5

Two-Point Boundary Value Problems

In linear problems, much more qualitative information is available than for nonlinear systems. Still, in general we cannot solve linear equations with variable coefficients explicitly. Therefore it is very surprising that in the case of singularly perturbed linear systems, we can often obtain asymptotic expansions to any accuracy required.

In most of this chapter, we shall look at rather general, linear two-point boundary value problems. To start, consider a second-order equation of the form

$$\varepsilon L_1 \phi + L_0 \phi = f(x), \quad x \in (0,1)$$

with boundary values $\phi_\varepsilon(0) = \alpha, \phi_\varepsilon(1) = \beta$. For the operator L_1, we write

$$L_1 = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$$

with $a_0(x), a_1(x), a_2(x)$ continuous functions in $[0,1]$; moreover,

$$a_2(x) > 0, \quad x \in [0,1].$$

We exclude that $a_2(x)$ vanishes in $[0,1]$, as this introduces singularities requiring a special approach.Taking the formal limit for $\varepsilon \to 0$, we retain $L_0\phi = f(x)$; for the operator L_0, we have

$$L_0 = b_1(x)\frac{d}{dx} + b_0(x).$$

We assume $b_0(x)$ and $b_1(x)$ to be continuous in $[0,1]$. It turns out that the analysis of the problem depends very much on the properties of b_0 and b_1.

5.1 Boundary Layers at the Two Endpoints

We first treat the case where $b_1(x)$ vanishes everywhere in $[0,1]$, so we are considering the equation

$$\varepsilon L_1\phi + b_0(x)\phi = f(x).$$

Assume that $b_0(x)$ does not change sign; for instance,

$$b_0(x) < 0, \quad x \in [0,1].$$

If $b_0(x)$ changes sign, the analysis is more complicated; we shall deal with such a case later on. As before, we suppose that in a subdomain of $[0,1]$ there exists a regular expansion of the solution of the form

$$\phi_\varepsilon(x) = \sum_{n=0}^{m} \varepsilon^n \phi_n(x) + O(\varepsilon^{m+1}).$$

On substituting this into the preceding equation, we find

$$\phi_0(x) = f(x)/b_0(x)$$

$$\phi_n = -L_1\phi_{n-1}(x)/b_0(x), \quad n = 1, 2, \cdots .$$

It is clear that to perform this construction $b_0(x)$ and $f(x)$ have to be sufficiently differentiable. By these recurrency relations, the terms ϕ_n of the expansion are completely determined and, of course, in general they will not satisfy the boundary conditions. Therefore we make the assumption that this regular expansion will exist in the subdomain $[d, 1-d]$ with d a constant, $0 < d < \frac{1}{2}$. Near the boundary points $x = 0$ and $x = 1$, we expect the presence of boundary layers to make the transitions to the boundary values possible.

First we use the "subtraction trick" by putting

$$\phi_\varepsilon(x) = \sum_{n=0}^{m} \varepsilon^n \phi_n(x) + \psi_\varepsilon(x)$$

to obtain, using the relations for ϕ_n,

$$\varepsilon L_1\psi + b_0(x)\psi = -\varepsilon^{m+1} L_1\phi_m(x).$$

The subtraction trick is not essential, to use it is a matter of taste. Its advantage is that in the transformed problem the regular (outer) expansion has coefficients zero, which facilitates matching. The boundary conditions become

$$\psi_\varepsilon(0) = \alpha - \textstyle\sum_{n=0}^{m} \varepsilon^n \phi_n(0),$$

$$\psi_\varepsilon(1) = \beta - \textstyle\sum_{n=0}^{m} \varepsilon^n \phi_n(1).$$

We start with the analysis of the boundary layer near $x = 0$. In a neighbourhood of $x = 0$, we introduce the local variable

$$\xi = \frac{x}{\delta(\varepsilon)}, \quad \delta(\varepsilon) = o(1),$$

and write $\psi_\varepsilon(\delta\xi) = \psi^*_\varepsilon(\xi)$. The orderfunction $\delta(\varepsilon)$ and correspondingly the size of the boundary layer still have to be determined. The equation in the local variable ξ becomes

$$L^*\psi^* = \frac{\varepsilon}{\delta^2} a_2(\delta\xi)\frac{d^2\psi^*}{d\xi^2} + \frac{\varepsilon}{\delta} a_1(\delta\xi)\frac{d\psi^*}{d\xi} + \varepsilon a_0(\delta\xi)\psi^* + b_0(\delta\xi)\psi^* = O(\varepsilon^{m+1}).$$

A significant degeneration L_0^* arises if $\delta(\varepsilon) = \sqrt{\varepsilon}$, which yields

$$L_0^* = a_2(0)\frac{d^2}{d\xi^2} + b_0(0).$$

The equation with this local variable has terms with coefficients containing ε and $\sqrt{\varepsilon}$. This suggests that we look for a regular expansion in ξ of the form

$$\psi^*_\varepsilon(\xi) = \sum_{n=0}^{2m} \varepsilon^{n/2}\psi_n(\xi) + O(\varepsilon^{m+\frac{1}{2}}).$$

We find at the lowest order

$$L_0^*\psi_0 = a_2(0)\frac{d^2\psi_0}{d\xi^2} + b_0(0)\psi_0 = 0.$$

Assuming that the coefficients have Taylor expansions to sufficiently high order near $x = 0$, we can expand a_2, a_1, a_0, and b_0, rewritten in the local variable ξ, in the equation for ψ^*. So we have for instance

$$a_2(\sqrt{\varepsilon}\xi) = a_2(0) + \sqrt{\varepsilon}\xi a_2'(0) + \varepsilon\cdots.$$

