Chapter 4

Fundamental Concepts

Probably the most important question that we need to address when starting a study of compressible turbulence is to determine what is qualitatively different in compressible and incompressible turbulence. Here, we try to define the new elements associated with compressibility, and the links between compressible and incompressible turbulence. We examine if it is possible to characterize these elements quantitatively using simple parameters, at least for some particular flows, and try to decide if the existing approaches are correct. Do they provide the right answers in practice? Do they predict the measured trends? Is it possible to observe the pertinent parameters in the existing experiments, or, conversely, can simple theories predict the measured quantities?

In view of the difficulties in finding general solutions to the Navier-Stokes equations, most research work is confined to finding properties of the solutions under some particular conditions. Two approaches can be taken. First, we can try to find the general properties of the equations, independent of the boundary conditions. Kolmogorov's theory, predicting the $-5/3$ law for the slope of energy spectra for incompressible turbulence at large enough Reynolds number is a good illustration. Unfortunately, this approach is successful only on rare occasions. The second possibility is to investigate the influence of the boundary conditions on the solutions. A classical example in low-speed flows is the analysis of turbulent boundary layers, where the log law exists as a dimensional requirement for matching the viscous region to the high Reynolds number region. The second approach forms the basis for all subsequent chapters, but here we try to summarize the attempts made to characterize compressible turbulence by the first approach, together with some examples to help define and quantify possible trends in the behavior of compressible turbulence. In particular, we consider the linearized equations of motion, which introduce the concept of modes, the interaction of these modes in a higher-order theory (especially with respect to the compressibility of the velocity fluctuations and the role of pressure fluctuations), the effects of rapid distortion of turbulence in compressible flows and the implications for the Reynolds stresses, and, finally, the physical bases for using different Mach numbers of turbulence and the connections among them.

4.1 Kovasznay's Modes

Kovasznay (1953) suggested that, intuitively, we may expect that sound waves will accompany vorticity fluctuations, and that fluid particles, having passed through different shear regions, may suffer different changes in entropy. To understand how the vortical, compressible, and acoustic motions are connected together, and how the dynamic and thermodynamic aspects are linked, Kovasznay performed a small perturbation analysis for fluctuations developing in a medium at rest ($U_0 = 0$), with uniform temperature T_0, density ρ_0, and pressure p_0. The fluid was assumed to be a perfect gas, with constant specific heats, viscosity, and heat conductivity. More precisely, the conditions on the fluctuations are:

$$\frac{T'}{T_0}, \quad \frac{\rho'}{\rho_0}, \quad \frac{p'}{p_0} \ll 1.$$

The condition on the velocity fluctuations u_i' is not obvious because the reference velocity is zero. However, because the only velocity scale in the problem is the reference sound speed a_0, the condition

$$M_t = \frac{u_i'}{a_0} \ll 1$$

is imposed on the magnitude of the velocity fluctuations. This point is discussed in more detail in what follows. It is important to note that although Kovasznay's analysis does not provide a predictive tool, it is very useful for classifying the mechanisms that operate in compressible turbulent flows.

The continuity equation can be written as:

$$\nabla \cdot \mathbf{V} = -\frac{D \ln \rho}{Dt} = -\frac{1}{\gamma}\frac{D \ln p}{Dt} + \frac{1}{C_p}\frac{Ds}{Dt}, \tag{4.1}$$

where the density has been written in terms of the pressure and entropy. Linearizing Equation 4.1 gives:

$$\frac{\partial u_i'}{\partial x_i} = \frac{1}{C_p}\frac{\partial s'}{\partial t} - \frac{1}{\gamma p_0}\frac{\partial p'}{\partial t}, \tag{4.2}$$

which shows that the divergence of velocity fluctuations (the *fluctuating divergence*) can be changed by pressure and entropy fluctuations.

The linearization of the momentum equation presents no real difficulty: the nonlinear transport terms are neglected, and only the contribution of the mean viscosity to the viscous stress is retained. Hence:

$$\frac{\partial u_i'}{\partial t} = -a_0^2 \frac{1}{\gamma p_0} \frac{\partial p'}{\partial x_i} + \nu_0 \left(\frac{\partial^2 u_i'}{\partial x_k^2} + \frac{1}{3} \frac{\partial^2 u_i'}{\partial x_i \partial x_k} \right). \tag{4.3}$$

The entropy equation has the usual form:

$$\frac{Ds}{Dt} = \frac{\mathbf{\Phi}}{T} + \frac{1}{\rho T} \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{\partial x_j} \right), \tag{4.4}$$

where $\mathbf{\Phi}$ is the dissipation rate per unit mass and k the coefficient of heat conduction (see Equation 2.24). The transport terms are linearized as in the previous two equations, and only the partial derivative with respect to time is retained. The dissipation is quadratic in velocity gradients, and therefore it is neglected. As a result:

$$\rho_0 \frac{\partial s'}{\partial t} = \frac{k_0}{C_p} \frac{\partial^2 s'}{\partial x_i^2} + \frac{k_0 (\gamma - 1)}{\gamma p_0} \frac{\partial^2 p'}{\partial x_i^2}. \tag{4.5}$$

By taking the curl and the divergence of Equation 4.3, two different forms of the momentum equation can be obtained. Taking the curl leads to an equation for the vorticity where the pressure does not appear because interactions between pressure and density fluctuations are not taken into account. The result has the form of a diffusion equation:

$$\frac{\partial \omega_i}{\partial t} = \nu_0 \nabla^2 \omega_i. \tag{4.6}$$

Taking the divergence of the momentum equation and eliminating the divergence of the velocity using continuity gives an equation for the pressure:

$$\begin{aligned} \frac{\partial^2 p'}{\partial t^2} - a_0^2 \frac{\partial^2 p'}{\partial x_i^2} &= \left(\frac{4\nu_0}{3} + \frac{k_0 (\gamma - 1)}{\rho_0 C_p} \right) \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial p'}{\partial t} \right) \\ &\quad - \frac{\gamma p_0}{C_p} \left(\frac{4\nu_0}{3} - \frac{k_0}{\rho_0 C_p} \right) \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial s'}{\partial t} \right). \end{aligned} \tag{4.7}$$

Equations 4.2, 4.6, 4.7, and 4.5 constitute a complete set of equations for the mass, vorticity, pressure, and entropy.

In fact, determining the velocity field with the present method implies the use of the Helmholtz decomposition in which the velocity field is separated into a solenoidal part with zero divergence and nonzero curl, and an irrotational part with zero vorticity and nonzero divergence. This decomposition is nonunique. In a homogeneous case such as that considered by Kovasznay, a solution can be obtained by specifying the proper boundary conditions at the limits of the domain.

