4

Boundary Layer Behaviour

In this chapter, we take a closer look at boundary layer phenomena. The tools we shall develop for our analysis are local boundary layer variables and degenerations of operators.

4.1 Regular Expansions and Boundary Layers

In Chapter 1, we considered two examples of first-order differential equations. For the solutions, we substituted a formal expansion of the form

$$\phi_\varepsilon(x) = \sum_{n=0}^{\infty} \varepsilon^n \phi_n(x).$$

In the case of the problem

$$\phi' + \varepsilon\phi = \cos x, \phi(0) = 0,$$

this led to a consistent formal expansion. In the problem

$$\varepsilon\phi' + \phi = \cos x, \phi(0) = 0,$$

by comparing this with the exact solution, we have shown that the formal expansion is far too simple to represent the solution. The difficulty with the formal expansion arises when applying the boundary condition; a problem of this type is often called a *singular perturbation* or *boundary layer problem.* The asymptotic expansion in the second case looks like

$$\phi_\varepsilon(x) = \sum_{n=0}^{\infty} \delta_n(\varepsilon)\psi_n(x, \varepsilon).$$

The simpler expansion that we used in the first case will be called *regular.* Note that in the literature the term "regular" is used in many ways.

Definition
Consider the function $\phi_\varepsilon(x)$ defined on $D \subset \mathbb{R}^n$; an asymptotic expansion for $\phi_\varepsilon(x)$ will be called regular if it takes the form

$$\phi_\varepsilon(x) = \sum_{n=0}^{m} \delta_n(\varepsilon)\psi_n(x) + 0(\delta_{m+1})$$

with $\delta_n(\varepsilon), n = 0, 1, \cdots$ an asymptotic sequence and $\psi_n(x), n = 0, 1, \cdots$ functions on D.

It turns out that in studying a function $\phi_\varepsilon(x)$ on a domain D we have to take into account that regular expansions of $\phi_\varepsilon(x)$ often only exist on subdomains of D.

Example 4.1
Consider the function

$$\phi_\varepsilon(x) = e^{-x/\varepsilon} + e^{\varepsilon x}, x \in [0, 1].$$

On any subdomain $[d, 1]$ with $0 < d < 1$ a constant independent of ε, we have the regular expansion

$$\phi_\varepsilon(x) = \sum_{n=0}^{m} \varepsilon^n \frac{x^n}{n!} + 0(\varepsilon^{m+1}).$$

However, there exists no regular expansion in functions of x on $[0, d]$ or $[0, 1]$.

Example 4.2
The function $\phi_\varepsilon(x)$ is for $x \in [0, 1]$ defined as the solution of the two-point boundary value problem

$$\varepsilon\phi'' + \phi' = 0, \phi(0) = 1, \phi(1) = 0.$$

The solution is

$$\phi_\varepsilon(x) = \frac{e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}} - \frac{e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

Ignoring this exact solution and looking for a regular expansion by substituting into the differential equation

$$\phi_\varepsilon(x) = \sum_{n=0}^{m} \varepsilon^n \phi_n(x),$$

we find

$$\phi_0' = 0,$$

$$\phi_0'' + \phi_1' = 0, \text{ etc.}$$

This leads to $\phi_0(x)$ = constant, $\phi_1(x)$ = constant, etc. There is no way to satisfy the boundary conditions with the regular expansion.

The exact solution can be written as

$$\phi_\varepsilon(x) = e^{-x/\varepsilon} + 0(e^{-1/\varepsilon}),$$

which shows that the behaviour of the solutions is different in two subdomains of $[0, 1]$: in a small region of size $0(\varepsilon)$ near $x = 0$, the solution decreases very rapidly from 1 towards 0; in the remaining part of $[0, 1]$, the solution is very near 0. The regular expansion is valid here with, rather trivially, $\phi_n(x) = 0, n = 0, 1, \cdots$. Note that even in the domain where the regular expansion is valid, the choice to expand with respect to order functions of the form ε^n is not a fortunate one as $e^{-1/\varepsilon} = o(\varepsilon^n), n = 1, 2, \cdots$.

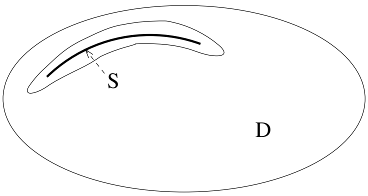

Fig. 4.1. Domain D with boundary layer near S.