Collecting terms of equal order of ε, we deduce equations for $\psi_1, \psi_2, \cdots$. Keeping terms to order $\sqrt{\varepsilon}$, this looks like

$$\begin{aligned}(a_2(0) + &\sqrt{\varepsilon}\,\xi a_2'(0) + \cdots)\left(\frac{d^2\psi_0}{d\xi^2} + \sqrt{\varepsilon}\frac{d^2\psi_1}{d\xi^2} + \cdots\right)\\ &+ \sqrt{\varepsilon}\,(a_1(0) + \cdots)\left(\frac{d\psi_0}{d\xi} + \cdots\right)\\ +(b_0(0) + &\sqrt{\varepsilon}\,\xi b_0'(0) + \cdots)(\psi_0 + \sqrt{\varepsilon}\psi_1 + \cdots) = O(\varepsilon^{m+1}),\end{aligned}$$

where the dots stand for $O(\varepsilon)$ terms. For ψ_1, we find the equation

$$L_0^*\psi_1 = a_2(0)\frac{d^2\psi_1}{d\xi^2} + b_0(0)\psi_1 = -\xi a_2'(0)\frac{d^2\psi_0}{d\xi^2} - a_1(0)\frac{d\psi_0}{d\xi} - \xi b_0'(0)\psi_0.$$

In fact, it is easy to see that the coefficients ψ_n all satisfy the same type of differential equation (with different right-hand sides) of the form

$$L_0^*\psi_n = F_n(\psi_0(\xi), ..., \psi_{n-1}(\xi), \xi).$$

The boundary values at $x = 0$ are

$$\psi_0 = \alpha - \phi_0(0),$$

$$\psi_1(0) = 0,$$

$$\psi_2(0) = -\phi_1(0), \quad \text{etc.}$$

So we have for $\psi_n, \quad n = 0, 1, ...$ a second-order equation and one boundary condition at $\xi = 0$, which is not enough to determine ψ_n completely. As in Chapter 3, we determine the functions ψ_n by requiring the boundary layer functions $\psi_n(\xi)$ to vanish outside the boundary layer. This means that we will add the matching relation

$$\lim_{\xi \to \infty} \psi_n(\xi) = 0.$$

That we have to match towards zero is a result of the subtraction trick. Of course, for $\xi \to \infty$ the variable ξ leaves the domain we are considering. However, if we let x tend to 1, ξ tends to $1/\sqrt{\varepsilon}$, which is very large, and we use $+\infty$ instead.

The solution of the equation for ψ_0 is

$$\psi_0(\xi) = Ae^{-\omega_0 \xi} + Be^{\omega_0 \xi},$$

where we have abbreviated $\omega_0 = (-b_0(0)/a_2(0))^{\frac{1}{2}}$; note that we assumed $b_0(0) < 0$. From the boundary condition, we find

$$A + B = \alpha - \phi_0(0).$$

The matching condition yields $B = 0$, so ψ_0 is now determined completely. The determination of ψ_1 runs along the same lines. We find

$$\psi_1(\xi) = A_1(\xi)e^{-\omega_0 \xi} + B_1(\xi)e^{\omega_0 \xi},$$

where A_1 and B_1 are polynomial functions of ξ that are determined completely by the boundary condition and the matching relation.

To satisfy the boundary conditions of the original problem, we have to repeat this analysis near the boundary at $x = 1$; the calculations mirror the preceding analysis. We give the results; the details are left as an exercise for the reader.

A suitable local variable is

$$\eta = \frac{1 - x}{\sqrt{\varepsilon}}.$$

The operator degenerates into

$$a_2(1)\frac{d^2}{d\eta^2} + b_0(1).$$

We propose a regular expansion of the form

$$\sum_{n=0}^{2m} \varepsilon^{n/2} \bar{\psi}_n(\eta),$$

which yields for the first term

$$\bar{\psi}_0(\eta) = \bar{A}e^{-\omega_1 \eta} + \bar{B}e^{\omega_1 \eta}$$

with $\omega_1 = (-b_0(1)/a_2(1))^{\frac{1}{2}}$. The boundary condition at $x = 1$ produces

$$\bar{A} + \bar{B} = \beta - \phi_0(1).$$

The matching relation becomes

$$\lim_{\eta \to \infty} \bar{\psi}_0(\eta) = 0,$$

so $\bar{B} = 0$. Collecting the first terms of the expansion with respect to x, ξ, and η in the three domains, we find

$$\phi_\varepsilon(x) = \frac{f(x)}{b_0(x)} + \left(\alpha - \frac{f(0)}{b_0(0)}\right) e^{-\omega_0 x/\sqrt{\varepsilon}} + \left(\beta - \frac{f(1)}{b_0(1)}\right) e^{-\omega_1(1-x)/\sqrt{\varepsilon}} + O(\sqrt{\varepsilon}).$$

For an illustration see Fig. 5.1.

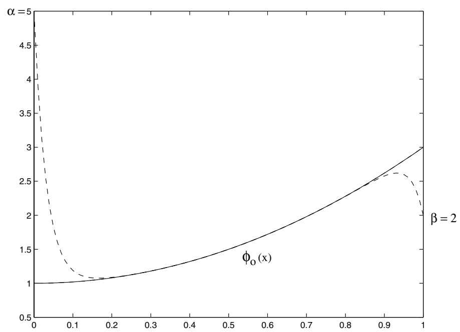

Fig. 5.1. Matching at two endpoints for the equation $\varepsilon d^2\phi/dx^2 + \phi = 1 + 2x^2, \phi(0) = 5, \phi(1) = 2, \varepsilon = 0.001$.

Remark

Omitting the $O(\sqrt{\varepsilon})$ term in the preceding expression and substituting the result in the differential equation, after checking the boundary conditions, we find that we have obtained a formal approximation of the solution of the boundary value problem. Because of the construction, this seems like a natural result; in the next section, we shall see, however, that this is due to the absence of the term $b_1(x)$.

Once we knew how to proceed, the calculation itself was as simple as in Section 4.2 even though the problem is much more general. We shall now consider other boundary value problems and shall encounter some new phenomena.

