2   THE BASIC CCR MODEL

2.1   INTRODUCTION

This chapter deals with one of the most basic DEA models, the *CCR model*, which was initially proposed by Charnes, Cooper and Rhodes in 1978. Tools and ideas commonly used in DEA are also introduced and the concepts developed in Chapter 1 are extended. There, for each DMU, we formed the virtual input and output by (yet unknown) weights (v_i) and (u_r):

$$\text{Virtual input} = v_1 x_{1o} + \cdots + v_m x_{mo}$$
$$\text{Virtual output} = u_1 y_{1o} + \cdots + u_s y_{so}.$$

Then we tried to determine the weight, using linear programming so as to maximize the ratio

$$\frac{\text{virtual output}}{\text{virtual input}}.$$

The optimal weights may (and generally will) vary from one DMU to another DMU. Thus, the "weights" in DEA are derived from the data instead of being fixed in advance. Each DMU is assigned a best set of weights with values that may vary from one DMU to another. Additional details and the algorithms used to implement these concepts are explained in succeeding chapters.

2.2   DATA

In DEA, the organization under study is called a *DMU* (Decision Making Unit). The definition of DMU is rather loose to allow flexibility in its use over a wide range of possible applications. Generically a DMU is regarded as the entity responsible for converting inputs into outputs and whose performances are to be evaluated. In managerial applications, DMUs may include banks, department stores and supermarkets, and extend to car makers, hospitals, schools, public libraries and so forth. In engineering, DMUs may take such forms as airplanes or their components such as jet engines. For the purpose of securing *relative* comparisons, a group of DMUs is used to evaluate each other with each DMU having a certain degree of managerial freedom in decision making.

Suppose there are n DMUs: DMU_1, DMU_2,..., and DMU_n. Some common input and output items for each of these $j = 1, ..., n$ DMUs are selected as follows:

1. Numerical data are available for each input and output, with the data assumed to be positive[1] for all DMUs.

2. The items (inputs, outputs and choice of DMUs) should reflect an analyst's or a manager's interest in the components that will enter into the relative efficiency evaluations of the DMUs.

3. In principle, smaller input amounts are preferable and larger output amounts are preferable so the efficiency scores should reflect these principles.

4. The measurement units of the different inputs and outputs need not be congruent. Some may involve number of persons, or areas of floor space, money expended, etc.

Suppose m input items and s output items are selected with the properties noted in 1 and 2. Let the input and output data for DMU_j be $(x_{1j}, x_{2j}, ..., x_{mj})$ and $(y_{1j}, y_{2j}, ..., y_{sj})$, respectively. The input data matrix X and the output data matrix Y can be arranged as follows,

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \tag{2.1}$$

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ y_{s1} & y_{s2} & \cdots & y_{sn} \end{pmatrix} \tag{2.2}$$

where X is an $(m \times n)$ matrix and Y an $(s \times n)$ matrix. For example, the hospital case in Section 1.5 has the data matrices:

$$X = \begin{pmatrix} 20 & 19 & 25 & 27 & 22 & 55 & 33 & 31 & 30 & 50 & 53 & 38 \\ 151 & 131 & 160 & 168 & 158 & 255 & 235 & 206 & 244 & 268 & 306 & 284 \end{pmatrix}$$

$$Y = \begin{pmatrix} 100 & 150 & 160 & 180 & 94 & 230 & 220 & 152 & 190 & 250 & 260 & 250 \\ 90 & 50 & 55 & 72 & 66 & 90 & 88 & 80 & 100 & 100 & 147 & 120 \end{pmatrix}$$

so x_{1j} = number of doctors and x_{2j} = number of nurses used by hospital j in servicing (= producing) y_{1j} = number of outpatients and y_{2j} = number of inpatients.

2.3   THE CCR MODEL

Given the data, we measure the efficiency of each DMU once and hence need n optimizations, one for each DMU$_j$ to be evaluated. Let the DMU$_j$ to be evaluated on any trial be designated as DMU$_o$ where o ranges over $1, 2, ..., n$. We solve the following fractional programming problem to obtain values for the input "weights" (v_i) $(i = 1, ..., m)$ and the output "weights" (u_r) $(r = 1, ..., s)$ as variables.

$$(FP_o) \qquad \max_{\boldsymbol{v},\boldsymbol{u}} \; \theta = \frac{u_1 y_{1o} + u_2 y_{2o} + \cdots + u_s y_{so}}{v_1 x_{1o} + v_2 x_{2o} + \cdots + v_m x_{mo}} \qquad (2.3)$$

$$\text{subject to} \qquad \frac{u_1 y_{1j} + \cdots + u_s y_{sj}}{v_1 x_{1j} + \cdots + v_m x_{mj}} \leq 1 \;\; (j = 1, \ldots, n) \qquad (2.4)$$

$$v_1, v_2, \ldots, v_m \geq 0 \qquad (2.5)$$

$$u_1, u_2, \ldots, u_s \geq 0. \qquad (2.6)$$

The constraints mean that the ratio of "virtual output" vs. "virtual input" should not exceed 1 for every DMU. The objective is to obtain weights (v_i) and (u_r) that maximize the ratio of DMU$_o$, the DMU being evaluated. By virtue of the constraints, the optimal objective value θ^* is at most 1. Mathematically, the nonnegativity constraint (2.5) is not sufficient for the fractional terms in (2.4) to have a positive value. We do not treat this assumption in explicit mathematical form at this time. Instead we put this in managerial terms by assuming that all outputs and inputs have some nonzero worth and this is to be reflected in the weights u_r and v_i being assigned some positive value.