Consider now the form of the equations. The vorticity mode is governed by the same diffusion equation as in incompressible flow, and it produces no pressure fluctuations. However, the pressure or acoustic mode, the entropy mode, and the velocity divergence are coupled together. Kovasznay (1953) considered the particular case when the Prandtl number $P = \mu_0 C_p/k_0 = 3/4$, which is close to that of air where P is about 0.72 over a wide range of temperatures. The acoustic mode equation then reduces to an equation for a pressure wave propagating at the speed of sound but damped by the action of viscosity. In this case, the equations for the acoustic and entropy modes are independent of each other. Up to first order, the entropy varies only by heat conduction, and velocity divergence follows pressure and entropy variations.

The equation for the vorticity can be solved directly to give:

$$\omega_j(t) = \omega_j(0)e^{-\nu_0 k^2 t}, \tag{4.8}$$

where k is the modulus of the wave number vector. Also, the linear system governing pressure, divergence and entropy can be solved in Fourier space to give the amplitude of sinusoidal fluctuations as a function of time. The influence of the diffusion terms (the viscous decay of the vorticity, entropy, and sound fields) depends on the length and time scales over which the observation is made. Order-of-magnitude arguments can be used to show that for some important cases, such as when a hot-wire is used to measure turbulence in the freestream of a supersonic wind tunnel, these terms are often small. Setting the viscosity and the heat conduction to negligibly small values in the solution shows that the amplitudes of the modes are governed by the following equations:

$$\frac{d\omega_j'(t)}{dt} = 0, \tag{4.9}$$

$$\frac{ds'}{dt} = 0, \tag{4.10}$$

$$\frac{\partial^2 p'}{\partial t^2} = a_0^2 \frac{\partial^2 p'}{\partial x_i^2}, \tag{4.11}$$

$$\frac{1}{\gamma p_0}\frac{\partial p'}{\partial t} = -\nabla.\mathbf{u}'. \tag{4.12}$$

This set of equations represents a consistent zeroth-order approximation for weak fluctuation fields when the time of observation is short. The vorticity and entropy are constant, and the pressure obeys a wave equation with a speed of propagation equal to the speed of sound. Equation 4.9 represents a "frozen pattern" of vorticity (or solenoidal velocity) and when it is applied to homogeneous turbulence in a wind tunnel it predicts an unchanged flow pattern carried by the mean velocity, as suggested by Taylor's hypothesis. Equation 4.10 indicates a similar frozen pattern behavior for temperature spots, and Equation 4.11 is simply the wave equation for propagation of the pressure field. The only compressibility effect, that is, the creation of velocity divergence, is related to the pressure fluctuations. Because the fluctuating pressure and divergence are in quadrature, the result is the superposition of acoustic waves on an incompressible vortical field, with no coupling.

Kovasznay assumed that the Mach number of the fluctuations was small, which allowed the nonlinear term $u'_j \partial u'_i / \partial x_j$ in the momentum equation to be neglected. Because this nonlinear term is known to be responsible for the energy cascade from large to small scales in three-dimensional turbulence, neglecting it implies some limitations. For example, the characteristic time scale derived for viscous diffusion in Equation 4.8 is several orders of magnitude larger than any turbulent time scale. The only possible conclusion is to assume that the nonlinear term is small compared to the pressure gradient $\partial p' / \partial x_i$. For a given domain size, defined as the volume or time interval over which the small perturbation approximations can be applied, a necessary condition is

$$\frac{\rho_0 u'^2}{p'} \ll 1 \quad \text{or} \quad \frac{\gamma M_t^2 p_0}{p'} \ll 1.$$

It was assumed that $p'/p \ll 1$, so that the condition is $M_t \ll 1$, which is always satisfied by acoustic waves.

Because this is a linear theory the modes do not interact, at least within this domain. An interaction may take place outside the domain, either at solid boundaries or in regions where the fluctuations are not small. If the fluid at some earlier time had passed through a grid, for example, the high-intensity turbulence created by the grid would have produced strong sound waves and the viscous dissipation would have produced entropy spots. The sound waves will propagate away according to the wave equation, but the entropy spots travel with the fluid, diffusing slowly by heat conduction. The small perturbation analysis applies when the modes are sufficiently small for nonlinear interactions between any two modes to be negligible, although nonlinearities may have had an important role in the creation of the modes. Furthermore, if these noninteracting modes pass through a region of strong gradient, such as a shock wave, they may interact with the strong field and then conversion from one mode to another may take place.

Nonlinear interactions between modes were considered in more detail by Chu and Kovasznay (1958). As to the generation of the modes, they found that mass addition produces the sound mode, in that the injected fluid displaces other fluid, and the movement of the displaced fluid generates pressure waves that propagate into the surrounding medium. A body force generates the vorticity and sound modes. The irrotational force component produces only sound, and the solenoidal component produces only vorticity. Heat addition generates the entropy and sound modes, because it increases the temperature of the gas and causes expansion of the heated fluid element. The possible interactions between the modes are summarized in Table 4.1. Several

	Sound source	Vorticity source	Entropy source
Sound–sound	'steepening' and 'self-scattering' $\frac{\partial^2(v_{pi}v_{pj})}{\partial x_i \partial x_j} + a_0^2 \nabla^2(P_p^2) + \frac{\gamma-1}{2}\frac{\partial}{\partial t^2}P_p^2$	$O(\alpha^2\epsilon)$	$O(\alpha^2\epsilon)$
Vorticity–vorticity	'generation' $\frac{\partial^2(v_{\Omega i}v_{\Omega j})}{\partial x_i \partial x_j}$	'self-convection' $-v_{\Omega j}\frac{\partial \Omega_i}{\partial x_j} + \Omega_j \frac{\partial v_{\Omega i}}{\partial x_j}$	$O(\alpha^2\epsilon)$
Entropy–entropy	$O(\alpha^2\epsilon)$	$O(\alpha^2\epsilon^2)$	$O(\alpha^2\epsilon)$
Sound–vorticity	'scattering' $2\frac{\partial^2(v_{\Omega i}v_{pj})}{\partial x_i \partial x_j}$	'vorticity convection' $-v_{pj}\frac{\partial \Omega_i}{\partial x_j} + \Omega_j \frac{\partial v_{pi}}{\partial x_j} - \Omega_i \frac{\partial v_{pj}}{\partial x_j}$	$O(\alpha^2\epsilon)$
Sound–entropy	'scattering' $\frac{\partial^2}{\partial t \partial x_i}(S_s v_{pi})$	'generation' $-\alpha_0^2(\nabla S_s)\times(\nabla P_p)$	'heat convection' $-v_{pi}\frac{\partial S_s}{\partial x_i}$
Vorticity–entropy	$O(\alpha^2\epsilon)$	$O(\alpha^2\epsilon)$	'heat convection' $-v_{\Omega i}\frac{\partial S_s}{\partial x_i}$