5.2 A Boundary Layer at One Endpoint

We consider again the boundary value problem formulated at the beginning of this chapter,

$$\varepsilon L_1\phi + L_0\phi = f(x),\ \phi_\varepsilon(0) = \alpha, \phi_\varepsilon(1) = \beta,$$

but now with

$$L_0 = b_1(x)\frac{d}{dx} + b_0(x).$$

As before, we have

$$L_1 = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$$

with $a_2(x) > 0,\ x \in [0,1]$; all coefficients are assumed to be sufficiently differentiable. Suppose, moreover, that $b_1(x)$ does not change sign, say

$$b_1(x) < 0, \quad x \in [0,1].$$

The case in which $b_1(x)$ vanishes in the interior of the interval is called a turning-point problem. For such problems, see Sections 5.4 and 5.5 and some of the exercises.

Here we assume again the existence of a regular expansion in a subdomain of $[0,1]$ of the form

$$\phi_\varepsilon(x) = \sum_{n=0}^{m} \varepsilon^n \phi_n(x) + O(\varepsilon^{m+1}).$$

After substitution of this expansion into the equation, we find

$$\begin{aligned} L_0\phi_0 &= f(x), \\ L_0\phi_n &= -L_1\phi_{n-1}(x), \quad n = 1, 2, \cdots . \end{aligned}$$

For $\phi_0(x)$, we find the first-order equation

$$b_1(x)\frac{d\phi_0}{dx} + b_0(x)\phi_0 = f(x),$$

which can be solved by variation of constants. We find, abbreviating,

$$g(x) = \int_0^x \frac{b_0(t)}{b_1(t)}\, dt,$$

$$\phi_0(x) = Ae^{-g(x)} + e^{-g(x)} \int_0^x e^{g(t)} f(t)\, dt.$$

In contrast with the preceding boundary value problem, we have one free constant in the expression for $\phi_0(x)$; we can verify that the same holds for $\phi_1, \phi_2, \cdots$. This means that the regular expansion in the variable x can be made to satisfy the boundary condition at $x = 0$ or at $x = 1$. Which one do we choose?

Suppose that we choose to satisfy the boundary condition at $x = 1$ and that we expect the existence of a boundary layer near $x = 0$. Does this lead to a consistent construction of a formal expansion? First we perform the subtraction trick

$$\phi_\varepsilon(x) = \sum_{n=0}^{m} \varepsilon^n \phi_n(x) + \psi_\varepsilon(x).$$

The equation for $\psi_\varepsilon(x)$ becomes

$$\varepsilon L_1 \psi + L_0 \psi = O(\varepsilon^{m+1})$$

with the boundary conditions

$$\psi_\varepsilon(0) = \alpha - \textstyle\sum_{n=0}^{m} \varepsilon^n \phi_n(0),$$
$$\psi_\varepsilon(1) = \beta - \textstyle\sum_{n=0}^{m} \varepsilon^n \phi_n(1).$$

With our assumption that the regular expansion satisfies the boundary condition at $x = 1$, we have $\phi_0(1) = \beta$ so $\psi_\varepsilon(1) = 0$. Introduce the local variable

$$\xi = \frac{x}{\delta(\varepsilon)}.$$

The differential operator written in the variable ξ takes the form

$$L^* = \frac{\varepsilon}{\delta^2} a_2(\delta\xi) \frac{d^2}{d\xi^2} + \frac{\varepsilon}{\delta} a_1(\delta\xi) \frac{d}{d\xi} + \varepsilon a_0(\delta\xi) + \frac{1}{\delta} b_1(\delta\xi) \frac{d}{d\xi} + b_0(\delta\xi).$$

Looking for a significant degeneration, we find $\delta(\varepsilon) = \varepsilon$ and the degeneration

$$L_0^* = a_2(0) \frac{d^2}{d\xi^2} + b_1(0) \frac{d}{d\xi}.$$

Expanding

$$\psi_\varepsilon(\varepsilon\xi) = \sum_{n=0}^{m} \varepsilon^n \psi_n(\xi) + O(\varepsilon^{m+1}),$$

we find

$$L_0^* \psi_0 = a_2(0) \frac{d^2 \psi_0}{d\xi^2} + b_1(0) \frac{d\psi_0}{d\xi} = 0.$$

The solution is

$$\psi_0(\xi) = B + Ce^{-\frac{b_1(0)}{a_2(0)}\xi}.$$

Because of the subtraction trick, the matching relation will again be

$$\lim_{\xi\to\infty} \psi_0(\xi) = 0.$$

As we assumed $b_1(0)/a_2(0)$ to be negative, this implies

$$B = C = 0.$$

A similar result holds for $\psi_1, \psi_2, \cdots$, namely all boundary layer terms (functions of ξ) vanish from the expansion. We conclude that the assumption of the existence of a boundary layer near $x = 0$ is not correct.

We consider now the other possibility, which is assuming that the regular expansion in the variable x satisfies the boundary condition at $x = 0$ that produces $\psi_\varepsilon(0) = 0$; we then expect the existence of a boundary layer near $x = 1$. Introduce the local variable

$$\eta = \frac{1-x}{\delta(\varepsilon)}.$$

Looking for a significant degeneration of the operator written in the variable η, we find $\delta(\varepsilon) = \varepsilon$. Expanding

$$\psi_\varepsilon(1 - \varepsilon\eta) = \sum_{n=0}^{m} \varepsilon^n \bar{\psi}_n(\eta) + O(\varepsilon^{m+1}),$$

we find

$$\bar{L}_0^*\bar{\psi}_0 = a_2(1)\frac{d^2\bar{\psi}_0}{d\eta^2} - b_1(1)\frac{d\bar{\psi}_0}{d\eta} = 0,$$

$$\bar{L}_0^*\bar{\psi}_n = F_n(\bar{\psi}_0, ..., \bar{\psi}_{n-1}, \eta), \quad n = 1, 2, \cdots .$$

