10

The Continuation Method

In Chapter 1 we considered an equation $L_\varepsilon y = 0$ that contains a small parameter ε. We associate with this equation an "unperturbed" problem, the equation $L_0 y = 0$. If the difference between the solutions of both equations in an appropriate norm does not tend to zero as ε tends to zero, we call the problem a singular perturbation problem.

Actually, when formulated with suitable norms, most perturbation problems are singular. This does not only apply to boundary layer problems as considered in the preceding chapters, but also to slow-time problems, as we shall see in what follows. In all of these cases, the solutions of the unperturbed problem can not be simply continued with a Taylor expansion in the small parameter to obtain an approximation of the full problem.

Simple perturbation examples can be found in the exercises of Chapter 2, where we looked at solutions of algebraic equations of the form

$$a(\varepsilon)x^2 + b(\varepsilon)x + c(\varepsilon) = 0,$$

in which a, b, and c depend smoothly on ε. Associating with this problem the "unperturbed" or "reduced" equation

$$a(0)x^2 + b(0)x + c(0) = 0,$$

we found that in the analysis the implicit function theorem plays a fundamental part. If x_0 is a solution of the unperturbed problem and if

$$2a(0)x_0 + b(0) \neq 0,$$

the implicit function theorem tells us that the solutions of the full problem depend smoothly on ε and we may expand in integral powers of ε: $x_\varepsilon = x_0 + \varepsilon \cdots$. If this condition is not satisfied, we cannot expect a Taylor series with respect to ε and there may be bifurcating solutions.

We shall discuss this application of the implicit function theorem for initial value problems for ordinary differential equations.

10.1 The Poincaré Expansion Theorem

We start with a few examples.

Example 10.1
$x(t)$ is the solution of the initial value problem

$$\dot{x} = -x + \varepsilon,\ x(0) = 1.$$

The unperturbed problem is

$$\dot{y} = -y,\ y(0) = 1.$$

We have $x(t) = \varepsilon + (1 - \varepsilon)e^{-t}, y(t) = e^{-t}$, so

$$|x(t) - y(t)| = \varepsilon - \varepsilon e^{-t} \leq \varepsilon,\ t \geq \varepsilon.$$

The approximation of $x(t)$ by $y(t)$ is valid for all time.

Example 10.2
$x(t)$ is the solution of the initial value problem

$$\dot{x} = x + \varepsilon,\ x(0) = 1.$$

The unperturbed problem is

$$\dot{y} = y,\ y(0) = 1,$$

and we have

$$|x(t) - y(t)| = \varepsilon(e^t - 1),$$

so we have an approximation that is valid for $0 \leq t \leq 1$ (or any positive constant that does not depend on ε).

We might conjecture that in the second example the approximation breaks down because the solutions are not bounded. However, to require the solutions to be bounded is not sufficient, as the following example shows.

Example 10.3
Consider the initial value problem

$$\ddot{x} + (1 + \varepsilon)^2 x = 0,\ x(0) = 1, \dot{x}(0) = 0,$$

with solution $x(t) = \cos(1 + \varepsilon)t$. The unperturbed problem is

$$\ddot{y} + y = 0,\ y(0) = 1, \dot{y}(0) = 0$$

which is solved by $y(t) = \cos t$. So we conclude that $|x(t) - y(t)| = 2|\sin(t + \frac{1}{2}\varepsilon t)\sin(\frac{1}{2}\varepsilon t)|$, which does not vanish with ε; take for instance $t = \pi/\varepsilon$. One might expect an improvement of the timescale where the approximation is valid when expanding to higher order. However, this is generally not the case. Expand $x(t) = \cos t + \varepsilon x_1(t) + \varepsilon^2 \cdots$ and substitute this expression into the differential equation. To $O(\varepsilon)$ we obtain the problem

$$\ddot{x}_1 + 2\cos t + x_1 = 0, \ x_1(0) = \dot{x}_1(0) = 0.$$

The solution is $x_1(t) = -t\sin t$ and is an unbounded expression, which we cannot call an improvement.

We shall now formulate the general perturbation procedure and state what we can expect in general of the accuracy of the approximation. Consider the initial value problem for $x \in \mathbb{R}^n$

$$\dot{x} = f(t, x, \varepsilon), \ x(t_0) = a(\varepsilon).$$

We assume that $f(t, x, \varepsilon)$ can be expanded in a Taylor series with respect to x (in a neighbourhood of the initial value) and with respect to ε. We also assume that $a(\varepsilon)$ can be expanded in a Taylor series with respect to ε. The solution of the initial value problem is $x_\varepsilon(t)$.

We associate with this problem the unperturbed (or reduced) problem

$$\dot{x_0} = f(t, x_0, 0), \ x(t_0) = a(0),$$

with solution $x_0(t)$. The Poincaré expansion theorem tells us that

$$||x_\varepsilon(t) - x_0(t)|| = O(\varepsilon), \ t_0 \leq t \leq t_0 + C,$$

with C a constant independent of ε. As C is $O(1)$ with respect to ε, this is sometimes called an approximation valid on the *timescale* 1.

We can improve the result by expanding to higher order. More generally the Poincaré expansion theorem asserts that $x_\varepsilon(t)$ can be expanded in a convergent Taylor series of the form

$$x_\varepsilon(t) = x_0(t) + \varepsilon x_1(t) + \cdots + \varepsilon^n x_n(t) + \cdots .$$

The terms $x_n(t)$ are obtained by substituting the series in the differential equation and expanding the vector function f with respect to ε as

$$\dot{x}_0 + \varepsilon\dot{x}_1 + \varepsilon^2 \cdots = f(t, x_0 + x_1 + \varepsilon^2 \cdots, \varepsilon), \ a(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 \cdots ,$$

which after expansion and collecting equations at the same power of ε produces

$$\begin{aligned}
\dot{x}_0 &= f(t, x_0, 0), \ x_0(t_0) = a_0, \\
\dot{x}_1 &= \frac{\partial f}{\partial x}(t, x_0, 0)x_1 + \frac{\partial f}{\partial \varepsilon}(t, x_0.0), \ x_1(t_0) = a_1,
\end{aligned}$$

and similar equations at higher order. Note that, apart from the unperturbed equation for x_0, all equations are linear. For the nth-order partial sum, we have the estimate

$$||x_\varepsilon(t) - (x_0(t) + \varepsilon x_1(t) + \cdots + \varepsilon^n x_n(t))|| = O(\varepsilon^{n+1})$$

for $t_0 \le t \le t_0 + C$. *So by this type of expansion we can improve the accuracy but not the timescale!*