2.4   FROM A FRACTIONAL TO A LINEAR PROGRAM

We now replace the above fractional program (FP_o) by the following linear program (LP_o),

$$(LP_o) \qquad \max_{\boldsymbol{\mu},\boldsymbol{\nu}} \; \theta = \mu_1 y_{1o} + \cdots + \mu_s y_{so} \qquad (2.7)$$

subject to $\nu_1 x_{1o} + \cdots + \nu_m x_{mo} = 1$ (2.8)

$\mu_1 y_{1j} + \cdots + \mu_s y_{sj} \leq \nu_1 x_{1j} + \cdots + \nu_m x_{mj}$ (2.9)

$(j = 1, \ldots, n)$

$\nu_1, \nu_2, \ldots, \nu_m \geq 0$ (2.10)

$\mu_1, \mu_2, \ldots, \mu_s \geq 0.$ (2.11)

Theorem 2.1 *The fractional program* (FP_o) *is equivalent to* (LP_o).

Proof.   Under the nonzero assumption of v and $X > 0$, the denominator of the constraint of (FP_o) is positive for every j, and hence we obtain (2.3) by multiplying both sides of (2.4) by the denominator. Next, we note that a fractional number is invariant under multiplication of both numerator and denominator by the same nonzero number. After making this multiplication, we set the denominator of (2.3) equal to 1, move it to a constraint, as is done in (2.8), and maximize the numerator, resulting in (LP_o). Let an optimal solution of (LP_o) be $(\boldsymbol{\nu} = \boldsymbol{\nu}^*, \boldsymbol{\mu} = \boldsymbol{\mu}^*)$ and the optimal objective value θ^*. The solution $(\boldsymbol{v} = \boldsymbol{\nu}^*, \boldsymbol{u} = \boldsymbol{\mu}^*)$ is also optimal for (FP_o), since the above transformation is reversible under the assumptions above. (FP_o) and (LP_o) therefore have the same optimal objective value θ^*.   □

We also note that the measures of efficiency we have presented are "units invariant" — i.e., they are independent of the units of measurement used in the sense that multiplication of each input by a constant $\delta_i > 0$, $i = 1, \ldots, m$, and each output by a constant $p_r > 0$, $r = 1, \ldots, s$, does not change the obtained solution. Stated in precise form we have

Theorem 2.2 (Units Invariance Theorem)   *The optimal values of max* $\theta = \theta^*$ *in (2.3) and (2.7) are independent of the units in which the inputs and outputs are measured provided these units are the same for every DMU.*

Thus, one person can measure outputs in miles and inputs in gallons of gasoline and quarts of oil while another measures these same outputs and inputs in kilometers and liters. They will nevertheless obtain the same efficiency value from (2.3) or (2.7) when evaluating the same collection of automobiles, say. See Note 2 for proof.[2]

Before proceeding we note that (LP_o) can be solved by the simplex method of linear programming. The optimal solution can be more easily obtained by dealing with the dual side of (LP_o), however, as will be explained in detail in Chapter 3.

In any case let us suppose we have an optimal solution of (LP_o) which we represent by $(\theta^*, \boldsymbol{v}^*, \boldsymbol{u}^*)$[3] where $\boldsymbol{v}^*$ and $\boldsymbol{u}^*$ are values with constraints given in (2.10) and (2.11). We can then identify whether *CCR-efficiency* has been achieved as follows:

Definition 2.1 (CCR-Efficiency)
1.   *DMU$_o$ is CCR-efficient if* $\theta^* = 1$ *and there exists at least one optimal* $(\boldsymbol{v}^*, \boldsymbol{u}^*)$, *with* $\boldsymbol{v}^* > \boldsymbol{0}$ *and* $\boldsymbol{u}^* > \boldsymbol{0}$.
2.   *Otherwise, DMU$_o$ is CCR-inefficient.*

Thus, CCR-inefficiency means that either (i) $\theta^* < 1$ or (ii) $\theta^* = 1$ and at least one element of $(\boldsymbol{v}^*, \boldsymbol{u}^*)$ is zero for every optimal solution of (LP_o). We will explain the latter case using an example in Section 2.6.2, and a detailed description of CCR-efficiency will be given in Chapter 3.

Now we observe the case where DMU_o has $\theta^* < 1$ (CCR-inefficient). Then there must be at least one constraint (or DMU) in (2.9) for which the weight $(\boldsymbol{v}^*, \boldsymbol{u}^*)$ produces equality between the left and right hand sides since, otherwise, θ^* could be enlarged. Let the set of such $j \in \{1, \ldots, n\}$ be

$$E_o' = \{ \, j : \sum_{r=1}^{s} u_r^* y_{rj} = \sum_{i=1}^{m} v_i^* x_{ij} \}. \tag{2.12}$$

The subset E_o of E_o', composed of CCR-efficient DMUs, is called the *reference set* or the *peer group* to the DMU_o. It is the existence of this collection of efficient DMUs that forces the DMU_o to be inefficient. The set spanned by E_o is called the *efficient frontier* of DMU_o.

2.5   MEANING OF OPTIMAL WEIGHTS

The $(\boldsymbol{v}^*, \boldsymbol{u}^*)$ obtained as an optimal solution for (LP_o) results in a set of optimal weights for the DMU_o. The ratio scale is evaluated by :

$$\theta^* = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{i=1}^{m} v_i^* x_{io}}. \tag{2.13}$$

From (2.8), the denominator is 1 and hence

$$\theta^* = \sum_{r=1}^{s} u_r^* y_{ro}. \tag{2.14}$$

As mentioned earlier, $(\boldsymbol{v}^*, \boldsymbol{u}^*)$ are the set of most favorable weights for the DMU_o in the sense of maximizing the ratio scale. v_i^* is the optimal weight for the input item i and its magnitude expresses how highly the item is evaluated, relatively speaking. Similarly, u_r^* does the same for the output item r. Furthermore, if we examine each item $v_i^* x_{io}$ in the virtual input

$$\sum_{i=1}^{m} v_i^* x_{io} \; (= 1), \tag{2.15}$$

then we can see the relative importance of each item by reference to the value of each $v_i^* x_{io}$. The same situation holds for $u_r^* y_{ro}$ where the u_r^* provides a measure of the relative contribution of y_{ro} to the overall value of θ^*. These values not only show which items contribute to the evaluation of DMU_o, but also to what extent they do so.

