9

Evolution Equations with Boundary Layers

Evolution equations form a rich problem field; the purpose of this chapter is to introduce the reader to some prototype problems illustrating the analysis of expansions for parabolic and hyperbolic equations.

The analysis in Sections 9.5 and 9.2 for wave equations is well-known; see Kevorkian and Cole (1996) or De Jager and Jiang Furu (1996). We include these problems because they display fundamental aspects.

9.1 Slow Diffusion with Heat Production

In this section, we consider a case where the ideas we have met thus far can be extended directly to a spatial problem with time evolution. Slow spatial diffusion takes place in various models, for instance in material where heat conduction is not very good; see the model in this section. Other examples are the slow spreading of pollution in estuaries or seas and, in mathematical biology, the slow spreading of a population in a domain.

First, we consider a one-dimensional bar described by the spatial variable x and length of the bar L; the temperature of the bar is $T(x, t)$, and we assume slow heat diffusion. The endpoints of the bar are kept at a constant temperature. Heat is produced in the bar and is also exchanged with its surroundings, so we have that $T = T(x, t)$ is governed by the equation and conditions

$$\frac{\partial T}{\partial t} = \varepsilon \frac{\partial^2 T}{\partial x^2} - \gamma(x)[T - s(x)] + g(x)$$

with boundary conditions $T(0, t) = t_0, T(L, t) = t_1$, and initial temperature distribution $T(x, 0) = \psi(x)$. The temperature of the neighbourhood of the bar is $s(x); \gamma(x)$ is the exchange coefficient of heat and $\gamma(x) \geq d$ with d a positive constant so the bar is nowhere isolated.

In the analysis we shall follow Van Harten (1979). From Chapter 5, we know how to determine an asymptotic expansion for the stationary (time-independent) problem. Starting with the approximation of the stationary problem, we will analyse the time-dependent problem. One of the questions is: does the time-dependent expansion converge to the stationary one?

Inspired by our experience in Chapter 5, we expect boundary layers near $x = 0$ and $x = L$ with local variables

$$\xi = \frac{x}{\sqrt{\varepsilon}}, \eta = \frac{L - x}{\sqrt{\varepsilon}}.$$

The approximation of the stationary solution will be of the form

$$T(x) = V_0(x) + \varepsilon V_1(x) + \varepsilon^2 \cdots + Y_0(\xi) + \varepsilon^{1/2} Y_1(\xi) + \bar{Y}_0(\eta) + \varepsilon^{1/2} \bar{Y}_1(\eta) + \varepsilon \cdots ,$$

where $V_0 + \varepsilon V_1$ is the first part of the regular expansion and the boundary layer approximations satisfy

$$\frac{d^2 Y_0}{d\xi^2} - \gamma(0) Y_0 = 0, Y_0(0) = t_0 - V_0(0), \lim_{\xi \to \infty} Y_0(\xi) = 0.$$

$\bar{Y}_0(\eta)$ will satisfy

$$\frac{d^2 \bar{Y}_0}{d\eta^2} - \gamma(L) \bar{Y}_0 = 0, \bar{Y}_0(L) = t_L - V_0(L), \lim_{\eta \to \infty} \bar{Y}_0(\eta) = 0.$$

We find

$$Y_o(\xi) = (t_0 - V_0(0)) e^{-\sqrt{\gamma(0)}\xi}, \bar{Y}_0(\eta) = (t_L - V_0(L)) e^{-\sqrt{\gamma(L)}\eta}.$$

9.1.1 The Time-Dependent Problem

Substituting a regular expansion of the form $T(x,t) = U_0(x,t) + \varepsilon U_1(x,t) \cdots$, we find in lowest order

$$\frac{\partial U_0}{\partial t} = -\gamma(x) U_0 + \gamma(x) s(x) + g(x), U_0(x, 0) = \psi(x),$$

with solution

$$U_0(x,t) = V_0(x) + (\psi(x) - V_0(x)) e^{-\gamma(x) t}.$$

This first-order regular expansion satisfies the initial condition and, as $\gamma(x) > 0$, the regular part of the time-dependent expansion tends to the regular part of the stationary solution as t tends to infinity.

This suggests proposing for the full expansion

$$T(x,t) = U_0(x,t) + \varepsilon \cdots + X_0(\xi, t) + \bar{X}_0(\eta, t) + \sqrt{\varepsilon} \cdots .$$

Substituting this expansion in the equation produces

$$\frac{\partial X_0}{\partial t} = \frac{\partial^2 X_0}{\partial \xi^2} - \gamma(0) X_0$$

with boundary condition, matching condition, and initial condition

$$X(0,t) = t_0 - U_0(0,t), \lim_{\xi\to\infty} X_0(\xi,t) = 0, X_0(\xi,0) = 0.$$

This problem is solved by putting $X_0 = Z_0 \exp(-\gamma(0)t)$, which yields

$$\frac{\partial Z_0}{\partial t} = \frac{\partial^2 Z_0}{\partial \xi^2}.$$

Applying Duhamel's principle (see, for instance, Strauss, 1992), we find

$$X_0(\xi,t) = \frac{2}{\sqrt{\pi}} e^{-\gamma(0)t} \int_{\frac{\xi}{2\sqrt{t}}}^{\infty} \phi\left(t - \frac{\xi^2}{4\tau^2}\right) e^{-\tau^2} d\tau$$

with $\phi(z) = (t_0 - U(0,z))\exp(\gamma(0)z)$.

In the same way, we find for $\bar{X}_0(\eta,t)$ the problem

$$\frac{\partial \bar{X}_0}{\partial t} = \frac{\partial^2 X_0}{\partial \eta^2} - \gamma(L)\bar{X}_0$$

with boundary condition, matching condition, and initial condition

$$\bar{X}(0,t) = t_L - U_0(L,t), \lim_{\eta\to\infty} \bar{X}_0(\eta,t) = 0, \bar{X}_0(\eta,0) = 0.$$

The solution is obtained as before and becomes

$$\bar{X}_0(\eta,t) = \frac{2}{\sqrt{\pi}} e^{-\gamma(L)t} \int_{\frac{\eta}{2\sqrt{t}}}^{\infty} \bar{\phi}\left(t - \frac{\eta^2}{4\tau^2}\right) e^{-\tau^2} d\tau$$

with $\bar{\phi}(z) = (t_L - U_0(L,z))\exp(\gamma(L)z)$.