Example 10.4
Consider the damped harmonic oscillator

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0,\ x(0) = 1,\ \dot{x}(0) = 0.$$

Ignoring the known, exact solution, we put $x_\varepsilon(t) = x_0(t) + \varepsilon x_1(t) + \cdots$ to obtain the initial value problems

$$\begin{aligned}\ddot{x}_0 + x_0 &= 0, \quad x_0(0) = 1,\ \dot{x}_0(0) = 0,\\ \ddot{x}_1 + x_1 &= -2\dot{x}_0,\ x_1(0) = 0,\ \dot{x}_1(0) = 0.\end{aligned}$$

We find $x_0(t) = \cos t$ and for x_1

$$\ddot{x}_1 + x_1 = -2\cos t,\ x_1(0) = 0,\ \dot{x}_1(0) = 0,$$

with solution

$$x_1(t) = \sin t - t\cos t.$$

This looks bad. The solutions of the damped harmonic oscillator are bounded for all time and even tend to zero, while the approximation has an increasing amplitude. However, the approximation is valid only on the timescale 1.

Such terms that are increasing with time were called "secular terms" in astronomy. The discussion of how to avoid them played an important part in the classical perturbation problems of celestial mechanics. Poincaré adapted the approximation scheme to get rid of secular terms.

10.2 Periodic Solutions of Autonomous Equations

To a certain extent, periodic solutions are determined by their behaviour on a timescale 1. This characteristic enables us to obtain a fruitful application of the Poincaré expansion theorem. Usually this is called the Poincaré-Lindstedt method.

In this section, we consider autonomous equations of the form

$$\dot{x} = f(x, \varepsilon),\ x \in \mathbb{R}^n,$$

for which we assume that the conditions of the expansion theorem have been satisfied. In addition, we assume that the unperturbed equation $\dot{x} = f(x, 0)$ has one or more periodic solutions. Can we continue such a periodic solution for the equation if $\varepsilon > 0$?

The reason to make a distinction between autonomous and nonautonomous equations is that in autonomous equations the period is not a priori fixed. As a consequence, a period T_0 of a periodic solution of the unperturbed equation will in general also be perturbed. To fix the idea consider the two-dimensional equation

$$\ddot{x} + x = \varepsilon f(x, \dot{x}, \varepsilon).$$

If $\varepsilon = 0$, we have a rather degenerate case, as all solutions are periodic and even have the same period, 2π. We cannot expect that all of these periodic solutions can be continued for $\varepsilon > 0$, but maybe some periodic solutions will branch off. For such a solution, we will have a period $T(\varepsilon)$ with $T(0) = T_0 = 2\pi$. A priori we do not know for which initial conditions periodic solutions branch off (if they do!), so we assume that we find them starting at

$$x(0) = a(\varepsilon),\ \dot{x}(0) = 0.$$

For a two-dimensional autonomous equation, putting $\dot{x}(0) = 0$ is no restriction. The expansion theorem tells us that on the timescale 1 we have

$$\lim_{\varepsilon \to 0} = a(0) \cos t.$$

It is convenient to fix the period by the transformation

$$\omega t = \theta,\ \omega^{-2} = 1 - \varepsilon \eta(\varepsilon),$$

where $\eta(\varepsilon)$ can be expanded in a Taylor series with respect to ε and is chosen such that any periodic solution under consideration has period 2π. With the notation $x' = dx/d\theta$, the equation becomes

$$x'' + x = \varepsilon \eta(\varepsilon) x + \varepsilon (1 - \varepsilon \eta(\varepsilon)) f(x, (1 - \varepsilon \eta(\varepsilon))^{-\frac{1}{2}} x', \varepsilon)$$

with initial values $x(0) = a(\varepsilon), x'(0) = 0$.

Abbreviating the equation to

$$x'' + x = \varepsilon F(x, x', \varepsilon, \eta(\varepsilon)),$$

we are now looking for a suitable initial value $a(\varepsilon)$ and scaling of ω (or η) to obtain 2π-periodic solutions in θ. This problem is equivalent with solving the integral equation

$$x(\theta) = a(\varepsilon) \cos \theta + \varepsilon \int_0^{\theta} F(x(s), x'(s), \varepsilon, \eta(\varepsilon)) \sin(\theta - s) ds$$

with the periodicity condition $x(\theta + 2\pi) = x(\theta)$ for each value of θ.

Applying the periodicity condition, we find

$$\int_{\theta}^{\theta + 2\pi} F(x(s), x'(s), \varepsilon, \eta(\varepsilon)) \sin(\theta - s) ds = 0.$$

Expanding $\sin(\theta - s) = \sin\theta\cos s - \cos\theta\sin s$ and using that the sin- and cos-functions are independent, we find the conditions

$$I_1(a(\varepsilon), \eta(\varepsilon)) = \int_0^{2\pi} F(x(s), x'(s), \varepsilon, \eta(\varepsilon)) \sin s ds = 0,$$

$$I_2(a(\varepsilon), \eta(\varepsilon)) = \int_0^{2\pi} F(x(s), x'(s), \varepsilon, \eta(\varepsilon)) \cos s ds = 0.$$

For each value of ε, this is a system of two equations with two unknowns, $a(\varepsilon)$ and $\eta(\varepsilon)$. According to the implicit function theorem, this system is uniquely solvable in a neighbourhood of $\varepsilon = 0$ if the corresponding Jacobian J does not vanish for the solutions:

$$J = \left| \frac{\partial(I_1, I_2)}{\partial(a(\varepsilon), \eta(\varepsilon))} \right| \neq 0.$$

