14

Wave Equations on Unbounded Domains

In the study of evolution equations describing wave phenomena on unbounded domains, one is confronted with a great many concepts and methods but usually only formal results. This is not a good reason to avoid the subject, as there are many interesting mathematical questions and physical phenomena in this field. Also, many parts of physics and engineering require a practical approach to real-life problems that cannot wait until rigorous mathematical methods are available.

So, this chapter will be different from the preceding ones because in some of the results discussed here a mathematical justification is lacking. This holds in particular when we are discussing perturbations of strongly nonlinear partial differential equations. The interest of the problems and the elegance of the methods will hopefully make up for this. Also, it may inspire exploration of the mathematical foundations of the methods discussed in this chapter.

14.1 The Linear Wave Equation with Dissipation

Consider as a simple example the wave equation with weak energy dissipation (damping)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \varepsilon \frac{\partial u}{\partial t},\ -\infty < x < \infty,\ t > 0,$$

with initial values $u(x, 0) = f(x)$, $u_t(x, 0) = 0$.

It is not difficult to see that a regular expansion of the form $u(x, t) = u_0(x, t) + \varepsilon u_1(x, t) + \cdots$ produces secular terms (Exercise 14.1). If $\varepsilon = 0$, it is useful to use characteristic coordinates

$$\xi = x - t,\ \eta = x + t.$$

From earlier experiences, we expect that the perturbation will also involve a timescale $\tau = \varepsilon t$ and maybe a spatial scale εx. For simplicity, we will consider a multiple-timescale expansion in three variables: ξ, η, and τ. Assuming that the solutions are C^2, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \ \frac{\partial}{\partial t} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + \varepsilon\frac{\partial}{\partial \tau},$$
$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2},$$
$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} + 2\varepsilon\left(-\frac{\partial^2}{\partial \xi \partial \tau} + \frac{\partial^2}{\partial \eta \partial \tau}\right) + \varepsilon^2\frac{\partial^2}{\partial \tau^2}.$$

The wave equation transforms to

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\varepsilon}{2}\left(-\frac{\partial^2 u}{\partial \xi \partial \tau} + \frac{\partial^2 u}{\partial \eta \partial \tau}\right) + \frac{\varepsilon}{4}\left(-\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \varepsilon^2 \cdots .$$

The multiple-timescale expansion is of the form

$$u = u_0(\xi, \eta, \tau) + \varepsilon u_1(\xi, \eta, \tau) + \varepsilon^2 \cdots .$$

Substitution in the transformed wave equation yields equations for u_0 and u_1:

$$\frac{\partial^2 u_0}{\partial \xi \partial \eta} = 0, \ \frac{\partial^2 u_1}{\partial \xi \partial \eta} = \frac{1}{2}\left(-\frac{\partial^2 u_0}{\partial \xi \partial \tau} + \frac{\partial^2 u_0}{\partial \eta \partial \tau}\right) + \frac{1}{4}\left(-\frac{\partial u_0}{\partial \xi} + \frac{\partial u_0}{\partial \eta}\right).$$

From the first equation, we obtain

$$u_0(\xi, \eta, \tau) = F(\xi, \tau) + G(\eta, \tau)$$

with F and G arbitrary C^2 functions. Integration of the equation for u_1 yields

$$u_1(\xi, \eta, \tau) = \frac{1}{2}\left(-\eta\frac{\partial F}{\partial \tau} + \xi\frac{\partial G}{\partial \tau}\right) + \frac{1}{4}(-F\eta + G\xi) + A(\xi) + B(\eta)$$

with A, B arbitrary C^2 functions. There are secular terms that can be eliminated by putting

$$-\frac{\partial F}{\partial \tau} - \frac{1}{2}F = 0, \ \frac{\partial G}{\partial \tau} + \frac{1}{2}G = 0.$$

In the next section, we shall see that the secularity conditions are obtained from an averaging process. The initial conditions require that

$$F(x, 0) + G(x, 0) = f(x), \ -\frac{\partial F}{\partial x}(x, 0) + \frac{\partial G}{\partial x}(x, 0) = 0,$$

so that

$$u_0(\xi, \eta, \tau) = \frac{1}{2}(f(\xi) + f(\eta))e^{-\tau/2}.$$

If $f(x)$ is localised (compact support), this corresponds with two waves, initially half the size of $f(x)$, moving respectively to the right and to the left but slowly damped in time.

14.2 Averaging over the Characteristics

The multiple-scale expansion of the preceding section is a simple example of a more general method developed by Chikwendu and Kevorkian (1972) that can also be called “averaging over the characteristics”. They consider problems of the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \varepsilon H\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right), \ -\infty < x < \infty, \ t > 0,$$

with initial values $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

The nonlinearity H is chosen such that the solutions of the nonlinear wave equation are bounded. However, with the actual constructions, one can consider various useful generalisations of H and other wave equations. We return to this later.