Table 4.1. Second-order nonlinear interaction between modes of order α. Here, $S = s'/C_p$, $P = p'/\gamma p_0$, and the subscripts p, Ω, and s denote the sound, vorticity, and entropy modes, respectively. The parameter α is a nondimensional measure of the intensity of the disturbance, and $\varepsilon = \nu_0 k/a_0$. (Adapted from Chu and Kovasznay (1958).)

well-known mechanisms are described by these interactions: vortex stretching (the production of vorticity by the interaction of vorticity with itself), the production of vorticity through baroclinic torques (interaction of acoustic and entropy modes, which can sometimes promote early laminar-to-turbulent transition in noisy wind tunnels), the wave-steepening mechanism (interaction between entropy and acoustic modes producing acoustic modes, which can lead to the formation of shock waves), and the scattering of incident sound waves by vorticity fluctuations (the vorticity-sound interaction). As pointed out by Gaviglio (1976), the baroclinic effect is the only generation process in which two modes interact to produce a third one. The other interactions only modify pre-existing modes. The role of pressure fluctuations is discussed in greater detail in the next section.

4.2 Velocity Divergence in Shear Flows

We saw that in very simple flows with small amplitude fluctuations, pressure fluctuations and compressible turbulence are linked. In general, a non-solenoidal velocity field can have many origins (see, for example, Batchelor (1967)). It can be produced by gravity in a stratified medium, by heat release, viscous dissipation, or pressure gradients, although for pressure gradients to produce significant velocity divergence the turbulence Mach number should be of order unity. Supersonic flows have significant kinetic energy, large enough to produce significant changes in temperature when it is converted into heat. Turbulence is a very dissipative process, and therefore contributions to the fluctuating divergence can come from adiabatic processes, where pressure gradients (and Mach number) are important, and dissipative processes, where entropy production is important. The main concern of this section is to examine if simple order-of-magnitude analyses can give some insight on the importance of compressibility effects on turbulence in known flows. We assume that there are significant compressibility effects when $\nabla \cdot \mathbf{u}'$ becomes large compared to the turbulent velocity gradient $\partial u_i'/\partial x_j$. For the energy-containing motions, this gradient is of order u'/Λ, where Λ is a length scale characteristic of these motions. We begin by deriving simple criteria to assess the importance of compressibility effects, as a function of the flow properties. For most common cases, the criteria can be expressed in terms of pressure fluctuations (isentropic processes), or entropy fluctuations (dissipation or heat conduction), as shown later. For this purpose, we consider the full perturbation forms of the continuity and entropy equations. The approach given by Dussauge et al. (1989) is followed here, in a somewhat expanded form.

If velocity divergence is produced by isentropic pressure variations, the equation for pressure is a wave equation, which is hyperbolic. If the divergence of velocity is produced by entropy variations, the pressure obeys a Poisson equation, and an elliptic problem is obtained (see, for instance, Equation 4.7). This can have many consequences on the overall nature of the flow. For example, for the supersonic flow of a reacting mixture in a nozzle, the heat release will increase the flow entropy, and if the nozzle is not properly designed it can choke. As the flow becomes subsonic there is a transition from hyperbolic to elliptic behavior. Also, if the pressure is governed by a Poisson equation, then it seems likely that in modeling the pressure strain terms in the Reynolds stress equation models similar to the ones used in subsonic flow can probably be used.

By combining the continuity and entropy equations (Equations 4.1 and 4.4) we have:

$$\nabla \cdot \mathbf{V} = -\frac{1}{\gamma}\frac{d \ln p}{dt} + \frac{\varepsilon}{C_p T} + \frac{1}{\rho C_p T}\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right), \tag{4.13}$$

with $\mathbf{\Phi} = \rho\varepsilon$. The problem is to find reasonable estimates for the amplitude of the pressure fluctuations and the corresponding (Lagrangian) time scale τ.

We can now derive the equation for divergence of the instantaneous velocity fluctuations. The pressure term in Equation 4.13 becomes:

$$\frac{1}{\gamma}\left(\frac{d\,(\ln p)'}{dt} - \overline{u_i'\frac{\partial\,(\ln p)'}{\partial x_i}} + u'\frac{\partial\overline{(\ln p)}}{\partial x_i}\right).$$

Here, the Reynolds decomposition is used, and d/dt denotes the derivative along the instantaneous motion. The last term can be neglected if we restrict ourselves to zero pressure gradient flows. The dissipation and conduction terms are more complicated because they are nonlinear. It is assumed that a sufficient approximation is obtained by assuming small temperature fluctuations, and by linearizing the dissipation and conduction terms with respect to temperature fluctuations. For $T'/T \ll 1$, therefore, the fluctuating part of the dissipation term ε/T becomes:

$$\frac{\varepsilon'}{\bar{T}} - \frac{\bar{\varepsilon}}{\bar{T}}\frac{T'}{\bar{T}}.$$

The bars denote mean quantities. The heat-conduction term can be written as the sum of a flux term and a term that is quadratic in temperature gradients (see Section 3.2.3). That is,

$$\frac{1}{\rho T}\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(k\frac{\partial \ln T}{\partial x_j}\right) + \frac{k}{\rho T^2}\left(\frac{\partial T}{\partial x_j}\right)^2.$$

The perturbation form for small temperature fluctuations may now be found, and then we obtain an equation for the fluctuating divergence:

$$\begin{aligned}\nabla.\mathbf{u}' \;=\; & -\frac{1}{\gamma}\left(\frac{d\,(\ln p)'}{dt} - \overline{u_i'\frac{\partial\,(\ln p)'}{\partial x_i}}\right) - \frac{T'}{T}\frac{\bar{\varepsilon}}{C_p T} - \frac{T'}{T}\frac{2\bar{\varepsilon}_\theta}{\gamma T^2} \\ & + \frac{\varepsilon'}{C_p T} + \frac{\varepsilon_\theta'}{\gamma T^2} + \frac{1}{\rho C_p}\frac{\partial}{\partial x_i}\left(k\frac{\partial (T'/T)}{\partial x_i}\right),\end{aligned} \tag{4.14}$$

where $\bar{\varepsilon}$ is the mean rate of dissipation per unit volume, and the term $\bar{\varepsilon}_\theta = \overline{k(\partial T'/\partial x_i)^2}/(\rho C_v)$ is sometimes called the *second dissipation.* It acts as a dissipation for the variance of temperature (energy) fluctuations. Because it represents irreversibility due to molecular effects, it is a local source term for entropy. In Equation 4.14, an approximation for small quantities is obtained. The terms that are cubic in velocity and temperature fluctuations were neglected (terms such as $\epsilon' T' \big/ \bar{\epsilon}\bar{T}$) because at high Reynolds numbers they involve wave numbers belonging to different domains. Quadratic terms were retained. The reason is that we have linear terms such as $(T'/T)(\bar{\epsilon}/C_pT)$ where $T'/T < 1$, and quadratic terms such as $\epsilon'/C_pT)$. It is known (see below) that the fluctuating dissipation can be an order of magnitude larger than the mean dissipation rate, so that the term involving ϵ' has been retained.