The way to handle the periodicity conditions and the corresponding Jacobian is to expand the solution $x_\varepsilon(t)$ and the parameters $a(\varepsilon)$, $\eta(\varepsilon)$ with respect to ε. At the lowest order, we find

$$F(x_0(t), x_0'(t), 0, \eta(0)) = \eta(0)a(0)\cos t + f(a(0)\cos t, -a(0)\sin t, 0)$$

and the equations

$$\int_0^{2\pi} f(a(0)\cos s, -a(0)\sin s, 0) \sin s ds = 0,$$

$$\pi\eta(0)a(0) + \int_0^{2\pi} f(a(0)\cos s, -a(0)\sin s, 0) \cos s ds = 0.$$

These equations have to be satisfied (necessary condition) to obtain a periodic solution. If the corresponding Jacobian does not vanish, a nearby periodic solution really exists. The condition derived from the lowest-order equations is

$$J_0 = \left| \frac{\partial(I_1, I_2)}{\partial(a(0), \eta(0))} \right| \neq 0.$$

We shall study this condition in a number of examples. Note that if we can satisfy the periodicity conditions but the Jacobian vanishes at lowest order, we have to calculate the Jacobian at the next order. Assuming that the Poincaré expansion theorem applies, we have the convergent series $J = J_0 + \varepsilon J_1 + \cdots + \varepsilon^n J_n + \cdots$. This calculation may decide the existence of a unique periodic solution, but it is possible that the Jacobian vanishes at all orders. This happens for instance if we have a family of periodic solutions so instead of having the existence of a unique solution, vanishing of the Jacobian can in some cases imply that there are many more periodic solutions. We shall meet examples of these phenomena.

Example 10.5
(Van der Pol equation)
A classical example is the Van der Pol equation

$$\ddot{x} + x = \varepsilon\dot{x}(1 - x^2),$$

which has a unique periodic solution for *each positive* value of ε. Of course, we consider only small values of ε. Transforming time by $\omega t = \theta,\ \omega^{-2} = 1 - \varepsilon\eta(\varepsilon)$, we find

$$x'' + x = \varepsilon\eta(\varepsilon)x + \varepsilon(1 - \varepsilon\eta(\varepsilon))^{\frac{1}{2}}x'(1 - x^2)$$

with unknown initial conditions $x(0) = a(\varepsilon), x' = 0$. The periodicity conditions at lowest order become

$$-\int_0^{2\pi} a(0)\sin s(1 - a^2(0)\cos^2 s)\sin s ds = 0,$$

$$\pi\eta(0)a(0) - \int_0^{2\pi} a(0)\sin s(1 - a^2(0)\cos^2 s)\cos s ds = 0.$$

After integration, we find

$$a(0)\left(1 - \frac{1}{4}a^2(0)\right) = 0,$$

$$\eta(0)a(0) = 0.$$

Apart from the trivial solution, we find $a(0) = 2, \eta(0) = 0$ (the solution $a(0) = -2$ produces the same approximation), so a periodic solution branches off at amplitude 2. The existence has been given, but we check this independently by computing the Jacobian at $(a(0), \eta(0)) = (2, 0)$: $J_0 = 4$ so the implicit function theorem applies.

Andersen and Geer (1982) used a formal manipulation of the expansions and obtained for the Van der Pol equation the expansion to $O(\varepsilon^{164})$.

Example 10.6
Consider the equation

$$\ddot{x} + x = \varepsilon x^3.$$

It is well-known that all the solutions of this equation are periodic in a large neighbourhood of $(0, 0)$. We follow the construction by again putting $\omega t = \theta,\ \omega^{-2} = 1 - \varepsilon\eta(\varepsilon)$ to find

$$x'' + x = \varepsilon\eta(\varepsilon)x + \varepsilon x^3$$

and at lowest order the periodicity conditions

$$\int_0^{2\pi} a^3(0)\cos^3 s\sin s ds = 0, \pi\eta(0)a(0) + \int_0^{2\pi} a^3(0)\cos^3 s\cos s ds = 0,$$

so that the first condition is always satisfied and the second condition gives

$$a(0)\left(\eta(0)+\frac{3}{4}a^2(0)\right)=0.$$

We conclude that $a(0)$ can be chosen arbitrarily and that $\eta(0)=-\frac{3}{4}a^2(0)$. For the Jacobian, we find $J_0=0$, as there exists an infinite number of periodic solutions.

Remark (on the importance of existence results)
Suppose that one can apply the periodicity conditions but that the Jacobian J vanishes. Why bother about this existence question? The reason to worry about this is that higher-order terms may destroy the periodic solution. A simple example is the equation

$$\ddot{x}+x=\varepsilon x^3-\varepsilon^n\dot{x}$$

with n a natural number ≥ 2. We can satisfy the periodicity conditions to $O(\varepsilon^{n-1})$, but the equation has no periodic solution.

10.3 Periodic Nonautonomous Equations

In this section, we consider nonautonomous, periodic equations so the period is a priori fixed. There are still many subtle problems here, as the period T_0 of a periodic solution of the unperturbed equation can be near the period of the perturbation or quite distinct. To fix the idea, we consider the two-dimensional equation

$$\ddot{x}+x=\varepsilon f(x,\dot{x},t,\varepsilon).$$

We shall look for periodic solutions that can be continued for $\varepsilon>0$. Let us assume that the perturbation is T-periodic with a period near 2π. To apply the periodicity condition, it is convenient to have periodicity 2π so we transform time with a factor

$$\omega^{-2}=1-\varepsilon\beta(\varepsilon),\ \ \beta(\varepsilon)=\beta_0+\varepsilon\beta_1+\cdots$$

with *known* constants $\beta_0,\beta_1,\cdots$.

In autonomous equations, we have the translation property that if $y(t)$ is a solution, $y(t-a)$ with a an arbitrary constant is also a solution. This is not the case in nonautonomous equations, so it is natural to introduce a phase ψ that will in general depend on ε: $\psi(\varepsilon)=\psi_0+\varepsilon\psi_1+\varepsilon^2\cdots$. The time transformation becomes

$$\omega t=\theta-\psi(\varepsilon)$$

and the equation transforms with $x'=dx/d\theta$ to

$$x''+x=\varepsilon\beta(\varepsilon)x+\varepsilon(1-\varepsilon\beta(\varepsilon))f(x,(1-\varepsilon\beta(\varepsilon))^{\frac{1}{2}}x',(1-\varepsilon\beta(\varepsilon))^{\frac{1}{2}}(\theta-\psi(\varepsilon)),\varepsilon).$$

We shall look for 2π-periodic solutions starting at

$$x(0) = a(\varepsilon),\ \dot{x}(0) = 0,$$

with the expansion $a(\varepsilon) = a_0 + \varepsilon a_1 + \cdots$, which still has to be determined.