9.2 Slow Diffusion on a Semi-infinite Domain

A different problem arises on considering for $t \geq 0$ the semi-infinite domain $x \geq 0$ for the equation

$$\frac{\partial \phi}{\partial t} = \varepsilon \frac{\partial^2 \phi}{\partial x^2} - p(t)\frac{\partial \phi}{\partial x}$$

with initial condition $\phi(x,0) = f(x)$ and boundary condition $\phi(0,t) = g(t)$ (boundary input). The functions $p(t), f(x), g(t)$ are assumed to be sufficiently smooth, and $p(t)$ does not change sign. Moreover, we assume continuity at $(0,0)$ and, for physical reasons, decay of the boundary input to zero:

$$f(0) = g(0), \quad \lim_{t\to\infty} = 0.$$

We start again with a regular expansion of the form

$$\phi(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 \cdots$$

to find for the first two terms after substitution

$$\begin{aligned}\frac{\partial u_0}{\partial t} + p(t)\frac{\partial u_0}{\partial x} &= 0,\\ \frac{\partial u_1}{\partial t} + p(t)\frac{\partial u_1}{\partial x} &= \frac{\partial^2 u_0}{\partial x^2}.\end{aligned}$$

Introducing $P(t) = \int_0^t p(s)ds$, the equation for u_0 has the characteristic $x - P(t)$, which implies that any differentiable function of $x - P(t)$ solves the equation for u_0. (The reader who is not familiar with the characteristics of first-order partial differential equations can verify this by substitution.) At this stage, the easiest way is to satisfy the initial condition by choosing

$$u_0(x,t) = f(x - P(t)).$$

A consequence is that for u_1 we have to add the initial condition $u_1(x,0) = 0$. Solving the equation

$$\frac{\partial u_1}{\partial t} + p(t)\frac{\partial u_1}{\partial x} = \frac{\partial^2}{\partial x^2} f(x - P(t))$$

with the initial condition, we find

$$u_1(x,t) = \int_0^t \frac{\partial^2}{\partial x^2}(x + P(s) - 2P(t))ds.$$

This regular expansion does not satisfy the boundary condition at $x = 0$, so we expect the presence of a spatial boundary layer there. Introducing the boundary layer variable

$$\xi = \frac{x}{\varepsilon^\nu},$$

we expect for the solution $\phi(x,t)$ an expansion of the form

$$\phi(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 \cdots + \psi(\xi,t)$$

with initial and boundary conditions

$$\psi(\xi,0) = 0, \psi(0,t) = g(t) - u_0(0,t) - \varepsilon u_1(0,t) - \varepsilon^2 \cdots$$

and matching condition

$$\lim_{\xi\to\infty} \psi(\xi,t) = 0.$$

Introducing the boundary layer variable into the equation for ϕ yields

$$\frac{\partial \psi}{\partial t} = \varepsilon^{1-2\nu}\frac{\partial^2 \phi}{\partial \xi^2} - \varepsilon^{-\nu} p(t)\frac{\partial \phi}{\partial \xi} = 0.$$

A significant degeneration arises if $1 - 2\nu = -\nu$ or $\nu = 1$. Assuming that we can expand $\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2 \cdots$, we have

$$-\frac{\partial^2 \psi_0}{\partial \xi^2} + p(t)\frac{\partial \psi_0}{\partial \xi} = 0,\ \psi_0(0,t) = g(t) - f(-P(t)),$$
$$-\frac{\partial^2 \psi_1}{\partial \xi^2} + p(t)\frac{\partial \psi_1}{\partial \xi} = \frac{\partial \psi_0}{\partial t}, \psi_1(0,t) = -\int_0^t \frac{\partial^2}{\partial x^2}(P(s) - 2P(t))ds.$$

For ψ_0, we find the expression

$$\psi_0(\xi,t) = A(t)e^{p(t)\xi} + B(t)\xi + C(t)$$

with A, B, C suitably chosen functions.

From the matching condition, we find $B(t) = C(t) = 0$ and the condition

$$p(t) < 0.$$

With this condition, we have

$$\psi_0(\xi,t) = (g(t) - f(-P(t))e^{p(t)\xi},$$

which satisfies the initial and boundary conditions. It is easy to calculate ψ_1 and higher-order approximations.

9.2.1 What Happens if $p(t) > 0$?

This problem was analysed by Shih (2001); see also this paper for related references. We recall that the regular expansion starts with $u_0(x,t) = f(x - P(t))$. If $p(t) < 0$, the characteristic $x - P(t) =$ constant is not located in the quarter-plane $x \geq 0, t \geq 0$, but if $p(t) > 0$, the characteristics $x - P(t) =$ constant extend into this domain. We have $f(0) = g(0)$, but the derivatives of these functions are generally not compatible. This causes jump discontinuities along the characteristic curve $x - P(t) = 0$, which can be compensated by a boundary layer along this curve. It turns out that the appropriate boundary layer variable in this case is

$$\eta = \frac{x - P(t)}{\sqrt{\varepsilon}}.$$

For more details of the analysis, see Shih (2001).

9.3 A Chemical Reaction with Diffusion

A number of chemical reaction problems can be formulated as singularly perturbed equations. Following Vasil'eva, Butuzov, and Kalachev (1995), we consider the problem

$$\varepsilon\left(\frac{\partial u}{\partial t}-a(x,t)\frac{\partial^2 u}{\partial x^2}\right)=f(u,x,t)+\varepsilon\cdots,$$

where $0\le x\le 1, 0\le t\le T$, and $a(x,t)>0$. We shall consider this as a prototype problem for the more complicated cases where u and x are vectors and we consider a system of equations.

The initial condition is

$$u(x,0)=\phi(x).$$

Natural boundary conditions in this case are the Neumann conditions

$$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(1,t)=0.$$

Putting $\varepsilon=0$, we have $f(u,x,t)=0$, and we note a similarity with the initial value problems of Chapter 8. We will indeed impose a similar condition as in the Tikhonov theorem (Section 8.2): assume that $f(u,x,t)=0$ has a unique solution $u_0(x,t)$ and that

$$\frac{\partial f}{\partial u}(u_0(x,t),x,t)<0$$

uniformly for $0\le x\le 1, 0\le t\le T$.

