10 SUPER-EFFICIENCY MODELS

10.1 INTRODUCTION

In this chapter we introduce a model proposed by Andersen and Petersen (1993),[1] that leads to a concept called "super-efficiency." The efficiency scores from these models are obtained by eliminating the data on the DMUo to be evaluated from the solution set. For the input model this can result in values which are regarded as according DMU_o the status of being "super-efficient." These values are then used to rank the DMUs and thereby eliminate some (but not all) of the ties that occur for efficient DMUs.

Other uses of this approach have also been proposed. Wilson (1993)[2], for example, suggests two uses of these measures in which each DMU_o is ranked according to its "influence" in either (or both) of the following two senses: (1) the number of observations that experience a change in their measure of technical efficiency as a result of these eliminations from the solution set and (2) the magnitude of these changes. Still other interpretations and uses are possible, and we shall add further to such possibilities. See also Ray (2000).[3]

There are troubles with these "super-efficiency" measures, as we shall see. These troubles can range from a lack of units invariance for these measures and extend to non-solution possibilities when convexity constraints are to be dealt with — as in the BCC model. However, the underlying concept is important, so we shall first review this approach to ranking in the form suggested by Andersen and Petersen (1993) and then show how these deficiencies may be eliminated by using other non-radial models that we will suggest.

10.2 RADIAL SUPER-EFFICIENCY MODELS

We start with the model used by Andersen and Petersen which takes the form of a CCR model and thereby avoids the possibility of non-solution that is associated with the convexity constraint in the BCC model. In vector form this model is

$$\text{[Super Radial-I-C]} \quad \theta^* = \min_{\theta, \boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+} \theta - \varepsilon \boldsymbol{e}\boldsymbol{s}^+ \tag{10.1}$$

$$\text{subject to} \quad \theta \boldsymbol{x}_o = \sum_{j=1, \neq o}^{n} \lambda_j \boldsymbol{x}_j + \boldsymbol{s}^-$$

$$\boldsymbol{y}_o = \sum_{j=1, \neq o}^{n} \lambda_j \boldsymbol{y}_j - \boldsymbol{s}^+$$

where all components of the $\boldsymbol{\lambda}$, $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$ are constrained to be non-negative, $\varepsilon > 0$ is the usual non-Archimedean element and $\boldsymbol{e}$ is a row vector with unity for all elements.

We refer to (10.1) as a "Radial Super-Efficiency" model and note that the vectors, $\boldsymbol{x}_o$, $\boldsymbol{y}_o$ are omitted from the expression on the right in the constraints. The data associated with the DMU_o being evaluated on the left is therefore omitted from the production possibility set. However, solutions will always exist so long as all elements are positive in the matrices X, $Y > 0$. (For weaker conditions see Charnes, Cooper and Thrall (1991)[4]).

The above model is a member of the class of Input Oriented-CCR (CCR-I) models. The output oriented version (Radial Super-O-C) has an optimal $\phi^* = 1/\theta^*$ and $\boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*}$ adjusted by division with θ^*, so we confine our discussion to the input oriented version, after which we will turn to other models, such as the BCC class of models, where this reciprocal relation for optimal solutions does not hold.

We illustrate the use of (10.1) with the data in Table 10.1 that is represented geometrically in Figure 10.1. Taken from Andersen and Petersen, these data are modified by adding DMU F, (which lies halfway between B and C), to portray all possibilities.

Table 10.1. Test Data

DMU	A	B	C	D	E	F
Input 1	2.0	2.0	5.0	10.0	10.0	3.5
Input 2	12.0	8.0	5.0	4.0	6.0	6.5
Output 1	1.0	1.0	1.0	1.0	1.0	1.0

Source: Andersen and Petersen (1993)