2.6   EXPLANATORY EXAMPLES

We illustrate the use of the CCR model via the following small-scale examples.

2.6.1   Example 2.1 (1 Input and 1 Output Case)

Table 2.1 shows 8 DMUs with 1 input and 1 output. (The first example in Chapter 1).

Table 2.1.   Example 2.1

DMU	A	B	C	D	E	F	G	H
Input	2	3	3	4	5	5	6	8
Output	1	3	2	3	4	2	3	5

We can evaluate the efficiency of DMU A, by solving the LP problem below:

$$
\begin{aligned}
<A>\quad \max \quad & \theta = u \\
\text{subject to}\quad & 2v = 1 \\
& u \le 2v \quad (A) \qquad 3u \le 3v \quad (B) \\
& 2u \le 3v \quad (C) \qquad 3u \le 4v \quad (D) \\
& 4u \le 5v \quad (E) \qquad 2u \le 5v \quad (F) \\
& 3u \le 6v \quad (G) \qquad 5u \le 8v \quad (H)
\end{aligned}
$$

where all variables are constrained to be nonnegative.

The optimal solution, easily obtained by simple ratio calculations, is given by $(v^* = 0.5,\ u^* = 0.5,\ \theta^* = 0.5)$. Thus, the CCR-efficiency of A is $\theta^* = u^* = 0.5$. The reference set for A is found to be $E_A = \{B\}$ by inserting $u^* = 0.5$ and $v^* = 0.5$, the best possible weights for DMU A, in each of the above constraints. Thus the performance of B is used to characterize A and rates it as inefficient even with the best weights that the data admit for A.

The efficiency of B can be similarly evaluated from the data in Table 2.1 by:

$$
\begin{aligned}
<B>\quad \max \quad & \theta = 3u \\
\text{subject to}\quad & 3v = 1 \\
& u \le 2v \quad (A) \qquad 3u \le 3v \quad (B) \\
& 2u \le 3v \quad (C) \qquad 3u \le 4v \quad (D) \\
& 4u \le 5v \quad (E) \qquad 2u \le 5v \quad (F) \\
& 3u \le 6v \quad (G) \qquad 5u \le 8v \quad (H)
\end{aligned}
$$

The optimal solution is $(v^* = 0.3333,\ u^* = 0.3333,\ \theta^* = 1)$ and B is CCR-efficient. See Definition 2.1.

We can proceed in a similar way with the other DMUs to obtain the results shown in Table 2.2. Only DMU B is efficient and is in the reference set of all of the other DMUs. (See Figure 2.1.) Figure 2.1 portrays the situation geometrically. The efficient frontier represented by the solid line passes through B and no other point. The θ^* values in Table 2.2 show what is needed to bring each DMU onto the efficient frontier. For example, the value of $\theta^* = 1/2$ applied

Table 2.2.   Results of Example 2.1

DMU	CCR(θ^*)	Reference Set
A	0.5000	B
B	1.0000	B
C	0.6667	B
D	0.7500	B
E	0.8000	B
F	0.4000	B
G	0.5000	B
H	0.6250	B

to A's input will bring A onto the efficient frontier by reducing its input 50% while leaving its output at its present value. Similarly $0.6667 \times 3 = 2$ will position C on the frontier. And so on.
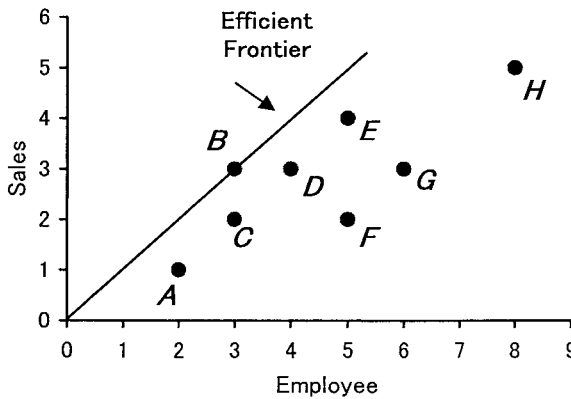
Figure 2.1.   Example 2.1

2.6.2 Example 2.2 (2 Inputs and 1 Output Case)

Table 2.3 shows 6 DMUs with 2 inputs and 1 output where the output value is unitized to 1 for each DMU.

(1) The linear program for DMU A is:

$$< A > \quad \max \quad \theta = u$$
$$\text{subject to} \quad 4v_1 + 3v_2 = 1$$

$u \leq 4v_1 + 3v_2$	(A)		$u \leq 7v_1 + 3v_2$	(B)
$u \leq 8v_1 + v_2$	(C)		$u \leq 4v_1 + 2v_2$	(D)
$u \leq 2v_1 + 4v_2$	(E)		$u \leq 10v_1 + v_2$	(F)

where all variables are constrained to be nonnegative.

Table 2.3.   Example 2.2

DMU		A	B	C	D	E	F
Input	x_1	4	7	8	4	2	10
	x_2	3	3	1	2	4	1
Output	y	1	1	1	1	1	1

This problem can be solved by a linear programming code. It can also be solved by simply deleting v_2 from the inequalities by inserting $v_2 = (1 - 4v_1)/3$ and observing the relationship between v_1 and u. The (unique) optimal solution is ($v_1^* = 0.1429$, $v_2^* = 0.1429$, $u^* = 0.8571$, $\theta^* = 0.8571$) and the CCR-efficiency of A is 0.8571. By applying the optimal solution to the above constraints, the reference set for A is found to be $E_A = \{D, E\}$.