We shall determine the lowest-order terms of the appropriate expansion. A regular expansion of the form $u(x,t)=u_0(x,t)+\varepsilon u_1\cdots$ will in general not satisfy the initial and boundary conditions. This suggests the presence of an initial layer near $t=0$ and boundary layers near $x=0$ and $x=1$. We subtract the regular expansion by putting

$$u(x,t)=u_0(x,t)+\varepsilon u_1\cdots+v(x,t,\tau,\xi,\eta),$$

in which we assume that v is of the form

$$v=P_\varepsilon(x,\tau)+Q_\varepsilon(\xi,t)+R_\varepsilon(\eta,t),$$

where $\tau=t/\varepsilon$ and ξ and η are the boundary layer variables near $x=0$ and $x=1$, respectively. Substitution produces for v

$$\varepsilon\left(\frac{\partial u_0}{\partial t}+\varepsilon\frac{\partial u_1}{\partial t}+\frac{\partial v}{\partial t}-a(x,t)\frac{\partial^2 u_0}{\partial x^2}-\varepsilon a(x,t)\frac{\partial^2 u_1}{\partial x^2}-a(x,t)\frac{\partial^2 v}{\partial x^2}\right)=$$

$$=f_u(u_0(x,t),x,t)(\varepsilon u_1+v+\cdots)$$

We assume expansions for P,Q,R such as $P_\varepsilon(x,\tau)=P_0(x,\tau)+\varepsilon P_1\cdots$. After putting $t=\varepsilon\tau$, we find for P_0

$$\frac{\partial P_0}{\partial\tau}=f(u_0(x,0)+P_0(x,\tau)+Q_0(\xi,0)+R_0(\eta,0),x,0)$$

with an initial condition for $P_0(x,0)$. It is natural to assume that $Q_0(\xi,0)=R_0(\eta,0)=0$ so that

$$P_0(x,0) = \phi(x) - u_0(x,0).$$

We solve the equation for P_0 with x as a parameter, noting that $P_0(x,\tau) = 0$ is an equilibrium solution; we have to assume that the initial condition is located in the domain of attraction of this equilibrium solution, as we did in the Tikhonov theorem. In that case, we have immediately that the matching condition

$$\lim_{\tau \to \infty} P_0(x,\tau) = 0$$

is satisfied. As x is a parameter, we still have the freedom to put $P_0(0,\tau) = 0$. To determine the boundary layer variables, we have to rescale the equation. We abbreviate again $P + Q + R = v$ and put $\xi = x/\varepsilon^\nu$ to find near $x = 0$

$$\varepsilon \frac{\partial u_0}{\partial t} + \varepsilon \frac{\partial v}{\partial t} - \varepsilon^{1-2\nu} a(\varepsilon^\nu \xi, t) \frac{\partial^2 u_0}{\partial \xi^2} - \varepsilon^{1-2\nu} a(\varepsilon^\nu \xi, t) \frac{\partial^2 v}{\partial \xi^2} = f(u_0 + v, \varepsilon^\nu \xi, t) + \varepsilon \cdots .$$

A significant degeneration arises if $\nu = \frac{1}{2}$. We will require the boundary layer solution $R_\varepsilon(\eta, t)$ to vanish outside the boundary layer near $x = 1$, so the equation for $Q_0(\xi, t)$ becomes

$$-a(0,t) \frac{\partial^2 Q_0}{\partial \xi^2} = f(u_0(0,t) + Q_0, 0, t).$$

From the Neumann condition at $x = 0$, we have at lowest-order

$$\frac{\partial u_0}{\partial x}(0,t) + \frac{\partial P_0}{\partial x}(0,\tau) + \varepsilon^{-\frac{1}{2}} \frac{\partial Q_0}{\partial \xi}(0,t) = 0,$$

which yields

$$\frac{\partial Q_0}{\partial \xi}(0,t) = 0$$

with the matching condition

$$\lim_{\xi \to \infty} Q_0(\xi, t) = 0.$$

We conclude that $Q_0(\xi,t) = 0$ and that nontrivial solutions $Q_1(\xi,t), Q_2(\xi,t)$, etc., arise at higher order.

In the same way, we conclude that $R_0(\xi,t) = 0$ and finally that the lowest-order expansion of the solution is of the form

$$u_0(x,t) + P_0\left(x, \frac{t}{\varepsilon}\right).$$

Remark

The computation of higher-order approximations leads to linear equations for Q_1, R_1, etc. An additional difficulty is that there will be corner boundary layers at $(x,t) = (0,0)$ and $(1,0)$ involving boundary layer functions of the forms $Q^*(\xi,\tau)$ and $R^*(\eta,\tau)$ at higher order. For more details, see Vasil'eva, Butuzov, and Kalachev (1995), where a proof of asymptotic validity is also presented.

9.4 Periodic Solutions of Parabolic Equations

Following Vasil'eva, Butuzov, and Kalachev (1995), we will look for 2π-periodic solutions of the equation

$$\varepsilon \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + s(x,t)u + f(x,t) + \varepsilon g(x,t,u) + \varepsilon \cdots$$

with Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0$$

and periodicity condition

$$u(x,t) = u(x, t+2\pi)$$

for $0 \leq x \leq 1$ and $t \geq 0$. All time-dependent terms in the differential equation are supposed to be 2π-periodic. To illustrate the technique, we shall analyse an example first and discuss the more general case later.

9.4.1 An Example

Consider the equation

$$\varepsilon \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + a(x)u + b(x)\sin t$$

with $a(x)$ and $b(x)$ smooth functions; $a(x) > 0$ for $0 \leq x \leq 1$.

Based on our experience with problems in the preceding sections, we expect that the boundary layer variables

$$\xi = \frac{x}{\sqrt{\varepsilon}},\ \eta = \frac{1-x}{\sqrt{\varepsilon}},$$

will play a part, so we will look for a solution of the form

$$u(x,t) = u_0(x,t) + Q_0(\xi,t) + \sqrt{\varepsilon}Q_1(\xi,t) + R_0(\eta,t) + \sqrt{\varepsilon}R_1(\eta,t) + \varepsilon \cdots,$$

where $u_0(x,t)$ is the first term of a regular expansion of the form $u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 \cdots$. The equation for $u_0(x,t)$ is

$$\frac{\partial u_0}{\partial t} + a(x)u_0 + b(x)\sin t = 0,$$

which has the general solution

$$c(x)e^{-a(x)t} - b(x)e^{-a(x)t}\int_0^t e^{a(x)s}\sin s ds.$$

After integration and applying the periodicity condition to determine the function $c(x)$, we find

$$c(x) = \frac{b(x)}{a^2(x)+1}$$

and

$$u_0(x,t) = -\frac{a(x)b(x)}{a^2(x)+1}\sin t + \frac{b(x)}{a^2(x)+1}\cos t.$$

Note that $u_0(x,t)$ is 2π-periodic but u_0 will in general not satisfy the boundary conditions. Introducing the boundary layer variable ξ, we find for $Q_0(\xi,t)$ the equation

$$\frac{\partial^2 Q_0}{\partial \xi^2} = \frac{\partial Q_0}{\partial t} + a(0)Q_0,$$

where we have used that $u_0(x,t)$ satisfies the inhomogeneous equation and that $R_0(\eta,t)$ vanishes outside a boundary layer near $x=1$. For $Q_0(\xi,t)$, we have the boundary, matching, and periodicity conditions

$$Q_0(0,t) = -u_0(0,t), \lim_{\xi\to\infty} Q_0(\xi,t) = 0, Q_0(\xi,t) = Q_0(\xi,t+2\pi).$$