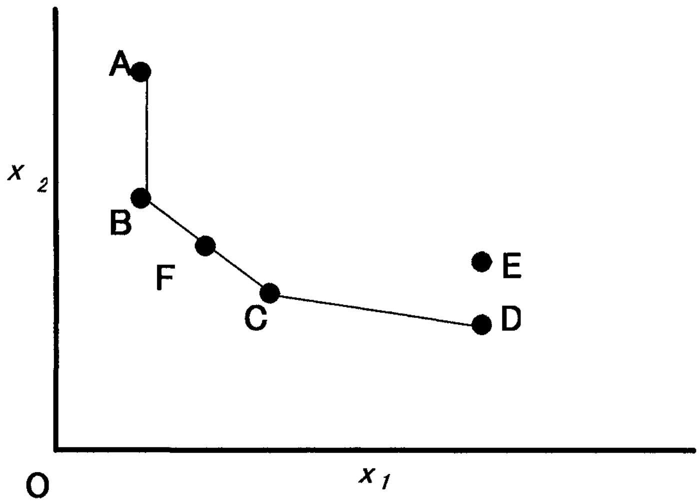

Figure 10.1. The Unit Isoquant Spanned by the Test Data in Table 10.1

Consider, first, the evaluation of DMU A as determined from the following adaptation of (10.1) where, as can be seen, the data for A are represented on the left but not on the right.

$$
\begin{aligned}
&z^* = \min \theta - \varepsilon(s_1^- + s_2^-) - \varepsilon s^+ \\
&\text{subject to} \\
&2\theta = 2\lambda_B + 5\lambda_C + 10\lambda_D + 10\lambda_E + 3.5\lambda_F + s_1^- \\
&12\theta = 8\lambda_B + 5\lambda_C + 4\lambda_D + 6\lambda_E + 6.5\lambda_F + s_2^- \\
&1 = \lambda_B + \lambda_C + \lambda_D + \lambda_E + \lambda_F + s^+ \\
&0 \le \lambda_B, \ldots, \lambda_F, s_1^-, s_2^-, s^+.
\end{aligned}
$$

The solution to this problem is $\theta^* = 1$, $\lambda_B^* = 1$ $s_2^{-*} = 4$. All other variables are zero so $z^* = 1 - 4\varepsilon$. This is the same solution that would be obtained if the data for DMU A were included on the right. The evaluation of an inefficient point, like A, is not affected by this omission from the production possibility set because the efficient points that enter into the evaluation are unaffected by such a removal.

The latter condition is also present in the case of F. That is, the efficient points used in the evaluation of F are not removed from the production possibility set. Thus, the solution $\theta^* = 1$ and $\lambda_B^* = \lambda_C^* = 1/2$ confirms the efficiency of F both before and after such a removal. The important consideration is that the same efficient points that enter into (or can enter into) the evaluation of a DMU_o are present in both cases. This will also be the case for any inefficient point. For example the evaluation of E is unaffected with $\theta^* = 3/4$ and[5] $\lambda_C^* = \lambda_D^* = 1/2$ both before and after removal of the point E from the production possibility set. A point like A, which is on a part of the frontier that is not efficient will have the value $\theta^* = 1$, as is also true for the efficient point F which is on a part of the efficient frontier but is not an extreme point. See Charnes, Cooper and Thrall (1991) for a discussion of all possible classes of points and their properties. Now consider the rankings. The slacks are represented in the objective of (10.1) so we have the ranking: DMU F > DMU A since $1 > 1-4\varepsilon$ while $\theta^* = 3/4$ lies below both values by the definition of $\varepsilon > 0$ so this ranking will be F > A > E.

We next delete C from the production possibility set and obtain the solution $\theta^* = 1.2$ and $\lambda_B^* = \lambda_D^* = 1/2$ so that DMU C ranks as a super-efficient convex combination of B and D. This solution gives the coordinates of C with $x_1 = x_2 = 6$ so the elimination of C results in a replacement that yields the same unit of output with a 1 unit (=20%) increase in each input.

The Andersen-Petersen rankings are exhibited in Table 10.2 with B ranked first followed by D. A possible interpretation of what kind of managerial decision might be involved is to assume that the rankings are directed to studying the consequences of eliminating some DMUs. For this purpose, D might be preferred over B because of the large (7.5 units) reduction in input1 — a reduction that is given the negligible weight of $\varepsilon > 0$ in the rankings associated with the Andersen-Petersen measure.

Table 10.2. Andersen-Petersen Ranking*

Rank Order	B >	D >	C >	F >	A >	E
z^*	1.32	$1.25 - 7.5\varepsilon$	1.2	1	$1 - 4\varepsilon$	0.75
Reference Set	$\lambda_A = 0.79$ $\lambda_C = 0.21$	$\lambda_C = 1$	$\lambda_A = 0.5$ $\lambda_D = 0.5$	$\lambda_B = 0.5$ $\lambda_C = 0.5$	$\lambda_B = 1$	$\lambda_C = 0.5$ $\lambda_D = 0.5$
s	0	$s_1^- = 7.5$	0	0	$s_2^- = 4$	0

* Source: Andersen and Petersen (1993)

For purposes like the possible removal of DMUs the Andersen-Petersen measure can be regarded as deficient in its treatment of the nonzero slacks. It is also deficient because its treatment of the slacks does not yield a measure that is "units invariant." We therefore turn to SBM (the slacks based measure) to eliminate these deficiencies. We will do this in a way that eliminates the non-solution possibilities that are present when the convexity condition $\sum_{j=1}^{n} \lambda_j = 1$ is adjoined to the models we will suggest.