We assume again that the characteristic coordinates $\xi = x - t$, $\eta = x + t$ play a part and that the perturbation will also involve the timescale $\tau = \varepsilon t$. In fact, Chikwendu and Kevorkian (1972) introduce a more general fast timescale T by putting $dT/dt = 1 + \omega_1(\tau)\varepsilon + \omega_2(\tau)\varepsilon^2 + \varepsilon^3 \cdots$. They show that $\omega_1(\tau) = 0$ so that, restricting ourselves to first order and $O(\varepsilon)$ expansions, we may as well use t. Transforming the equation as in Section 14.1, we have

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\varepsilon}{2}\left(-\frac{\partial^2 u}{\partial \xi \partial \tau} + \frac{\partial^2 u}{\partial \eta \partial \tau}\right) + \frac{\varepsilon}{4} H\left(-\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} + \varepsilon \frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right).$$

The multiple-timescale expansion is again of the form

$$u = u_0(\xi, \eta, \tau) + \varepsilon u_1(\xi, \eta, \tau) + \varepsilon^2 \cdots ,$$

and substitution in the wave equation yields equations for u_0 and u_1:

$$\frac{\partial^2 u_0}{\partial \xi \partial \eta} = 0,$$
$$\frac{\partial^2 u_1}{\partial \xi \partial \eta} = \frac{1}{2}\left(-\frac{\partial^2 u_0}{\partial \xi \partial \tau} + \frac{\partial^2 u_0}{\partial \eta \partial \tau}\right) + \frac{1}{4} H\left(-\frac{\partial u_0}{\partial \xi} + \frac{\partial u_0}{\partial \eta}, \frac{\partial u_0}{\partial \xi} + \frac{\partial u_0}{\partial \eta}\right).$$

As before, we obtain from the first equation

$$u_0(\xi, \eta, \tau) = F(\xi, \tau) + G(\eta, \tau)$$

with F and G arbitrary C^2 functions. The equation for u_1 can then be written as

$$\frac{\partial^2 u_1}{\partial \xi \partial \eta} = \frac{1}{2}\left(-\frac{\partial^2 F}{\partial \xi \partial \tau} + \frac{\partial^2 G}{\partial \eta \partial \tau}\right) + \frac{1}{4} H\left(-\frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta}, \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta}\right).$$

Integration gives for the first derivatives

$$\frac{\partial u_1}{\partial \xi} = \frac{1}{2}\left(-\eta\frac{\partial^2 F}{\partial \xi \partial \tau} + \frac{\partial G}{\partial \tau}\right) + \frac{1}{4}\int^{\eta} H(\cdots,\cdots)d\eta,$$
$$\frac{\partial u_1}{\partial \eta} = \frac{1}{2}\left(-\frac{\partial F}{\partial \tau} + \xi\frac{\partial^2 G}{\partial \eta \partial \tau}\right) + \frac{1}{4}\int^{\xi} H(\cdots,\cdots)d\xi.$$

As $\xi, \eta \to \infty$, the derivatives must be bounded, which results in the conditions

$$\frac{\partial^2 F}{\partial \xi \partial \tau} = \lim_{\eta\to\infty} \frac{1}{2\eta}\int^{\eta} H\left(-\frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta}, \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta}\right) d\eta,$$
$$\frac{\partial^2 G}{\partial \eta \partial \tau} = -\lim_{\xi\to\infty} \frac{1}{2\xi}\int^{\xi} H\left(-\frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta}, \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta}\right) d\xi.$$

These secularity conditions are partial differential equations for F and G corresponding with (general) averaging over ξ and η. The averaged equations have to be solved while applying the initial conditions. The next step is to solve the equation for u_1. This again produces arbitrary functions that are determined by considering the equation for u_2 and again applying secularity conditions. If we have no a priori estimates for boundedness of the solutions, we still have to check whether the resulting approximation for $u_0 + \varepsilon u_1$ is bounded, as the secularity conditions on the derivatives are necessary but not sufficient.

Remark

The actual boundedness of the solutions is not essential for the constructions as long as the solutions are bounded on a timescale of the order $1/\varepsilon$. In this respect, the secularity conditions are misleading. As we have seen for ordinary differential equations in Chapters 11 and 12, the averaging process is the basic technique producing a normal form for the original equation. After obtaining a normal form, one can usually estimate the error introduced by normalisation on a long timescale.

Following Chikwendu and Kevorkian (1972), we give some examples.

Example 14.1

Suppose that we have nonlinear damping $H = u_t^3$ and so

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \varepsilon\left(\frac{\partial u}{\partial t}\right)^3, \ -\infty < x < \infty,\ t > 0.$$

Assume that if $\varepsilon = 0$ we have a progressive wave, initially $u(x, 0) = f(x), u_t(x, 0) = -f_x(x)$. This gives a drastic simplification, as in this case

$$u_0 = F(\xi, \tau).$$

The secularity conditions reduce to

$$\frac{\partial^2 F}{\partial \xi \partial \tau} = \lim_{\eta \to \infty} \frac{1}{2\eta} \int^{\eta} \left(-\frac{\partial F}{\partial \xi} \right)^3 d\eta = -\frac{1}{2} \left(\frac{\partial F}{\partial \xi} \right)^3 .$$

Considering this as an ordinary differential equation of the form $w_\tau = -\frac{1}{2}w^3, w = F_\xi$, we find

$$\frac{\partial F}{\partial \xi} = \frac{1}{(\tau + A(\xi))^{1/2}}$$

with $A(\xi)$ still arbitrary. As $\xi = x$ when $t = \tau = 0$, we can apply the initial condition so that

$$A(\xi) = (f_\xi(\xi))^{-2}.$$

The approximation to first order becomes finally

$$u_0(\xi, \tau) = \int_0^{\xi} \frac{f_s(s)}{(1 + f_s^2(s)\tau)^{1/2}} ds + f(0).$$

For a number of elementary functions $f(x)$, we can evaluate the integral explicitly.

