ROUND ROBIN TOURNAMENTS WITH ONE BYE AND NO BREAKS IN HOME–AWAY PATTERNS ARE UNIQUE

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Abstract We examine round robin tournaments with m teams and m rounds, for $m \geq 3$, with the property that every team plays no game in one round and exactly one game in each of the remaining $m - 1$ rounds. We show that for every such m there exists a unique schedule in which no team plays two consecutive home or away games.

Keywords: scheduling tournaments, round robin tournament, home–away pattern, complete graph factorization.

1. INTRODUCTORY NOTES AND DEFINITIONS

Many sport competitions are played as round robin tournaments. A *round* is a collection of games in which every team plays at most one game. A *k-round round robin tournament of m teams*, denoted $RRT(m, k)$, is a tournament in which each team meets every other team exactly once and the games are divided into k rounds. A schedule, which is played in the minimum number of rounds possible is called *compact*; if more than the minimum number of rounds is used the schedule is *non-compact*. Although tournaments where every pair of opponents meets exactly l times (called *l-leg tournaments*) are very common, we will discuss only 1-leg tournaments here. An l-leg tournament can be indeed scheduled as a 1-leg tournament repeated l times with teams exchanging their respective home fields regularly. There are many different models that are widely used. In some competitions, like North-American NHL, NBA, NFL and others, the teams are divided into several divisions and it is required that games "inside" the divisions (called *intradivisional* games) and "across" the divisions (called *interdivisional* games) are distributed according to some rules. These rules often take into account travel distances. Therefore a team usually plays several games in a row with teams of another division at their fields. Then there follow several games played at the team's home field or with teams of the same division. Many other constraints are also considered. These can include TV schedules, availability of fields/stadiums, traditional rivals, etc. However, the schedule is usually not strictly divided into rounds and the number of days when the games are played is therefore larger than the necessary minimum. Construction of schedules of this kind is usually based on optimization methods like integer programming or finite-domain constraint programming (see e.g. Henz, 1999, 2001; Henz, *et al.*, 2003; Nemhauser and Trick, 1998; Schaerf, 1999; Schreuder, 1992; Trick, 2000). The result is then an exact schedule in which the dates and fields of all games between particular pairs of opponents are assigned. A graph-theoretic approach can be used for leagues with a small number of teams (see e.g. Dinitz and Froncek, 2000).

In other cases the rules are based on certain restrictions resulting from a limited number of available fields and/or suitable time slots. Schedules of this type were studied among others by Finizio (1993) and Straley (1983). On the other hand, most European national football (soccer) leagues are scheduled as 2-leg compact round robin tournaments (see e.g. Griggs and Rosa, 1996; UEFA, 2004). These tournaments are usually scheduled in such a way that a schedule for a 1-leg $RRT(2n, 2n-1)$ is repeated twice. It is then required that for each team the home and away games should interchange as regularly as possible provided that each team meets every opponent in one leg at its own field and in the other leg at the opponent's field.

In competitions that are played in regular rounds it is usually desirable that for each team the home games and away games interchange as regularly as possible. The leagues often have fixed *draw tables* (or *generic schedules*) with teams denoted just $1, 2, \ldots, m$ that are used repeatedly every season. The teams then draw their numbers either from the whole pool of m numbers (if they have no specific requirements) or from a limited pool (if they have some specific constraints). In this paper we actually present such generic schedules. Fundamental theoretical results concerning such generic schedules were studied by de Werra (1981) and Schreuder (1980).

2. SCHEDULES WITH ONE BYE

In what follows we consider $RRT(2n, 2n-1)$. The *home–away pattern* of a team i, denoted $HAP(i)$, is a sequence $a_1(i), a_2(i), \ldots, a_{2n-1}(i)$, where $a_j(i) = H$ if team i plays in round j a game in the home field and $a_j(i) = A$ if team i plays in round j a game in the opponent's field. If the regularity of the home–away patterns is our top priority, then the most desirable HAP is indeed either $AHAH \ldots AH$ or $HAHA \ldots HA$ in which no subsequence AA or HH appears. Obviously, one can never find a schedule in which all teams would have one of these two HAPs. In this case the teams, which start the season with a home game would never meet. A natural way to measure how "good" a given schedule is is to count the number of breaks in HAPs. A *break* in the HAP of team i is a subsequence AA or HH. Therefore, if we concentrate only on HAPs, we can say that the best schedule is the one with the least number of breaks. By a *break game* we mean the second game in any sequence AA or HH. For instance, in the sequence $HHHAA$ the break games are the games in rounds $2, 3, 5$. Two teams i_1 and i_2 have *complementary HAPs* if $a_j(i_1) = A$ if and only if $a_j(i_2) = H$.