10.3 NON-RADIAL SUPER-EFFICIENCY MODELS

Here, we discuss the super-efficiency issues under the assumption that the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-efficient, i.e., it is strongly efficient. (See Definition 4.6 in Chapter 4.)

Let us define a production possibility set $P \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$ spanned by (X, Y) excluding $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, i.e.

$$P \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o) = \left\{ (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) |\ \bar{\boldsymbol{x}} \geq \sum_{j=1, \neq o}^{n} \lambda_j \boldsymbol{x}_j,\ \ \bar{\boldsymbol{y}} \leq \sum_{j=1, \neq o}^{n} \lambda_j \boldsymbol{y}_j,\ \ \bar{\boldsymbol{y}} \geq \mathbf{0},\ \ \boldsymbol{\lambda} \geq \mathbf{0} \right\}. \tag{10.2}$$

Further, we define a subset $\bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$ of $P \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$ as

$$\bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o) = P \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o) \bigcap \{\bar{\boldsymbol{x}} \geq \boldsymbol{x}_o \text{ and } \bar{\boldsymbol{y}} \leq \boldsymbol{y}_o\}, \tag{10.3}$$

where $P \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$ means that point $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is excluded. By assumption $X > 0$ and $Y > 0$, $\bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$ is not empty.

As a weighted l_1 distance from $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ and $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$, we employ the index δ as defined by

$$\delta = \frac{\frac{1}{m} \sum_{i=1}^{m} \bar{x}_i / x_{io}}{\frac{1}{s} \sum_{r=1}^{s} \bar{y}_r / y_{ro}}. \tag{10.4}$$

From (10.3), this distance is not less than 1 and attains 1 if and only if $(\boldsymbol{x}_o, \boldsymbol{y}_o) \in \bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$, i.e. exclusion of the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ has no effect on the original production possibility set P.

We can interpret this index as follows. The numerator is a weighted l_1 distance from $\boldsymbol{x}_o$ to $\bar{\boldsymbol{x}}(\geq \boldsymbol{x}_o)$, and hence it expresses an average expansion rate of $\boldsymbol{x}_o$ to $\bar{\boldsymbol{x}}$ of the point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$. The denominator is a weighted l_1 distance from $\boldsymbol{y}_o$ to $\bar{\boldsymbol{y}}(\leq \boldsymbol{y}_o)$, and hence it is an average reduction rate of $\boldsymbol{y}_o$ to $\bar{\boldsymbol{y}}$ of $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \bar{P} \setminus (\boldsymbol{x}_o, \boldsymbol{y}_o)$. The smaller the denominator is, the farther $\boldsymbol{y}_o$ is positioned relative to $\bar{\boldsymbol{y}}$. Its inverse can be interpreted as an index of the distance from $\boldsymbol{y}_o$ to $\bar{\boldsymbol{y}}$. Therefore, δ is a product of two indices: one, the distance in the input space, and the other in the output space. Both indices are dimensionless.

10.3.1 Definition of Non-radial Super-efficiency Measure

Based on the above observations, we define the super-efficiency of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ as the optimal objective function value δ^* from the following program:

$$
\begin{aligned}
\text{[SuperSBM-C]} \quad & \delta^* = \min_{\bar{\boldsymbol{x}},\bar{\boldsymbol{y}},\boldsymbol{\lambda}} \frac{\frac{1}{m}\sum_{i=1}^{m} \bar{x}_i / x_{io}}{\frac{1}{s}\sum_{r=1}^{s} \bar{y}_r / y_{ro}} \qquad (10.5)\\
\text{subject to} \quad & \bar{\boldsymbol{x}} \geq \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{x}_j \\
& \bar{\boldsymbol{y}} \leq \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{y}_j \\
& \bar{\boldsymbol{x}} \geq \boldsymbol{x}_o \text{ and } \bar{\boldsymbol{y}} \leq \boldsymbol{y}_o \\
& \bar{\boldsymbol{y}} \geq \mathbf{0}, \quad \boldsymbol{\lambda} \geq \mathbf{0}.
\end{aligned}
$$