(2) The linear program for DMU B is:

$$< B > \quad \max \quad \theta = u$$
$$\text{subject to} \quad 7v_1 + 3v_2 = 1$$
$$u \leq 4v_1 + 3v_2 \quad (A) \qquad u \leq 7v_1 + 3v_2 \quad (B)$$
$$u \leq 8v_1 + v_2 \quad (C) \qquad u \leq 4v_1 + 2v_2 \quad (D)$$
$$u \leq 2v_1 + 4v_2 \quad (E) \qquad u \leq 10v_1 + v_2 \quad (F)$$

The (unique) optimal solution is ($v_1^* = 0.0526$, $v_2^* = 0.2105$, $u^* = 0.6316$, $\theta^* = 0.6316$), the CCR-efficiency of B is 0.6316, and the reference set is $E_B = \{C, D\}$.

Now let us observe the difference between the optimal weights $v_1^* = 0.0526$ and $v_2^* = 0.2105$. The ratio $v_2^*/v_1^* = 0.2105/0.0526 = 4$ suggests that it is advantageous for B to weight *Input* x_2 four times more than *Input* x_1 in order to maximize the ratio scale measured by virtual input vs. virtual output. These values have roles as measures of the sensitivity of efficiency scores in reference to variations in input items. This topic will be dealt with in detail in Chapter 9 of this book where a systematic basis for conducting such sensitivity analyses will be provided. Here we only note that our analysis shows that a reduction in *Input* x_2 has a bigger effect on efficiency than does a reduction in *Input* x_1.

(3) An optimal solution for C is ($v_1^* = 0.0833, v_2^* = 0.3333, u^* = 1, \theta^* = 1$) and C is CCR-efficient by Definition 2.1. However, the optimal solution is not uniquely determined, as will be observed in the next section.

Likewise, D and E are CCR-efficient.

(4) The linear program for DMU F is:

$$< F > \quad \max \quad \theta = u$$
$$\text{subject to} \quad 10v_1 + v_2 = 1$$
$$u \leq 4v_1 + 3v_2 \quad (A) \qquad u \leq 7v_1 + 3v_2 \quad (B)$$
$$u \leq 8v_1 + v_2 \quad (C) \qquad u \leq 4v_1 + 2v_2 \quad (D)$$
$$u \leq 2v_1 + 4v_2 \quad (E) \qquad u \leq 10v_1 + v_2 \quad (F)$$

The optimal solution for F is $(v_1^* = 0,\ v_2^* = 1,\ u^* = 1,\ \theta^* = 1)$ and with $\theta^* = 1$, F looks efficient. However, we notice that $v_1^* = 0$. We therefore assign a small positive value ε to v_1 and observe the change in θ^*. That is, we use the data for F and with $10\varepsilon + v_2 = 1$ to obtain $v_2 = 1 - 10\varepsilon$. By inserting this value in the above inequalities, the following constraints are obtained:

$$
\begin{array}{llll}
u \leq 3 - 26\varepsilon & (A) & \qquad u \leq 3 - 23\varepsilon & (B) \\
u \leq 1 - 2\varepsilon & (C) & \qquad u \leq 2 - 16\varepsilon & (D) \\
u \leq 4 - 38\varepsilon & (E) & \qquad u \leq 1 & (F)
\end{array}
$$

Noting that ε is a small positive value, the minimum of the right-hand terms is attained with

$$u = 1 - 2\varepsilon.$$

Therefore, for any $\varepsilon > 0$, it follows that $\theta^* = 1 - 2\varepsilon < 1$. Thus, v_1 must be zero in order for F to have $\theta^* = 1$. We therefore conclude that F is CCR-inefficient by Definition 2.1.

Furthermore, let us examine the inefficiency of F by comparing F with C. C has *Input* $x_1 = 8$ and *Input* $x_2 = 1$, while F has *Input* $x_1 = 10$ and *Input* $x_2 = 1$. F has 2 units of excess in *Input* x_1 compared with C. This deficiency is concealed because the optimal solution forces the weight of *Input* x_1 to zero $(v_1^* = 0)$. C is in the reference set of F and hence by direct comparison we can identify the fact that F has used an excessive amount of this input.

It is not always easy to see such an excess in an input (or a shortage in output) from the optimal solution of the CCR model. In the next chapter, we will approach the CCR model from the dual side of the linear program and this will enable us to determine the excesses and shortfalls explicitly by the nonzero values with which these are identified.

A DMU such as F, with $\theta^* = 1$ and with an excess in inputs and/or a shortage in outputs, is called *ratio* efficient but *mix* inefficient.

Table 2.4 shows the CCR-efficiency (θ^*) of Example 2.2 and Figure 2.2 depicts the efficient frontier.

Table 2.4.   Results of Example 2.2

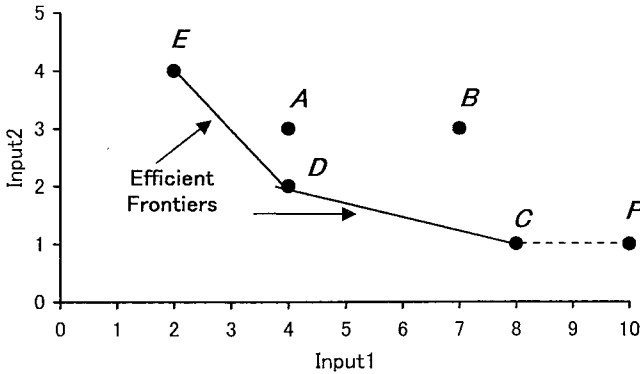| DMU	x_1	x_2	y	CCR(θ^*)	Reference Set	v_1	v_2	u
A	4	3	1	0.8571	$D\ E$	.1429	.1429	.8571
B	7	3	1	0.6316	$C\ D$	.0526	.2105	.6316
C	8	1	1	1	C	.0833	.3333	1
D	4	2	1	1	D	.1667	.1667	1
E	2	4	1	1	E	.2143	.1429	1
F	10	1	1	1	C	0	1	1