Putting $\omega = -b_1(1)/a_2(1)$, we have

$$\bar{\psi}_0(\eta) = B + Ce^{-\omega\eta}.$$

The matching relation is

$$\lim_{\eta\to\infty} \bar{\psi}_0(\eta) = 0$$

so that $B = 0$. (Note that $\omega > 0$.) The boundary condition yields

$$C = \beta - \phi_0(1).$$

We compose an expansion from regular expansions in two subdomains, in the variables x and η, respectively, to obtain

$$\phi_\varepsilon(x) = \phi_0(x) + (\beta - \phi_0(1))e^{-\omega(1-x)/\varepsilon} + O(\varepsilon),$$

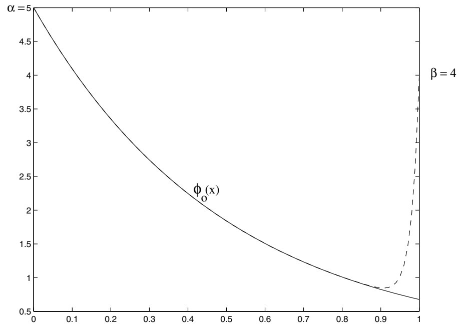

Fig. 5.2. Matching at one endpoint for the equation $\varepsilon d^2\phi/dx^2 - d\phi/dx - 2\phi = 0, \phi(0) = 5, \phi(1) = 4, \varepsilon = 0.02$.

where

$$\begin{aligned}\phi_0(x) &= \alpha e^{-g(x)} + e^{-g(x)} \int_0^x e^{g(t)} f(t)\,dt,\\ g(x) &= \int_0^x \frac{b_0(t)}{b_1(t)}\,dt.\end{aligned}$$

For an illustration, see Fig. 5.2.

Remark

When omitting the $O(\varepsilon)$ terms, do we have a formal approximation of the solution? The boundary condition at $x = 0$ is satisfied with an exponentially small error. However, on calculating $(\varepsilon L_1 + L_0)\bar{\psi}_0$, we find a result that is $O_s(1)$, so we have not obtained a formal approximation. It is easy to see that to obtain a formal approximation we have to include the $O(\varepsilon)$ terms of the expansion. This is in contrast with the calculation in the preceding section.

On the other hand, it can be proved (see Section 5.6 for references) that on omitting the $O(\varepsilon)$ terms we have an *asymptotic* approximation of the solution! This looks like a paradox, but one should realise that a second-order linear ODE is characterised by a two-dimensional solution space. In the case of a scalar equation this is a space spanned by the solution and its derivative. In the problem at hand, omitting the $O(\varepsilon)$ terms produces an asymptotic approximation of the solution but not of the derivative.

Remark

Note that the location of the boundary layer is determined by the sign of $b_1(x)/a_2(x)$. If we were to choose $b_1(x) > 0, \quad x \in [0, 1]$, the boundary layer would be located near $x = 0$, while the regular expansion in the variable x would extend to the boundary $x = 1$. Note, however, that it is not necessary to know this a priori, as the location of the boundary layer is determined while constructing the formal approximation.

5.3 The WKBJ Method

The method ascribed to Wentzel, Kramers, Brillouin, and Jeffreys plays a part in theoretical physics, in particular in quantum mechanics. One of the simplest examples is the analysis of the one-dimensional Schrödinger equation

$$\frac{d^2\psi}{dx^2} + (k^2 - U(x))\psi = 0.$$

$U(x)$ is the potential associated with the problem and k the wave number ($k/2\pi = 1/\lambda$ with λ the wavelength). We are looking for solutions of the Schrödinger equation with short wavelength (i.e., k is large). Over a few (short) wavelengths, $U(x)$ will not vary considerably, so it seems reasonable to introduce an effective wave number $q(x)$ by

$$q(x) = \sqrt{k^2 - U(x)}$$

and propose as a first approximation of the Schrödinger equation

$$\tilde{\psi} = e^{\pm i \int q(x)\, dx}.$$

One expects that this type of formal approximation may break down if, after all, $U(x)$ changes very quickly ($dU/dx \gg 1$) or if $k^2 - U(x)$ has zeros. Both situations occur in practice; in the case of zeros of $k^2 - U(x)$, one usually refers to turning points. For a discussion of a number of applications of the WKBJ method in physics, the reader may consult Morse and Feshbach (1953, Vol. II, Chapter 9.3). Here we shall explore the method from the point of view of asymptotic analysis for one-dimensional problems. Consider the two-point boundary value problem

$$\varepsilon \frac{d^2\phi}{dx^2} - w(x)\phi = 0, \quad 0 < x < 1.$$

$w(x)$ is sufficiently smooth and positive in $[0, 1]$ with boundary values $\phi(0) = \alpha, \quad \phi(1) = \beta$. We analysed this problem in this chapter to find boundary layers near $x = 0$ and $x = 1$. We propose to interpret the WKBJ method as a regularising transformation in the following sense. We try to find solutions in the form

$$\exp.(Q(x)/\delta(\varepsilon)).$$

The regularisation assumption implies that we expect Q to have a regular expansion that is valid in the whole domain. In the case of the two-point boundary value problem, we substitute

$$\phi = \exp.(Q/\sqrt{\varepsilon})$$

to find

$$\sqrt{\varepsilon}Q'' + (Q')^2 - w(x) = 0.$$

This does not look like an equation with a regular expansion. However, if it has one, and because of $\sqrt{\varepsilon}$ in the equation, we expect such a regular expansion to take the form

$$Q_\varepsilon(x) = \sum_{n=0}^{2m} \varepsilon^{n/2} q_n(x) + O(\varepsilon^{m+\frac{1}{2}}).$$

We find after substitution

$$\begin{aligned} (q_0')^2 &= w(x), \\ 2q_0'q_1' &= -q_0'', \quad \text{etc.} \end{aligned}$$

with solutions

$$\begin{aligned} q_0(x) &= \pm \int_0^x \sqrt{w(t)}\, dt + C_0, \\ q_1(x) &= -\ln w^{\frac{1}{4}}(x) + C_1. \end{aligned}$$