Let us introduce $\boldsymbol{\phi} \in R^m$ and $\boldsymbol{\psi} \in R^s$ such that

$$
\bar{x}_i = x_{io}(1+\phi_i) \ (i = 1, \ldots, m) \text{ and } \bar{y}_r = y_{ro}(1+\psi_r) \ (r = 1, \ldots, s). \qquad (10.6)
$$

Then, this program can be equivalently stated in terms of $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ and $\boldsymbol{\lambda}$ as follows:

$$
\begin{aligned}
\text{[SuperSBM-C}'\text{]} \quad & \delta^* = \min_{\boldsymbol{\phi},\boldsymbol{\psi},\boldsymbol{\lambda}} \frac{1 + \frac{1}{m}\sum_{i=1}^{m} \phi_i}{1 - \frac{1}{s}\sum_{r=1}^{s} \psi_r} \qquad (10.7)\\
\text{subject to} \quad & \sum_{j=1,\neq o}^{n} x_{ij}\lambda_j - x_{io}\phi_i \leq x_{io} \ (i = 1, \ldots, m) \\
& \sum_{j=1,\neq o}^{n} y_{rj}\lambda_j + y_{ro}\psi_r \geq y_{ro} \ (r = 1, \ldots, s) \\
& \phi_i \geq 0 \ (\forall i), \ \psi_r \geq 0 \ (\forall r), \ \lambda_j \geq 0 \ (\forall j)
\end{aligned}
$$

We have the following two propositions (Tone (2002)[6].

Proposition 10.1 *The super-efficiency score δ^* is units invariant, i.e. it is independent of the units in which the inputs and outputs are measured provided these units are the same for every DMU.*

Proof : This proposition holds, since both the objective function and constraints are units invariant. □

Proposition 10.2 *Let $(\alpha \boldsymbol{x}_o, \beta \boldsymbol{y}_o)$ with $\alpha \leq 1$ and $\beta \geq 1$ be a DMU with reduced inputs and enlarged outputs than $(\boldsymbol{x}_o, \boldsymbol{y}_o)$. Then, the super-efficiency score of $(\alpha \boldsymbol{x}_o, \beta \boldsymbol{y}_o)$ is not less than that of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$.*

Proof: The super-efficiency score $(\hat{\delta}^*)$ of $(\alpha \boldsymbol{x}_o, \beta \boldsymbol{y}_o)$ is evaluated by solving the following program:

$$
\text{[SuperSBM-C-2]} \quad \hat{\delta}^* = \min \frac{\frac{1}{m}\sum_{i=1}^{m} \hat{x}_i / (\alpha x_{io})}{\frac{1}{s}\sum_{r=1}^{s} \hat{y}_r / (\beta y_{ro})}
$$

$$= \min \frac{\beta}{\alpha} \frac{\frac{1}{m}\sum_{i=1}^{m} \hat{x}_i / x_{io}}{\frac{1}{s}\sum_{r=1}^{s} \hat{y}_r / y_{ro}} \tag{10.8}$$

$$\begin{aligned} \text{subject to} \quad & \hat{\boldsymbol{x}} \geq \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{x}_j \\ & \hat{\boldsymbol{y}} \leq \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{y}_j \\ & \hat{\boldsymbol{x}} \geq \alpha \boldsymbol{x}_o \ \text{ and } \mathbf{0} \leq \hat{\boldsymbol{y}} \leq \beta \boldsymbol{y}_o \\ & \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned}$$

It can be observed that, for any feasible solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ for [SuperSBM-C-2], $(\hat{\boldsymbol{x}}/\alpha, \hat{\boldsymbol{y}}/\beta)$ is feasible for [SuperSBM-C]. Hence it holds

$$\delta^* \leq \frac{\frac{1}{m}\sum_{i=1}^{m} (\hat{x}_i/\alpha)/x_{io}}{\frac{1}{s}\sum_{r=1}^{s} (\hat{y}_r/\beta)/y_{ro}} = \frac{\beta}{\alpha} \frac{\frac{1}{m}\sum_{i=1}^{m} \hat{x}_i/x_{io}}{\frac{1}{s}\sum_{r=1}^{s} \hat{y}_r/y_{ro}}. \tag{10.9}$$