Remark
We have started with an initial progressive wave, which simplifies the calculation. For more general initial conditions, we have to put $u_0 = F(\xi, \tau) + G(\eta, \tau)$, which enables the presence of another wave moving to the left.

Another classical example is the Rayleigh wave equation.

Example 14.2
Choosing $H = -u_t + \frac{1}{3}u_t^3$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \varepsilon \left(-\frac{\partial u}{\partial t} + \frac{1}{3} \left(\frac{\partial u}{\partial t} \right)^3 \right), \ -\infty < x < \infty, \ t > 0.$$

Starting again with a progressive wave $u(x, 0) = f(x), u_t(x, 0) = -f_x(x)$, we find with $u_0 = F(\xi, \tau)$ from the secularity condition

$$\frac{\partial^2 F}{\partial \xi \partial \tau} = \lim_{\eta \to \infty} \frac{1}{2\eta} \int^{\eta} \left(\frac{\partial F}{\partial \xi} - \frac{1}{3} \left(\frac{\partial F}{\partial \xi} \right)^3 \right) d\eta = \frac{1}{2} \left(\frac{\partial F}{\partial \xi} - \frac{1}{3} \left(\frac{\partial F}{\partial \xi} \right)^3 \right).$$

We consider this as an ordinary differential equation of the form $w_\tau = \frac{1}{2}(w - \frac{1}{3}w^3), w = F_\xi$, with solution

$$w(\tau) = \left(\frac{C(\xi)e^\tau}{1 + \frac{1}{3}C(\xi)e^\tau} \right)^{\frac{1}{2}},$$

where $C(\xi)$ will be determined by the initial conditions so that

$$\frac{\partial F}{\partial \xi} = \frac{|f_\xi(\xi)|}{(\frac{1}{3}f_\xi^2(\xi) + (1 - \frac{1}{3}f_\xi^2(\xi))e^{-\tau})^{\frac{1}{2}}}.$$

Choosing certain elementary functions $f(x)$, we can explicitly integrate the equation for $F(\xi, \tau)$.

Chikwendu and Kevorkian (1972) note that an interesting result is obtained by letting τ tend to infinity. We find

$$\lim_{\tau \to \infty} \frac{\partial F}{\partial \xi} = \sqrt{3}\frac{f_\xi(\xi)}{|f_\xi(\xi)|},$$

which corresponds with a sawtooth oscillation. Choosing for instance that initially $u(x,0) = A \sin px, u_t(x,0) = -Ap \cos px$, we have a sawtooth limit with amplitude $\sqrt{3}/p$ (depending on the wave number only) with spatial oscillation period (wavelength) $2\pi/p$. This limiting behaviour where an initially smooth wave train evolves towards a nonsmooth, generalised solution is confirmed by numerical analysis. Also, it is easy to see that there exist an infinite number of exact sawtooth solutions with slopes $\pm\sqrt{3}$. Their stability for general initial conditions is still an open question.

14.3 A Weakly Nonlinear Klein-Gordon Equation

We return again to the cubic Klein-Gordon equation discussed earlier with boundary values (Chapter 13) but now on an unbounded domain,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = \varepsilon u^3, \ -\infty < x < \infty, \ t > 0.$$

This is an example of a nonlinear dispersive wave equation displaying slowly varying wave trains. It is well-known that if $\varepsilon = 0$ we can substitute functions of the form $f(kx \pm \omega t)$, with k and ω constants, to obtain Fourier (trigonometric) wave trains satisfying the equation

$$(\omega^2 - k^2)f'' + f = 0.$$

With dispersion relation $\omega^2 - k^2 = 1$, this produces for instance wave solutions of the form

$$A \cos(kx - \omega t) + B \sin(kx - \omega t).$$

In this nonlinear case, we shall take a more restricted approach than in Section 14.1. We want to investigate for instance what happens to the wave trains found moving to the right for $\varepsilon = 0$ when the nonlinearity is turned on. We put $\theta = kx - \omega t$, with k and ω constants, and assume the dispersion relation $\omega^2 - k^2 = 1$ for $\varepsilon > 0$ and moreover that the modulated wave train, at least to first order, only depends on θ and $\tau = \varepsilon t$. With these assumptions, transforming the equation, we find

$$\frac{\partial^2 u}{\partial \theta^2} - 2\omega\varepsilon\frac{\partial^2 u}{\partial\theta\partial\tau} + \varepsilon^2\frac{\partial^2 u}{\partial\tau^2} + u = \varepsilon u^3.$$

