405.

8. This term is taken from the literature on dimensional analysis. See pp.123-125, in R.M. Thrall "Duality, Classification and Slacks in DEA," *Annals of Operations Research* 66, 1996.

9. See R.S. Färe and C.A.K. Lovell (1978), "Measuring the Technical Efficiency of Production," *Journal of Economic Theory* 19, pp.150-162. See also R.R. Russell (1985) "Measures of Technical Efficiency," *Journal of Economic Theory* 35, pp.109-126.

10. See the Note 9 references.

11. J.T. Pastor, J.L. Ruiz and I. Sirvent (1999), "An Enhanced DEA Russell Graph Efficiency Measure," *European Journal of Operational Research* 115, pp.596-607.

12. R.S. Färe, S. Grosskopf and C.A.K Lovell (1985), *The Measurement of Efficiency of Production* Boston: Kluwer-Nijhoff.

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