As before, the differential equation can be transformed to an integral equation, in this case of the form

$$x(\theta) = a(\varepsilon)\cos\theta + \varepsilon \int_0^\theta F(x(s), x'(s), \psi(\varepsilon), s, \varepsilon,)\sin(\theta - s)ds$$

with $F = F(x, x', \psi, \theta, \varepsilon)$ or

$$F = \beta(\varepsilon)x + (1 - \varepsilon\beta(\varepsilon))f(x, (1 - \varepsilon\beta(\varepsilon))^{\frac{1}{2}}x', (1 - \varepsilon\beta(\varepsilon))^{\frac{1}{2}}(\theta - \psi(\varepsilon)), \varepsilon)$$

and with the periodicity condition $x(\theta + 2\pi) = x(\theta)$ for each value of θ. So

$$\int_0^{2\pi} F(x(s), x'(s), \psi(\varepsilon), s, \varepsilon)\sin(\theta - s)ds = 0.$$

We find the two conditions

$$\int_0^{2\pi} F(x(s), x'(s), \psi(\varepsilon), s, \varepsilon)\sin s ds = 0,$$

$$\int_0^{2\pi} F(x(s), x'(s), \psi(\varepsilon), s, \varepsilon)\cos s ds = 0.$$

This is a system of two equations with two unknowns, $a(\varepsilon)$ and $\psi(\varepsilon)$, which we shall study in a number of examples.

Example 10.7
(forced Van der Pol equation)
Consider the case of the Van der Pol equation with a small forcing

$$\ddot{x} + x = \varepsilon\dot{x}(1 - x^2) + \varepsilon h \cos\omega t.$$

Using the transformations outlined above, we have

$$F = \beta(\varepsilon)x(\theta) + (1 - \varepsilon\beta(\varepsilon))[(1 - \varepsilon\beta(\varepsilon))^{\frac{1}{2}}x'(1 - x^2(\theta)) + h\cos(\theta - \psi(\varepsilon))],$$

which can be expanded to

$$F = \beta_0 a_0 \cos\theta - a_0 \sin\theta(1 - a_0^2\cos^2 s) + h\cos(\theta - \psi_0) + \varepsilon\cdots .$$

From the periodicity conditions, to first order we find

$$I_1 = -a_0\left(1 - \frac{1}{4}a_0^2\right) + h\sin\psi_0 = 0,$$
$$I_2 = \beta_0 a_0 + h\cos\psi_0 = 0.$$

For the Jacobian, we find at lowest order

$$J_0 = \left| \frac{\partial(I_1, I_2)}{\partial(a_0, \psi_0)} \right| = |h| \left| \left(1 - \frac{3}{4}a_0^2\right) \sin\psi_0 - \beta_0 \cos\psi_0 \right| .$$

Exact 2π-periodic forcing means $\beta_0 = 0$.

Exploring this case first, we find $\psi_0 = \pi/2, 3\pi/2$, and

$$a_0 \left(1 - \frac{1}{4}a_0^2\right) = \pm h.$$

If $|h| > h^* = 4/(3\sqrt{3})$, we have one solution; see Fig. 11.3. When $|h|$ passes the critical value h^*, there is a bifurcation producing three solutions. At the value $h = 0$, we have returned to the "ordinary" Van der Pol equation that has one periodic solution with $a_0 = 2$.

If $\beta_0 = 0$, we have $J_0 = |(1 - \frac{3}{4}a_0^2)h|$. We observe that at the bifurcation values $h = h^*, 0$, the Jacobian J_0 vanishes.

If $\beta_0 \neq 0$, we have the relation

$$\tan\psi_0 = \frac{\frac{1}{4}a_0^2 - 1}{\beta_0}$$

and a similar analysis can be made.

Example 10.8

(damped and forced Duffing equation)

A fundamental example of mechanics is an oscillator built out of a Hamiltonian system with damping and forcing added. A relatively simple but basic nonlinear case is

$$\ddot{x} + \varepsilon\mu\dot{x} + x + \varepsilon\gamma x^3 = \varepsilon h \cos\omega t$$

with damping coefficient $\mu > 0$. The equation of motion of the underlying Hamiltonian system is obtained when $\mu = h = 0$. Putting $\varepsilon\gamma = -1/6$ produces the first nonlinear term of the mathematical pendulum equation.

Using the formulas derived above, we have in this case

$$f = \mu\dot{x} - \gamma x^3 + h\cos\omega t$$

and after transformation

$$F = \beta(\varepsilon)x(\theta) + (1 - \varepsilon\beta(\varepsilon))[-(1 - \varepsilon\beta(\varepsilon))^{\frac{1}{2}}\mu x' - \gamma x^3 + h\cos(\theta - \psi(\varepsilon))],$$

which can be expanded to

$$F = \beta_0 a_0 \cos\theta + \mu a_0 \sin\theta - \gamma a_0^3 \cos^3\theta + h\cos(\theta - \psi_0) + \varepsilon\cdots .$$

The periodicity conditions to first order produce

$$I_1 = \mu a_0 + h \sin \psi_0 = 0,$$
$$I_2 = \beta_0 a_0 - \frac{3}{4}\gamma a_0^3 + h \cos \psi_0 = 0.$$

The Jacobian to first order becomes

$$J_0 = \left| \frac{\partial(I_1, I_2)}{\partial(a_0, \psi_0)} \right| = |h| \left| \mu \sin \psi_0 + \left(\beta_0 - \frac{9}{4}\gamma a_0^2 \right) \cos \psi_0 \right|.$$

Easiest to analyse is the case without damping, $\mu = 0$. We have the periodicity conditions

$$h \sin \psi_0 = 0, \beta_0 a_0 - \frac{3}{4}\gamma a_0^3 + h \cos \psi_0 = 0,$$

so that $\psi_0 = 0, \pi$, and two possibilities,

$$a_0 \left(\beta_0 - \frac{3}{4}\gamma a_0^3 \right) = \pm h.$$

Interestingly, the product $\beta_0 \gamma$ also plays a part in the bifurcations (the existence of periodic solutions). If $\beta_0 \gamma > 0$, we have one or three solutions depending on the value of h, and if $\beta_0 \gamma < 0$, there is only one solution. This picture also emerges from the Jacobian

$$J_0 = \left| h \left(\beta_0 - \frac{9}{4}\gamma a_0^2 \right) \cos \psi_0 \right|.$$