The small region near $x = 0$ where no regular expansion exists in Example 4.2, is called a boundary layer. We now characterise such regions.

4.1.1 The Concept of a Boundary Layer

Consider the function $\phi_\varepsilon(x)$ defined on $D \subseteq \mathbb{R}^n$. Suppose there exists a connected subset $S \subset D$ of dimension $\leq n$, with the property that $\phi_\varepsilon(x)$ has no regular expansion in each subset of D containing points of S (see Fig. 4.1). Then a neighbourhood of S in D with a size to be determined, will be called a boundary layer of the function $\phi_\varepsilon(x)$.

In Examples 4.1 and 4.2 the domain is one-dimensional. A boundary layer, corresponding with the subset S, has been found near the boundary point $x = 0$ of the domain. Note that in applications we also may find a boundary layer in the interior of the domain and, if we have an evolution equation, the boundary layer may even be moving in time.

To study the behaviour of a function $\phi_\varepsilon(x)$ in a boundary layer, a fundamental technique is to use a *local analysis*. This is a technique that we shall meet again and again. In this chapter, we consider local analysis in a one-dimensional context. Later we shall meet higher-dimensional problems.

Suppose that near a point $x_0 \in S$ the boundary layer is characterised in size by an order function $\delta(\varepsilon)$. We "rescale" or "stretch" the variable x by introducing the local variable

$$\xi = \frac{x - x_0}{\delta(\varepsilon)}.$$

If $\delta(\varepsilon) = o(1)$, we call ξ a local (stretched or boundary layer) variable. The function $\phi_\varepsilon(x)$ transforms to

$$\begin{aligned} \phi_\varepsilon(x) &= \phi_\varepsilon(x_0 + \delta(\varepsilon)\xi) \\ &= \phi^*_\varepsilon(\xi). \end{aligned}$$

It is then natural to continue the local analysis by expanding the function ϕ^*_ε with respect to the local variable ξ; we hope to find again a *regular* expansion. To be more precise, assume that

$$\phi^*_\varepsilon(\xi) = 0_s(1) \ \text{ near } \ \xi = 0.$$

We wish to find local approximations of ϕ^*_ε by a regular expansion of the form

$$\phi^*_\varepsilon(\xi) = \sum_n \delta^*_n(\varepsilon)\psi_n(\xi)$$

with $\delta^*_n(\varepsilon), n = 0, 1, 2, \cdots$ an asymptotic sequence.

When solving the problem of approximating a function $\phi_\varepsilon(x)$ in a domain D, our program will then be as follows.

1. Try to construct a regular expansion in the original variable x. This is possible outside the boundary layers (by definition), and this expansion is usually called the *outer expansion.*
2. Construct in the boundary layer(s) a local expansion in an appropriate local variable. Such a regular expansion is usually called the *inner expansion* or *boundary layer expansion.*
3. The inner and outer expansions should be matched to obtain a formal expansion for the whole domain D. As we shall see later, in a number of problems, techniques have been developed to combine the three stages, which makes the process more efficient. This formal expansion, which is valid in the whole domain, is sometimes called a uniform expansion. Note, however, that in the literature and also in this book, expressions that are called "uniformly valid expansion" are more often than not formal expansions. So we come to the next point.
4. Prove that formal expansions, obtained in the stages 1–3, represent valid asymptotic approximations of the function $\phi_\varepsilon(x)$ that we set out to study.

In many problems, the function $\phi_\varepsilon(x)$ has been implicitly defined as the solution of a system of differential equations with initial and/or boundary conditions. To study such a problem by perturbation theory, we have to be more precise in the use of the expressions "formal expansion" and "formal approximation". Suppose that we have to study the perturbation problem

$$L_\varepsilon \phi = f(x), x \in D + \quad \text{other conditions.}$$

L_ε is an operator containing a small parameter ε. For instance, in Example 4.2 we have

$$L_\varepsilon = \varepsilon \frac{d^2}{dx^2} + \frac{d}{dx}$$

$D = [0, 1]$, $f(x) = 0$ and boundary conditions $\phi(0) = 1, \phi(1) = 0$.