The best possible schedule with respect to the number of breaks is given by the following theorem, which was proved by de Werra (1981).

Theorem 1 *In an $RRT(2n, 2n-1)$, the least number of breaks is $2n-2$. It can be attained in such a way that there are exactly $n-1$ teams with a home break, $n-1$ teams with an away break and 2 teams with no break. There are exactly $n-1$ rounds with break games, each of them containing exactly one home break game and one away break game.*

2.1 Odd Number of Teams

It is well known that a schedule for an odd number of teams, $2n-1$, can be constructed by taking a schedule for $2n$ teams and leaving out one team (called the *dummy team*). Then the team i that was scheduled to play the dummy team in round j plays no game in that round and is said to have a *bye*. We denote a bye in $HAP(i)$ by $a_j(i) = B$. It is also well known that the most commonly used schedule, sometimes called the *canonical* or *1-rotational* schedule has the nice property that if we let the dummy team be the team $2n$, then the remaining teams have no breaks in their schedules. This includes also no breaks around byes, that is, there is no sequence AA, HH, HBH, or ABA in any HAP. The schedule is described in the following construction.

Construction 2 We construct an $RRT(2n+1, 2n+1)$. First we introduce some necessary notation. When a game between teams i and k is scheduled for round j, we denote it by $g(i,k) = g(k,i) = j$. Set $g(i,k) = g(k,i) = i+k-1$. Obviously, $g(i,i) = 2i-1$ means that the team i has a bye in the round $2i-1$.

Table 1. $RRT(7,7)$.

R 1	R 2	R 3	R 4	R 5	R 6	R 7
$7-2$	$4-6$	$1-3$	$5-7$	$2-4$	$6-1$	$3-5$
$6-3$	$3-7$	$7-4$	$4-1$	$1-5$	$5-2$	$2-6$
$5-4$	$2-1$	$6-5$	$3-2$	$7-6$	$4-3$	$1-7$
1 bye	5 bye	2 bye	6 bye	3 bye	7 bye	4 bye

The addition is modulo $2n+1$ with the exception that 0 is replaced by $2n+1$. Home field is determined as follows. In the first round, team 1 has a bye, teams $2, 3, \ldots, n+1$ play home and teams $n+2, n+3, \ldots, 2n+1$ play away. We observe that having scheduled a round j, we can obtain opponents for round $j+1$ by adding $n+1$ to each team number. That is, if $j = g(i,k) = g(k,i) = i+k-1$ with i playing home and k away, then

$$g(i+n+1, k+n+1) = (i+n+1) + (k+n+1) - 1 = i+k = j+1$$

and the team $(i+n+1)$ plays home while $(k+n+1)$ plays away.

An example of the schedule for seven teams is shown in Table 1. A game between teams i and k with i playing home is denoted by $k-i$.

Surprisingly, this schedule is the only one with this property. Notice that for schedules with byes the definition of complementary HAPs of teams i_1, i_2 requires the following: If $a_j(i_1) = B$ for some j, then also $a_j(i_2) = B$.

Theorem 3 *For every $n \geq 1$ there exists an $RRT(2n+1, 2n+1)$ such that no $HAP(i)$ contains any sequence AA, HH, HBH, or ABA. Moreover, for each such n, the schedule is unique up to permutation of team numbers.*

Proof. The existence was proved in Construction 2. Now we prove the uniqueness. First we observe that there is exactly one team with a bye in each round. In an $RRT(2n+1, 2n+1)$ we need to play $n(2n+1)$ games. Because we can schedule at most n games in each of the $2n+1$ rounds, it is easy to see that there must be *exactly* n games in each round.

As opposed to the notation used in Construction 2, we will assume that a team $i, i = 1, 2, \ldots, 2n+1$, has a bye in the round i. That is, $a_i(i) = B$. For clarity, we present in Table 2 the schedule for seven teams again following the notation used in this proof. We can observe that the schedule here can be obtained from Construction 2 by the permutation $\pi(i) = 2i-1$.

We can without loss of generality (WLOG) assume that $a_2(1) = H, a_3(1) = A$ and so on. Then, because $a_2(2) = B$ and teams 1 and 2 cannot play in either

Table 2. $RRT(7,7)$.