We assume that we may substitute the expansion

$$u = u_0(\theta,\tau) + \varepsilon u_1(\theta,\tau) + \varepsilon^2 \cdots .$$

To order 1, we find

$$\frac{\partial^2 u_0}{\partial\theta^2} + u_0 = 0$$

with solution $u_0 = a(\tau)\cos(\theta + \phi(\tau))$. To $O(\varepsilon)$, we find

$$\frac{\partial^2 u_1}{\partial\theta^2} + u_1 = 2\omega\frac{\partial^2 u_0}{\partial\theta\partial\tau} + u_0^3$$

or

$$\frac{\partial^2 u_1}{\partial\theta^2} + u_1 = -2\omega\left(\frac{da}{d\tau}\sin(\theta+\phi) + a(\tau)\cos(\theta+\phi)\frac{d\phi}{d\tau}\right)$$
$$+a^3(\tau)\left(\frac{3}{4}\cos(\theta+\phi) + \frac{1}{4}\cos(3\theta+3\phi)\right).$$

To avoid secular terms, we put

$$\frac{da}{d\tau} = 0,\ -2\omega a(\tau)\frac{d\phi}{d\tau} + \frac{3}{4}a^3(\tau) = 0,$$

with solutions $a(\tau) = a_0, \phi(\tau) = \frac{3}{8\omega}a_0^2\tau$, where $a(0) = a_0$. We conclude that

$$u(x,t) = a_0\cos\left(kx - \omega t + \frac{3}{8\omega}a_0^2\varepsilon t\right) + \cdots$$

represents the first-order (formal) approximation of the solution. Note that in this approximation the amplitude is still constant but there is a modulation of the phase speed.

Remark

In this problem, we have fixed k and ω and allowed for slow variations of amplitude and phase. Another classical approach is to look for solutions of θ, or explicitly $U = U(kx - \omega t)$. Again putting $\omega^2 - k^2 = 1$, we find after substitution

$$\frac{d^2U}{d\theta^2} + U = \varepsilon U^3.$$

We can solve this equation in terms of elliptic functions or alternatively we can approximate U. In the latter case, the perturbation scheme again must allow for variations of amplitude and phase.

14.4 Multiple Scaling and Variational Principles

A large number of equations, in particular conservative ones, can be derived from a variational principle. Consider for instance the function $u : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ characterised by the Lagrangian

$$L = L(u_t, u_x, u)$$

and the variational principle

$$\delta \int \int L(u_t, u_x, u) dt dx = 0.$$

This so-called first-order variation leads to a Euler equation for $u(x,t)$ of the form

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} - \frac{\partial L}{\partial u} = 0.$$

Note that u_x and $\partial L/\partial u_x$ are vectors with components $\partial u/\partial x_i$ and $\partial L/\partial u_{x_i}$, $i = 1, \cdots, n$. Assuming that the Euler equation corresponds with a dispersive wave problem, we can look for special solutions of the form

$$u = U(\theta),\ \theta = k_i x_i - \omega t,$$

where $k_i, i = 1, \cdots, n$ and ω are constants, respectively called wave numbers and frequency. Substitution in the Euler equation produces a second-order ordinary differential equation, in general nonlinear, which upon integration will contain two free constants, the amplitude and the phase. The free parameters k_i, ω, amplitude and phase, are not arbitrary but must satisfy a so-called dispersion relation. We shall see examples later on. This approach to Euler (wave) equations has been applied by many scientists since the end of the nineteenth century.

In a number of papers starting in 1965, Whitham gave a new perturbation approach; in the description we will follow Whitham (1970, 1974) and Luke (1966). To fix the idea, consider the strongly nonlinear, one-dimensional Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + V'(u) = 0,$$

which can be derived as the Euler equation generated by the Lagrangian

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - V(u).$$

Exact periodic wave trains can be produced by substituting $u = U(\theta)$ as discussed above. We will study the slowly varying behaviour of the wave train over large distances and for large times by introducing the slow variables $X = \varepsilon x$ and $\tau = \varepsilon t$. The quantity θ will depend on X and τ; moreover the rescaling

$$\theta = \frac{\Theta(X,\tau)}{\varepsilon}$$

is sometimes used. The wave number and the frequency will be slowly varying:

$$k = \theta_x \to k(X,\tau) = \Theta_X, \; \omega = -\theta_t \to \omega(X,\tau) = -\Theta_\tau.$$

Consistency then requires that

$$k_\tau + \omega_X = 0.$$

The solution describing the slowly varying wave train is supposed to have the expansion

$$u = U(\theta, X, \tau) + \varepsilon U_1(\theta, X, \tau) + \varepsilon^2 \cdots .$$

Transformation of the differential operators as in the previous sections yields

$$\begin{aligned} u_{xx} &\to U_{\theta\theta}k^2 + \varepsilon(U_\theta k_X + 2U_{\theta X}k + U_{1\theta\theta}k^2) + \varepsilon^2 \cdots , \\ u_{tt} &\to U_{\theta\theta}\omega^2 + \varepsilon(-U_\theta \omega_\tau - 2U_{\theta\tau}\omega + U_{1\theta\theta}\omega^2) + \varepsilon^2 \cdots , \\ V' &\to V'(U) + \varepsilon U_1 V''(U) + \varepsilon^2 \cdots . \end{aligned}$$