Comparing (10.8) with (10.9) we see that:

$$\delta^* \leq \hat{\delta}^*.$$

Thus, the super-efficiency score of $(\alpha \boldsymbol{x}_o, \beta \boldsymbol{y}_o)$ ($\alpha \leq 1$ and $\beta \geq 1$) is not less than that of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$. □

10.3.2 Solving Super-efficiency

The fractional program [SuperSBM-C′] can be transformed into a linear programming problem using the Charnes-Cooper transformation as:

$$\text{[LP]} \qquad \tau^* = \min t + \frac{1}{m}\sum_{i=1}^{m} \Phi_i \tag{10.10}$$

$$\begin{aligned} \text{subject to} \quad & t - \frac{1}{s}\sum_{r=1}^{s} \Psi = 1 \\ & \sum_{j=1,\neq o}^{n} x_{ij}\Lambda_j - x_{io}\Phi_i - x_{io}t \leq 0 \ (i = 1, \ldots, m) \\ & \sum_{j=1,\neq o}^{n} y_{rj}\Lambda_j + y_{ro}\Psi_r - y_{ro}t \geq 0 \ (r = 1, \ldots, s) \\ & \Phi_i \geq 0 \ (\forall i), \ \Psi_r \geq 0 \ (\forall r), \ \Lambda_j \geq 0 \ (\forall j) \end{aligned}$$

Let an optimal solution of [LP] be $(\tau^*, \boldsymbol{\Phi}^*, \boldsymbol{\Psi}^*, \boldsymbol{\Lambda}^*, t^*)$. Then we have an optimal solution of [SuperSBM-C′] expressed by

$$\delta^* = \tau^*, \ \boldsymbol{\lambda}^* = \boldsymbol{\Lambda}^*/t^*, \ \boldsymbol{\phi}^* = \boldsymbol{\Phi}^*/t^*, \ \boldsymbol{\psi}^* = \boldsymbol{\Psi}^*/t^*. \tag{10.11}$$

Furthermore, the optimal solution of [Super-SBM-C] is given by:

$$\bar{x}^*_{io} = x_{io}(1 + \phi^*_i) \ (i = 1, \ldots, m) \text{ and } \bar{y}^*_{ro} = y_{ro}(1 - \psi^*_r) \ (r = 1, \ldots, s). \tag{10.12}$$

10.3.3 Input/Output-Oriented Super-efficiency

In order to adapt the non-radial super-efficiency model to input (output) orientation, we can modify the preceding program as follows. For input orientation, we deal with the weighted l_1-distance only in the input space. Thus, the program turns out to be:

$$\text{[SuperSBM-I-C]} \quad \delta_I^* = \min_{\phi,\boldsymbol{\lambda}} 1 + \frac{1}{m}\sum_{i=1}^{m}\phi_i \tag{10.13}$$

$$\begin{aligned} \text{subject to} \quad & \sum_{j=1,\neq o}^{n} x_{ij}\lambda_j - x_{io}\phi_i \leq x_{io} \ (i=1,\ldots,m) \\ & \sum_{j=1,\neq o}^{n} y_{rj}\lambda_j \geq y_{ro} \ (r=1,\ldots,s) \\ & \phi_i \geq 0 \ (\forall i), \ \lambda_j \geq 0 \ (\forall j) \end{aligned}$$

In a similar way we can develop the output-oriented super-efficiency model as follows:

$$\text{[SuperSBM-O-C]} \quad \delta_O^* = \min_{\psi,\boldsymbol{\lambda}} \frac{1}{1-\frac{1}{s}\sum_{r=1}^{s}\psi_r} \tag{10.14}$$

$$\begin{aligned} \text{subject to} \quad & \sum_{j=1,\neq o}^{n} x_{ij}\lambda_j \leq x_{io} \ (i=1,\ldots,m) \\ & \sum_{j=1,\neq o}^{n} y_{rj}\lambda_j + y_{ro}\psi_r \geq y_{ro} \ (r=1,\ldots,s) \\ & \psi_r \geq 0 \ (\forall r), \ \lambda_j \geq 0 \ (\forall j) \end{aligned}$$

Since the above two models have the same restricted feasible region as [SuperSBM-C], we have:

Proposition 10.3 *$\delta_I^* \geq \delta^*$ and $\delta_O^* \geq \delta^*$, where δ^* is defined in (10.4).*