Figure 2.2.   Example 2.2

2.7   ILLUSTRATION OF EXAMPLE 2.2

In order to demonstrate the role of weights (v, u) for identifying the CCR-efficiency of DMUs, we will show graphically the efficient frontier of Example 2.2 in the weight variables (=multiplier) space. Example 2.2 has 2 inputs and 1 output, whose value is unitized to 1. For this simple example we can illustrate the situations using a two dimensional graph. The linear programming constraints for each DMU have the following inequalities in common with all variables being constrained to be nonnegative.

$$u \leq 4v_1 + 3v_2 \quad (A) \qquad u \leq 7v_1 + 3v_2 \quad (B)$$
$$u \leq 8v_1 + v_2 \quad (C) \qquad u \leq 4v_1 + 2v_2 \quad (D)$$
$$u \leq 2v_1 + 4v_2 \quad (E) \qquad u \leq 10v_1 + v_2 \quad (F)$$

Dividing these expressions by $u > 0$, we obtain the following inequalities:

$$1 \leq 4(v_1/u) + 3(v_2/u) \quad (A) \qquad 1 \leq 7(v_1/u) + 3(v_2/u) \quad (B)$$
$$1 \leq 8(v_1/u) + (v_2/u) \quad (C) \qquad 1 \leq 4(v_1/u) + 2(v_2/u) \quad (D)$$
$$1 \leq 2(v_1/u) + 4(v_2/u) \quad (E) \qquad 1 \leq 10(v_1/u) + (v_2/u) \quad (F)$$

These inequalities are depicted in Figure 2.3 by taking v_1/u and v_2/u as axes. The area denoted by P then shows the feasible region for the above constraints. The boundary of P consists of three line segments and two axes. The three line segments correspond to the efficient DMUs C, D and E.

We explain this situation using D as an example and we also explain the relationship between this region and the inefficient DMUs using A as an example.
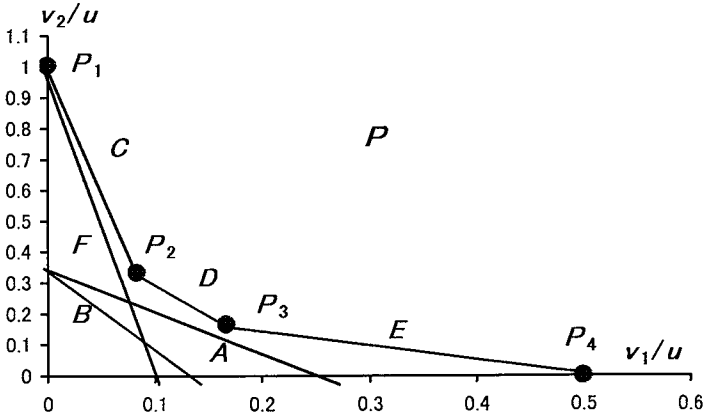
Figure 2.3.   Region P

1. **Example D**

   The linear program for D consists of the preceding inequalities plus the following,

   $$\text{max} \quad u$$
   $$\text{subject to} \quad 4v_1 + 2v_2 = 1. \tag{2.16}$$

   Dividing (2.16) by u, we have

   $$4(v_1/u) + 2(v_2/u) = 1/u. \tag{2.17}$$

   The objective function $u \rightarrow$ max yields the same solutions as $1/u \rightarrow$ min, so the problem is to find the minimum t for which the following line touches the region P:

   $$4(v_1/u) + 2(v_2/u) = t. \tag{2.18}$$

   From Figure 2.3 we see that $t = 1$ (and hence $u = 1$) represents the optimal line for D, showing that D is efficient. It is also easy to see that D is efficient for *any* weight (v_1, v_2) on the line segment (P_2, P_3). This observation leads to the conclusion that the optimal (v_1, v_2) for D is not unique. In fact, the value $(v_1 = .1667, v_2 = .1667)$ for D in Table 2.4 is an example, and actually corresponds to P_3 in Figure 2.3. See Problem 2.1 at the end of this Chapter.

   Similarly, any (v_1, v_2) on the line segment (P_1, P_2) expresses the optimal weight for C and any (v_1, v_2) on the line segment (P_3, P_4) for E.

   Thus, the optimal weights for an efficient DMU need not be unique and we should be careful to keep this in mind.

2. **Example A**

   Next, we consider the inefficient DMUs, taking A as an example. The linear

program for A consists of the following expressions added to the inequalities above:

$$\max \quad u$$
$$\text{subject to} \quad 4v_1 + 3v_2 = 1. \tag{2.19}$$

As with example D, (2.19) can be transformed into

$$4(v_1/u) + 3(v_2/u) = t. \tag{2.20}$$

Then the problem is to find the minimum t within the region P. Referring to Figure 2.4, we can see that the solution is given by the point P_3, where the line parallel to line A touches the region P for the first time. P_3 is the intersection of lines D and E and this is the geometric correspond of the fact that the reference set to A consists of D and E. A simple calculation finds that $t = 1/0.8571$ and hence the efficiency of A is $u = .8571$. The value of (v_1, v_2) at P_3 is :

$$v_1 = .1667 \times .8571 = .1429, \quad v_2 = .1667 \times .8571 = .1429, \tag{2.21}$$

which are the optimal weights for A. The optimal weights for A are unique. Usually, the optimal weights for inefficient DMUs are unique, the exception being when the line of the DMU is parallel to one of the boundaries of the region P.
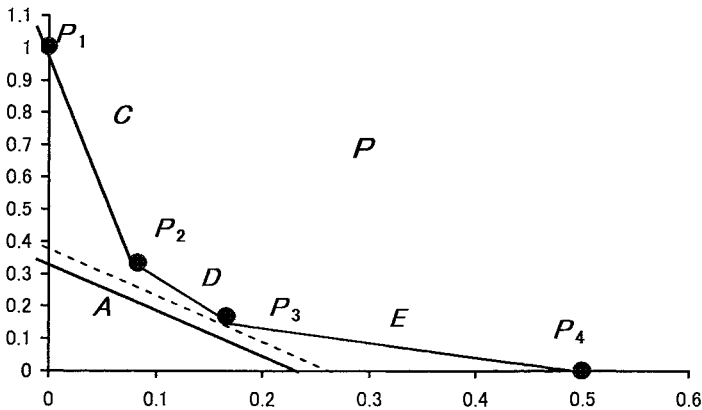

Figure 2.4.   The Case of DMU A

2.8   SUMMARY OF CHAPTER 2

In this chapter, we introduced the CCR model, which is a basic DEA model.