We see from Equation 4.14 that divergence fluctuations come from pressure fluctuations, local source terms, and heat fluxes. The sources are proportional to the fluctuations of the dissipation rate, or to the temperature fluctuations themselves. In the following, a primed quantity denotes an *rms* value, rather than the fluctuation itself. The sum of the two pressure terms is assumed to be of order $(p'/p)/\tau$, where the time scale τ depends strongly on whether the pressure wave propagates with respect to the medium. Even if simple estimates can be found for the external flow, the behavior of the near field, or the interaction between the pressure waves and the turbulent field, is not at all clear. Some aspects of the problem are related to problems of wave propagation in a random medium, where the speed of propagation can vary dramatically in amplitude and direction. So two cases are considered: one for which the time scale is acoustic, $\tau_a = \Lambda/a$, and another for which the time scale is turbulent, $\tau_t = \Lambda/u'$, where Λ is a length scale, a the speed of sound and u' the *rms* velocity. The ratio of the two time scales turns out to be the turbulence Mach number:

$$\frac{\tau_a}{\tau_t} = \frac{u'}{a} = M_t.$$

As for the other terms, it is assumed for simplicity that the terms involving the first and second (mean) dissipation rates are all of comparable order, so that only the viscous dissipation needs to be considered. An example is given for supersonic flows without heat sources, which indicates the limits of this approximation. The fluctuating dissipation rates ε' (and ${\varepsilon'}_\theta$) are related to the phenomenon of internal intermittency, according to which the dissipation is distributed randomly in space. This results in a random occurrence in time of strongly dissipative events, at a given fixed point. In subsonic flows, ε' is often an order of magnitude larger than $\bar{\varepsilon}$ (Sreenivasan et al., 1977), so that in Equation 4.14 we expect that the term containing ε' can be larger than the term involving $\bar{\varepsilon}$. An important difference is that the term in ε' probably contains more smaller scales than the term in $\bar{\varepsilon}$, which is proportional to T'/T. The internal intermittency is certainly modified by compressibility because if shock waves appear they introduce regions of strong dissipation with a very particular shape. If the results of recent numerical simulations are a good indication, the shock waves produced by compressible turbulence have a large aspect ratio: they are thin (small-scale), corrugated sheets of long extent (large-scale), located randomly in space, and therefore they are important for the evolution of large and small scales. However, no measurements of the fluctuating dissipation rates are available for supersonic flows, and we need to assume that their orders of magnitude are not greatly changed by compressibility. Then the contribution of dissipation to the divergence is estimated by considering the fluctuating dissipation rate:

$$\frac{\varepsilon'}{C_p T} \approx 10\,(\gamma - 1)\, M_t^2 \frac{u'}{\Lambda} = 10\,(\gamma - 1)\, \frac{M_t^2}{\tau_t}, \tag{4.15}$$

where it is assumed that, typically, $\varepsilon' \approx 10\bar{\varepsilon}$. For large scales, the important contribution to the divergence is probably the one involving the mean dissipation rate:

$$\frac{T'}{T}\frac{\bar{\varepsilon}}{C_p T} \approx \frac{T'}{T}\,(\gamma - 1)\,\frac{M_t^2}{\tau_t}. \tag{4.16}$$

T'/T is generally less than 1, and so the fluctuating dissipation term is probably more important than the mean dissipation term, implying that the major contribution to the velocity divergence occurs at small scales.

The last term, which is related to the fluctuating heat flux, is not considered here. For constant mean temperature at large Reynolds and Peclet numbers it involves no large scales. It is probably not the leading mechanism in flows without heat sources (such as adiabatic or nonreacting flows) or in flows without chemical reactions, because it follows the temperature fluctuations produced by dissipation or pressure fluctuations and just smooths out the smaller scales.

We can now examine some particular situations, beginning with the case of weak compressibility, as found in boundary layers at moderate Mach numbers. Figure 4.1 shows temperature and pressure fluctuation measurements in a zero pressure gradient boundary layer at Mach 1.8. It was possible to measure pressure at only two places: at the wall using a pressure transducer, and in the outer flow using a hot-wire anemometer (where it was assumed that in the freestream the fluctuations were isentropic: see Laufer (1961)). The wall pressure fluctuations measurements are affected by spurious signals such as mechanical vibrations or electronic noise, and they probably underestimate the *rms* level because of spatial integration. For the measurements shown in Figure 4.1, this error was estimated at ±20%. Despite these difficulties, the levels at the wall and in the outer flow are comparable, and we expect a smooth variation between these two levels within the layer so that it is clear that adiabatic flows develop temperature fluctuations which are much larger than pressure fluctuations. The entropy fluctuation is $s'/C_p = T'/T$. If the time scales of pressure and entropy are of the same order, Equations 4.14 and 4.15 show that the magnitude of the velocity divergence is given by $10(\gamma - 1)M_t^2/\tau_t$. The divergence can be compared to the order of magnitude of the instantaneous velocity gradients for the energy-containing motions u'/Λ by

$$\frac{\nabla . \mathbf{u}'}{u'/\Lambda} \approx 10(\gamma - 1) M_t^2, \tag{4.17}$$

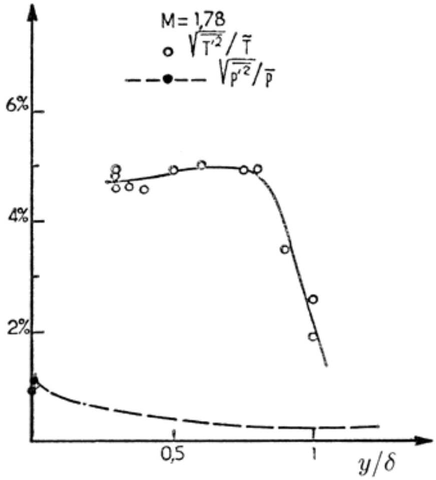

Figure 4.1. Temperature and pressure fluctuation measurements in a zero pressure gradient boundary layer at Mach 1.8. (From Dussauge (1986), with permission.)