The function $\tilde{\phi}_\varepsilon(x)$ will be called a *formal approximation* or *formal expansion* of $\phi_\varepsilon(x)$ if $\tilde{\phi}$ satisfies the boundary conditions to a certain approximation and if

$$L_\varepsilon \tilde{\phi} = f(x) + o(1).$$

We shall see that to require $\tilde{\phi}$ to satisfy the boundary conditions in full is asking too much and in practice it suffices for $\tilde{\phi}$ to satisfy the boundary conditions to a certain approximation.

To prove that if $\tilde{\phi}$ is a formal approximation it also is an asymptotic approximation of ϕ is in general a difficult problem. Moreover, one can give simple and realistic examples in which this is not true. Remarkably enough, we shall later also meet cases where we have an asymptotic approximation that is not a formal approximation.

Example 4.3
(a formal approximation that is not asymptotic)
Consider the harmonic oscillator

$$\ddot{\phi} + (1 + \varepsilon)^2 \phi = 0, t \geq 0$$

with initial conditions $\phi(0) = 1, \dot{\phi}(0) = 0$. The solutions of the equation are bounded, and with these initial conditions we have

$$\phi(t) = \cos((1 + \varepsilon)t).$$

On the other hand, $\tilde{\phi}(t) = \cos t$ satisfies the initial conditions and moreover

$$\ddot{\tilde{\phi}} + (1 + \varepsilon)^2 \tilde{\phi} = 2\varepsilon \cos t + \varepsilon^2 \cos t = O(\varepsilon).$$

However, using the sup norm, it is easy to see that $\phi(t) - \tilde{\phi}(t) = 0_s(1), t \geq 0$.

4.2 A Two-Point Boundary Value Problem

We shall now illustrate the process of constructing a formal expansion for a simple boundary value problem.

Example 4.4
Consider the equation

$$\varepsilon \frac{d^2\phi}{dx^2} - \phi = f(x), x \in [0,1]$$

with boundary values $\phi_\varepsilon(0) = \phi_\varepsilon(1) = 0$. The function $f(x)$ is sufficiently smooth on $[0,1]$ to allow for the construction that follows. We note that the choice of boundary values equal to zero (homogeneous boundary values) is not a restriction: aif $\phi_\varepsilon(0) = \alpha, \phi_\varepsilon(1) = \beta$, we introduce $\psi_\varepsilon(x) = \phi_\varepsilon(x) - \alpha - (\beta - \alpha)x$, which produces zero boundary values for the problem in ψ_ε (while of course changing the right-hand side).

In the spirit of Section 3.1, we assume that in some subset D_0 of $[0,1]$ the solution has a regular expansion of the form

$$\phi_\varepsilon(x) = \sum_{n=0}^{m} \varepsilon^n \phi_n(x).$$

Substitution in the preceding equation produces successively

$$\phi_0(x) = -f(x),$$

$$\phi_n(x) = \tfrac{d^2\phi_{n-1}}{dx^2}, n = 1, 2, \cdots .$$

The expansion coefficients ϕ_n are determined completely by the recurrency relation so that we *cannot impose* the boundary conditions to the regular expansion. (Even if accidentally $f(0) = f(1) = 0$, the next order will change the boundary conditions again.) We conclude that for the regular expansion to make sense in D_0, this subset should not contain the boundary points $x = 0$ and $x = 1$ (see Fig. 4.2). Before analysing what is going on near the boundary

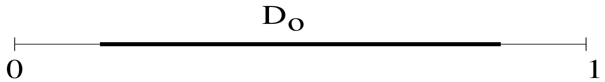

Fig. 4.2. Boundary layers near the end points.

points, we carry out the *subtraction trick*; this trick is of no fundamental importance but is computationally convenient, as it shifts the equation to a homogeneous one. For nonlinear equations, the subtraction trick can be less convenient. Introduce

$$\psi_\varepsilon(x) = \phi_\varepsilon(x) - \sum_{n=0}^{m} \varepsilon^n \phi_n(x).$$

The two-point boundary value problem for ψ_ε becomes

$$\varepsilon \frac{d^2\psi}{dx^2} - \psi = 0(\varepsilon^{m+1})$$

and

$$\psi_\varepsilon(0) = -\sum_{n=0}^{m} \varepsilon^n \phi_n(0), \psi_\varepsilon(1) = -\sum_{n=0}^{m} \varepsilon^n \phi_n(1).$$

Near $x = 0$, we introduce the local variable

$$\xi = \frac{x}{\delta(\varepsilon)}$$

with $\delta(\varepsilon) = o(1)$; at this point, we have no a priori knowledge of a suitable choice of $\delta(\varepsilon)$.