As $Q_0(\xi,t)$ is 2π-periodic in t, we propose a Fourier series for Q_0 and, the boundary condition only having two Fourier terms, we postulate

$$Q_0(\xi,t) = f_1(\xi)\sin t + f_2(\xi)\cos t.$$

Substitution in the equation for Q_0 yields

$$\begin{aligned} f_1'' &= -f_2 + a(0)f_1, \\ f_2'' &= -f_1 + a(0)f_2. \end{aligned}$$

This is a linear system with characteristic equation $(\lambda^2 - a(0))^2 + 1 = 0$ and with eigenvalues

$$\lambda = \pm\sqrt{a(0)+i}, \pm\sqrt{a(0)-i}.$$

We assumed $a(0) > 0$; the matching condition requires us to discard the solutions corresponding with $+\sqrt{a(0)}$, and we retain the independent solutions

$$e^{-\sqrt{a(0)}\xi}\cos\xi, e^{-\sqrt{a(0)}\xi}\sin\xi.$$

We find

$$\begin{aligned} f_1(\xi) &= e^{-\sqrt{a(0)}\xi}\left(\frac{a(0)b(0)}{a(0)^2+1}\cos\xi + \alpha\sin\xi\right), \\ f_2(\xi) &= e^{-\sqrt{a(0)}\xi}\left(-\frac{b(0)}{a(0)^2+1}\cos\xi + \beta\sin\xi\right), \end{aligned}$$

where α and β can be determined by substitution in the equations for f_1 and f_2.

Introducing the boundary layer variable η, we find for $R_0(\eta, t)$ the equation

$$\frac{\partial^2 R_0}{\partial \eta^2} = \frac{\partial R_0}{\partial t} + a(1)R_0,$$

where we have again used that $u(x, t)$ satisfies the inhomogeneous equation and that $Q_0(\xi, t)$ vanishes outside a boundary layer near $x = 0$. For $R_0(\eta, t)$, we have the boundary, matching, and periodicity conditions

$$R_0(1, t) = -u_0(1, t), \lim_{\eta \to \infty} R_0(\eta, t) = 0, R_0(\eta, t) = R_0(\eta, t + 2\pi).$$

Again we retain a finite Fourier series, and the calculation runs in the same way.

9.4.2 The General Case with Dirichlet Conditions

We return to the general case

$$\varepsilon \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + s(x, t)u + f(x, t) + \varepsilon g(x, t, u) + \varepsilon \cdots$$

with Dirichlet boundary conditions $u(0, t) = u(1, t) = 0$ and all time-dependent terms 2π-periodic in t. Again we expect the same boundary layer variables ξ and η and an expansion of the form

$$u(x, t) = u_0(x, t) + Q_0(\xi, t) + \sqrt{\varepsilon} Q_1(\xi, t) + R_0(\eta, t) + \sqrt{\varepsilon} R_1(\eta, t) + \varepsilon \cdots ,$$

where $u_0(x, t)$ is the first term of the regular expansion. The equation for $u_0(x, t)$ is

$$\frac{\partial u_0}{\partial t} + s(x, t)u_0 + f(x, t) = 0,$$

which has to be solved with x as a parameter. After integration by variation of constants, we have a free constant - dependent on x - to apply the periodicity condition to u_0.

As before, we can derive the equation for the boundary layer solution near $x = 0$,

$$\frac{\partial^2 Q_0}{\partial \xi^2} = \frac{\partial Q_0}{\partial t} + s(0, t)Q_0,$$

with boundary, matching, and periodicity conditions

$$Q_0(0, t) = -u_0(0, t), \lim_{\xi \to \infty} Q_0(\xi, t) = 0, Q_0(\xi, t) = Q_0(\xi, t + 2\pi).$$

A Fourier expansion for Q_0 is again appropriate, and we expand

$$s(0, t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$$

and in the same way $u_0(0,t)$. We assume

$$a_0 > 0.$$

For Q_0, we substitute

$$Q_0(\xi,t) = \alpha_0(\xi) + \sum_{k=1}^{\infty}(\alpha_k(\xi)\cos kt + \beta_k(\xi)\sin kt).$$

The coefficients are obtained by substitution of the Fourier series for Q_0 and $s(0,t)$ into the differential equation for Q_0, which produces an infinite set of equations; they can be solved as they are ODE's with constant coefficients. In the same way we determine the boundary layer function $R_0(\eta,t)$. Note that higher-order approximations can be obtained by extending the regular expansion and subsequently deriving higher-order equations for $Q_k, R_k, k = 1, 2, \cdots$. These equations are linear.

9.4.3 Neumann Conditions

The problem

$$\varepsilon\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + s(x,t)u + f(x,t) + \varepsilon g(x,t,u) + \varepsilon^2\cdots$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$$

and all terms in the equation 2π-periodic in t is slightly easier to handle.

Determine again a regular expansion of the form $u_0(x,t)+\varepsilon\cdots$ and assume again a full expansion of the solution of the form

$$u(x,t) = u_0(x,t) + Q_0(\xi,t) + \sqrt{\varepsilon}Q_1(\xi,t) + R_0(\eta,t) + \sqrt{\varepsilon}R_1(\eta,t) + \varepsilon\cdots.$$

The regular expansion will in general not satisfy the Neumann conditions, and we require for instance at $x = 0$

$$\frac{\partial u_0}{\partial x}(0,t) + \varepsilon\frac{\partial u_1}{\partial x}(0,t) + \varepsilon^2 + \cdots + \frac{1}{\varepsilon}\frac{\partial Q_0}{\partial \xi}(0,t) + \frac{\partial Q_1}{\partial \xi}(0,t) + \varepsilon\cdots = 0.$$

Multiplying with ε and comparing coefficients, we have

$$\frac{\partial Q_0}{\partial \xi}(0,t) = 0, \frac{\partial Q_1}{\partial \xi}(0,t) = -\frac{\partial u_0}{\partial x}(0,t), \cdots.$$

The equation for Q_0 will again be

$$\frac{\partial^2 Q_0}{\partial \xi^2} = \frac{\partial Q_0}{\partial t} + s(0,t)Q_0,$$

which is satisfied by the trivial solution. A similar result holds for $R_0(\eta,t)$. Nontrivial boundary layer solutions will generally arise at higher order.