10.3.4 An Example of Non-radial Super-efficiency

Using [SuperSBM-I-C] model in (10.13), we solved the same data set described in Table 10.1 and obtained the results exhibited in Table 10.3. Since DMUs A and E are enveloped respectively by B and C, their SBM scores are less than unity and SBM-inefficient. Compared with the radial model, DMU A dropped its efficiency from $1-4\varepsilon$ to 0.833. This is caused by the slack $s_2^- = 4$. We need to solve [SuperSBM-I-C] for DMUs B, C, D and F which have SBM score unity. DMU B has the optimal solution $\lambda_A^* = 1$, $s_2^* = x_{2A}\phi_2^* = 4$, ($\phi_2^* = 4/8 = 0.5$) with all other variables zero, and hence its super-efficiency score is $1 + 0.5/2 = 1.25$. Similarly, DMUs C, D and F have super-efficiency scores 1.092, 1.125 and 1, respectively. In this example, the ranking is the same as the Andersen-Petersen model, although they are not always same (see Tone (2002)).

Table 10.3. Non-radial Super-efficiency

DMU	A	B	C	D	E	F
Super-eff.	0.833	1.25	1.092	1.125	0.667	1
Rank	5	1	3	2	6	4
Reference set	$\lambda_B = 1$	$\lambda_A = 1$	$\lambda_D = 0.23$ $\lambda_F = 0.77$	$\lambda_C = 1$	$\lambda_C = 1$	$\lambda_B = 0.5$ $\lambda_C = 0.5$
Slacks						
$s_1^- = x_1\phi_1$	0	0	0	0	5	0
$s_2^- = x_2\phi_2$	4	4	0.923	1	1	0

10.4 EXTENSIONS TO VARIABLE RETURNS-TO-SCALE

In this section, we extend our super-efficiency models to the variable returns-to-scale models and discuss the infeasible LP issues that then arise.

We extend our analysis to the variable returns-to-scale case by adjoining the following convexity constraints to the models:

$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geq 0, \ \forall j. \tag{10.15}$$

We observe two approaches as follows:

10.4.1 Radial Super-efficiency Case

We have two models [SuperRadial-I-V] (Input-oriented Variable RTS) and [SuperRadial-O-V] (Output-oriented Variable RTS) with models represented respectively as follows.

$$[\text{SuperRadial-I-V}] \qquad \theta^* = \min \theta \tag{10.16}$$

$$\text{subject to} \qquad \theta x_{io} \geq \sum_{j=1,\neq o}^{n} \lambda_j x_{ij} \ (i = 1, \ldots, m) \tag{10.17}$$

$$y_{ro} \leq \sum_{j=1,\neq o}^{n} \lambda_j y_{rj} \ (r = 1, \ldots, s) \tag{10.18}$$

$$\sum_{j=1,\neq o}^{n} \lambda_j = 1 \tag{10.19}$$

$$\lambda_j \geq 0 \ (\forall j).$$

$$
\begin{aligned}
&[\text{SuperRadial-O-V}] \quad && 1/\eta^* = \min 1/\eta && (10.20)\\
&\text{subject to} && x_{io} \geq \sum_{j=1,\neq o}^{n} \lambda_j x_{ij} \ (i = 1, \ldots, m) && (10.21)\\
& && \eta y_{ro} \leq \sum_{j=1,\neq o}^{n} \lambda_j y_{rj} \ (r = 1, \ldots, s) && (10.22)\\
& && \sum_{j=1,\neq o}^{n} \lambda_j = 1 && (10.23)\\
& && \lambda_j \geq 0 \ (\forall j).
\end{aligned}
$$

The above two programs may suffer from infeasibility under the following conditions. Suppose, for example, y_{1o} is larger than the other y_{1j} $(j \neq o)$, i.e.,

$$y_{1o} > \max_{j=1,\neq o}^{n} \{y_{1j}\}.$$

Then, the constraint (10.18) in [SuperRadial-I-V] is infeasible for $r = 1$ by dint of the constraint (10.19).

Likewise, suppose, for example, x_{1o} is smaller than the other x_{1j} $(j \neq o)$, i.e.,

$$x_{1o} < \min_{j=1,\neq o}^{n} \{x_{1j}\}.$$

Then, the constraint (10.21) in [SuperRadial-O-V] is infeasible for $i = 1$ by dint of the constraint (10.23).