The original differential equation is linear and has two independent solutions. Using the first two terms q_0, q_1 to determine Q, we find from the calculation up to now two expressions that we propose to use as approximations for the independent solutions:

$$\begin{aligned} \psi_1(x) &= \frac{1}{w^{\frac{1}{4}}(x)} e^{-\frac{1}{\sqrt{\varepsilon}} \int_0^x \sqrt{w(t)}\, dt}, \\ \psi_2(x) &= \frac{1}{w^{\frac{1}{4}}(x)} e^{-\frac{1}{\sqrt{\varepsilon}} \int_x^1 \sqrt{w(t)}\, dt}. \end{aligned}$$

Note that $\psi_1(1)$ and $\psi_2(0)$ are exponentially small. A linear combination of ψ_1 and ψ_2 should represent a formal approximation of the boundary value problem; we put

$$\tilde{\phi}_\varepsilon(x) = A\psi_1(x) + B\psi_2(x).$$

Imposing the boundary values, we have

$$\begin{aligned} A &= \alpha w^{\frac{1}{4}}(0) + O(e^{-\Omega/\sqrt{\varepsilon}}), \\ B &= \beta w^{\frac{1}{4}}(1) + O(e^{-\Omega/\sqrt{\varepsilon}}), \end{aligned}$$

with $\Omega = \int_0^1 w(t)\, dt$. In Section 5.1, we found a formal approximation $\tilde{\phi}_\varepsilon(x)$ of the boundary value problem with two boundary layers and a regular expansion identically zero. To compare the results, we expand $\tilde{\phi}_\varepsilon(x)$ with respect to x and $1-x$:

$$\tilde{\phi}_\varepsilon(x) = \alpha e^{-\sqrt{w(0)}x/\sqrt{\varepsilon}} + \beta e^{-\sqrt{w(1)}(1-x)/\sqrt{\varepsilon}} + o(1), \quad x \in [0,1].$$

So, to a first approximation, the results of the boundary layer method in Section 5.1 (which can be proved to be asymptotically valid) agree with the results of the WKBJ method. The quantitative difference between the methods can be understood in terms of relative and absolute errors. One can show that in this case one can write for the independent solutions ϕ_1 and ϕ_2 of the original differential equation solutions ϕ_1 and ϕ_2 of the original differential equation

$$\phi_1 = \psi_1(1 + O(\sqrt{\varepsilon})),$$
$$\phi_2 = \psi_2(1 + O(\sqrt{\varepsilon})), \quad x \in [0, 1].$$

The error outside the boundary layers is in the case of our boundary layer expansion an *absolute* one of order $\sqrt{\varepsilon}$; in the case of the WKBJ expansion, we have a *relative* error $O(\sqrt{\varepsilon})$, and as ψ_1, ψ_2 are exponentially decreasing, this is a much better result. However, this advantage of the WKBJ method is lost in slightly more general perturbation problems, as it rests on the regular expansion being identically zero. As soon as we find nontrivial regular expansions, the corresponding errors destroy the nice exponential estimates. Finally, we note that proofs of asymptotic validity involving WKBJ expansions are still restricted to relatively simple cases.

5.4 A Curious Indeterminacy

If we omit some of the assumptions of the preceding sections, the expansion and matching techniques that we have introduced may fail to determine the approximation. We shall demonstrate this for an example where we have an exact solution. The phenomenon itself is interesting but, even more importantly, it induced Grasman and Matkowsky (1977) to develop a new method to resolve the indeterminacy. We shall discuss this at the end of this section.

Consider the boundary value problem (see also Kevorkian and Cole, 1996, Section 2.3.4)

$$\varepsilon \frac{d^2\phi}{dx^2} - x\frac{d\phi}{dx} + \phi = 0, -1 < x < +1,$$
$$\phi(-1) = \alpha, \phi(+1) = \beta.$$

Assuming the existence of a regular expansion of the form

$$\psi_\varepsilon(x) = \sum_{n=0}^{m} \varepsilon^n \phi_n(x) + O(\varepsilon^{m+1}),$$

we find

$$-x\frac{d\phi_0}{dx} + \phi_0 = 0$$

so that $\phi_0(x) = c_0 x$ and actually to any order $\phi_n(x) = c_n x$ with $c_n, n = 0, \cdots, m$ arbitrary constants. We can satisfy one of the boundary conditions by setting either $-c_0 = \alpha$ or $+c_0 = \beta$.

In fact, if $\alpha = -\beta$, $\phi_0(x)$ solves the boundary value problem exactly. In the following we assume that $\alpha \neq -\beta$ with the presence of a boundary layer near $x = -1$ or $x = +1$. We expect that one of the choices will lead to an obstruction when trying to match the boundary layer solution to the regular expansion. Subtraction of the regular expansion by

$$\phi_\varepsilon(x) = \sum_{n=0}^{m} \varepsilon^n c_n x + \psi_\varepsilon(x)$$

leads to

$$\varepsilon\frac{d^2\psi}{dx^2} - x\frac{d\psi}{dx} + \psi = O(\varepsilon^{m+1}), \psi(-1) = \alpha + c_0 - \varepsilon\cdots, \psi(+1) = \beta - c_0 + \varepsilon\cdots .$$

Suppose we have a boundary layer near $x = -1$ with local variable

$$\xi = \frac{x+1}{\varepsilon^\nu}.$$

We find

$$\varepsilon^{1-2\nu}\frac{d^2\psi}{d\xi^2} - \frac{\varepsilon^\nu\xi - 1}{\varepsilon^\nu}\frac{d\psi}{d\xi} + \psi = O(\varepsilon^{m+1}),$$

with a significant degeneration for $\nu = 1$; expanding ψ_ε produces to first order

$$\frac{d^2\psi_0}{d\xi^2} + \frac{d\psi_0}{d\xi} = 0$$

with solution

$$\psi_0(\xi) = A_1 + A_2 e^{-\xi}$$

with A_1, A_2 constants. The matching relation will be

$$\lim_{\xi\to\infty}\psi_0(\xi) = 0$$

so that $A_1 = 0$; the boundary condition yields $A_2 = \alpha + c_0$.