Substitution into the Klein-Gordon equation produces at lowest order

$$(\omega^2 - k^2)U_{\theta\theta} + V'(U) = 0$$

and to $O(\varepsilon)$

$$(\omega^2 - k^2)U_{1\theta\theta} + V''(U)U_1 = 2\omega U_{\theta\tau} + 2kU_{\theta X} + \omega_\tau U_\theta + k_X U_\theta.$$

As before, we have obtained a system of ordinary differential equations, only the first one is nonlinear. The constants of integration may depend on τ and X. In general, the nonlinear equation will have an infinite number of solutions periodic in θ (see, for instance, Verhulst, 2000). We choose one, $U_0(\theta)$, and normalise the period to 2π. The lowest order equation has the integral

$$\frac{1}{2}(\omega^2 - k^2)U_\theta^2 + V(U) = E(X,\tau),$$

where the parameter E ("energy") still depends on τ and X. From the energy integral, we can extract U_θ and integrate

$$\theta = \sqrt{\frac{1}{2}(\omega^2 - k^2)} \int \frac{dU_0}{\sqrt{E - V(U_0)}}.$$

Integration for $U_0(\theta)$ over the whole period in the phase-plane yields

$$2\pi = \sqrt{\frac{1}{2}(\omega^2 - k^2)} \oint \frac{dU_0}{\sqrt{E - V(U_0)}}.$$

This is a relation between ω, k, and E that we will again call the *dispersion relation*. Whitham (1970, 1974) introduces the averaged Lagrangian $\bar{L}$ by substituting $U_0(\theta)$ into the expression for the Lagrangian and averaging over the period

$$\bar{L} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}(\omega^2 - k^2)U_0'^2 - V(U_0)\right)d\theta.$$

We eliminate $V(U_0)$ using the energy integral so that

$$\bar{L} = \frac{1}{2\pi}\int_0^{2\pi}(\omega^2 - k^2)U_0'^2 d\theta - E = \frac{1}{2\pi}\oint(\omega^2 - k^2)U_0' dU_0 - E,$$

which can also be written as

$$\bar{L} = \frac{1}{2\pi}\sqrt{2(\omega^2 - k^2)}\oint\sqrt{E - V(U_0)}dU_0 - E.$$

The last expression for the averaged Lagrangian $\bar{L}$ depends for a given potential V on ω, k, and E only; for the integration, U_0 is a dummy variable.

Whitham proposes to use this averaged Lagrangian to obtain appropriate Euler equations for the unknown quantities ω, k, and E. So far, we did not apply the secularity conditions to the equation for U_1. Remarkably enough, these are tied in with the variations of the averaged Lagrangian. For technical details, see the literature cited.

Example 14.3
Consider the strongly nonlinear, one-dimensional Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u + au^3 = 0,$$

corresponding with $V(u) = \frac{1}{2}u^2 + \frac{a}{4}u^4$ with a a constant; the equation can be derived as a Euler equation by variation of the Lagrangian

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u^2 - \frac{a}{4}u^4.$$

With the multiple-timescale expansion $u = U + \varepsilon U_1 + \varepsilon^2 \cdots$, we have

$$(\omega^2 - k^2)U_{\theta\theta} + U + aU^3 = 0$$

and to $O(\varepsilon)$

$$(\omega^2 - k^2)U_{1\theta\theta} + (1 + 3aU^2)U_1 = 2\omega U_{\theta\tau} + 2kU_{\theta X} + \omega_\tau U_\theta + k_X U_\theta.$$

The solutions for U can be obtained as elliptic functions that are periodic in θ. They oscillate between the two zeros of $E(X,\tau) - \frac{1}{2}U^2 - \frac{a}{4}U^4$. The dispersion relation among ω, k, and E takes the form

$$2\pi = \sqrt{\frac{1}{2}(\omega^2 - k^2)} \oint \frac{dU_0}{\sqrt{E - \frac{1}{2}U_0^2 - \frac{a}{4}U_0^4}}.$$

The averaged Lagrangian becomes

$$\bar{L} = \frac{1}{2\pi}\sqrt{2(\omega^2 - k^2)} \oint \sqrt{E - \frac{1}{2}U_0^2 - \frac{a}{4}U_0^4}dU_0 - E,$$

for which various series expansions are available.

In the case of weakly nonlinear problems, it is easier to obtain explicit expressions. We show this with another example that is used quite often in the literature.

Example 14.4
Consider Bretherton's model equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u = \varepsilon u^3,$$

which can be derived from the Lagrangian

$$L = \frac{1}{2}u_t^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4.$$

If $\varepsilon = 0$, substitution of $u = a\cos(kx - \omega t)$ produces the dispersion relation

$$\omega^2 - k^4 + k^2 = 1.$$

For $\varepsilon > 0$, we assume that the solution is slowly varying in $\theta = k(X,\tau)x - \omega(X,\tau)t, X = \varepsilon x$, and $\tau = \varepsilon t$:

$$u = a(X,\tau)\cos(k(X,\tau)x - \omega(X,\tau)t) + \varepsilon\cdots.$$

For the averaged Lagrangian at lowest order, we find

$$\bar{L} = \frac{1}{2\pi}\int_0^{2\pi} L d\theta = \frac{1}{4}a^2(\omega^2 - k^4 + k^2 - 1) + \frac{1}{32}\varepsilon a^4 + \cdots.$$

The Euler-Lagrange equation with respect to the amplitude a is $\partial\bar{L}/da = 0$; this produces the dispersion relation

$$\omega^2 - k^4 + k^2 = 1 + \frac{3}{4}\varepsilon a^2,$$

which is an extension of the "linear" dispersion relation. Other variations of $\bar{L}$ will produce relations among amplitude a, frequency ω, and wave number k that play a part in wave mechanics. For explicit calculations, see for instance Shivamoggi (2003).