which suggests that in the weak compressibility regime, where the dissipation rate is solenoidal, the relative importance of velocity divergence varies according to M_t^2. In a boundary layer at Mach 1.8 with a Reynolds number Re_θ of 5000, M_τ is about 0.07, so that M_t is about 0.1, and the fluctuating divergence is certainly negligible. In a mixing layer with a convective Mach number of 0.6, where compressibility effects are just becoming important, the peak value of M_t is about 0.25, and the ratio in Equation 4.17 is also about 0.25. So $\nabla.\mathbf{u}'$ begins to become significant because of the high dissipation rate. The dissipation rate in the averaged equations, $\bar{\varepsilon}$, may be split into two parts, $\bar{\varepsilon}_s$ and $\bar{\varepsilon}_d$. The solenoidal dissipation rate $\bar{\varepsilon}_s$ is the dissipation in the solenoidal part of the fluctuating motion. In the homogeneous case $\bar{\varepsilon}_s = \nu\overline{\omega_i'\omega_i'}$, where ω_i' is the fluctuating vorticity and $\overline{\omega_i'\omega_i'}$ is the enstrophy. Similarly, the dilatational dissipation rate $\bar{\varepsilon}_d$ is the dissipation in the nonsolenoidal part of the fluctuating motion. In the homogeneous case, $\bar{\varepsilon}_d = \frac{4}{3}\nu\,\overline{(\nabla.\mathbf{u}')^2} = \frac{4}{3}\nu\overline{(\partial u_i'/\partial x_i)^2}$, so that in general, the ratio $\bar{\varepsilon}_d/\bar{\varepsilon}_s$ is something like $(\nabla.\mathbf{u}'/(u'/\Lambda))^2$, which suggests that $\bar{\varepsilon}_d$ is not large in this case. Hence, this simple analysis of dilatation produced by solenoidal dissipation indicates the following dependences for the variance of velocity divergence and for the dilatation dissipation:

$$\overline{(\nabla\dot{\mathbf{u}}')^2} \propto M_t^4 \qquad \text{and} \qquad \frac{\epsilon_d}{\epsilon_s} \propto M_t^4.$$

More recent studies (Ristorcelli, 1997; Fauchet, 1998; Fauchet and Bertoglio, 1999) have found the same dependence for weak compressibility cases, using different methods. They both refer to the concept of pseudo-sound used by Ribner (1962). In the regime of pseudo-sound, the acoustic fluctuations (with characteristic velocity scale a) and the turbulent fluctuations (with characteristic velocity scale u') involve distinct ranges of time scales, such as $\tau_a/\tau_t = M_t \ll 1$. Ristorcelli uses the notion of compact sources of sound. The acoustic sources (the turbulent eddies) radiate acoustic waves in the far field, but the near field has a subsonic behavior: for small Mach numbers, information is felt simultaneously in the whole source. Moreover, the "pseudo-pressure" developed in the sources obeys a Poisson equation, as in subsonic flows. The work of Fauchet and Bertoglio (1999) uses simulations of isotropic turbulence with a two-point closure adapted from Kraichnan's DIA model. Their computations reproduce the pseudo sound regime and the M_t^4 dependence. These studies use formulations more elaborate than our simple order-of-magnitude analysis. However, even if the details of the closure are very different in each case, the primary hypotheses are very similar, and the convergence of the results suggests that the scaling in M_t^4 is a rather robust result, which can be used with some confidence.

In flows with temperature inhomogeneities, the magnitude of the second dissipation should be estimated. This implies that we can specify the link between kinetic and thermal energy. We will examine the case of flows without heat sources. Because the flow is adiabatic, it is assumed that the temperature fluctuations are given by the Strong Reynolds Analogy (Equation 5.15), and that ε' and ε'_θ are linked in the same way as usually assumed in models for low-speed flows. That is,

$$\frac{\varepsilon'_\theta}{\varepsilon'} \approx \frac{\bar{\varepsilon}_\theta}{\bar{\varepsilon}} = \frac{r_\varepsilon T'^2}{\frac{1}{2}q'^2},$$

where $q'^2/2$ is the turbulent kinetic energy and $r_\varepsilon = 1.25$ is a constant determined from experiments. For weak compressibility, if the anisotropy of turbulent stresses is not altered,

$$\begin{aligned} \frac{\nabla.\mathbf{u}'}{u'/\Lambda} &\approx 10(\gamma-1)M_t^2\left(1+\frac{r_\varepsilon(\gamma-1)}{\gamma}M^2\right) \\ &\approx 10(\gamma-1)M_t^2\left(1+\frac{M^2}{2\gamma}\right). \end{aligned} \tag{4.18}$$

We can now examine the typical values taken by this ratio in simple flows. If we consider adiabatic supersonic boundary layers for $M < 5$ and mixing layers for $M_c \leq 1$ (see Chapters 6 and 8), maximum values of M_t are at most 0.2. Equation 4.17 indicates that the divergence of fluctuations produced by solenoidal heating is two orders of magnitude lower than the individual component of the fluctuating gradient, and therefore is negligible. A comparison of Equations 4.17 and 4.18 shows that ε'_θ can become the leading term even under adiabatic conditions (for air), and it can produce significant divergence levels with an elliptic overall behavior if M_t is large enough.

The large-scale estimate of velocity divergence, which is the companion to Equation 4.16, is given by:

$$\begin{aligned}\frac{\nabla.\mathbf{u}'}{u'/\Lambda} &\approx (\gamma-1)^2 M_t^3 M \left(1+\frac{r_\varepsilon(\gamma-1)}{\gamma}M^2\right)\\ &\approx (\gamma-1)^2 M_t^3 M \left(1+\frac{M^2}{2\gamma}\right),\end{aligned}$$

which suggests analogous conclusions: because the ratio is proportional to M_t^3, it remains small for $M_t < 1$ (implying that M remains moderate at the same time).

A second important case is the production of fluctuations by weak shock waves. The entropy increase through the wave is proportional to the cube of the pressure variation (Liepmann and Roshko, 1957; Landau and Lifshitz, 1987):

$$\frac{s'}{C_p} \propto \left(\frac{p'}{p}\right)^3 .$$

Because in this case the acoustic and turbulence time scales are the same, the ratio of the pressure and entropy terms in Equation 4.13 is $(p'/p)^2$, which shows that the leading term is the pressure term and in the acoustic approximation it is proportional to M_t. In this case, velocity divergence is produced by quasi-isentropic pressure waves.