9.4.4 Strongly Nonlinear Equations

Thus far, we have considered weakly nonlinear equations. It is of interest to consider more difficult equations of the form

$$\varepsilon \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + f(x, t, u) + \varepsilon \cdots$$

with Dirichlet or Neumann boundary conditions.

The main obstruction for the construction of an expansion is the solvability of the lowest-order equations. For the regular expansion, we have

$$\frac{\partial u_0}{\partial t} + f(x, t, u_0) = 0,$$

where, after solving the equation, we have to apply the periodicity condition. For the boundary layer contribution, this is even nastier. Assuming again an expansion for the periodic solution $u(x, t)$ of the form $u(x, t) = u_0(x, t) + Q_0(\xi, t) + R_0(\eta, t) + \sqrt{\varepsilon} \cdots$, we have

$$\frac{\partial^2 Q_0}{\partial \xi^2} = \frac{\partial Q_0}{\partial t} + f(0, t, u_0(0, t) + Q_0) - f(0, t, u_0(0, t)),$$

which looks nearly as bad as the original problem in the case of Dirichlet conditions. At higher order, the equations are linear.

In the case of Neumann conditions, we have again that the trivial solution $Q_0(\xi, t) = 0$ satisfies the equation and the boundary condition.

9.5 A Wave Equation

As a prototype of a hyperbolic equation with initial values, we consider the equation

$$\varepsilon \left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} \right) - \left(a \frac{\partial \phi}{\partial x} + b \frac{\partial \phi}{\partial t} \right) = 0,$$

where $t \geq 0, -\infty < x < +\infty$. The initial values are

$$\phi(x, 0) = f(x), \phi_t(x, 0) = g(x), -\infty < x < +\infty.$$

The functions $f(x), g(x)$ are sufficiently smooth; a and b are constants, and in a dissipative system $b > 0$. As we shall see, the constants have to satisfy the conditions $0 < |a| < b$.

The wave operator $\partial^2/\partial x^2 - \partial^2/\partial t^2$ has real characteristics

$$r = t - x, s = t + x.$$

The reduced ($\varepsilon = 0$) equation

$$a\frac{\partial \phi}{\partial x} + b\frac{\partial \phi}{\partial t} = 0$$

has characteristics of the form $bx - at =$ constant; we call them subcharacteristics of the original equation. Any differentiable function of $(bx - at)$ or $(x - \frac{a}{b}t)$ satisfies the reduced equation.

We start by calculating a formal approximation. This will give us the practical experience that will lead to more general insight into hyperbolic problems. Again we start by assuming that in some part of the half-plane $t \geq 0, -\infty < x < +\infty$; a regular expansion exists of the form

$$\phi_\varepsilon(x,t) = \sum_{n=0}^{m} \varepsilon^n \phi_n(x,t) + 0(\varepsilon^{m+1}).$$

The coefficients ϕ_n satisfy

$$\begin{aligned} a\frac{\partial \phi_0}{\partial x} + b\frac{\partial \phi_0}{\partial t} &= 0, \\ a\frac{\partial \phi_n}{\partial x} + b\frac{\partial \phi_n}{\partial t} &= \frac{\partial^2 \phi_{n-1}}{\partial x^2} - \frac{\partial^2 \phi_{n-1}}{\partial t^2}, n = 1, 2, \cdots . \end{aligned}$$

We find $\phi_0 = h(z), z = x - \frac{a}{b}t$, and $h(z)$ a differentiable function of its argument. The equation for ϕ_1 becomes

$$a\frac{\partial \phi_1}{\partial x} + b\frac{\partial \phi_1}{\partial t} = \frac{\partial^2 h}{\partial x^2} - \frac{\partial^2 h}{\partial t^2} = \left(1 - \frac{a^2}{b^2}\right) h'' \left(x - \frac{a}{b}t\right).$$

Transforming $t, x \to t, z$, we find

$$b\frac{\partial \phi_1}{\partial t} = \left(1 - \frac{a^2}{b^2}\right) h''(z),$$

so that

$$\phi_1(z,t) = \frac{b^2 - a^2}{b^3} h''(z)t + k(z),$$

where $h(z)$ and $k(z)$ still have to be determined. It would be natural to choose $h(z) = f(z)$, but we leave this decision until later.

We cannot satisfy both initial values, so we expect boundary layer behaviour near $t = 0$. Subtracting the regular expansion

$$\psi(x,t) = \phi(x,t) - \sum_{n=0}^{m} \varepsilon^n \phi_n(x,t)$$

and substitution in the original wave equation yields

$$\varepsilon\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial t^2}\right) - \left(a\frac{\partial \psi}{\partial x} + b\frac{\partial \psi}{\partial t}\right) = 0(\varepsilon^{m+1}).$$

The initial values are

$$\psi(x,0) = f(x) - \sum_0^m \varepsilon^n \phi_n(x,0),$$

$$\psi_t(x,0) = g(x) - \sum_0^m \varepsilon^n \phi_{n_t}(x,0).$$

We introduce the boundary layer variable

$$\tau = \frac{t}{\varepsilon^\nu}.$$

Transforming $x, t \to x, \tau$, the equation for ψ becomes

$$\left(\varepsilon \frac{\partial^2}{\partial x^2} - \varepsilon^{1-2\nu} \frac{\partial^2}{\partial \tau^2} - a \frac{\partial}{\partial x} - \varepsilon^{-\nu} b \frac{\partial}{\partial \tau}\right) \psi^* = 0(\varepsilon^{m+1}).$$

A significant degeneration arises when $1 - 2\nu = -\nu$ or $\nu = 1$. Near $t = 0$, we clearly have an ordinary boundary layer. Assuming the existence of a regular expansion

$$\psi_\varepsilon^* = \sum_{n=0}^m \varepsilon^n \psi_n(x,\tau) + 0(\varepsilon^{m+1}),$$

we find

$$L_0^* \psi_0 = 0,$$

$$L_0^* \psi_1 = a \frac{\partial \psi_0}{\partial x}, \quad \text{etc.},$$

with

$$L_0^* = \frac{\partial^2}{\partial \tau^2} + b \frac{\partial}{\partial \tau}.$$