R 1	R 2	R 3	R 4	R 5	R 6	R 7
$6-3$	$7-4$	$1-5$	$2-6$	$3-7$	$4-1$	$5-2$
$4-5$	$5-6$	$6-7$	$7-1$	$1-2$	$2-3$	$3-4$
$2-7$	$3-1$	$4-2$	$5-3$	$6-4$	$7-5$	$1-6$
1 bye	2 bye	3 bye	4 bye	5 bye	6 bye	7 bye

Table 3. HAP for $RRT(2n+1, 2n+1)$.

	1	2	3	4	...	$2n-2$	$2n-1$	$2n$	$2n+1$
1	**B**	H	A	H	...	H	A	H	A
2	A	**B**	H	A	...	A	H	A	H
3	H	A	**B**	H	...	H	A	H	A
4	A	H	A	**B**	...	A	H	A	H
...					...				
...					...				
...					...				
$2n-2$	A	H	A	H	...	**B**	H	A	H
$2n-1$	H	A	H	A	...	A	**B**	H	A
$2n$	A	H	A	H	...	H	A	**B**	H
$2n+1$	H	A	H	A	...	A	H	A	**B**

round 1 or 2 since one of them has a bye in each of these rounds, we can see that $a_1(2) = A, a_3(2) = H, a_4(2) = A$ and so on. For similar reasons, because $a_3(3) = B$, we have $a_1(3) = H, a_2(3) = A, a_4(3) = H, \ldots$ or otherwise the teams 2 and 3 can never play against each other. Inductively, we can see that for teams i and $i+1$ one of them has to start the schedule with a home game while the other one with an away game otherwise they never meet. An example is shown in Table 3.

We introduce some more notation. By $S(i,k)$ we denote the set of all rounds in which teams i and k can possibly meet. In other words, $j \in S(i,k)$ if and only if $a_j(i) = H$ and $a_j(k) = A$ or $a_j(i) = A$ and $a_j(k) = H$.

We now proceed inductively. First we observe that the teams 1 and 3 can meet only in round 2 as after round 3 they have both the home games in even rounds and away games in odd rounds. In general, for any team i, $S(i, i+2) = i+1$ and hence there is a unique round in which the game between i and $i+2$ can be scheduled (team numbers are taken modulo $2n+1$ with the exception that 0 is replaced by $2n+1$). In particular, for $i = 1, 2, \ldots, 2n+1$ we have to set $g(i, i+2) = i+1$. Now the teams 1 and 5 can play each other only in round 3: in rounds 1 and 5 one of them has a bye, in round 2 the team 1 plays the game against the team 3, and in round 4 the team 5 plays the game against the team 3. After round 5 their HAPs are equal. We can also check that for $i = 1, 2, \ldots, n$ we have $S(i, i+4) = \{i+1, i+2, i+3\}$ as the respective HAPs are equal before round i and after round $i+4$. But the game between i and $i+4$ cannot be played in round $i+1$, since there is the uniquely determined game $g(i, i+2)$. Or, in our notation, $i+1 = g(i, i+2)$. Also, this game cannot be scheduled for round $i+3$, as $i+3 = g(i+2, i+4)$. Therefore, we must have $g(i, i+4) = i+2$.

We continue inductively and suppose that for every $i = 1, 2, \ldots, 2n+1$ all values $g(i, i+2), g(i, i+4), \ldots, g(i, i+2s)$ have been uniquely determined. This indeed means that also the values $g(i, i-2), g(i, i-4), \ldots, g(i, i-2s)$ have been uniquely determined. We want to show that subsequently the game between i and $i+2s+2$ is also uniquely determined. We can assume here that $2s \leq 2n-1$ because of modularity. Then $S(i, i+2s+2) = \{i+1, i+2, \ldots, i+2s-1\}$. From our assumption it follows that $g(i, i+2) = i+1, g(i, i+4) = i+2, \ldots, g(i, i+2s) = i+s$. Also $g(i+2s, i+2s+2) = i+2s+1, g(i+2s-2, i+2s+2) = i+2s, \ldots, g(i+2, i+2s+2) = i+s+2$, and hence the game between i and $i+2s+2$ must be scheduled for round $i+s+1$. We notice here that because of modularity we get here also all games between teams i and $i+2t+1$, since $i+2t+1 \equiv i+2t-2n \pmod{2n+1}$. □

2.2 Even Number of Teams

One can now ask an obvious question: When it is possible to play an $RRT(2n+1, 2n+1)$ with no breaks, is it possible for an $RRT(2n, 2n)$ as well? The answer is affirmative. Although it may seem unnatural to construct a schedule that needs one more round than the necessary minimum, we can find a motivation in North-American collegiate competitions. The teams are divided into many conferences and it is required that conference games and non-conference games are distributed according to certain rules. Sometimes the non-conference games are scheduled before and after a block of conference games. However, some conferences have schedules where one or more non-conference games are scattered among conference games. Thus, the schedule of the conference games is usually non-compact.