The equation with respect to this local variable becomes

$$\frac{\varepsilon}{\delta^2} \frac{d^2\psi^*}{d\xi^2} - \psi^* = 0(\varepsilon^{m+1}),$$

where $\psi^* = \psi_\varepsilon(\delta(\varepsilon)\xi)$. How to choose $\delta(\varepsilon)$ will be the subject of discussion later on; here we just remark that

$$\delta^2(\varepsilon) = \varepsilon \ \text{ or } \ \delta(\varepsilon) = \sqrt{\varepsilon}$$

seems to be a well-balanced choice. We now assume that there exists a regular expansion of ψ_ε^*, so

$$\lim_{\varepsilon \downarrow 0} \psi_\varepsilon^*(\xi) = \psi_0(\xi) \ \text{ exists}$$

and satisfies the formal limit equation

$$\frac{d^2\psi_0}{d\xi^2} - \psi_0 = 0.$$

ψ_0 would be the first term in a formal regular expansion in ξ near the point $x = 0$. Solving this limit equation, we find

$$\psi_0(\xi) = ae^{-\xi} + be^{+\xi}.$$

Imposing the boundary condition at $\xi = 0$, we have

$$a + b = -\phi_0(0) = f(0).$$

We have to find a second relation to determine a and b; for this we observe that the solution in the boundary layer near $x = 0$ should be matched with the regular expansion in D_0. Rewriting $\psi_0(\xi)$ in x, we note that the term $\exp(x/\sqrt{\varepsilon})$ becomes exponentially large with x in D_0 unless its coefficient b vanishes, so we propose the *matching relation* $\lim_{\xi\to\infty} \psi_0(\xi) = 0$ or $b = 0$ and we have

$$\psi_0(\xi) = f(0)e^{-\xi}.$$

We can proceed to calculate higher-order terms by assuming a regular expansion of the form

$$\psi_\varepsilon^*(\xi) = \sum_{n=0}^{m} \varepsilon^{n/2}\psi_n(\xi),$$

where for each term in the expansion we have again one boundary condition and one matching relation. The equations for $\psi_1, \psi_2, \cdots$ will become increasingly complicated. We can repeat this local analysis near $x = 1$ by introducing the local variable

$$\eta = \frac{1-x}{\sqrt{\varepsilon}}.$$

The calculation runs along exactly the same lines and is left to the reader. Adding the terms from the outer expansion and the two local (boundary layer) expansions, we find to first order the formal uniform expansion

$$\tilde{\phi}_\varepsilon(x) = -f(x) + f(0)e^{-x/\sqrt{\varepsilon}} + f(1)e^{-(1-x)/\sqrt{\varepsilon}} + \cdots .$$

The dots are standing for terms such as $\varepsilon\phi_1(x)$, $\sqrt{\varepsilon}\psi_1(\xi)$, etc. It follows from the construction that $\tilde{\phi}_\varepsilon(x)$ satisfies the equation to a certain approximation and it is a formal approximation of the solution, as we allow for exponentially small deviations of the boundary conditions. Later we shall return to the question of whether $\tilde{\phi}$ is an asymptotic approximation of ϕ. At this stage, it is interesting to note that one easily finds an affirmative answer by analysing the exact solution, which can be found by variation of constants.

4.3 Limits of Equations and Operators

We consider differential operators L_ε parametrised by ε. We are interested in discussing the limit as $\varepsilon \to 0$ of L_ε while keeping in mind our experience of the preceding sections where in various subdomains different variables have played a part. Consider again Example 4.4 of Section 4.2,

$$L_\varepsilon\phi = f(x), \phi_\varepsilon(0) = \phi_\varepsilon(1) = 0,$$

with

$$L_\varepsilon = \varepsilon\frac{d^2}{dx^2} - 1.$$

Taking the formal limit of L_ε as $\varepsilon \to 0$, we obtain

$$L_0 = -1.$$

Introducing local variables of course changes the limiting behaviour of the operator. Consider for instance the boundary layer near $x = 0$ and introduce

$$\xi = \frac{x}{\delta(\varepsilon)} \text{ with } \delta(\varepsilon) = o(1).$$