14.5 Adiabatic Invariants and Energy Changes

In Chapter 12, we looked at adiabatic invariants that can be seen as asymptotic integrals or asymptotic conservation laws of a system. In contrast with "classical integrals of motion," these adiabatic invariants represent a relation between phase variables and time that is conserved with a certain precision on a certain timescale. This is of particular interest when we want to characterise changes of energy or angular momentum without having to integrate the complete equations of motion.

In this section, we explore the ideas for ordinary differential equations, after which, in the next section, we give an application to the Korteweg-de Vries equation.

Example 14.5
Consider initial value problems for the equation

$$\ddot{x} + x = \varepsilon f(x, \dot{x}, \varepsilon t)$$

with sufficiently smooth right-hand side. Applying averaging as in Chapter 11, we can introduce amplitude-phase variables of the form (11.5) by putting $x(t) = r(t)\cos(t + \phi(t)), \dot{x}(t) = -r(t)\sin(t + \phi(t))$. We will add two variables,

$$E(t) = \frac{1}{2}\dot{x}^2(t) + \frac{1}{2}x^2(t),\ \tau = \varepsilon t,$$

and, after differentiation of $E(t)$, the two equations

$$\frac{dE}{dt} = \varepsilon \dot{x} f(x, \dot{x}, \tau),\ \dot{\tau} = \varepsilon.$$

Note that $E(t) = \frac{1}{2}r^2(t)$. The four equations (for r, ϕ, E, τ) that we can derive, are all slowly varying; we only average the equation for E:

$$\frac{dE_a}{dt} = -\frac{\varepsilon}{2\pi}\int_0^{2\pi} r\sin(t + \phi) f(r\cos(t + \phi), -r\sin(t + \phi), \tau)dt.$$

As in Chapter 11, we can put $s = t + \phi$ with the result that ϕ does not occur in the averaged equation. We find after averaging an expression of the form

$$\frac{dE_a}{dt} = \varepsilon F(r_a(t), \tau) = \varepsilon F(\sqrt{2E_a}, \tau).$$

This is a first-order differential equation for $E_a(t)$ that we can study without solving the averaged equations of motion. $E_a(t)$ is an adiabatic invariant with $E_a(t) - E(t) = O(\varepsilon)$ on the timescale $1/\varepsilon$.

Consider as an example the Rayleigh equation

$$\ddot{x} + x = \varepsilon \dot{x}\left(1 - \frac{1}{3}\dot{x}^2\right)$$

with

$$\frac{dE}{dt} = \varepsilon \dot{x}^2 \left(1 - \frac{1}{3}\dot{x}^2\right).$$

Amplitude-phase variables produce

$$\frac{dE}{dt} = \varepsilon r^2(t)\sin^2(t+\phi(t))\left(1 - \frac{1}{3}r^2(t)\sin^2(t+\phi(t))\right)$$

and, after averaging over t,

$$\frac{dE_a}{dt} = \frac{1}{2}\varepsilon r_a^2(t)\left(1 - \frac{1}{4}r_a^2(t)\right) = \varepsilon E_a\left(1 - \frac{1}{2}E_a\right).$$

If we choose $E_a(t) = 2$, the energy does not change in time. This is the energy value of the well-known limit cycle of the Rayleigh equation. If we start with $E(0) < 2$, the energy grows to the value 2 as the solution tends to the limit cycle, and if we start with $E(0) > 2$, the energy decreases to this value.

Integrating the equation for E_a, we obtain an expression that we can interpret as an adiabatic invariant for the Rayleigh equation.

We consider now a strongly nonlinear problem based on Huveneers and Verhulst (1997).

Example 14.6
Consider the equation

$$\ddot{x} + x = a(\varepsilon t)x^2$$

with $a(0) = 1$ and $a(\varepsilon t)$ a smooth, positive function decreasing towards zero. This is a simple model exemplifying a Hamiltonian system with asymmetric potential that by some evolution process tends towards a symmetric one. Transforming $y = a(\varepsilon t)x$, we obtain the equation

$$\ddot{y} + y = y^2 + 2\varepsilon\frac{a'(\varepsilon t)}{a(\varepsilon t)}\dot{y} + \varepsilon^2 \cdots .$$