The schedule is actually very simple and as in the case of an odd number of teams, it is also unique up to permutation of team numbers and reflection of the order of rounds. We first construct such a schedule and then prove the uniqueness.

Construction 4 Set $g(i, k) = g(k, i) = i+k-1$. The addition is modulo $2n$ with the exception that 0 is replaced by $2n$. Obviously, $g(i, i) = 2i-1$

Table 4. $RRT(8,8)$.

R 1	R 2	R 3	R 4	R 5	R 6	R 7	R 8
8 − 2	5 − 6	1 − 3	6 − 7	2 − 4	7 − 8	3 − 5	8 − 1
7 − 3	4 − 7	8 − 4	5 − 8	1 − 5	6 − 1	2 − 6	7 − 2
6 − 4	3 − 8	7 − 5	4 − 1	8 − 6	5 − 2	1 − 7	6 − 3
	2 − 1		3 − 2		4 − 3		5 − 4
1, 5 bye		2, 6 bye		3, 7 bye		4, 8 bye	

means that the team i has a bye in the round $2i - 1$. So the teams with byes in the first rounds are 1 and $n + 1$, and we choose as home teams for the first round the teams $2, 3, \ldots, n$. Notice that for $i = 1, 2, \ldots, n$ the teams i and $i + n$ have complementary home–away patterns with a bye in round $2i - 1$. By setting $g'(i, k) = 2n + 1 - g(i, k)$ we get a tournament with byes in even rounds.

An example for eight teams is shown in Table 4.

Theorem 5 *For every $n \geq 2$ there exists an $RRT(2n, 2n)$ such that no $HAP(i)$ contains any sequence AA, HH, HBH, or ABA. Moreover, for each such n, the schedule is unique up to permutation of team numbers and reflection of the order of rounds.*

Proof. The existence was proved in Construction 4. Now we prove the uniqueness. Clearly, each team has exactly one bye, as there are $2n$ teams and $2n$ rounds. First we observe that there are at most two teams with a bye in each round. Obviously, the number of bye teams in each round must be even. Suppose there are at least four teams, i_1, i_2, i_3, and i_4, having a bye in round j. At least two teams of the quadruple i_1, i_2, i_3, i_4 play their first game either both away or both home. This is either in round 1 (if $j > 1$) or in round 2 (if $j = 1$). Suppose i_1 and i_2 play both an away game. Then their HAPs are equal and they can never play each other, because they play in each round either both a home game or both an away game. This contradicts our definition of a round robin tournament. We also observe that the two teams i, k that have a bye in a week j (recall that this is denoted by $a_j(i) = a_j(k) = B$) must have complementary schedules.

Now we show that there are at most two teams with a bye in any two consecutive rounds. Suppose it is not the case and there are teams i_1 and i_2 with a bye in a round j and k_1 and k_2 with a bye in the round $j + 1$. Let $a_m(i_1) = A$ for some $m \neq j, j+1$. Then from the complementarity of HAPs of k_1 and k_2 it follows that $a_m(k_1) = A$ and $a_m(k_2) = H$ or vice versa. Suppose the former

Table 5. HAP for $RRT(2n, 2n)$.

	1	2	3	4	...	$2n-3$	$2n-2$	$2n-1$	$2n$
1	**B**	H	A	H	...	A	H	A	H
2	H	A	**B**	H	...	A	H	A	H
3	H	A	H	A	...	A	H	A	H
4	H	A	H	A	...	A	H	A	H
...					...				
...					...				
...					...				
$n-1$	H	A	H	A	...	**B**	H	A	H
n	H	A	H	A	...	H	A	**B**	H
$n+1$	**B**	A	H	A	...	H	A	H	A
$n+2$	A	H	**B**	A	...	H	A	H	A
...					...				
...					...				
...					...				
$2n-3$	A	H	A	H	...	H	A	H	A
$2n-2$	A	H	A	H	...	H	A	H	A
$2n-1$	A	H	A	H	...	**B**	A	H	A
$2n$	A	H	A	H	...	A	H	**B**	A

holds. Then the HAPs of the teams i_1 and k_1 are equal with the exception of rounds j and $j+1$. Therefore, they cannot play each other except possibly in round j or $j+1$. But $a_j(i_1) = B$ and $a_{j+1}(k_1) = B$ and hence they cannot play in rounds j or $j+1$ either. This is the desired contradiction.