We conclude that a small detuning of the forcing from exact 2π-periodicity is essential to obtain interesting bifurcations. On taking $\beta_0 = 0$, we find to first order the periodicity conditions

$$h \sin \psi_0 = -\mu a_0, h \cos \psi_0 = \frac{3}{4}\gamma a_0^3.$$

Again there are two possible solutions for the phase shift ψ_0 corresponding with one periodic solution each for whatever the values of h, μ, and $\gamma (\neq 0)$ are. This is also illustrated by the Jacobian, which at an exact 2π-periodic forcing, using the periodicity conditions, becomes

$$J_0 = |h| \left| \mu \sin \psi_0 - \frac{9}{4}\gamma a_0^2 \cos \psi_0 \right| = |a_0| \left| \mu^2 + \frac{27}{16}\gamma^2 a_0^4 \right|.$$

Example 10.9
(Mathieu equation)
Linear equations with periodic coefficients play an important part in physics and engineering. A typical example is the π-periodic Mathieu equation that can be written in the form

$$\ddot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0.$$

Of particular interest is usually the question of for which values of ω and ε the solutions are stable (i.e., decreasing to zero) or unstable. The answer to this question is guided by Floquet theory, which tells us that for ω fixed, the (two) independent solutions of the equation can be written in the form $\exp(\lambda(\varepsilon)t)p(t)$ with $p(t)$ a π-periodic function. So the two possible expressions for the so-called characteristic exponents $\lambda(\varepsilon)$ - call them λ_1 and λ_2 - determine the stability of the solutions. Also from Floquet theory we have that $\lambda_1(\varepsilon) + \lambda_2(\varepsilon) = 0$. For extensive introductions to Floquet theory, see Magnus and Winkler (1966) and Yakubovich and Starzhinskii (1975), for summarising introductions Hale (1963) or Verhulst (2000).

The expansion theorem tells us that the exponents $\lambda_{1,2}(\varepsilon)$ can be expanded in a Taylor series with respect to ε with $\lambda_1(0) = \omega i$, $\lambda_2(0) = -\omega i$. This has important consequences. If ω is not close to a natural number, any perturbation of $\lambda_{1,2}(0)$ will cause it to move along the imaginary axis, so for the possibility of instability we only have to consider the cases of ω near $1, 2, 3, \cdots$.

It turns out that in an ω, ε-diagram, we have domains emerging from the ω-axis, called Floquet tongues, where the solutions are unstable; see Fig. 10.1. The boundaries of these tongues correspond with the values of ω, ε where the solutions are periodic (i.e., they are neutrally stable). We shall determine these tongues in two cases.

One should note that if we have for a linear homogeneous equation one periodic solution, we have a one-parameter family of periodic solutions and definitely no uniqueness. In this case, the boundaries of the tongues correspond with periodic solutions, so the two independent solutions are periodic and have the same period. However, this uniqueness question need not bother us, as we know a priori that in this case families of periodic solutions exist.

We assume that $\omega^2 = m^2 - \varepsilon\beta(\varepsilon)$ with $m = 1, 2, \cdots$ and $\beta(\varepsilon) = \beta_0 + \beta_1\varepsilon + \cdots$ a known Taylor series in ε. The equation becomes

$$\ddot{x} + m^2 x = \varepsilon\beta(\varepsilon)x - \varepsilon \cos 2tx.$$

It will turn out that in the cases $m = 2, 3, \cdots$ we have to perform higher-order calculations. In the case of nonautonomous equations, it is then more convenient to drop the phase-amplitude representation of the solutions and use the transformation $x, \dot{x} \to y_1, y_2$:

$$x(t) = y_1(t)\cos mt + y_2(t)\sin mt,$$
$$\dot{x}(t) = -my_1(t)\sin mt + my_2(t)\cos mt.$$

The expansion of $x(t)$ will take the form $x(t) = y_1(0)\cos mt + y_2(0)\sin mt + \varepsilon\cdots$. For y_1, y_2, we find the equations

$$\dot{y}_1 = -\frac{\varepsilon}{m}(\beta(\varepsilon) - \cos 2t)(y_1 \cos mt + y_2 \sin mt)\sin mt,$$
$$\dot{y}_2 = \frac{\varepsilon}{m}(\beta(\varepsilon) - \cos 2t)(y_1 \cos mt + y_2 \sin mt)\cos mt.$$

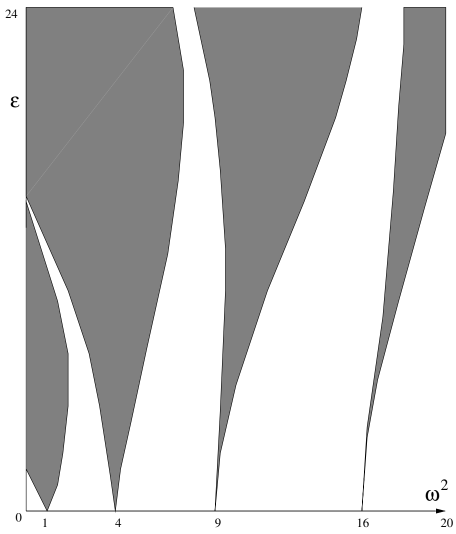

Fig. 10.1. Instability (or Floquet) tongues of the Mathieu equation $\ddot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0$. The shaded domains correspond with instability.

The Mathieu equation can be transformed to an integral equation by formally integrating the equations for y_1 and y_2 and substituting them into the expression for $x(t)$.
We shall now use the periodicity conditions in various cases.