1. For each DMU, we formed the virtual input and output by (yet unknown) weights (v_i) and (u_r):

$$\text{Virtual input} = v_1 x_{1o} + \cdots + v_m x_{mo}$$
$$\text{Virtual output} = u_1 y_{1o} + \cdots + u_s y_{so}.$$

Then we tried to determine the weight, using linear programming so as to maximize the ratio
$$\frac{\text{virtual output}}{\text{virtual input}}.$$

The optimal weights may (and generally will) vary from one DMU to another DMU. Thus, the "weights" in DEA are derived from the data instead of being fixed in advance. Each DMU is assigned a best set of weights with values that may vary from one DMU to another. Here, too, the DEA weights differ from customary weightings (e.g., as in index number constructions) so we will hereafter generally use the term "multiplier" to distinguish these DEA values from the other commonly used approaches.

2. CCR-efficiency was defined, along with the reference sets for inefficient DMUs.

3. Details of the linear programming solution procedure and the production function correspondence are given in Chapter 3.

2.9   SELECTED BIBLIOGRAPHY

The term 'Decision Making Unit' (DMU) was used for the first time in the CCR model proposed in Charnes, Cooper and Rhodes (1978).[4] The term DEA (Data Envelopment Analysis) was introduced in their report "A Data Envelopment Analysis Approach to Evaluation of the Program Follow Through Experiment in U.S. Public School Education," (1978),[5] Rhodes (1978)[6] and appeared in Charnes, Cooper and Rhodes' subsequent paper (1979).[7] DEA originated from efforts to evaluate results from an early 1970's project called "Program Follow Through"—a huge attempt by the U.S. Office (now Department) of Education to apply principles from the statistical design of experiments to a set of matched schools in a nationwide study. The purpose of the study was to evaluate educational programs designed to aid disadvantaged students in U.S. public schools. The data base was sufficiently large that issues of degrees of freedom, etc., were not a serious problem despite the numerous input and output variables used in the study. Nevertheless, unsatisfactory and even absurd results were secured from all of the statistical-econometric approaches that were tried. While trying to respond to this situation, Rhodes called Cooper's attention to Farrell's seminal article, "The Measurement of Productive Efficiency," in the *Journal of the Royal Statistical Society* (1957). Charnes, Cooper and Rhodes extended Farrell's work and succeeded in establishing DEA as a basis for efficiency analysis. Details of the project are described in Charnes, Cooper and Rhodes (1981).[8] A brief history of DEA can be found in Charnes and Cooper (1985).[9]

2.10   PROBLEM SUPPLEMENT FOR CHAPTER 2

Problem 2.1

In Example 2.2, determine the region of (v_1, v_2) that makes each of DMUs C, D and E efficient, by referring to Table 2.4 and Figure 2.3.

Suggested Answer : For C: line segment $\overline{P_1 P_2}$. This is the line segment stretching from $(v_1/u = 0,\ v_2/u = 1)$ at P_1 to $(v_1/u = .08333,\ v_2/u = .3333)$ at P_2. For D: line segment $\overline{P_2 P_3}$. This is the line segment stretching from $(v_1/u = .08333,\ v_2/u = .3333)$ at P_2 to $(v_1/u = .1667,\ v_2/u = .1667)$ at P_3. For E: line segment $\overline{P_3 P_4}$. This is the line segment stretching from $(v_1/u = .1667,\ v_2/u = .1667)$ at P_3 to $(v_1/u = .5,\ v_2/u = 0)$ at P_4.

Problem 2.2

Use the data of Tables 2.3 and 2.4 to relate Figures 2.2 and 2.3.

Suggested Answer : The relation between Figures 2.2 and 2.3 is an example of what is called "the point-to-hyperplane correspondence" in mathematical programming. This means that the coordinates of the points for the representation in one of the spaces correspond to the coefficients for the hyperplanes in the dual (here=multiplier) space. The spaces in these figures are 2-dimensional, so the hyperplanes take the form of lines. For example, the coordinates for A in Figure 2.2 as obtained from Table 2.3 correspond to the coefficients in $4v_1 + 3v_2 \geq 1u$, the expression associated with DMU A as obtained from Table 2.4. In this single output case, we can use (2.19) to effect a further simplification by moving to homogenous coordinates and then try to minimize t, as given in (2.20). This minimization is undertaken subject to the similar transformation (to homogenous coordinates) for all of the other constraints obtained from Table 2.4. The result, as given in the discussion of (2.21), is $t = 1/0.8571$, so $u = 0.8571$ and $v_1 = v_2 = 0.1429$. Substitution in the expression with which we began then gives

$$1.00 \doteq 4v_1 + 3v_2 > 1u = 0.8571.$$

Hence DMU A is found to be inefficient. It should have produced more output, or used less input (or both).

This evaluation, as noted in Table 2.4, is determined relative to the corresponding expressions obtained for D and E — both of which achieve equality between both sides of their expressions, while using these same (best) weights for A — viz.

$$D:\quad 0.8574\ = 4v_1 + 2v_2\quad \approx 0.8571$$
$$E:\quad 0.8574\ = 2v_1 + 4v_2\quad \approx 0.8571.$$

We now note that the points for A and B in Figure 2.2 lie *above* the efficient frontier whenever the corresponding hyperplanes (here=lines) lie *below* the hyperplanes for D and E in Figure 2.3. The situation for F, which is also inefficient, differs because its hyperplane (=line) intersects the efficiency

frontier at one point, P_1, but otherwise lies everywhere below it. This is the dual reflection of the hyperplane-to-point correspondence while examination of Figure 2.2 shows that F has the same coordinate as C for Input 2 but F's coordinate for Input 1 exceeds the value of the same coordinate for C.