We find

$$L_\varepsilon^* = \frac{\varepsilon}{\delta^2}\frac{d^2}{d\xi^2} - 1.$$

Our choice of $\delta(\varepsilon)$ determines which operator L_ε^* degenerates into as $\varepsilon \to 0$. For instance, if $\delta^2(\varepsilon) = \varepsilon$, then

$$L_0^* = \frac{d^2}{d\xi^2} - 1.$$

If $\varepsilon = o(\delta^2(\varepsilon))$, and for instance $\delta = \varepsilon^{1/4}$, we have

$$L_0^* = -1.$$

If $\delta^2(\varepsilon) = o(\varepsilon)$, the limit does not exist and it makes sense to rescale,

$$\frac{\delta^2}{\varepsilon}L_\varepsilon^* = L_\varepsilon^{**}.$$

We find the degeneration

$$L_0^{**} = \frac{d^2}{d\xi^2}.$$

Making these calculations, we observe that, taking the formal limits, the degenerations of the operator in the cases $\varepsilon = o(\delta^2)$ and $\delta^2 = o(\varepsilon)$ are *contained* in the degeneration obtained on choosing $\delta^2 = \varepsilon$. We call L_0^* in this case,

$$L_0^* = \frac{d^2}{d\xi^2} - 1,$$

a *significant degeneration* of the operator L_ε near $x = 0$. Put in a different way, a significant degeneration implies a well-balanced choice of the local variable ξ such that the corresponding operator as $\varepsilon \to 0$ contains as much information as possible.

Note that a number of authors are using the term *distinguished limit* instead of significant degeneration.

Definition

Consider the operator L_ε, written in the variable x, near the boundary layer point $x = x_0$ and the operator L_ε rewritten in all possible local variables of the form $(x - x_0)/\delta(\varepsilon)$ near x_0. L_0^* is called a significant degeneration of L_ε if L_0^* is obtained by writing L in the local variable ξ and taking the formal limit as $\varepsilon \to 0$ (possibly after rescaling), whereas the corresponding degenerations in the other local variables are contained in L_0^*.

Remark

In practice, many operators can be analysed for significant degenerations by considering the set of order functions $\delta(\varepsilon) = \varepsilon^\nu, \nu > 0$. For instance, in our example

$$L_\varepsilon = \varepsilon \frac{d^2}{dx^2} - 1,$$

we have, introducing the local variable $\xi_\nu = x/\varepsilon^\nu$,

$$L_\varepsilon = \varepsilon^{1-2\nu} \frac{d^2}{d\xi_\nu^2} - 1,$$

$$\nu = \frac{1}{2},\ L_0^* = \frac{d^2}{d\xi_{\frac{1}{2}}^2} - 1,$$

$$\nu > \frac{1}{2},\ L_0^* = \frac{d^2}{d\xi_\nu^2},$$

$$\nu < \frac{1}{2},\ L_0^* = -1.$$

Remark

This definition is too simple in some cases. For instance, it is possible that near a point $x = x_0$, more than one significant degeneration exists corresponding with a more complicated boundary layer structure. For these cases, we have to adjust the definition somewhat, but we omit this here.

By introducing the concept of significant degeneration, we have a formal justification of our choice of boundary layer variables in the problem of Example 3.3 in Section 3.2. The underlying assumption here and in the following is that on analysing the significant degenerations of an operator, we find the correct boundary layer variables and the corresponding expansions in these variables. It turns out that this asumption works very well in most problems.

Still we have to keep in mind the possibility of hidden pitfalls. One of the assumptions in the construction is that the problem obtained by taking the formal limit of the operator does have something to do with the original problem. That is, if we study the function $\phi_\varepsilon(x)$ in a domain D given by the equation

$$L_\varepsilon \phi = f_\varepsilon(x),$$

by taking the limit we have

$$L_0 \psi = f_0(x).$$

Does this mean that we have $\lim_{\varepsilon \downarrow 0} \phi_\varepsilon(x) = \psi(x)$ in some nontrivial subdomain $D_0 \subset D$?

In applications, the answer seems to be affirmative. The reason is that equations in practice have been obtained by modelling reality. In these equations, various distinct effects or forces (in mechanics) play a part. For a degeneration to make sense, we have that locally in space or time one or a few of these effects or forces is dominant. This is quite natural in applications. Mathematically, this is not a simple question, and we give an example, admittedly artificial, to show that we may have a serious problem here.