A prime denotes differentiation with respect to its argument. The result is surprising. The $O(\varepsilon)$ term represents a dissipative term, which means that our system in evolution towards symmetry is characterised by an autonomous Hamiltonian system with dissipation added. To see what happens, we consider a special choice: $a(\varepsilon t) = e^{-\varepsilon t}$. The equation becomes

$$\ddot{y} + y = y^2 - 2\varepsilon\dot{y} + \varepsilon^2 \cdots .$$

If $\varepsilon = 0$, we have a centre point at $(0, 0)$ and a saddle at $(1, 0)$ with a homoclinic loop emerging from the saddle and intersecting the y-axis at $(-\frac{1}{2}, 0)$. If $\varepsilon > 0$, the loop will break up but the saddle still has two stable and two unstable one-dimensional manifolds. If $\varepsilon = 0$, we can associate with the equation the energy

$$E = \frac{1}{2}\dot{y}^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3$$

and by differentiation and using the equation for $\varepsilon > 0$

$$\frac{dE}{dt} = -2\varepsilon\dot{y}^2 + \varepsilon^2 \cdots .$$

To see what happens to the homoclinic loop, we approximate E by E_a, omitting the ε^2 terms and using in the equation the unperturbed homoclinic loop behaviour of $\dot{y}$. Integrating from y_0 to y_1, we have

$$E_a = -2\varepsilon \int_{t(y_0)}^{t(y_1)} \dot{y}^2(t)dt = -2\varepsilon \int_{y_0}^{y_1} \dot{y}dy.$$

We now use that for $\varepsilon = 0$ the homoclinic loop is given by

$$\frac{1}{2}\dot{y}^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3 = \frac{1}{6}$$

and that the loop is symmetric with respect to the y-axis; we have

$$E_a = \frac{1}{6} - 4\varepsilon \int_{-\frac{1}{2}}^{1} \sqrt{\frac{2}{3}y^3 - y^2 + \frac{1}{3}}dy.$$

Fortunately, this is an elementary integral. (Use MATHEMATICA or an integral table.) We find for the first-order changed energy

$$E_a = \frac{1}{6} - \frac{12}{5}\varepsilon.$$

It is interesting to deduce from this the position where the stable manifold of the unstable equilibrium intersects the y-axis. The top half of the homoclinic loop is bent inwards with an energy change of $\frac{6}{5}\varepsilon$, so the stable manifold has energy (to a first approximation) $\frac{1}{6} + \frac{6}{5}\varepsilon$ and is approximately described by

$$\frac{1}{2}\dot{y}^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3 = \frac{1}{6} + \frac{6}{5}\varepsilon.$$

Introducing into this equation $\dot{y} = 0, y = -\frac{1}{2} + \varepsilon\alpha + \cdots$, we find $\alpha = -\frac{8}{5}$, so the intersection takes place at approximately $(-\frac{1}{2} - \varepsilon\frac{8}{5}, 0)$.

Remark

This type of energy change for one homoclinic loop can be found in Guckenheimer and Holmes (1997). In the interior of the homoclinic loop "ordinary" averaging is valid, but this is not the case in a boundary layer near the loop. In Huveneers and Verhulst (1997), the computations use elliptic functions and cover both the interior of the homoclinic loop and the boundary layer near the homoclinic loop and the saddle. The error analysis is subtle and involves various domains and different expressions for the adiabatic invariants; for the passage of the saddle, the analysis follows Bourland and Haberman (1990).

14.6 The Perturbed Korteweg-de Vries Equation

In the spirit of example 14.6 and following Scott (1999), we discuss perturbations of solitons in the Korteweg-de Vries (KdV) equation. The equation is

$$u_t + uu_x + u_{xxx} = \varepsilon f(\cdots), \; -\infty < x < \infty, t > 0,$$

where we have the KdV equation if $\varepsilon = 0$, and f is a perturbation that may depend on x, t, u and the derivatives of u. The well-known single soliton solution of the KdV equation is

$$u(x,t) = 3v \operatorname{sech}^2 \left(\frac{\sqrt{v}}{2}(x - vt) \right)$$

with v the constant soliton velocity. The perturbation f can have many consequences, but we will study the case with the assumption that we have only small variations of the velocity v, $v = v(\tau), \tau = \varepsilon t$.

As in the preceding section, we can directly derive an equation for the behaviour of the energy with time. It is convenient to introduce the function w by $w_x = u$. The KdV equation can be derived from the Lagrangian density

$$\mathcal{L} = \frac{1}{2} w_x w_t + \frac{1}{6} w_x^3 - \frac{1}{2} w_{xx}^2.$$

Instead of the Lagrangian, we will use the associated Hamiltonian density

$$\mathcal{H} = w_t \frac{\partial \mathcal{L}}{\partial w_t} - \mathcal{L} = -\frac{1}{6} w_x^3 + \frac{1}{2} w_{xx}^2.$$

The total energy (which we will later specify for a soliton) is

$$H = \int_{-\infty}^{\infty} \mathcal{H} dx.$$

From this energy functional, we find by differentiation

$$\frac{dH}{dt} = \int_{-\infty}^{\infty} \left(-\frac{1}{2} w_x^2 w_{xt} + w_{xx} w_{xxt} \right) dx.$$

The first term is partially integrated once and the second term twice; we also assume that the first three derivatives of w vanish as $x \to \pm\infty$. We find

$$\frac{dH}{dt} = \int_{-\infty}^{\infty} (w_{xxx} + w_{xxxx}) w_t dx$$

and finally, using that w_x satisfies the perturbed KdV equation,

$$\frac{dH}{dt} = \int_{-\infty}^{\infty} (-w_{xt} + \varepsilon f) w_t dx = \varepsilon \int_{-\infty}^{\infty} f w_t dx.$$

