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Introduction

Perturbation theory, defined as the theory of approximating solutions of mathematical problems, goes back to ancient times. An important example is the practice of measurement, where quantities such as distance and volume (such as the contents of wine barrels) have been estimated by professional people through the ages.

The theory of perturbations expanded very rapidly when mathematical analysis was founded in the eighteenth century, and many classical results in this field can be traced to Newton, Euler, Lagrange, Laplace, and others. One of the most stimulating fields of application of that time was celestial mechanics, where the controversies and excitement about Newton's gravitational theory triggered many detailed calculational studies.

The establishment of more rigorous foundations of perturbation theory had to wait until Poincaré (1886) and Stieltjes (1886) separately published papers on asymptotic series, which are in general divergent; see also the discussion on the literature at the end of Chapter 2. In the twentieth century, an additional stimulus came from other fields of application. In 1905, Prandtl published a paper on the motion of a fluid or gas with small viscosity along a body. In the case of an airfoil moving through air, the problem is described by the Navier-Stokes equations with large Reynolds number; see also Prandtl and Tietjens (1934) and, for modern developments, Van Ingen (1998). Ting (2000) and other authors discuss the boundary layer theory of fluids in a special issue of the Zeitschrift für Angewandte Mathematik und Mechanik dedicated to Ludwig Prandtl.

In this problem, there are two regions of interest: a boundary layer around the solid body, where the velocity gradient becomes large, and the region outside this layer, where we can neglect the velocity gradient and the viscosity. The mathematical analysis of the problem uses this insight to develop an appropriate perturbation theory in the case of the presence of boundary layers. Notes on the historical development of boundary layer theory are given by O'Malley (1991).

As mentioned above, the roots of classical perturbation theory, which are mainly in celestial mechanics, are quite old. A modern stimulus came from the theory of nonlinear oscillations in electronics and mechanics. The name of the Dutch physicist Balthasar van der Pol is connected with this field, for instance in the theory of relaxation oscillations. One can find historical remarks in the books by Bogoliubov and Mitropolsky (1961), Sanders and Verhulst (1985), and Grasman (1987).

We conclude this introduction by giving some examples. *Note that here and henceforth ε will always be a small positive parameter:*

$$0 < \varepsilon \ll 1.$$

Quantities and functions will be real unless explicitly stated otherwise.

Example 1.1
The first example is a series studied by Euler (1754) with partial sum

$$S_m(\varepsilon) = \sum_{n=0}^{m} (-1)^n n! \varepsilon^n.$$

It is clear that the series diverges as, denoting the terms of the series by a_n, we have

$$\left| \frac{a_n}{a_{n-1}} \right| = n\varepsilon.$$

However, the size of the terms for small values of ε does not increase much in the beginning (i.e. if $n\varepsilon \ll 1$), but growth seriously affects the partial sum for larger values of m. The question is, can we use a number of the first terms of such a divergent series to approximate a function in some sense? This looks like a wild idea, but consider the function $f(\varepsilon)$ defined by the convergent integral

$$f(\varepsilon) = \int_0^\infty e^{-t} \frac{dt}{1+\varepsilon t}.$$

Partial integration leads to the expression

$$f(\varepsilon) = S_m(\varepsilon) + (-1)^{m+1}(m+1)!\varepsilon^{m+1} \int_0^\infty e^{-t} \frac{dt}{(1+\varepsilon t)^{m+2}}.$$

The integral on the right-hand side converges, and we estimate

$$|f(\varepsilon) - S_m(\varepsilon)| \leq (m+1)!\varepsilon^{m+1}.$$

In some sense, to be made precise later on, for ε small enough, S_m constitutes an approximation of f. To be more explicit, we give some numerical details.

ε	$f(\varepsilon)$	$S_2(\varepsilon) = 1 - \varepsilon + 2\varepsilon^2$
.05	.9543	.9550
.10	.9156	.9200
.20	.8521	.8800

To see when the divergence becomes effective, we list $S_m(.10)$ for $m = 1, \cdots, 21$. The best approximation in this case is found for $m = 9$.