In this chapter we examined the multiplier model in which the maximizing objective tries to move each hyperplane "up," as far as the other constraints allow, with the highest hyperplanes used to determine the boundaries of P, as exhibited in Figure 2.3. In the next chapter we shall examine the minimizing model which moves in the opposite direction in order to move the boundaries of P (the production possibility set) as far "down" as possible. Then we shall use the duality theorem of linear programming to show that the optimal values yield the same production possibility sets so either (or both) problems may be used, as desired and, even better, a suitable reading of the solution yields the solution to both problems when either one is solved.

Problem 2.3

Can you relate the CCR ratio form given by (2.3)-(2.6) in this chapter to the "ratio of ratios" form given by (1.4) in Chapter 1 for engineering efficiency?

Suggested Response : The "ratio of ratios" form given for engineering efficiency in (1.4) is for use with a single input and a single output. This can be interpreted as "virtual inputs" and "virtual outputs" for the values used in the numerators and denominators of (2.3)-(2.6) when specific values are assigned to all of the variables u_r and v_i. To put these ratios in the form of a "ratio of ratios" we will use what is called the TDT (Thompson-Dharmapala-Thrall) measure of efficiency obtained from the following problem,

$$\max_{u,v} \quad \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \bigg/ \frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}} \tag{2.22}$$

$$\text{where} \quad \frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}} = \text{maximum}_{j=1,..,n} \left\{ \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \right\} \tag{2.23}$$

$$u_r, v_i \geq 0 \ \ \forall r, i. \tag{2.24}$$

Here "$\forall r, i$" means "for all r, i." Thus, for each choice of u_r and v_i, the maximum of the ratios in the braces on the right in the last expression is to be used as the denominator ratio in the first expression. Considering all allowable possibilities the problem is to maximize the ratio in (2.22) and "theoretical" as well as "observed" values may be used in these expressions if desired. A detailed discussion of the TDT measure and its properties may be found in W.W. Cooper, R.G. Thompson and R.M. Thrall "Extensions and New Development in DEA," *Annals of Operations Research*, 66, 1996, pp.3-45. Here we only note that no bounds are placed on the admissible values in the ratios of the last expression. The idea is to allow these to be specialized, if one wants to do so, by imposing such bounds on any or all these ratios. If we limit all the ratio values to a bound of unity, as is done in (2.4), we obtain the model given

in (2.3)-(2.6). The value of the objective, (2.3), can then be interpreted as a "ratio of ratios" because a necessary condition for a solution to be optimal in (2.3)-(2.6) is that at least one of the ratios must achieve the maximal allowable value of unity. Thus, the maximization for (2.3)-(2.6) obtains an optimal value in the form of "ratio of ratios" in which the maximal term in the braces gives

$$\frac{\sum_{r=1}^{s} u_r^* y_{rk}}{\sum_{i=1}^{m} v_i^* x_{ik}} = 1.$$

Problem 2.4

Background: "Evaluating Efficiency of Turbofan Jet Engines in Multiple Input-Output Contexts: A Data Envelopment Analysis Approach," by S. Bulla, W.W. Cooper, K.S. Park and D. Wilson(1999)[10] reports results from a study of 29 jet engines produced by different manufacturers. The engineering measure of efficiency is given by $\eta = TV/\dot{Q}$ where T = Thrust of Engine, V = Cruise Velocity and $\dot{Q}$ = Heat Input from Fuel. The DEA evaluations were based on these same two outputs but the single input (in the denominator) was replaced by 3 inputs: (1) Fuel Consumption (2) Weight of Engine and (3) Drag.

Using data from commercially available sources the results obtained from these two approaches to efficiency are portrayed in Figure 2.5 on the left. Coefficients for the resulting regression were found to be statistically significant. The value of R^2, however, suggested that the explanatory power of engineering efficiency was low. More input variables are needed to bring the engineering results into closer correspondence with the DEA efficiency scores.
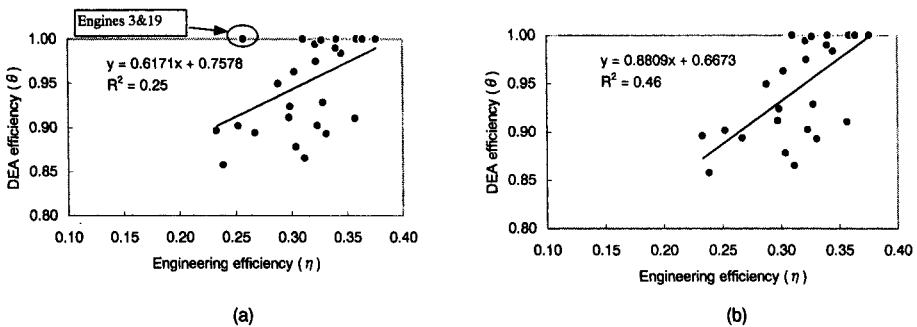
Figure 2.5.   A Scatter Plot and Regression of DEA and Engineering Efficiency Ratings

As shown at the top of Figure (a), Engines 3 and 19 have the same coordinates. Their "outlier" character suggests that their removal would yield a regression with a higher R^2. This is confirmed by the new regression and the new R^2 in Figure (b) where the estimates show statistically significant improve-

ments. Nevertheless, the same conclusion is reached: more inputs are needed for the engineering ratio to provide better "explanations" of the DEA scores, but it is not clear how this can be done.

Assignment : Is the removal of engines 3 and 19 justified or should the potential effects on the efficiency scores of other engines also be considered? Discuss.