Finally, more complicated situations can be considered, with strong pressure fluctuations and high rates of dissipation, by assuming that a typical entropy variation term is given by the fluctuating dissipation rate and that the time scale for pressure is either τ_a or τ_t. The following estimate is obtained,

$$\begin{aligned}&-\frac{1}{\gamma}\left(\frac{d\ln p'}{dt} - \overline{u_i'\frac{\partial \ln p'}{\partial x_i}}\right) + \frac{d\,(s'/C_p)}{dt} \approx \\ &\qquad \frac{p'}{p}\left\{10\,(\gamma-1)\,M_t^n\left(1+C_v r_\epsilon\left(\frac{T'}{T}\right)\left(\frac{T'}{\frac{1}{2}q'^2}\right)\right)\right\}^{-1},\end{aligned}$$

where $n = 2$ if $\tau_p = \tau_t$, and $n = 1$ if $\tau_p = \tau_a$, and where T' now denotes the *rms* temperature fluctuation level.

In this case, in the absence of heat sources or chemical reactions, pressure can be an important source of velocity divergence. This is consistent with the results of the previous section, where in simplified cases pressure and divergence fluctuations were found to be strongly linked. Also, the link between T' and p' is crucial, because they both appear in these estimates. There is no general rule to find the magnitude of these terms because they depend on the flow conditions. For the particular case studied by Blaisdell et al. (1993), who simulated homogeneous turbulence subjected to a shear with constant mean density, pressure, and temperature, it turned out that the fluctuations were produced by nearly isentropic processes, so that $p'/p \approx (\gamma/(\gamma-1))(T'/T)$. They found $p'/p \approx 0.3$ for $M_t \approx 3$. In that extreme case, the ratio of the rates of variation of pressure and entropy is of order 1, so that none of the terms may be neglected. This result is probably not very general, because most of the pressure fluctuations found in these simulations are due to shock waves produced by the interaction of vortices. It is not clear that the time scale for pressure can then be related directly to τ_t or τ_a. Moreover, this case is somewhat far from the more usual case of compressible shear flows where temperature gradients can produce strong temperature fluctuations. For example, with the same values for pressure fluctuations and turbulence Mach number, if the temperature and velocity fluctuations are still linked according to the SRA, then the entropy variations are much larger than the pressure variations. The continuity equation indicates that here the velocity divergence is due to dissipation. Neglecting the pressure terms in the continuity equation leads to a Poisson equation for the pressure fluctuations. The links among T', p', and u' are again important, and in the strongly compressible regimes found in practice they are not clearly identified.

These estimates confirm that turbulence in zero pressure gradient boundary layers at moderate Mach numbers ($M < 5$) may be assumed to be weakly compressible, as long as phenomena such as sound radiation are not of primary interest. Because the divergence of the fluctuating velocity is small, the behavior of the pressure strain terms is found to be similar to that found in the low-speed, variable density case. The fluctuating divergence appears to depend on the turbulence Mach number M_t. In flows where M_t is large, such as in mixing layers, significant levels of fluctuating divergence may be found. If the source of velocity divergence is dissipative heating it seems difficult to produce high levels of the dilatational dissipation, suggesting that significant values of $\bar{\varepsilon}_d$ will be produced by shocklets rather than by viscous heating. If it is possible to produce significant levels of velocity divergence in mixing layers at moderate convective Mach numbers ($M_c \geq 0.5$, say), the equation for the pressure fluctuations and the form of the pressure strain should be changed, and it may be expected that the anisotropy of the Reynolds stresses changes accordingly. This is consistent with an elliptic behavior of the energetic eddies. In addition, it seems difficult to develop significant fluctuations of velocity divergence by the usual levels of dissipation, that is, without shock waves and without heating (or cooling) the flow strongly. If no heat flux is considered, it is clear that the existence of shocklets must be assumed to characterize compressible turbulence.

4.3 Velocity Induced by a Vortex Field

In Chapter 2 we saw that the vorticity transport-equation and the theorems of Kelvin and Helmholtz are easily extended to compressible flows. What is not straightforward is the interpretation of vorticity concentrations in terms of the induced velocity field. In an incompressible flow, we can use the Biot-Savart relation to determine the velocity field induced by concentrated elements of vorticity. When the density is variable (due to the effects of compressibility or stratification), the Biot-Savart relation can strictly no longer be used. The communication among vortex elements is no longer global—it will be confined to directions lying along characteristics (see, for instance, Section 1.3). When the Mach number gradients are severe, the communication paths may have a very complicated geometry. However, in many parts of an otherwise compressible fluid, the mean and instantaneous Mach number gradients may be relatively small. A good example is the outer part of a turbulent boundary layer. Within certain limits, Biot-Savart law may still be used, depending on the importance of the fluctuating divergence.

Formally, we can use the Helmholtz decomposition, in which the velocity field is written as the sum of a rotational part ($\mathbf{V}^{\omega}$) and an irrotational part ($\mathbf{V}^{\phi}$) such that $\mathbf{V}^{\phi} = \nabla\phi$, where ϕ is a potential function (see, for example, Panton (1984)). Hence,

$$\mathbf{V} = \mathbf{V}^{\omega} + \mathbf{V}^{\phi}.$$

That is,

$$\begin{aligned} \nabla \times \mathbf{V} &= \mathbf{\Omega} = \nabla \times \mathbf{V}^{\omega} \\ \nabla \times \mathbf{V}^{\phi} &= 0, \end{aligned}$$

so the vorticity is contained in the rotational part. The appropriate boundary conditions provide the uniqueness of the decomposition. If we require that $\mathbf{V}^{\omega}$ is solenoidal, then $\nabla \cdot \mathbf{V}^{\omega} = 0$, and therefore

$$\nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{V}^{\phi} = \nabla^2\phi,$$

so the dilatation is contained in the irrotational part. For incompressible flow, $\nabla^2\phi = 0$, and ϕ is a harmonic function. For the rotational part, we note that any vector field that is solenoidal can be represented in terms of a vector potential β, so that

$$\mathbf{V}^{\omega} = \nabla \times \beta.$$

The vector identity $\nabla^{\mathbf{2}}\beta = -\nabla \times (\nabla \times \beta) + \nabla\,(\nabla \cdot \beta)$ shows that if $\nabla \cdot \beta = 0$, then

$$\nabla^{\mathbf{2}}\beta = -\mathbf{\Omega},$$

which can be solved to give the Biot-Savart law for the velocity induced by a vorticity distribution $\mathbf{\Omega}$:

$$\mathbf{V}^{\omega} = -\frac{1}{4\pi}\int \frac{\mathbf{r}\times\mathbf{\Omega}}{|\mathbf{r}|^{3}}d\upsilon, \tag{4.19}$$

where $\mathbf{r}$ is the position vector measured from the point of interest within the volume υ. This result is general, insofar as the condition $\nabla\cdot\beta = 0$ holds.