In Fig. 1.1 we show the behaviour of the error $|S_m - f(.1)|$ as a function of m. It is typical for an asymptotic approximation that there is an optimal choice of the number of terms that generates the best approximation. In approximations by a convergent series, there is not such a finite optimal choice and usually we take as many terms as possible.

m	$S_m(f(.1) = .9156)$	m	$S_m(f(.1) = .9156)$
0	1	11	.9154
1	.9000	12	.9159
2	.9200	13	.9153
3	.9140	14	.9161
4	.9164	15	.9148
5	.9152	16	.9169
6	.9159	17	.9134
7	.9154	18	.9198
8	.9158	19	.9076
9	.9155	20	.9319
10	.9158	21	.8809

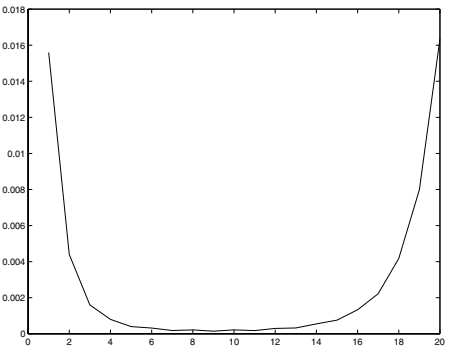

Fig. 1.1. The error $|S_m - f(.1)|$ as a function of m; this behaviour is typical for an asymptotic approximation that generally does not converge to the solution.

We shall now discuss some perturbation problems arising from differential equations.

Example 1.2
The function $\phi_\varepsilon(x)$ is defined for $x \in [0, 1]$ as the solution of the differential equation

$$\frac{d\phi}{dx} + \varepsilon\phi = \cos x$$

with initial value $\phi_\varepsilon(0) = 0$.

Solving the "unperturbed" problem means putting $\varepsilon = 0$ in the equation; we find, using the initial condition,

$$\phi_0(x) = \int_0^x \cos t dt = \sin x.$$

To solve the problem for $\varepsilon > 0$, we might try an expansion of the form

$$\phi_\varepsilon(x) = \sum_{n=0}^{\infty} \varepsilon^n \phi_n(x),$$

which, after substitution in the differential equation, leads to the recurrent system

$$\frac{d\phi_n}{dx} = -\phi_{n-1}, \phi_n(0) = 0, \qquad n = 1, 2, \cdots .$$

In this problem, it is natural to put all initial values for the higher-order equations equal to zero. We find for the first correction to $\phi_0(x), \phi_1(x) = \cos x - 1$, so we have

$$\phi_\varepsilon(x) = \sin x + \varepsilon(\cos x - 1) + \varepsilon^2 \cdots .$$

The expansion for $\phi_\varepsilon(x)$ is a so-called formal expansion, which leads to a consistent construction of the successive terms. (Note that it is strange that mathematicians call this a "formal" expansion when it is really "informal".) In this example, we can analyse the approximate character i.e., the validity of this expansion, by writing down the solution of the problem, obtained by variation of constants,

$$\phi_\varepsilon(x) = e^{-\varepsilon x} \int_0^x \cos t e^{\varepsilon t} dt.$$

We can study the relation between this solution and the formal expansion by partial integration of the integral, see Fig. 1.2. We find

$$\phi_\varepsilon(x) = \sin x + \varepsilon(\cos x - e^{-\varepsilon x}) - \varepsilon^2 \phi_\varepsilon(x)$$

so that we have

$$\phi_\varepsilon(x) = \frac{1}{1+\varepsilon^2}(\sin x + \varepsilon(\cos x - e^{-\varepsilon x})).$$

Expansion with respect to ε produces the validity of the formal approximation on $[0, 1]$.

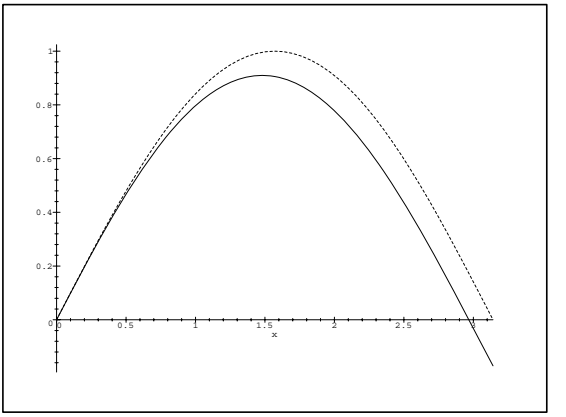

Fig. 1.2. Solution (full line) and formal expansion (dashed line) with $\varepsilon = 0.1$ in Example 1.2. Calculating higher-order approximations will improve the approximation.