The initial conditions have to be expanded and yield

$$\psi_0(x,0) = f(x) - h(x), \;\; \psi_1(x,0) = -k(x),$$

$$\frac{\partial \psi_0}{\partial \tau}(x,0) = 0, \;\; \frac{\partial \psi_1}{\partial \tau}(x,0) = g(x) + \frac{a}{b} h'(x).$$

The matching conditions take the form

$$\lim_{\tau \to \infty} \psi_n(x,\tau) = 0, n = 0, 1, 2, \cdots .$$

For ψ_0, we find

$$\psi_0(x,\tau) = A(x) + B(x) e^{-b\tau}.$$

The restriction $b > 0$ is necessary to satisfy the matching conditions and, to satisfy $\partial \psi_0 / \partial \tau = 0$, we are then left with the trivial solution, $\psi_0(x,\tau) = 0$. So this determines h, as we have to take $\psi_0(x,0) = 0$:

$$h(x) = f(x).$$

For ψ_1, we find the same expression as for ψ_0. Matching produces again $A(x) = 0$, leaving

$$\psi_1(x, \tau) = C(x)e^{-b\tau}.$$

The initial values yield

$$\begin{aligned} C(x) &= -k(x), \\ -bC(x) &= g(x) + \tfrac{a}{b}f'(x), \end{aligned}$$

which determines k:

$$k(x) = \frac{g(x)}{b} + \frac{a}{b^2}f'(x).$$

At this stage, we propose the formal expansion

$$\begin{aligned} \phi_\varepsilon(x,t) = f(x - \tfrac{a}{b}t) + \varepsilon[\tfrac{b^2-a^2}{b^3}tf''(x - \tfrac{a}{b}t) + \tfrac{1}{b}g(x - \tfrac{a}{b}t) + \tfrac{a}{b^2}f'(x - \tfrac{a}{b}t)] \\ -\varepsilon[\tfrac{a}{b^2}f'(x) + \tfrac{1}{b}g(x)]e^{-bt/\varepsilon} + \varepsilon^2 \cdots . \end{aligned}$$

It is clear that, outside the boundary layer, the solution is dominated by the initial values of ϕ that propagate along the *subcharacteristic* through a given point.

This, however, opens the possibility of the following situation. The solution at a point $P(x, t)$ is determined by the propagation of initial values along the *characteristics* (i.e., no information can reach P from the initial values in $x < A$ or $x > B$). If $|a| > b$, the formal expansion consists mainly of terms carrying information from these forbidden regions. This means that in this case the formal expansion cannot be correct, resulting in the condition $0 < |a| < b$; see Fig. 9.1.

We demonstrate this somewhat more explicitly by considering the problem where $\phi(x, 0) = f(x) = 0, -\infty < x < +\infty, \phi_t(x, 0) = g(x)$. The solution can be written as an integral using the Riemann function R:

$$\phi(x,t) = \int_{x-t}^{x+t} R_\varepsilon(x - \tau, t)g(\tau)d\tau.$$

Riemann functions are discussed in many textbooks on partial differential equations, such as Strauss (1992). For some special values of a, b, the Riemann function can be expressed in terms of elementary functions. Suppose now that we prescribe

$$\begin{aligned} g(x) > 0, x < A, x > B, \\ g(x) = 0, A \leq x \leq B. \end{aligned}$$

It is clear that in this case $\phi(x,t)|_P = 0$. On the other hand, the formal expansion yields

$$\phi_\varepsilon(x,t) = \frac{\varepsilon}{b}g\left(x - \frac{a}{b}t\right) - \frac{\varepsilon}{b}g(x)e^{-bt/\varepsilon} + \varepsilon^2 \cdots .$$

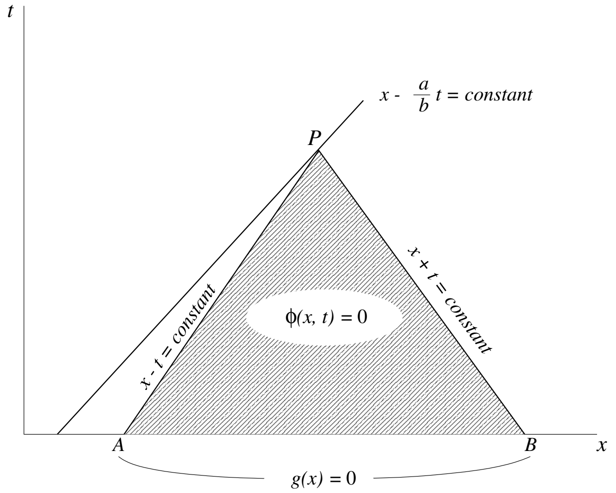

Fig. 9.1. Characteristics and subcharacteristics in the hyperbolic problem of Section 9.1.

If $|a| > b$, it is clear that $\phi_\varepsilon(x,t)|_P \neq 0$, which means that in this case the formal expansion does not produce correct results.

Remark
The condition $|a| < b$ means that the direction of the subcharacteristic is "contained" between the directions of the characteristics. In this case, one calls the subcharacteristic *time-like.*

9.6 Signalling

As an example of the part played by boundaries, we consider a so-called signalling or radiation problem. We have $x \geq 0, t \geq 0$. At $t = 0$, the medium is in a state of rest: $\phi(x,0) = \phi_t(x,0) = 0$. At the boundary, we have a source of signals or radiation:

$$\phi(0,t) = f(t), t > 0.$$

Wave propagation is described again by the equation

$$\varepsilon\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2}\right) - \left(a\frac{\partial\phi}{\partial x} + b\frac{\partial\phi}{\partial t}\right) = 0,$$

where $0 < a < b$. As ϕ is initially identically zero, the solution for $t > 0$ will be identically zero for $x > t$. (Information propagates along the characteristics.)

As before we start with a regular expansion.