10.3.1 Frequency ω near 1

The solutions of the unperturbed (harmonic) equation are near 2π-periodic and the forcing is π-periodic; such a forcing is called subharmonic. We will look for 2π-periodic solutions, and we have the periodicity conditions

$$\int_0^{2\pi} (\beta(\varepsilon) - \cos 2s)(y_1(s)\cos s + y_2(s)\sin s)\sin s ds = 0,$$
$$\int_0^{2\pi} (\beta(\varepsilon) - \cos 2s)(y_1(s)\cos s + y_2(s)\sin s)\cos s ds = 0.$$

Expanding, we find to first order the periodicity conditions

$$y_2(0)\left(\beta_0 + \frac{1}{2}\right) = 0,\ y_1(0)\left(\beta_0 - \frac{1}{2}\right) = 0.$$

For the boundaries of the Floquet tongue, we find $\beta_0 = \pm\frac{1}{2}$ or to first order $\omega^2 = 1 \pm \frac{1}{2}\varepsilon$.

10.3.2 Frequency ω near 2

The solutions of the unperturbed equation are π-periodic like the forcing, and so we are looking for π-periodic solutions of the Mathieu equation. We have

$$\int_0^\pi (\beta(\varepsilon) - \cos 2s)(y_1(s)\cos s + y_2(s)\sin s)\sin 2s ds = 0,$$

$$\int_0^\pi (\beta(\varepsilon) - \cos 2s)(y_1(s)\cos s + y_2(s)\sin s)\cos 2s ds = 0.$$

To first order, we find

$$y_2(0)\beta_0 = 0,\ y_1(0)\beta_0 = 0.$$

To have nontrivial solutions, we conclude that $\beta_0 = 0$ and we have to expand to higher order. For this we compute

$$\begin{aligned} y_1(t) &= y_1(0) - \frac{\varepsilon}{2}\int_0^t (-\cos 2s)(y_1(0)\cos 2s + y_2(0)\sin 2s)\cos 2s ds + O(\varepsilon^2) \\ &= y_1(0) + \varepsilon\frac{3}{16}\left[y_1(0)\left(\frac{4}{3} - \cos 2t - \frac{1}{3}\cos 6t\right)\right. \\ &\quad \left. + y_2(0)\left(\sin 2t - \frac{1}{3}\sin 6t\right)\right] + O(\varepsilon^2). \end{aligned}$$

Substituting this expression in the periodicity conditions produces $\beta_1 = \frac{1}{48}$ or $\beta_1 = -\frac{5}{48}$. Accordingly, the Floquet tongue is bounded by

$$\omega^2 = 4 - \frac{1}{48}\varepsilon^2 + \cdots, \omega^2 = 4 + \frac{5}{48}\varepsilon^2 + \cdots .$$

Subsequent calculations will show that the neglected terms are $O(\varepsilon^4)$.

The calculations for $m = 3, 4, \cdots$ will be even more laborious. The same holds when we want more precision (i.e., calculation of higher-order terms) for a particular value of m. In this case, computer algebra can be very helpful, especially as we know in advance here that the expansions are convergent. For a computer algebra approach, see for instance Rand (1994).

Remark

In Section 15.3, we will show that, for certain values of ω near 1, the solutions in the instability tongue are growing exponentially with $\varepsilon^{\frac{3}{2}}t$. This is not in contradiction with the Poincaré expansion theorem, as the theorem guarantees the existence of an expansion in integer powers of ε on the timescale 1. Such an expansion is clearly not valid on the timescale where the instability is developing. For periodic solutions the situation is different, as in this case timescale 1 implies "for all time" (assuming that the period does not depend on the small parameter). An extensive discussion of timescales is given in Chapter 11.

Example 10.10
(Mathieu equation with damping)
In general, dissipation effects play a part in mechanics, so it seems natural to look at the effect of damping on the Mathieu equation

$$\ddot{x} + \varepsilon\mu\dot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0,$$

where μ is a positive coefficient and we consider the simplest case of ω near 1: $\omega^2 = 1 - \varepsilon\beta_0 + O(\varepsilon^2)$. Omitting the $O(\varepsilon^2)$ terms, the equation becomes

$$\ddot{x} + x = \varepsilon\beta_0 x - \varepsilon\mu\dot{x} - \varepsilon \cos(2t)x.$$

In the periodicity conditions derived in the preceding example, we have to add the term

$$\mu(y_1(0)\sin s - y_2(0)\cos s)$$

to the integrand. This results in the periodicity conditions

$$\frac{1}{2}\mu y_1(0) + \left(\beta_0 + \frac{1}{2}\right) y_2(0) = 0,$$
$$\left(\beta_0 - \frac{1}{2}\right) y_1(0) - \frac{1}{2}\mu y_2(0) = 0.$$

To have nontrivial solutions, the determinant has to be zero or

$$\beta_0 = \pm\frac{1}{2}\sqrt{1 - \mu^2},$$

and for the boundaries of the instability domain, we find

$$\omega^2 = 1 \pm \frac{1}{2}\varepsilon\sqrt{1 - \mu^2}.$$

As a consequence of damping, the instability domain of the Mathieu equation shrinks and the tongue is lifted off the ω-axis.

10.4 Autoparametric Systems and Quenching

Consider a system consisting of weakly interacting subsystems. To fix the idea, we consider a system with two degrees of freedom (four dimensions). Suppose that in one degree of freedom stable motion is possible without interaction with the other subsystem; such motion is usually called "normal mode behaviour" and in many cases this will be a (stable) periodic solution. Is this normal mode stable in the four-dimensional system? If not, the corresponding instability phenomenon is called autoparametric resonance.

This question is of particular interest in engineering problems where the normal mode may represent undesirable behaviour of flexible structures such as vibrations of overhead transmission lines, connecting cables, or chimney pipes. In the engineering context, normal modes are often called “semitrivial solutions”. We may try to destabilise them by actually introducing a suitable interacting system. This may result in destabilisation or energy reduction of the undesirable normal mode; this process of permanent reduction of the amplitude of the normal mode is called “quenching”, and the second oscillator, which does the destabilisation, is called the “energy absorber”. For a survey and treatment of such problems, see the monograph by Tondl et al. (2000).

In the case of Hamiltonian systems, we may have autoparametric resonance and a normal mode may be destabilised; an example is the elastic pendulum. However, because of the recurrence of the phase flow, we have no quenching. For this we need energy dissipation in the second oscillator.