Suggested Answer : Statistical independence for all observations is generally assumed in the removal of outliers. However, engines 3 and 19 have DEA ratings of unity and this suggests that they may be members of reference sets used to evaluate *other* engines. To check this possibility we use Figure 2.6 which is known as an "envelopment map."[11]

| Eng.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	N*
1			✓												✓														✓	3
2			✓												✓														✓	3
3			✓																											0
4			✓													✓		✓											✓	4
5											✓				✓															2
6																✓													✓	2
7															✓	✓		✓												3
8															✓	✓		✓											✓	4
9														✓		✓		✓											✓	4
10															✓	✓		✓											✓	4
11											✓																			0
12																✓													✓	2
13														✓																1
14														✓																0
15															✓															0
16																✓														0
17														✓		✓		✓											✓	4
18																		✓												0
19																			✓											0
20															✓	✓														2
21														✓															✓	2
22																✓													✓	2
23			✓											✓				✓											✓	4
24															✓	✓		✓												3
25														✓		✓		✓												3
26															✓	✓		✓												3
27														✓		✓		✓												3
28															✓	✓		✓												3
29																													✓	0
TN*	0	0	4	0	0	0	0	0	0	0	1	0	0	7	10	15	0	12	0	0	0	0	0	0	0	0	0	0	12	61

TN* ≡ Total number of times that engine *j* (= 1, ..., 29 in column) was used to evaluate *other* engines.

N* ≡ Number of times that *other* engines were used to evaluate engine *i* (= 1, ..., 29 in row).

Figure 2.6.   Envelopment Map for 29 Jet Engines

The values at the bottom of each column show the number of times an engine identified at the top of this column entered into a reference set to evaluate *other* engines, and the total for each row shows the number of *other* engines used in an evaluation. Thus, engine 3 was used a total (net) of 4 times in evaluating *other* engines and the row total shows that no *other* engine was involved in the evaluation of engine 3 — a property of efficient engines when the latter are extreme points of the production possibility set. A removal of engine 3 will therefore affect the efficiency ratings of other engines — unless in each case there exists an alternate optimum in which engine 3 does *not* actively enter as a member of the basis (=reference set).

No similar interaction with other engines occurs for engine 19, however, as is clear from the fact that the row and column totals are both zero for this engine. Hence our analysis shows that 19 (but not 3) may be treated as an outlier *without affecting any other observations*. Proceeding in this manner produces Figure 2.7 which, though statistically significant, is closer to Figure 2.5 (a).
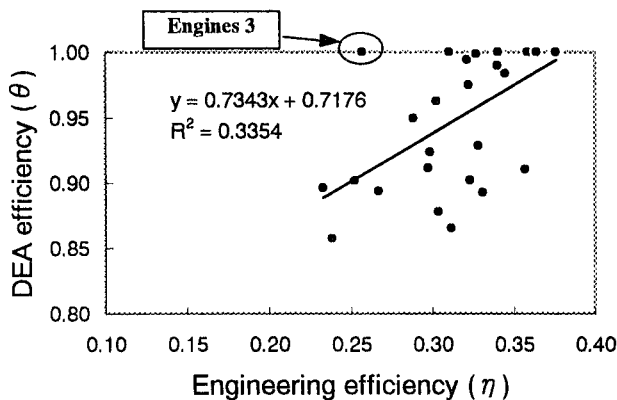
Figure 2.7. A Comparative Portrayal and Regression Analysis for 28 Engines (All Data Except for Engine 19)

Problem 2.5

Try to bring the engineering definition into play in a DEA evaluation.

Suggested Response : As noted in (1.4) the engineering "ratio of ratios" reduces to a comparison of actual to theoretically obtainable output from the amount of input actually used by any engine. This could be brought into play in a DEA evaluation by associating the theoretically obtainable output with the actual input for each engine (=DMU). This would produce n additional observations giving $2n$ DMUs from which evaluations would be made. However, there is no guarantee that a DMU will be evaluated relative to its own input. A DMU

evaluation of any engine (=DMU) could utilize a reference set yielding a *lower* input and a *higher* output. This could be checked and further analyzed, however, because the members of the reference set would be identified as well as the output shortfalls and input excesses for the actual DMU being evaluated.

Notes

1. This condition will be relaxed to allow nonnegative data in Chapter 3. Furthermore, in Chapter 5 (Section 5.2), we will introduce models which can also deal with negative data.

2. *Proof of Theorem 2.2.*   Let θ^*, u_r^*, v_i^* be optimal for (2.3)-(2.6). Now replace the original y_{rj} and x_{ij} by $\rho_r y_{rj}$ and $\delta_i x_{ij}$ for some choices of $\rho_r, \delta_i > 0$. But then choosing $u_r' = u_r^*/\rho_r$ and $v_i' = v_i^*/\delta_i$ we have a solution to the transformed problem with $\theta' = \theta^*$. An optimal value for the transformed problem must therefore have $\theta'^* \geq \theta^*$. Now suppose we could have $\theta'^* > \theta^*$. Then, however, $u_r = u_r'^*\rho_r$ and $v_i = v_i'^*\delta_i$ satisfy the original constraints so the assumption $\theta'^* > \theta^*$ contradicts the optimality assumed for θ^* under these constraints. The only remaining possibility is $\theta'^* = \theta^*$. This proves the invariance claimed for (2.3). Theorem 2.1 demonstrated the equivalence of (LP_o) to (FP_o) and thus the same result must hold and the theorem is therefore proved.   □

3. We use the notations v and u instead of ν and μ in (LP_o).

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11. See Research Report CCS 532(Austin, Texas: University of Texas at Austin, Graduate School of Business, 1986) "Data Envelopment Analysis Approaches to Policy Evaluation and Management of Army Recruiting Activities I: Tradeoff between Joint Service and Army Advertising" by A. Charnes, W.W. Cooper, B. Golany, Major R. Halek, G. Klopp, E. Schmitz and Captain D. Thomas.