Example 4.5
(Eckhaus, 1979)
Consider the following initial value problem in a neighbourhood of $x = 0$:

$$L_\varepsilon \phi = f_\varepsilon(x), \phi_\varepsilon(0) = 1, \phi'_\varepsilon(0) = 0,$$
$$L_\varepsilon = \varepsilon^3 \cos(\frac{x}{\varepsilon^2})\frac{d^2}{dx^2} + \varepsilon \sin(\frac{x}{\varepsilon^2})\frac{d}{dx} - 1,$$
$$f_\varepsilon(x) = -\varepsilon(1 - \cos(\frac{x}{\varepsilon^2})).$$

Taking the formal limit $L_0\psi = f_0(x)$, we find $\psi(x) = 0$, but the solution of the problem is

$$\phi_\varepsilon(x) = 1 + \varepsilon - \varepsilon \cos\left(\frac{x}{\varepsilon^2}\right)$$

with, for all $x \geq 0$,

$$\lim_{\varepsilon \to 0} \phi_\varepsilon(x) = 1.$$

Note that $\psi(x) = 0$ is not a formal approximation since it does not satisfy the boundary conditions. However, more dramatically, there is also no subdomain where $\psi(x)$ represents an asymptotic approximation of ϕ. On the other hand, the function $\phi_0(x) = 1$ does represent an asymptotic approximation, but this function does not satisfy the limit equation!

Fortunately, this example is not typical for applications, but it is instructive to keep it in mind as a possible phenomenon. It also illustrates the importance of giving proofs of asymptotic validity.

4.4 Guide to the Literature

The idea of a boundary layer orginates from physics, in particular fluid mechanics; see Prandtl (1905) and Prandtl and Tietjens (1934). Degenerations of operators, significant degenerations in local variables, and matching techniques are more recent concepts. Both the terminology and the techniques may take rather different forms.

One of the approaches to validate matching is the assumption of the so-called *overlap hypothesis*. This assumes that if one has two neighbouring local expansions or a neighbouring local and a regular expansion, there exists a common subdomain where both expansions are valid. This provides a sufficient condition to match the expansions by using in this subdomain intermediate variables. Such variables were considered by Kaplun and Lagerstrom (1957). More discussion of matching rules is found in Section 6.2 and Section 15.4.

The foundations are further discussed by Van Dyke (1964), Fraenkel (1969), Lagerstrom and Casten (1972), Eckhaus (1979) and Kevorkian and Cole (1996).

A different approach to identify scales and layers is to use blow-up transformations; see Krupa and Szmolyan (2001) and Popović and Szmolyan (2004).

The theory of singular perturbations as far as boundary layer theory is concerned, is still largely a collection of inductive methods in which taste and inventiveness play an important part.

4.5 Exercises

Exercise 4.1 Compute a second-order approximation of Example 4.4 treated in Section 4.2. Discuss the asymptotic character of the approximation.

Exercise 4.2 Consider the boundary value problem

$$\begin{cases} \varepsilon y'' + y' + y = 0, \\ y(0) = a, y(1) = b. \end{cases}$$

Compute a first-order approximation of the solution $y_\varepsilon(x)$ of the boundary value problem and compare this approximation with the exact solution. Is this first-order approximation a formal approximation?

Exercise 4.3 Consider the operator

$$L_\varepsilon = \varepsilon(\varepsilon^2 + x - 1)\frac{d}{dx} + \varepsilon(\varepsilon + 1) + x - 1.$$

a. Compute the significant degenerations of the operator L_ε in a neighbourhood of $x = 0$ and $x = 1$. (Show that the other degenerations are contained in them.)
b. To illustrate the result of (a), we solve the initial value problem

$$L_\varepsilon y = 0, y(1) = 1.$$

Compute the solution of this problem and compare it with the result in (a).

Exercise 4.4 Consider the boundary value problem

$$\begin{aligned} &\varepsilon L_1 y - y = 1, x \in (0, 1), \\ &y(0) = a, y(1) = b, a, b \neq 0, \\ &L_1 = (1 + x^2)\tfrac{d^2}{dx^2} + \tfrac{d}{dx} - x. \end{aligned}$$

Compute the first- and second-order terms of a formal approximation; show explicitly that this expansion is a formal approximation.