To find the velocity $\mathbf{V}$ from a given potential and rotational field, we need V^{ω} and V^{ϕ}. For incompressible flow, $\nabla\cdot\mathbf{V}^{\phi} = 0$, but for compressible flow the dilatation needs to be known. In the particular case of a boundary layer, where the turbulence is relatively weak, compressibility effects are weak and the results of the previous section can be used to determine the magnitude of the fluctuating divergence (Equations 4.17 to 4.19). For low levels of the mean and turbulent Mach number, Biot-Savart can still be used to understand vorticity interactions, and many of the intuitive concepts used in describing structures in boundary layers in subsonic flow carry over unchanged when the flow is supersonic, as long as we are careful in the neighborhood of vorticity sources within the fluid, places where barotropic torques are important (including curved shocks or shocklets).

4.4 Rapid Distortion Concepts

In this section, a particular class of flows is examined where distortions are imposed on the turbulence over a small period of time. The distortions considered here are such that fluid elements are stretched or compressed in some or all directions. A typical example of a "distorting constant area box" used in studies of rapidly distorted subsonic flows is sketched in Figure 4.2. In the experiment by Tucker and Reynolds (1968), for example, the turbulence in the initial section was isotropic and homogeneous. As the flow passed through the box, the spanwise scales were elongated and the vertical ones were compressed. Similar distortions occur frequently in practice, as when turbulence passes through a convergent or a divergent channel, or when a turbulent flow is deformed by the presence of an obstacle or pressure gradient. The characteristic time scale of the energy-containing motions may be estimated by:

$$T_t = \frac{\Lambda}{u'} \qquad \text{or} \qquad T_t = \frac{k}{\varepsilon}.$$

In the first estimate, the scale Λ is some average size of the energetic perturbations, for example, an integral scale, and u' is a characteristic value of the *rms* velocity. The second estimate, due to Townsend (1976), is commonly used in k–ε models, where k is the mean turbulent kinetic energy ($= \frac{1}{2}q'^{2}$) and ε is its mean dissipation rate per unit mass. As Bradshaw (1973) points out, if the

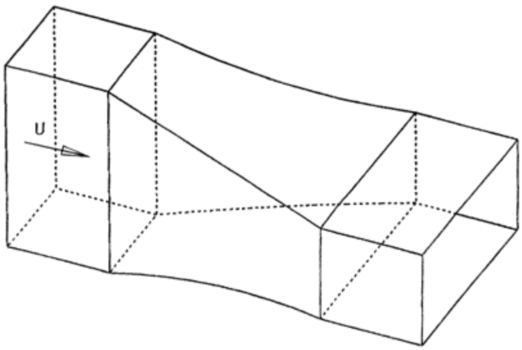

Figure 4.2. An example of a distorting constant area box used in studying rapid distortion effects in subsonic flows.

production were suddenly shut off the turbulent energy initially decays with this time constant. If T_d is the time during which the distortion is applied to turbulence, then if $T_d/T_t \ll 1$ the distortion can be formally classified as a *rapid distortion*.

Most of the work in rapid distortion has been confined to incompressible flows. Yet, compressible flows are potentially very attractive for the application of rapid distortion methods. Changes in the mean field can occur over very short distances, much shorter than is possible in subsonic flows, and the limits of a rapid distortion can often be more easily satisfied. For example, when a boundary layer passes through a Prandtl-Meyer fan or a short region of compression, including the case where the layer interacts with a shock wave, the perturbation can occur over a distance comparable to the boundary layer thickness. Typically, outside the viscous sublayer, the pressure gradients are much stronger than the other stress gradients, and rapid distortion methods become very appealing.

This situation is particularly attractive because analytical solutions can be found for some simple cases where the nonlinear terms in the equation of motion become negligible. The turbulence time scale T_t is related to the nonlinear terms in the momentum equation: after a time T_t, an eddy of size Λ loses its "identity." That is, the nonlinearities are responsible for transferring its energy to smaller scales. For times much smaller than T_t, these nonlinear effects can be neglected, and if the flow is subsonic it obeys a linear set of equations, as long as the mean flowfield is constant and does not depend on the turbulence. These cases represent the field of application of Rapid Distortion Theory (RDT).

The basic theory was developed by Ribner and Tucker (1952) and Batchelor and Proudman (1954) for homogeneous, initially isotropic turbulence in an irrotational mean flow. The theory was later extended to shear flows (see, for example, Moffatt (1968) and Townsend(1970)) and subsequent developments led to a wide variety of applications. The papers by Hunt (1977); Townsend (1980); Cambon (1982), and the reviews by Savill (1987) and Hunt and Caruthers (1990) provide a comprehensive picture of the current status of RDT for incompressible flows. What seems particularly encouraging is that despite the strict limits on the applicability of RDT, the theory often gives qualitatively useful results outside these bounds, as well as providing guidelines for distorted structure modeling (Savill, 1987).

To solve the linear rapid distortion problem for specific cases, some conditions often need to be satisfied, in addition to the usual rapid distortion criteria. For example, if Fourier transforms of the velocity field are used, the turbulence often needs to be homogeneous. Other solutions may require that the distortion is irrotational, or that the evolution is isentropic along streamlines. In most practical flow realizations these assumptions are not obeyed, and RDT cannot be applied. It is often more attractive to consider instead the Reynolds stress evolution, rather than the associated spectra, especially in supersonic flows where the scope of experimental data is more limited than in subsonic cases. The simplifications of second-order closure needed for rapid distortion approximations to the Reynolds stress equations are not straightforward, as pointed out by Hunt (1977), mainly because terms involving pressure fluctuations must be modeled, giving at best an approximation to the solution. This latter approach, where the scaling arguments and limiting processes employed in RDT are used to approximate the Reynolds stress equations, is what we term the Rapid Distortion Approximation (RDA) and its application to supersonic flows is discussed in Section 4.4.3.