We can explicitly compute the total soliton energy by substituting the expression for the soliton while neglecting variations of $v(\tau)$; they are of higher order. We find after integration

$$H = -\frac{36v^{\frac{5}{2}}(\tau)}{5}$$

and so derive variations of the energy from long-term variations of $v(\tau)$:

$$\frac{dH}{dt} = -18v^{\frac{3}{2}}\frac{dv}{dt}.$$

Combining the general expression for dH/dt with this specific one, we find for the velocity variation

$$\frac{dv}{dt} = -\varepsilon\frac{1}{18v^{\frac{3}{2}}}\int_{-\infty}^{\infty} fw_t dx.$$

For the soliton ($\varepsilon = 0$), we have the relation

$$w_t = -vw_x = -vu,$$

and assuming that for $\varepsilon > 0$ we have to first order $w_t = -v(\tau)u$, we obtain

$$\frac{dv}{dt} = \varepsilon\frac{1}{18v^{\frac{1}{2}}}\int_{-\infty}^{\infty} fw_t dx.$$

One of the simplest examples is the choice $f = -u$; inserting this and using the expression for a single soliton yields after integration

$$\frac{dv}{dt} = -\varepsilon\frac{4}{3}v$$

with solution

$$v(\varepsilon t) = v(0)e^{-\frac{4}{3}\varepsilon t}.$$

As we can observe, in the expression for the single soliton, the amplitude obeys the same variation with time. In an interesting discussion, Scott (1999) notes that this result is confirmed by numerical calculations. On the other hand, when analysing other conservation laws of the KdV equation with the same technique, the results are not always correct. The implication is that mathematical analysis of these approximation techniques is much needed.

14.7 Guide to the Literature

An early reference is Benney and Newell (1967), where multiple-scale methods are developed to study wave envelopes and interacting nonlinear waves. Around the same time, Luke (1966) gave a multiple-scale analysis for some

prominent wave equations; this is tied in with work started in 1965 and extensively described in Whitham (1970, 1974). In Whitham's work, multiple-scale analysis is imbedded in variational principles, employing averaged Lagrangians, which puts the computations in a more fundamental although still formal framework. If the medium is not homogeneous, Whitham shows that the wave action is conserved for strongly nonlinear dispersive waves analogous to the action being an adiabatic invariant for strongly nonlinear Hamiltonian oscillations.
Until the papers of Haberman and Bourland (1988) and Bourland and Haberman (1989), modulations of the phase shift for strongly dispersive waves got little attention. Using the equation for the wave action, they developed a modification of the approximation scheme to include higher-order effects and characterise at what order of the perturbation scheme such effects play a part.

Chikwendu and Kevorkian (1972) developed a general multiple-scale approach for wave equations that we have called "averaging over the characteristics". See also Kevorkian and Cole (1996), where wave equations and conservation laws are discussed. Chikwendu and Easwaran (1992) extended this for semi-infinite domains.

Van der Burgh (1979) has shown that in a large number of weakly nonlinear equations the multiple-scale analysis yields asymptotically valid results. This is based on the usual integral inequality estimates, and the asymptotic estimates may even be improved. See also the discussion in De Jager and Jiang Furu (1996). We are not aware of other proofs of asymptotic validity for unbounded domains.

Important extensions and applications have been found and are being implemented. Ablowitz and Benney (1970) extended Whitham's averaged variational principle to multiphase dispersive nonlinear waves. McLaughlin and Scott (1978) did this extension for solitary waves and solitons. In Scott's (1999) book, used in the previous section, applications to the sine-Gordon equation, the nonlinear Schrödinger equation, and other interesting examples can be found.

14.8 Exercises

Exercise 14.1 Consider the weakly damped Klein-Gordon equation in the form

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \varepsilon \frac{\partial u}{\partial t}, \ -\infty < x < \infty, \ t > 0,$$

with $u(x,0) = f(x)$, $u_t(x,0) = 0$.

a. Introduce a regular expansion $u(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \cdots$ and formulate the initial value problems for u_0 and u_1.
b. Reformulate the initial value problems for u_0 and u_1 in characteristic coordinates ξ, η.

c. Solve the initial value problems for u_0 and u_1 to show that secular terms arise.

Exercise 14.2 Compare the first-order multiple-timescale expansion of Section 14.1 with the exact solution. For comparison, see Chikwendu and Kevorkian (1972).

Exercise 14.3 Consider the weakly nonlinear wave equation from Section 14.3 with damping added:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u + \varepsilon \frac{\partial u}{\partial t} = \varepsilon u^3, \ -\infty < x < \infty, \ t > 0.$$

Consider a wave train moving to the right. Determine the change in the secularity conditions.

Exercise 14.4 Consider the wave equation with a Van der Pol perturbation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \varepsilon \frac{\partial u}{\partial t}(1 - u^2), \ -\infty < x < \infty, \ t > 0.$$

Initially we have a progressive wave $u(x, 0) = f(x), u_t(x, 0) = -f_x(x)$. Compute a first-order approximation as in Section 14.2. Consider a suitable initial $f(x)$ explicitly.