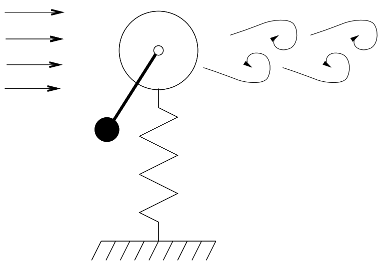

Fig. 10.2. Example of an autoparametric system with flow-induced vibrations. The system consists of a single mass on a spring to which a pendulum is attached as an energy absorber. The flow excites the mass and the spring but not the pendulum.

Example 10.11
(quenching of self-excited oscillations)
Consider the system

$$\ddot{x} - \varepsilon(1 - \dot{x}^2)\dot{x} + x = \varepsilon f(x, y),$$
$$\ddot{y} + \varepsilon\mu\dot{y} + q^2 y = \varepsilon y g(x, y),$$

where μ is the (positive) damping constant, $f(x, y)$ is an interaction term with expansion that starts with quadratic terms, and the interaction term $g(x, y)$ starts with linear terms. In Fig. 10.2, a pendulum is attached as an example of an energy absorber. The equation for x is typical for flow-induced vibrations, where $(1 - \dot{x}^2)\dot{x}$ is usually called Rayleigh self-excitation.

To fix the idea, assume that $f(x, y) = c_1x^2 + c_2xy + c_3y^2, g(x, y) = d_1x + d_2y$. (This is different from the case where a pendulum is attached.) Putting $y = 0$ produces normal mode self-excited oscillations described by

$$\ddot{x} - \varepsilon(1 - \dot{x}^2)\dot{x} + x = \varepsilon c_1 x^2.$$

As in Example 10.5 for the Van der Pol equation, we transform time by $\omega t = \theta,\ \omega^{-2} = 1 - \varepsilon\eta(\varepsilon)$ to find

$$x'' + x = \varepsilon\eta(\varepsilon)x + \varepsilon(1 - \varepsilon\eta(\varepsilon))^{\frac{1}{2}} x'(1 - (1 - \varepsilon\eta(\varepsilon))^{-1} x'^2) + \varepsilon(1 - \varepsilon\eta(\varepsilon))c_1 x^2$$

with unknown initial conditions $x(0) = a(\varepsilon), x' = 0$. The periodicity conditions at lowest order become

$$- \int_0^{2\pi} [a(0)\sin s(1 - a^2(0)\sin^2 s) + c_1 a^2(0)\cos^2 s]\sin s ds = 0,$$
$$\pi\eta(0)a(0) - \int_0^{2\pi} [a(0)\sin s(1 - a^2(0)\sin^2 s) + c_1 a^2(0)\cos^2 s]\cos s ds = 0.$$

After integration, we find as for the Van der Pol equation

$$a(0)\left(1 - \frac{1}{4}a^2(0)\right) = 0,$$
$$\eta(0)a(0) = 0,$$

so, apart from the trivial solution, we have $a(0) = 2, \eta(0) = 0$. (The solution $a(0) = -2$ produces the same approximation.) A periodic solution $\phi(t)$ branches off at amplitude 2 with first-order approximation $x_0(t) = 2\cos t$. For the periodic solution we have the estimate $\phi(t) = 2\cos t + O(\varepsilon)$.

To study the stability of this normal mode solution, we put $x(t) = \phi(t) + u$. Substitution in the equation for x and using that $\phi(t)$ is a solution, we find

$$\ddot{u} + u = \varepsilon(1 - \dot{\phi}^2)\dot{u} - \varepsilon(2\dot{\phi}\dot{u} + \dot{u}^2)(\dot{\phi} + \dot{u}) + \varepsilon(c_1(2\phi u + u^2) + c_2(\phi + u)y + c_3 y^2).$$

Also, we substitute $x(t) = \phi(t) + u$ in the equation for y. To determine the stability of $\phi(t)$, we linearise the system and replace $\phi(t)$ by its first-order approximation $x_0(t)$ to obtain

$$\ddot{u} + u = \varepsilon(1 - 12\sin^2 t)\dot{u} + \varepsilon 4c_1 u\cos t + \varepsilon 2c_2 y\cos t,$$
$$\ddot{y} + q^2 y = -\varepsilon\mu\dot{y} + \varepsilon 2d_1 y\cos t.$$

The equations are in a certain sense decoupled: first we can solve the problem for y, after which we consider the problem for u. This decoupling happens often in autoparametric systems.

As in Example 10.9, we use the transformation $y, \dot{y} \to y_1, y_2$:

$$y(t) = y_1(t)\cos qt + y_2(t)\sin qt,$$
$$\dot{y}(t) = -qy_1(t)\sin qt + qy_2(t)\cos qt.$$

The expansion will take the form $y(t) = y_1(0)\cos qt + y_2(0)\sin qt + \varepsilon\cdots$. For y_1, y_2, we find the equations

$$\dot{y}_1 = -\frac{\varepsilon}{q}[\mu q y_1(t) \sin qt - \mu q y_2(t) \cos qt + 2d_1(y_1 \cos qt + y_2 \sin qt) \cos t] \sin qt,$$
$$\dot{y}_2 = \frac{\varepsilon}{q}[\mu q y_1(t) \sin qt - \mu q y_2(t) \cos qt + 2d_1(y_1 \cos qt + y_2 \sin qt) \cos t] \cos qt.$$

Integration and application of the periodicity conditions leads at first order to nontrivial results if $q = \frac{1}{2}$. We find

$$\frac{1}{2}\mu y_1 - d_1 y_2 = 0,$$
$$d_1 y_1 - \frac{1}{2}\mu y_2 = 0.$$

We have nontrivial solutions if the determinant of the matrix of coefficients vanishes, or

$$\mu^2 = 4d_1^2.$$

If the damping coefficient μ satisfies $0 \le \mu \le 2|d_1|$, we have instability with respect to perturbations orthogonal to the normal mode (in the $y-\dot{y}$ direction); if $\mu > 2|d_1|$, we have stability.

It is an interesting question whether perturbations in the normal mode plane can destabilise the normal mode. For this we have to solve the equation for u. With the choice $q = \frac{1}{2}$ and if $c_2 \neq 0$, we find that $u = 0$ is unstable; the calculation is left to the reader.

10.5 The Radius of Convergence

When obtaining a power series expansion with respect to ε by the Poincaré-Lindstedt method, we have a convergent series for the periodic solution. So, in contrast with the results for most asymptotic expansions, it makes sense to ask the question of to what value of ε the series converges.