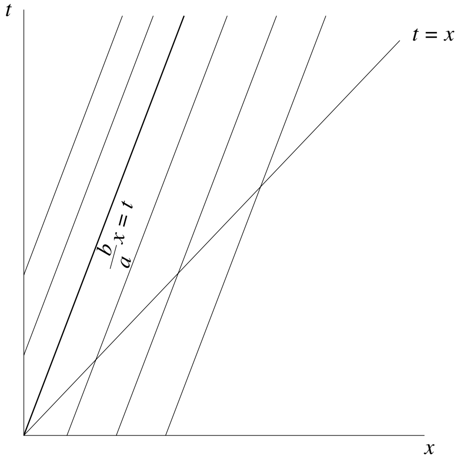

Fig. 9.2. Signalling: a source at the boundary $x = 0$.

$$\phi_\varepsilon(x,t) = \sum_{n=0}^{m} \varepsilon^n \phi_n(x,t) + 0(\varepsilon^{m+1}).$$

We find again $\phi_0 = h(z), z = x - \frac{a}{b}t$.

$$\phi_1 = \frac{b^2 - a^2}{b^3} h''(z)t + k(z)$$

with h, k sufficiently differentiable functions. We can satisfy the boundary condition by putting

$$\begin{aligned} \phi_0 &= 0, \qquad\qquad t < \tfrac{b}{a}x \\ &= f(t - \tfrac{b}{a}x),\ t > \tfrac{b}{a}x. \end{aligned}$$

(f is defined only for positive values of its argument.)

Note that a consequence of this choice of ϕ_0 is that, unless $f(0) = 0$, the regular expansion is discontinuous along the subcharacteristic $t = \frac{b}{a}x$, so we expect a discontinuity propagating from the origin as in general $f(0) \neq 0, f'(0) \neq 0$, etc. However, the theory of hyperbolic equations tells us that such a discontinuity propagates along the characteristics, in this case $x = t$, so we expect a boundary layer along the subcharacteristic to make up for this discrepancy; see Fig. 9.2.

We transform $x, t \to \xi, t$ with

$$\xi = \frac{x - \frac{a}{b}t}{\varepsilon^\nu},$$

whereas $\phi_\varepsilon(x,t) \to \psi_\varepsilon(\xi, t)$. We have

$$\frac{\partial}{\partial x} \to \varepsilon^{-\nu} \frac{\partial}{\partial \xi},$$
$$\frac{\partial}{\partial t} \to -\varepsilon^{-\nu} \frac{a}{b} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial t}.$$

The equation becomes $L_\varepsilon^* \psi = 0$ with

$$L_\varepsilon^* = \varepsilon^{1-2\nu} c \frac{\partial^2}{\partial \xi^2} + 2\varepsilon^{1-\nu} \frac{a}{b} \frac{\partial^2}{\partial \xi \partial t} - \varepsilon \frac{\partial^2}{\partial t^2} - b \frac{\partial}{\partial t},$$

where $c = 1 - a^2/b^2$. A significant degeneration arises if $\nu = \frac{1}{2}$, so that the operator becomes

$$L_0^* = c \frac{\partial^2}{\partial \xi^2} - b \frac{\partial}{\partial t}\ .$$

Note that $c > 0$; near the subcharacteristic $t = \frac{b}{a} x$, we have a parabolic boundary layer. For the boundary layer solution, an expansion of the form

$$\psi_\varepsilon(\xi, t) = \sum_{n=0}^{2m} \varepsilon^{n/2} \psi_n(\xi, t) + 0(\varepsilon^{m+\frac{1}{2}})$$

is taken, where ψ_0 is a solution of the diffusion equation

$$L_0^* \psi_0 = 0,$$

which has to be matched with the regular expansion. We have, moving to the right-hand side of the subcharacteristic, $(t < \frac{b}{a} x)$ that $\psi_0 \to 0$. On the left-hand side, the regular expansion becomes

$$\lim_{z \uparrow 0} f(z) = f(0^+)$$

on approaching the subcharacteristic, which should match with ψ_0 moving to the left. We find, omitting the technical details of matching, that

$$\psi_0(\xi, t) = \frac{1}{2} f(0^+) \mathrm{erfc} \left(\frac{\xi}{2\sqrt{kt/b}} \right).$$

where

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt; \;\; \mathrm{erfc}(0) = 1.$$

The boundary layer approximation ψ_0 satisfies the equation while $\psi_0(0, t) = \frac{1}{2} f(0^+)$; moreover, moving to the right-hand side of the characteristic $\xi \to \infty$ or $z \to \infty$, we have agreement with the expression above for $\mathrm{erfc}(z)$.

Note that we did not perform the subtraction trick before carrying out the matching, but, as we have seen, ψ_0 satisfies the required conditions. Note also that, according to this analysis, the formal approximation becomes zero in the region between the characteristics $t = \frac{b}{a} x$ and $t = x$.

9.7 Guide to the Literature.

A survey of results obtained in the former Soviet Union for parabolic equations has been given by Butuzov and Vasil'eva (1983), and Vasil'eva, Butuzov, and Kalachev (1995). For expansions and proofs, see also Shih and Kellogg (1987) and Shih (2001) and further references therein. The first section on a problem with slow diffusion and heat production is actually a simplified version of a more extensive problem (Van Harten, 1979). In all of these papers, proofs of asymptotic validity are given.

We shall not consider here the case of equilibrium solutions of parabolic equations, which reduces to the study of elliptic equations; for an introduction to singular perturbations of equilibrium solutions of systems with reaction and diffusion, see Aris (1975, Vol. 1, Chapter 3.7).

A classical example of a nonlinear diffusion equation is Burgers' equation

$$u_t + uu_x = \varepsilon u_{xx}$$

on an infinite domain, which is also a typical problem for studying shock waves. An introductory treatment can be found in Holmes (1998, Chapter 2).

Applications of slow manifolds (Fenichel theory) and extensions are described by Jones (1994), Kaper (1999) and Kaper and Jones (2001). These papers contain many references and applications. One possibility is to project the solutions of the evolution equation to a finite space, which reduces the problem to the analysis given in the preceding chapter (ODE's). Another possibility is to compute travelling or solitary waves; an example is the paper by Szmolyan (1992). It is typical to start with an evolution equation such as

$$u_t + uu_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0,$$

(see for instance Jones, 1994) and then look for solitary waves (i.e., solutions that depend on $x - ct$ alone). This results in a system of ordinary differential equations that contains one or more slow manifolds. The reason for omitting these examples in this chapter is that in what follows one needs an extensive dynamical systems analysis with subtle reasoning also to connect the results to the original evolution equation. The results, however, are very interesting.

Another important subject that we did not discuss is combustion, which leads to interesting boundary layer problems. A basic paper is Matkowsky and Sivashinsky (1979); introductory texts are Van Harten (1982), Buckmaster and Ludford (1983), and Fife (1988). More recent results can be found in Vasil'eva, Butuzov, and Kalachev (1995), and Class, Matkowsky, and Klimenko (2003).