Example 1.3
Suppose now that we change the interval in which we are interested in the behaviour of the solution of the equation in Example 1.2 to $[0, \infty]$. Note that the formal approximation to $O(\varepsilon)$,

$$\phi_\varepsilon(x) = \sin x + \varepsilon \cdots ,$$

still holds on the whole interval but the approximation to $O(\varepsilon^2)$ is not a formal approximation. For x in a neighbourhood of $x = 0$, we recover the formal approximation from the exact solution to $O(\varepsilon^2)$, but for x very large, we find from the exact solution the approximation

$$\sin x + \varepsilon \cos x.$$

This motivates us to be more precise in our notions of approximation; we shall return to this in Chapter 2.

Example 1.4
We consider for $x \in [0, 1]$ the function $\phi_\varepsilon(x)$ defined by the initial value problem

$$\varepsilon \frac{d\phi}{dx} + \phi = \cos x, \quad \phi_\varepsilon(0) = 0.$$

The equation is nearly the same as in Example 1.2 but, as will be apparent shortly, the different location of ε changes the problem drastically. Substituting again a formal expansion of the form

$$\phi_\varepsilon(x) = \sum_{n=0}^{\infty} \varepsilon^n \phi_n(x)$$

produces, after regrouping the terms,

$$\phi_0 - \cos x + \sum_{n=1}^{\infty} \varepsilon^n \left(\phi_n + \frac{d\phi_{n-1}}{dx} \right) = 0.$$

So we have $\phi_0 = \cos x, \phi_n = -\frac{d\phi_{n-1}}{dx}, n = 1, 2, \cdots$, and as a formal expansion

$$\phi_\varepsilon(x) = \cos x + \varepsilon \sin x - \varepsilon^2 \cos x + \cdots .$$

However, the expansion makes little sense, as we cannot satisfy the initial condition! To understand what is going on, we write down the solution obtained by variation of parameters,

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon} e^{-x/\varepsilon} \int_0^x e^{t/\varepsilon} \cos t dt.$$

We expand the integral by partial integration to find

$$\phi_\varepsilon(x) = \cos x - e^{-x/\varepsilon} + e^{-x/\varepsilon} \int_0^x e^{t/\varepsilon} \sin t dt.$$

The function $\exp(-x/\varepsilon)$ is quickly varying in a neighbourhood of $x = 0$, see Fig. 1.3. For say, $x \geq \sqrt{\varepsilon}$, this term is very small; we call it "exponentially small".

To order m, we find

$$\begin{aligned} \phi_\varepsilon(x) = {} & \textstyle\sum_{n=0}^{m} (-1)^n \varepsilon^n [\cos^{(n)}(x) - e^{-x/\varepsilon} \cos^{(n)}(0)] \\ & + (-1)^{m+1} \varepsilon^m e^{-x/\varepsilon} \textstyle\int_0^x e^{t/\varepsilon} \cos^{(m+1)}(t) dt. \end{aligned}$$

Introducing the expansion

$$S_m(x) = \sum_{n=0}^{m} (-1)^n \varepsilon^{(n)} [\cos^{(n)}(x) - e^{-\frac{x}{\varepsilon}} \cos^{(n)}(0)],$$

we have in $[0, 1]$

$$\begin{aligned} |\phi_\varepsilon(x) - S_m(x)| \ & \leq C \, \varepsilon^m e^{-\frac{x}{\varepsilon}} \textstyle\int_0^x e^{\frac{t}{\varepsilon}} dt \\ & \leq C \, \varepsilon^{m+1} \left(1 - e^{-\frac{x}{\varepsilon}} \right). \end{aligned}$$

Note that $S_m(x)$ satisfies the initial condition and represents an approximation of the solution. The structure of the expansion, however, is essentially different from the formal expansion. On the other hand, the formal expansion represents the solution well outside a neighbourhood of $x = 0$.

In the next chapter, we shall make our terminology more precise. This is essential to avoid confusion and to obtain a fair appraisal of the results to be obtained by expansion techniques.

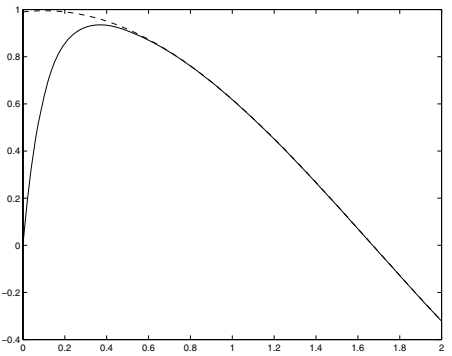

Fig. 1.3. Solution (full line) and formal expansion (dashed line) with $\varepsilon = 0.1$ in Example 1.4. The solution goes through a fast transition near $x = 0$.