In the paper by Andersen and Geer (1982), where 164 terms were calculated for the expansion of the periodic solution of the van der Pol equation, the numerics surprisingly suggests convergence until $\varepsilon = O_s(1)$. These results become credible when looking at the analytic estimates by Grebenikov and Ryabov (1983). After introducing majorising equations for the expansion, Grebenikov and Ryabov give some examples. First, for the Duffing equation with forcing,

$$\ddot{x} + x - \varepsilon x^3 = \varepsilon a \sin t.$$

Grebenikov and Ryabov show that the convergence of the Poincaré-Lindstedt expansion for the periodic solution holds for

$$0 \le \varepsilon \le 1.11|a|^{-\frac{2}{3}}.$$

In the case of the Mathieu equation

$$\ddot{x} + (a + \varepsilon \cos 2t)x = 0$$

near $a = 1$, they obtain convergence for

$$0 \leq \varepsilon \leq 5.65.$$

For the resonances $a = 4, 9$, $O_s(1)$ estimates are also found.

Finally, we note that the radius of convergence of the power series with respect to the small parameter ε does not exclude continuation of the periodic solution beyond the radius of convergence. A simple example is the equation

$$\ddot{x} + \frac{3}{2}\varepsilon\dot{x} + x = -3\sin 2t.$$

The equation contains a unique periodic solution

$$\phi(t) = \frac{1}{1+\varepsilon^2}\sin 2t + \frac{\varepsilon}{1+\varepsilon^2}\cos 2t$$

that has a convergent series expansion for $0 \leq \varepsilon < 1$ but exists for all values of ε.

10.6 Guide to the Literature

The techniques discussed in this chapter were already in use in the eighteenth and nineteenth centuries, but the mathematical formulation of such results for initial value problems for ordinary differential equations was given by Henri Poincaré. Usually his method of using the expansion theorem to construct periodic solutions is called the Poincaré-Lindstedt method, as Lindstedt produced a formal calculation of this type. The procedure itself is older, but Poincaré (1893, Vol. 2) was the first to present sound mathematics. His 1893 proof of the expansion theorem is based on majorising series and rather complicated, see also Roseau (1966) for an account. More recent proofs use a continuation of the problem into the complex domain in combination with contraction; an example of such a proof is given in Verhulst (2000, Chapter 9).

The analysis is in fact an example of a very general problem formulation. Consider an equation of the form

$$F(u, \varepsilon) = 0$$

with F a nonlinear operator on a linear space - a Hilbert or Banach space - and with known solution $u = u_0$ if $\varepsilon = 0$, so $F(u_0, 0) = 0$. The problem is then under what condition we can obtain for the solution a convergent series of the form

$$u = u_0 + \sum_{n \geq 1} \varepsilon^n u_n.$$

The operator F can be a function, a differential equation, or an integral equation and can take many other forms. Vainberg and Trenogin (1974) give a general discussion, with the emphasis on Lyapunov-Schmidt techniques, and many examples.

A survey of the implicit function theorem with modern extensions and applications is given by Krantz and Parks (2002). Application of the Poincaré-Lindstedt method to systems with dimension higher than two poses no fundamental problem but requires laborious formula manipulation. An example of an application to Hamiltonian systems with two degrees of freedom has been presented in two papers by Presler and Broucke (1981a, 1981b). They apply formal algebraic manipulation to obtain expansions of relatively high order. Rand (1994) gives an introduction to the use of computer algebra in nonlinear dynamics.

More general nonlinear equations, in particular integral equations, are considered by Vainberg and Trenogin (1974). The theoretical background and many applications to perturbation problems in linear continuum mechanics can be found in Sanchez Hubert and Sanchez Palencia (1989).

Apart from this well-founded work, there are applications to partial differential equations using the Poincaré-Lindstedt method formally. It is difficult to assess the meaning of these results unless one has a priori knowledge about the existence and smoothness of periodic solutions.

10.7 Exercises

Exercise 10.1 Consider the algebraic equation

$$x^3 - (3+\varepsilon)x + 2 = 0,$$

which has three real solutions for small $\varepsilon > 0$.
a. Can one obtain the solutions in a Taylor series with respect to ε?
b. Determine a two-term expansion for the solutions.

Exercise 10.2 Kepler's equation for the gravitational two-body problem is

$$E - e\sin E = M$$

with M (depending on the period) and e (eccentricity) given. Show that the angle E, $0 \leq E \leq 2\pi$, is determined uniquely.

Exercise 10.3 Consider the equation

$$\dot{x} = 1 - x + \varepsilon x^2.$$

If $\varepsilon = 0$, $x = 1$ is a stable equilibrium and it is easy to see that if $\varepsilon > 0$, a stable equilibrium exists in a neighbourhood of $x = 1$.

a. Apply the Poincaré expansion theorem to find the first two terms of the expansion $x_0(t) + \varepsilon x_1(t) + \cdots$.
b. Compute the limit for $t \to +\infty$ of this approximation. Does this fit with the observation about the stable equilibrium?

Exercise 10.4 In Example 10.4, we showed for the damped harmonic oscillator that secular (unbounded) terms arise in the straightforward expansion. One might argue that this is caused by the presence of linear terms in the perturbation, as these are the cause of linear resonance. Consider as an example the problem

$$\ddot{x} + x = \varepsilon x^2, \; x(0) = 1, \dot{x}(0) = 0.$$

a. Show that no secular terms arise for $x_0(t)$ and $x_1(t)$.
b. Do secular terms arise at higher order, for instance for $x_2(t)$?

Exercise 10.5 Consider again the Van der Pol equation (Example 10.5) and calculate the periodic solution to second order. The calculation to higher order becomes laborious but it is possible to implement the procedure in a computer programme. Andersen and Geer (1982) did this for the Van der Pol equation to compute the first 164 terms.

Exercise 10.6 It will be clear from Examples 10.5 and 10.6 that an interesting application arises in examples such as

$$\ddot{x} + x = \varepsilon x^3 + \varepsilon^2 \dot{x}(1 - x^2).$$

At lowest order, we find that the first-order Jacobian J_0 vanishes, but in this case a unique periodic solution exists. When going to second order in ε, the Jacobian does not vanish and this unique periodic solution is found. Check this statement and compute the approximation to this order.