Thus, we have:

Proposition 10.4 [SuperRadial-I-V] *has no feasible solution if there exists* r *such that* $y_{ro} > \max_{j\neq o}\{y_{rj}\}$, *and* [SuperRadial-O-V] *has no feasible solution if there exists* i *such that* $x_{io} < \min_{j\neq o}\{x_{ij}\}$.

We notice that Proposition 10.4 above is not a necessary condition for infeasibility, i.e., infeasibility may occur in other cases.

10.4.2 Non-radial Super-efficiency Case

The non-radial super-efficiency under variable returns-to-scale is evaluated by solving the following program:

$$
\begin{aligned}
&[\text{SuperSBM-V}] \quad && \delta^* = \min \delta = \frac{\frac{1}{m}\sum_{i=1}^{m} \bar{x}_i / x_{io}}{\frac{1}{s}\sum_{r=1}^{s} \bar{y}_r / y_{ro}} && (10.24)\\
&\text{subject to} && \bar{\boldsymbol{x}} \geq \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{x}_j\\
& && \bar{\boldsymbol{y}} \leq \sum_{j=1,\neq o}^{n} \lambda_j \boldsymbol{y}_j
\end{aligned}
$$

$$\bar{\boldsymbol{x}} \geq \boldsymbol{x}_o \text{ and } \bar{\boldsymbol{y}} \leq \boldsymbol{y}_o$$
$$\sum_{j=1,\neq o}^{n} \lambda_j = 1$$
$$\bar{\boldsymbol{y}} \geq \mathbf{0}, \quad \boldsymbol{\lambda} \geq \mathbf{0}.$$

Under the assumptions $X > O$ and $Y > O$, we demonstrate that the [SuperSBM-V] is always feasible and has a finite optimum in contrast to the radial super-efficiency. This can be shown as follows.

We choose a DMU $j(\neq o)$ with $(\boldsymbol{x}_j, \boldsymbol{y}_j)$ and set $\bar{\lambda}_j = 1$ and $\bar{\lambda}_k = 0$ $(k \neq j)$. Using this DMU $(\boldsymbol{x}_j, \boldsymbol{y}_j)$ we define:

$$\widetilde{x}_i = \max\{x_{io}, x_{ij}\} \ (i = 1, \ldots, m) \tag{10.25}$$
$$\widetilde{y}_r = \min\{y_{ro}, y_{rj}\} \ (r = 1, \ldots, s). \tag{10.26}$$

Thus, the set $(\bar{\boldsymbol{x}} = \widetilde{\boldsymbol{x}}, \bar{\boldsymbol{y}} = \widetilde{\boldsymbol{y}}, \boldsymbol{\lambda} = \bar{\boldsymbol{\lambda}})$ is feasible for the [SuperSBM-V]. Hence, [SuperSBM-V] is always feasible with a finite optimum. Thus, we have:

Theorem 10.1 (Tone (2002)) *The non-radial super-efficiency model under the variable returns-to-scale environment,*[SuperSBM-V]*, is always feasible and has a finite optimum.*

We can define [SuperSBM-I-V] (Input-oriented Variable RTS) and [SuperSBM-O-V] (Output-oriented Variable RTS) models similar to [SuperSBM-I-C] and [SuperSBM-O-C]. We notice that [SuperSBM-I-V] and [SuperSBM-O-V] models confront the same infeasible LP issues as the [SuperRadial-I-V] and [Super Radial-O-V]. See Problem 10.1 for comparisons of super-efficiency models.

10.5 SUMMARY OF CHAPTER 10

This chapter introduced the concept of super-efficiency and presented two types of approach for measuring super-efficiency: radial and non-radial. Super-efficiency measures are widely utilized in DEA applications for many purposes, e.g., ranking efficient DMUs, evaluating the Malmquist productivity index and comparing performances of two groups (the bilateral comparisons model in Chapter 7).

10.6 NOTES AND SELECTED BIBLIOGRAPHY

Andersen and Petersen (1993) introduced the first super-efficiency model for radial models. Tone (2002) introduced non-radial super-efficiency models using the SBM.

10.7 RELATED DEA-SOLVER MODELS FOR CHAPTER 10

Super-efficiency codes have the same data format with the CCR model.

Super-Radial-I(O)-C(V) (Input(Output)-oriented Radial Super-efficiency model).