We expect no boundary layer near $x = +1$; let's check this. Introduce the local variable

$$\eta = \frac{1-x}{\varepsilon^\nu}$$

so that we have locally

$$\varepsilon^{1-2\nu}\frac{d^2\bar\psi}{d\eta^2} + \frac{1-\varepsilon^\nu\eta}{\varepsilon^\nu}\frac{d\bar\psi}{d\eta} + \bar\psi = O(\varepsilon^{m+1})$$

with a significant degeneration for $\nu = 1$. To first order, we find

$$\frac{d^2\bar\psi_0}{d\eta^2} + \frac{d\bar\psi_0}{d\eta} = 0$$

with solution

$$\bar\psi_0(\eta) = B_1 + B_2 e^{-\eta}$$

with B_1, B_2 constants. The matching relation

$$\lim_{\eta\to\infty}\bar\psi_0(\eta) = 0$$

produces $B_1 = 0$, and the boundary condition yields $B_2 = \beta - c_0$.

It turns out there is no obstruction to the presence of boundary layers near $x = -1$ *and* near $x = +1$. The approximation obtained until now takes the form

$$\phi_\varepsilon(x) = c_0 x + (\alpha + c_0)e^{-\frac{x+1}{\varepsilon}} + (\beta - c_0)e^{-\frac{1-x}{\varepsilon}} + O(\varepsilon)$$

with undetermined constant c_0. It can easily be checked that introducing higher-order approximations does not resolve the indeterminacy.

This is an unsatisfactory situation. In what follows we analyse the exact solution, which luckily we have in this case. In general, this is not a possible option and we shall discuss a general method that enables us to resolve the indeterminacy.

As $\phi_\varepsilon(x) = x$ solves the equation, we can construct a second independent solution to obtain the general solution

$$\phi_\varepsilon(x) = C_1 x + C_2 \left(e^{\frac{x^2}{2\varepsilon}} - \frac{x}{\varepsilon} \int_{-1}^{x} e^{\frac{t^2}{2\varepsilon}} dt \right).$$

We assume again $\alpha \neq -\beta$ to avoid this simple case. C_1 and C_2 are determined by the boundary conditions. Analysing the exact solution is quite an effort; see Exercise 5.6 or Kevorkian and Cole (1996). The conclusion is that indeed near $x = -1$ and $x = +1$ a boundary layer of size $O(\varepsilon)$ exists. In the interior of the interval, there exists a regular expansion with first-order term $c_0 x, c_0 = (\beta - \alpha)/2$.

The method developed by Grasman and Matkowsky (1977) to resolve the indeterminacy is based on variational principles. The solution to our boundary value problem can be viewed as the element of the set

$$V = \{C^2(-1, +1) | y(-1) = \alpha, y(+1) = \beta\}$$

that extremalises the functional

$$I_\varepsilon = \int_{-1}^{+1} L(x, \phi, \phi'; \varepsilon) dx,$$

where L is a suitable Lagrangian function. Extremalisation of the functional leads to the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \phi'}\right) - \frac{\partial L}{\partial \phi} = 0.$$

For a general reference to variational principles, see Stakgold (2000). In the case of the equation

$$\varepsilon\phi'' - x\phi' + \phi = 0,$$

a suitable Lagrangian function is

$$L = \frac{1}{2}(\varepsilon\phi'^2 - \phi^2)e^{-\frac{x^2}{2\varepsilon}}.$$

The approximation that we derived can be seen as a one-parameter family of functions, a subset of V, parameterised by c_0. We can look for a member of this family that extremalises the functional I_ε by substituting the expression and looking for an extremal value by satisfying the condition

$$\frac{dI_\varepsilon}{dc_0} = 0.$$

Keeping the terms of L to leading order in ε, we find again $c_0 = (\beta - \alpha)/2$. This result has been obtained without any explicit knowledge of the exact solution.

We can apply this elegant method to many other boundary value problems where a combination of boundary layer and variational methods is fruitful.

5.5 Higher Order: The Suspension Bridge Problem

Following Von Kármán and Biot (1940), we consider a model for a suspension bridge consisting of a beam supported at the endpoints and by hangers attached to a cable. Without a so-called live load on the bridge, the cable assumes a certain shape while bearing the beam that forms the bridge (the dead weight position). Adding a live load, and upon linearising, we obtain an equation describing the deflection $w(x)$ from the dead weight position of the cable,

$$\varepsilon \frac{d^4 w}{dx^4} - \frac{d^2 w}{dx^2} = p(x).$$

On deriving the equation, we have assumed that the beam and cable axes are lined up with the x-axis and that the total tension in the cable is large relative to the flexural rigidity (Young's elasticity modulus times the inertial moment). Also, we have rescaled such that $0 \le x \le 1$; $p(x)$ represents the result of a dead weight and live load. Natural boundary conditions are clamped supports at the endpoints which means

$$w(0) = w(1) = 0; \quad w'(0) = w'(1) = 0.$$

We assume that in a subdomain of $[0, 1]$ a regular expansion exists of the form

$$w(x) = \sum_{n=0}^{m} \varepsilon^n w_n(x) + O(\varepsilon^{m+1}).$$

We find after substitution

$$-\frac{d^2 w_0}{dx^2} = p(x), \quad \frac{d^2 w_n}{dx^2} = \frac{d^4 w_{n-1}}{dx^4}, \quad n = 1, 2, \cdots .$$

The second derivative $d^2 w/dx^2$ is inversely proportional to the curvature and so proportional to the bending moment of the cable. In the domain of the regular expansion, tension induced by the load $p(x)$ dominates the elasticity effects. Solving the lowest-order equation, we have

$$w_0(x) = -\int_0^x \int_0^s p(t)dtds + ax + b$$

with a and b constants. On assuming $p(x)$ to be sufficiently differentiable, we obtain higher-order terms of the same form.