Next we show that there are *exactly* two teams with a bye in any two consecutive rounds. In other words, we prove that the byes occur either in all odd rounds, or in all even rounds. We again proceed by contradiction. Suppose to the contrary that there are two consecutive rounds j and $j+1$ without byes. As there are no consecutive rounds *with* byes, it must happen that j is even and the byes occur precisely in rounds $1, 3, \ldots, j-1, j+2, \ldots, 2n$. But then there are teams i_1 and i_2 with $HAP(i_1) = BAHA\ldots HA$ and $HAP(i_2) = BHAH\ldots AH$ and also teams k_1 and k_2 with $HAP(k_1) = AHA\ldots HAB$ and $HAP(k_2) = HAH\ldots AHB$. Obviously, teams i_1 and k_2 can never play each other since their HAPs are equal except for weeks 1 and $2n$, when one of them has a bye. This contradiction shows that we can WLOG assume that byes occur in weeks $1, 3, \ldots 2n-1$.

Therefore, we define HAPs of respective teams as follows. For $i = 1, 2, \ldots, n$ we have $a_{2i-1}(i) = a_{2i-1}(n+i) = B$. For $i = 2, 3, \ldots, n$ we have $a_1(i) = H$ and $a_1(n+i) = A$. An example is shown in Table 5.

We again proceed by induction. First we observe that for any team i, $S(i, i+1) = \{i\}$ and hence there is a unique round in which the game between i and $i+1$ can be scheduled (team numbers are taken modulo $2n$ with the exception that 0 is replaced by $2n$). In particular, for $i = 1, 2, \ldots, n$ we have to set $g(i, i+1) = 2i$ and $g(n+i, n+i+1) = 2n - 2i$. We can also check that for $i = 1, 2, \ldots, n$ we have $S(i, i+2) = S(n+i, n+i+2) = \{2i, 2i+1, 2i+2\}$. But the game between i and $i+2$ cannot be played in round $2i$, since there is the uniquely determined game $g(i, i+1)$. Or, in our notation, $2i = g(i, i+1)$. Also, this game cannot be scheduled for round $2i+2$, as $2i+2 = 2(i+1) = g(i+1, i+2)$. The games between $n+1$ and $n+i+2$ can be argued similarly. Therefore, we must have $g(i, i+2) = g(n+i, n+i+2) = 2i+1$.

We can now continue inductively and suppose that for every $i = 1, 2, \ldots, 2n$ all values $g(i, i+1), g(i, i+2), \ldots, g(i, i+s)$ are uniquely determined. This indeed means that also the values $g(i, i-1), g(i, i-2), \ldots, g(i, i-s)$ are uniquely determined. We want to show that subsequently the game between i and $i+s+1$ is also uniquely determined. We can assume here that $s \leq n-1$ because of modularity. Then $S(i, i+s+1) = \{2i, 2i+1, \ldots, 2i+2s\}$. From our assumption it follows that $g(i, i+1) = 2i, g(i, i+2) = 2i+1, \ldots, g(i, i+s) = 2i+s-1$. Also $g(i+1, i+s+1) = 2i+s+1, g(i+2, i+s+1) = 2i+s+2, g(i+s, i+1) = 2i+2s$, and hence the game between i and $i+s+1$ must be scheduled for round $2i+s$. □

We observe that even if we consider a non-conference game to be scheduled in each conference bye slot, a schedule with the perfect HAP without breaks for more than two teams again cannot be found. The reason is the same as when we considered the compact schedule. Suppose there are more than two teams with a perfect HAP. Then two of them begin with a home game and no matter when they play their respective non-conference games, they again never play against each other.

In this paper we focused on schedules for 1-leg tournaments. Although there are competitions where 1-leg tournaments are widely used (e.g., chess tournaments, North-American collegiate football conferences, etc), 2-leg tournaments are much more common. It is natural to examine extensions of our schedules to 2-leg tournaments. The extension for $2n$ teams is easy and natural, because after swapping the home and away games in the second leg we get no breaks. For $2n+1$ teams, however, each team has a break between the first and second leg, that is, between the rounds $2n+1$ and $2n+2$. This can be avoided only by reversing the order of rounds in the second leg. This indicates that the new schedule for $2n$ teams, which we have constructed here may find its way to real life and we certainly hope it will.

Finally, we observe that if we number the teams and rounds $0, 1, \ldots, 2n$ or $0, 1, \ldots, 2n-1$, respectively, and disregard the home and away games, the game assignment function can be now defined in both cases as $g'(i, k) =$

$g'(k,i) = i + k$ which is corresponding to the additive group of order $2n + 1$ or $2n$, respectively.

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