This code solves the oriented radial super-efficiency model under the constant (variable) returns-to-scale assumption. If the corresponding LP for a DMU is infeasible, we return a score 1 to the DMU.

Super-SBM-I(O)-C(V or GRS) (Input(Output)-oriented Non-radial Super-efficiency model)

This code solves the oriented non-radial super-efficiency model under the constant (variable or general) returns-to-scale assumption. If the corresponding LP for a DMU is infeasible, we return a score 1 to the DMU.

Super-SBM-C(V or GRS) (Non-radial and Non-oriented Super-efficiency model)

This code solves the non-oriented non-radial super-efficiency model under the constant (variable or general) returns-to-scale assumption.

10.8 PROBLEM SUPPLEMENT FOR CHAPTER 10

Problem 10.1

Table 10.4 displays data for 6 DMUs (A, B, C, D, E, F) with two inputs (x_1, x_2) and two outputs (y_1, y_2). Obtain and compare the super-efficiency scores using the attached DEA-Solver.

Table 10.4. Data for Super-efficiency

DMU	A	B	C	D	E	F
Input 1 (x_1)	2	2	5	10	10	3.5
Input 2 (x_2)	12	8	5	4	6	6.5
Output 1 (y_1)	4	3	2	2	1	1
Output 2 (y_2)	1	1	1	1	1	1

Suggested Answer : Tables 10.5 and 10.6 exhibit super-efficiency scores respectively under variable and constant returns-to-scale conditions. In the tables I (O) indicates the Input (Output) orientation and V (C) denotes the Variable (Constant) RTS. "NA" (not available) represents an occurrence of infeasible LP solution.

It is observed that, by dint of Proposition 10.4, Super-BCC-I and Super-SBM-I-V scores of DMU A are not available since it has $y_{1A} = 4$ which is strictly larger than other y_{1j} $(j \neq A)$, and Super-BCC-O and Super-SBM-O-V scores of DMU D are not available since it has $x_{2D} = 4$ which is strictly smaller than other x_{2j} $(j \neq D)$. The infeasibility of DMUs B and C for Super-BCC-O and Super-SBM-O-V is caused simply by non-existence of feasible solution for the corresponding linear programs. However, Super-SBM-V always has finite solutions as claimed by Theorem 10.1.

Under the constant RTS assumption, all DMUs have a finite score which resulted from exclusion of the convexity constraint on λ_j. The differences in scores between radial and non-radial models are due to non-zero slacks.

Table 10.5. Super-efficiency Scores under Variable RTS

DMU	A	B	C	D	E	F
Super-BCC-I	NA	1.26	1.17	1.25	0.75	1
Super-BCC-O	1.33	NA	NA	NA	1	1
Super-SBM-I-V	NA	1.25	1.12	1.13	0.66	1
Super-SBM-O-V	1.14	NA	NA	NA	0.57	0.57
Super-SBM-V	1.14	1.25	1.12	1.13	0.42	0.57

Table 10.6. Super-efficiency Scores under Constant RTS

DMU	A	B	C	D	E	F
Super-CCR-I	1.33	1.26	1.17	1.25	0.75	1
Super-CCC-O	1.33	1.26	1.17	1.25	0.75	1
Super-SBM-I-C	1.17	1.25	1.12	1.13	0.67	1
Super-SBM-O-C	1.14	1.2	1.09	1.25	0.5	0.57
Super-SBM-C	1.14	1.16	1.09	1.13	0.42	0.57

Notes

1. P. Andersen and N.C. Petersen (1993), "A Procedure for Ranking Efficient Units in Data Envelopment Analysis," *Management Science* 39, pp.1261-1264.

2. P. Wilson (1993), "Detecting Influential Observations in Data Envelopment Analysis," *Journal of Productivity Analysis* 6, pp. 27-46.

3. S. Ray (2004), *Data Envelopment Analysis* (Cambridge: Cambridge University Press).

4. A. Charnes, W.W. Cooper and R.M. Thrall (1991), "A Structure for Classifying and Characterizing Inefficiency in DEA," *Journal of Productivity Analysis* 2, pp. 197-237.

5. Andersen and Petersen report this as $\theta^* = 0.8333$, but their reported value is seen to be incorrect since $0.8333>0.75$.

6. K. Tone (2002), "A Slacks-based Measure of Super-efficiency in Data Envelopment Analysis," *European Journal of Operational Research* 143, pp.32-41