We cannot apply the four boundary conditions, but a good choice turns out to be

$$w_0(0) = w_0(1) = 0.$$

We could also leave this decision until matching conditions have to be applied, but we shall run ahead of this. We find

$$a = \int_0^1 \int_0^s p(t)dtds, b = 0.$$

Subtracting the regular expansion

$$\psi(x) = w(x) - \sum_{n=0}^{m} \varepsilon^n w_n(x)$$

produces

$$\varepsilon \frac{d^4\psi}{dx^4} - \frac{d^2\psi}{dx^2} = O(\varepsilon^{m+1})$$

with boundary conditions

$$\psi(0) = \psi(1) = 0, \quad \psi\prime(0) = -a, \psi\prime(1) = \int_0^1 p(s)ds - a.$$

Expecting boundary layers at $x = 0$ and $x = 1$, we analyse what happens near $x = 0$; near $x = 1$ the analysis is similar. Introduce the local variable

$$\xi = \frac{x}{\varepsilon^\nu},$$

and the equation becomes

$$\varepsilon^{1-4\nu} \frac{d^4\psi^*}{d\xi^4} - \varepsilon^{-2\nu} \frac{d^2\psi^*}{d\xi^2} = O(\varepsilon^{m+1}).$$

A significant degeneration arises if $1 - 4\nu = -2\nu$ or $\nu = \frac{1}{2}$. Expanding $\psi^* = \psi_0(\xi) + \varepsilon^{1/2}\psi_1(\xi) + \cdots$, we have

$$\frac{d^4\psi_0}{d\xi^4} - \frac{d^2\psi_0}{d\xi^2} = 0$$

with boundary conditions

$$\psi_0(0) = \frac{\psi_0(0)}{d\xi} = 0$$

and general solution

$$\psi_0(\xi) = c_0 + c_1\xi + c_2 e^{-\xi} + c_3 e^{+\xi}.$$

Applying the matching condition

$$\lim_{\xi \to +\infty} \psi_0(\xi) = 0,$$

we have $c_0 = c_1 = c_3 = 0$; the boundary conditions yield $c_2 = 0$ so we have to go to the next order to find a nontrivial boundary layer contribution. Note that this is not unnatural because of the clamping conditions of the cable ($w(0) = w\prime(0) = 0$). For ψ_1 (and to any order) we have the same equation; the boundary conditions are

$$\psi_1(0) = 0, \frac{d\psi_1(0)}{d\xi} = -a.$$

However, applying the matching condition, we find again $c_0 = c_1 = c_3 = 0$, and we cannot apply both boundary conditions.

What is wrong with our assumptions and construction? At this point, we have to realise that the matching rule

$$\lim_{\xi \to +\infty} \psi(\xi) = 0$$

is matching in its elementary form. What we expect of matching is what this terminology expresses: the boundary layer expansion should be smoothly fitted to the regular (outer) expansion. If the boundary layer expansion is growing exponentially as $\exp(+\xi)$, there is no way to fit this behaviour with a regular expansion. Polynomial growth, however, is a different matter; a term such as $\varepsilon^{1/2}\xi$ behaves as x outside the boundary layer and poses no problem for incorporation in the regular expansion.

To allow for polynomial growth, we have to devise slightly more general matching rules. This will not be a subject of this chapter, but it is important to realise that one may encounter these problems. See for more details Section 6.2 and Section 15.4.

To illustrate this here and to conclude the discussion, one can compute the exact solution of the problem by variation of constants and by applying the boundary conditions. It is easier to look at the equation for $\psi(x)$ obtained by the subtraction trick. Its general solution is

$$\psi(x) = c_0 + c_1 x + c_2 e^{-x/\sqrt{\varepsilon}} + c_3 e^{(x-1)/\sqrt{\varepsilon}}.$$

Applying the boundary conditions, one finds that in general $c_1 = O_s(\sqrt{\varepsilon})$.

5.6 Guide to the Literature

Linear two-point boundary value problems have been studied by a number of authors. An elegant technique to prove asymptotic validity is the use of maximum principles. It was introduced by Eckhaus and De Jager (1966) to study elliptic problems. Such problems will be considered in Chapter 7; we give an example of a proof in Section 15.6. The technique of using maximum principles was applied extensively by Dorr, Parter, and Shampine (1973). Other general references for two-point boundary value problems, including turning-point problems, are Wasow (1965), Eckhaus (1979), Smith (1985), O'Malley (1991), and De Jager and Jiang Furu (1996). A number of basic aspects were analysed by Ward (1992, 1999).

An interesting phenomenon involving turning points is called Ackerberg-O'Malley resonance and has inspired a large number of authors. De Groen (1977, 1980) clarified the relation of this resonance with spectral properties of the related differential operator; for other references, see his papers.

Higher-dimensional linear boundary value problems can present themselves in various shapes. One type of problem is the linear scalar equation

$$\varepsilon\phi_\varepsilon^{(n)} + L_{n-1}\phi_\varepsilon = f(x)$$

with L_{n-1} a linear operator of order $(n-1)$ and appropriate boundary values. Another formulation is for systems of first-order equations of the form

$$\begin{aligned}\dot{x} &= a(t)x + b(t)y,\\ \varepsilon\dot{y} &= c(t)x + d(t)y,\end{aligned}$$

with x an n-dimensional vector, y an m-dimensional vector, a, b, c and d matrices, and appropriate boundary values added. The vector form is also relevant for control problems. For a more systematic treatment, see O'Malley (1991).