For hyperbolic problems, the formal construction of expansions has been discussed in Kevorkian and Cole (1996). Constructions and proofs of asymptotic validity on a timescale $O(1)$ have been given in an extensive study by Geel (1981); see also De Jager and Jiang Furu (1996). The proofs are founded on energy integral estimates and fixed-point theorems in a Banach space. An extension to larger timescales for hyperbolic problems has been given by Eckhaus and Garbey (1990), who develop a formal approximation that is shown to be valid on timescales of the order $1/\varepsilon$.

9.8 Exercises

Exercise 9.1 Consider the hyperbolic initial value problem

$$\begin{aligned} \varepsilon(u_{tt} - u_{xx}) + a u_t + u_x &= 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= g(x) \quad (x \in \mathbb{R}), \end{aligned}$$

based on Kevorkian and Cole (1996), with $a \geq 1$. We will introduce ϕ by $u(x,t) = \phi(x,t) \exp(\frac{\alpha x - \beta t}{2\varepsilon})$.

a. Compute α and β such that

$$\varepsilon(\phi_{tt} - \phi_{xx}) - \left(\frac{a^2 - 1}{4\varepsilon}\right) \phi = 0.$$

Consider the case $a = 1$. In this special case the function $\phi(x,t)$ has to satisfy the wave equation, so write: $\phi(x,t) = \psi_l(x-t) + \psi_r(x+t)$.

b. Compute the functions ψ_l and ψ_r; use the initial conditions of $u(x,t)$.
c. Give the exact solution $u(x,t,\varepsilon)$ of the given problem for $a \geq 1$.
d. Construct an approximation $\tilde{U}(x,t,\varepsilon)$ of the form

$$\tilde{U}(x,t,\varepsilon) = U_0(x,t) + \varepsilon U_1(x,t) + \varepsilon W_1(x,\tau) + 0(\varepsilon^2)$$

with $\tau = t/\varepsilon$.

e. Compare the approximation $\tilde{U}(x,t,\varepsilon)$ with the exact solution $u(x,t,\varepsilon)$.

Exercise 9.2 Consider the equation

$$\varepsilon u_{xx} = u_t - u_x, x \geq 0, t \geq 0,$$

with initial condition $u(x,0) = 1$ and boundary condition $u(0,t) = \exp(-t)$.

a. Produce the first two terms of a regular expansion and locate the boundary layer(s).
b. Repeat the analysis when we change the boundary condition to $u(0,t) = \sin t$.

Exercise 9.3 As an example of a parabolic problem on an unbounded domain, consider Fisher's equation (KPP)

$$u_t = \varepsilon u_{xx} + u(1-u), \ -\infty < x < \infty, t > 0,$$

with $u(x,0) = g(x)$, $0 \leq u \leq 1$.

a. Determine the stationary solutions in the case $\varepsilon = 0$. Which one is clearly unstable?
b. Introduce the regular expansion $u = u_0(x,t) + \varepsilon u_1(x,t) + \cdots$ and give the equations and conditions for u_0, u_1.
c. Solve the problem for $u_0(x,t)$.

Exercise 9.4 Consider the parabolic initial boundary value problem

$$\begin{aligned} \varepsilon(u_t - au_{xx}) + bu &= f(x), \\ u(x,0) &= 0,\ a > 0, b > 0, \\ u(0,t) &= 0,\ (x,t) \in \mathbb{R}^2, \\ u(1,t) &= 0,\ 0 < x < 1, 0 < t \leq T. \end{aligned}$$

We construct an approximation in four steps.

a. Compute the regular expansion $U(x,t)$:

$$U(x,t,\varepsilon) = U_0(x,t) + \varepsilon U_1(x,t) + 0(\varepsilon^2).$$

b. Compute the initial layer correction $\Pi_0(x,\tau), \tau = t/\varepsilon$, and give the equation (with initial value) for $\Pi_1(x,\tau)$ such that $V(x,t,\varepsilon)$ satisfies the initial condition at $t = 0$:

$$V(x,t,\varepsilon) = U(x,t,\varepsilon) + \Pi_0(x,\tau) + \Pi_1(x,\tau) + 0(\varepsilon^2).$$

c. Compute the boundary layer correction

$$Q_0(\xi,t), \xi = \frac{x}{\sqrt{\varepsilon}},$$

and give the equations for $Q_1(\xi,t)$ and $Q_2(\xi,t)$ such that

$$W(x,t,\varepsilon) = V(x,t,\varepsilon) + Q_0(\xi,t) + \sqrt{\varepsilon}Q_1(\xi,t) + \varepsilon Q_2(\xi,t) + 0(\varepsilon^{3/2})$$

satisfies the boundary condition at $x = 0$. In the same way, boundary layer corrections can be constructed at the boundary $x = 1$ with ($\xi^* = \frac{(1-x)}{\sqrt{\varepsilon}}$); we omit this.

Also, at the corner boundary points $(0,0)$ and $(1,0)$, corrections are needed.
d. Give the equations for the corner boundary layer corrections $P_0(\xi,\tau)$, $P_1(\xi,\tau)$, and $P_2(\xi,\tau)$ at $(0,0)$ ($\xi^* = \frac{x}{\sqrt{\varepsilon}}, \tau = t/\varepsilon$). Give the initial boundary values.

Combining (a), (b), (c) and (d) produces an approximation $\tilde{u}(x,t,\varepsilon)$ of the solution $u(x,t,\varepsilon)$ of the given problem of the form

$$
\begin{aligned}
\tilde{u}(x,t,\varepsilon) = {} & U_0(x,t) + \Pi_0(x,\tau) + Q_0(\xi,t) \\
& + P_0(\xi,\tau) + \sqrt{\varepsilon}[Q_1(\xi,t) + P_1(\xi,\tau)] \\
& + \varepsilon[U_1(x,t) + \Pi_1(x,\tau) + Q_2(\xi,t) + P_2(\xi,\tau)] + 0(\varepsilon^2),
\end{aligned}
$$

where $\tau = t/\varepsilon, \xi = x/\sqrt{\varepsilon}$. Note that corner boundary layer corrections at $(1,0)$ can be constructed in the same way with $\xi^* = (1-x)/\sqrt{\varepsilon}$. No correction terms are added for the boundary $x = 1$ and the corner point $(1,0)$, so the approximation only holds on

$$
\{(x,t) \in \mathbb{R}^2 | 0 < x < 1-d, \ \ (0 < d < 1) \ \text{ and } 0 < t \leq T\}.
$$