More details about the WKBJ method can be found in Eckhaus (1979), Vainberg (1989), O'Malley (1991) and Holmes (1998). For turning points, see Wasow (1984), Smith (1985), and De Jager and Jiang Furu (1996).

5.7 Exercises

Exercise 5.1 We consider the following boundary problem on $[0, 1]$:

$$\varepsilon\left(\frac{d^2\phi}{dx^2} + \arctan(x)\frac{d\phi}{dx} - e^{x^2}\cos(x)\phi\right) - \cos(x)\phi = x^2,$$

$$\phi(0) = \alpha, \phi(1) = \beta.$$

Compute a first-order approximation. Is this a formal approximation?

Exercise 5.2 Consider the following boundary value problem on $[0, 1]$:

$$\varepsilon \frac{d^2\phi}{dx^2} + \frac{d\phi}{dx} + \cos x\phi = \cos x,$$

$$\phi(0) = \alpha, \phi(1) = \beta.$$

Compute a first-order approximation. Is the approximation formal?

Exercise 5.3 Consider the boundary value problem

$$\begin{aligned}&\varepsilon \tfrac{d^2y}{dx^2} + (1+2x)\tfrac{dy}{dx} - 2y = 0, \quad x \in (0,1),\\ &y(0) = \alpha, \quad y(1) = \beta.\end{aligned}$$

a. Compute a first-order approximation of $y(x)$ using Section 5.2.
b. Compute a first-order approximation of $y(x)$ by the WKBJ method and compare the results of (a) and (b).

Note that for a one-parameter set of boundary values no boundary layer is present.

Exercise 5.4 Consider the so-called "turning-point problem":

$$\begin{aligned}&L_\varepsilon y = (\varepsilon L_1 + L_0)(y) = 0, x \in [0,1],\\ &L_1 = a_2(x)\tfrac{d^2}{dx^2} + a_1(x)\tfrac{d}{dx} + a_0(x), a_2(x) \neq 0,\\ &L_0 = b_1(x)\tfrac{d}{dx} + b_0(x).\end{aligned}$$

Suppose $b_1(x)$ has a simple zero $x_0 \in (0,1)$ and $b_0(x) \neq 0$ in $[0,1]$. This is usually called a turning-point problem.

a. Compute the significant degenerations of L_ε in a neighbourhood of $x = x_0$. Take for instance $b_1(x) = \beta_0(x - x_0) + \cdots$, where β_0 is a nonzero constant and the dots indicate the higher-order terms in $(x - x_0)$ so that $b_1(x)$ has a simple zero.
b. Does a significant degeneration arise if $b_1 \equiv 0$ in $[0,1]$ and b_0 has a simple zero?

Exercise 5.5 To recognise some of the difficulties arising with singular differential equations mentioned in the introduction to this chapter, we consider the Euler equation

$$\varepsilon(x^2y'' + 3xy') - y = 0, x \in (0,1),$$

$$y(0) = \alpha, y(1) = \beta, \alpha^2 + \beta^2 \neq 0, p \in \mathbb{R}.$$

a. Try to find a suitable local variable near $x = 0$.
b. Show that the boundary value problem has no solution.

Exercise 5.6 In Section 5.4 we obtained an exact solution,

$$\phi_\varepsilon(x) = C_1 x + C_2 \left(e^{\frac{x^2}{2\varepsilon}} - \frac{x}{\varepsilon} \int_{-1}^{x} e^{\frac{t^2}{2\varepsilon}} \, dt \right),$$

which has to satisfy the boundary conditions $\phi_\varepsilon(-1) = \alpha, \phi_\varepsilon(-1) = \beta$. We wish to determine the first-order term of the regular expansion in the interior of $[-1, +1]$. We also want to show that there exist boundary layers near $x = -1$ and $x = +1$. The calculation closely follows Kevorkian and Cole (1996), Section 2.3.4.

a. Apply the boundary conditions to find

$$C_1 = \frac{(\beta - \alpha)e^{1/2\varepsilon} + \alpha A(\varepsilon)}{2e^{1/2\varepsilon} - A(\varepsilon)}, C_2 = \frac{\beta + \alpha}{2e^{1/2\varepsilon} - A(\varepsilon)},$$

with

$$A(\varepsilon) = \frac{1}{\varepsilon} \int_{-1}^{+1} e^{t^2/2\varepsilon} dt.$$

In the following, we assume $\beta + \alpha \neq 0$. Putting $\beta + \alpha = 0$ eliminates the boundary layers and produces the exact solution $\phi_\varepsilon(x) = \beta x$.

b. Use Laplace's method (Chapter 3) to evaluate

$$e^{-1/2\varepsilon} A(\varepsilon) = 2(1 + \varepsilon + 3\varepsilon^2) + O(\varepsilon^3),$$

$$C_1 = -\frac{\beta + \alpha}{2\varepsilon} + \frac{3\beta + \alpha}{2} + O(\varepsilon), C_2 = (\beta + \alpha)e^{-1/2\varepsilon} \left(-\frac{1}{2\varepsilon} + \frac{3}{2} + O(\varepsilon) \right).$$

c. To expand $\phi_\varepsilon(x)$ in the interior of $[-1, +1]$, note that, away from the boundary layers, $C_2 e^{x^2/2\varepsilon}$ is exponentially small.

d. Again with Laplace's method show that in the interior of $[-1, +1]$ with $x \neq 0$

$$\int_{-1}^{x} e^{\frac{t^2}{2\varepsilon}} \, dt = \varepsilon e^{1/2\varepsilon}(1 + \varepsilon + O(\varepsilon^2)) + \frac{2\varepsilon}{x} \left(1 + \frac{\varepsilon}{x^2} + O(\varepsilon^2) \right) e^{x^2/2\varepsilon}.$$

e. Conclude that in the interior of $[-1, +1]$

$$\phi_\varepsilon(x) = \frac{\beta - \alpha}{2} x + O(\varepsilon).$$

f. Introduce local variables to analyse the exact solution near the endpoints.