4.4.1 Linearizing the Equations for the Fluctuations

The main consideration in linearizing the equation of momentum comes from the form of the fluctuating acceleration a_i'. That is,

$$a_i' = \underset{(1)}{\frac{\partial u_i'}{\partial t}} + \underset{(2)}{\bar{u}_j \frac{\partial u_i'}{\partial x_j}} + \underset{(3)}{u_j' \frac{\partial \bar{u}_i}{\partial x_j}} + \underset{(4)}{u_j' \frac{\partial u_i'}{\partial x_j}}.$$

Terms 1 and 2 represent the particle derivative of u_i' ($= Du_i'/Dt$), and it is assumed that they are of the same order as the other terms in the acceleration. A necessary condition for the linearization of the acceleration terms is that the second term is much larger than the fourth term. If q' and U are representative scales for u_i' and U_i, respectively, this simply requires that the fluctuations are small:

$$\frac{q'}{U} \ll 1. \tag{4.20}$$

By comparing the third and fourth terms, another condition related to time scales is obtained. If S is the scale for the mean velocity gradient, we require:

$$\frac{q'}{\Lambda S} \ll 1. \tag{4.21}$$

We should also make sure that Du'_i/Dt is large compared to the nonlinear term. If the scale of variation of the fluctuation is the fluctuation itself q', and if the time scale is the transport time L/U, where L is the size of the distorted zone and U a typical mean velocity, we obtain the condition:

$$\frac{q'}{U}\frac{L}{\Lambda} \ll 1. \tag{4.22}$$

This condition may be more stringent than condition 4.21, depending on the flow. Because we have neglected the nonlinear inertial terms, we need to consider only the low wave number part of the spectrum. In this case, the viscous terms in the momentum equation may be neglected, which requires large Reynolds numbers.

The equation for the velocity fluctuation now has the form,

$$\frac{Du'_i}{Dt} + u'_j\frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i}, \tag{4.23}$$

which indicates that the time constant of the linear system is $1/(\partial U_i/\partial x_j)$, and that the nature of the pressure terms depends on the velocity and density fields.

We must also consider the fluctuating continuity equation:

$$\underset{(1)}{\frac{\partial \rho}{\partial t}} + \underset{(2)}{U_i\frac{\partial \rho'}{\partial x_i}} + \underset{(3)}{\bar{\rho}\frac{\partial u'_i}{\partial x_i}} + \underset{(4)}{u'_i\frac{\partial \bar{\rho}}{\partial x_i}} + \underset{(5)}{\rho'\frac{\partial U_i}{\partial x_i}} + \underset{(6)}{\rho'\frac{\partial u'_i}{\partial x_i}} + \underset{(7)}{u'_i\frac{\partial \rho'}{\partial x_i}} = 0.$$

If condition 4.20 is fulfilled, it is clear that the seventh term in this equation is small compared to the second term, and if the density fluctuations are small

$$\frac{\rho'}{\bar{\rho}} \ll 1, \tag{4.24}$$

the sixth term is much less than the third.

The form of the continuity equation is therefore not a problem by itself because the hypothesis of small fluctuations is sufficient to yield a linear equation. The density can be replaced by a function of pressure and entropy. However, as noted in the previous section, there is a contribution of the energy dissipation to the low wave numbers through temperature fluctuations. A linearized form of the energy equation, for example, Equation 4.5, combined with the linearized equation of state, closes the system. This constitutes a complete set of equations for the RDT problem, and a flow case can formally be called a rapid distortion when conditions 4.20 to 4.22 and 4.24 are fulfilled.

4.4.2 Application to Supersonic Flows

As noted earlier, RDT problems for subsonic flows have been studied by many authors. In particular, it is possible to find mean fields that produce homogeneous turbulence fields. Fourier transforms can then be used, and the evolution of the three-dimensional spectra of turbulence can be derived. After integration over all wave numbers, the turbulent stresses can be calculated, analytically if the mean distortion is irrotational.

In trying to formulate a similar approach for compressible flows, a difficulty is encountered with the energy equation. The solution now depends on the source terms in the energy equation, or, if the entropy fluctuation level is constant, from source terms in the distortion. The groundwork for the application of RDT to compressible flows was laid by Ribner and Tucker (1952) who studied the evolution of a solenoidal turbulent field subjected to an irrotational compressible mean field. Another important contribution was made by Goldstein (1978) who separated the velocity fluctuations into vortical and potential parts. He applied this decomposition to the case of isentropic fluctuations with an irrotational mean distortion. This yields a wave equation for pressure, with a source term depending on the vortical part of velocity. Debiève (1986) (reported also in Dussauge et al., 1989) gave a classification of the different possible source terms in this equation.

The applications of RDT to homogeneous flows by Durbin and Zeman (1992) and Jacquin et al. (1993) are of particular interest. Homogeneity makes possible some analytical results, and Fourier transforms can be used. For these particular distortions, the mean velocity gradients, the pressure and the density were spatially uniform but time dependent. Jacquin et al. used the Helmholtz decomposition of the velocity fluctuations into a solenoidal and a dilatational part, while Durbin and Zeman used Goldstein's decomposition into a vortical and a potential part. For the particular case of irrotational distortion they concluded that the nature of the pressure field depends critically on the gradient Mach number $M_g = \lambda S^*/a$. Here Λ is an integral space scale, S^* is a characteristic value of the velocity gradient ($= |\partial U_i/\partial x_j|$), and a is the speed of sound. This Mach number is therefore based on the velocity difference occurring over a distance equal to the integral scale in the distortion. For subsonic velocity differences, the velocity divergence depends only on pressure fluctuations generated by a purely acoustic mode, and it does not depend on the solenoidal part. However, if this velocity difference becomes supersonic, there is a direct coupling with the dilatational mode, without damping by pressure. M_g can also be interpreted as the ratio of the acoustic time scale of the pressure fluctuations Λ/a to the time scale of the distortion $1/S^*$. If M_g is large, the pressure does not have time to develop fully during the distortion. This suggests that in very rapid interactions, such as in shock waves, the effect of pressure can be neglected. For moderate M_g, the contributions of the vortical and potential modes to the pressure variance and to the pressure divergence term in the turbulent kinetic energy equation are both small. This analysis has recently been extended to the case of a pure shear by Simone and Cambon (1995) and Simone et al. (1997).

4.4.3 Rapid Distortion Approximations

It is usually difficult to apply RDT to shear flows, because in general the distortions of such flows are vortical and inhomogeneous, the entropy fluctuations are not constant, and the initial three-dimensional spectra are not known. The theory has not been developed for such cases. Nevertheless, the analysis of the Reynolds stress equations and the physical hypothesis used in RDT can still provide useful information on practical shear flows. This approach is what we call Rapid Distortion Approximation (RDA).

Consider the Reynolds stress equations. In the equation for the velocity fluctuations (Equation 4.23), it was possible to identify linear and nonlinear interactions with appropriate time scales. This is not so obvious in the Reynolds stress equations, because they are derived by multiplying the equation for the velocity fluctuations by the fluctuations themselves, integrating over all wave numbers and averaging. Consequently, some properties related to the length scales of the fluctuations have been lost. However, we can still apply some of the same physical arguments used in RDT. For example,

(i) The characteristic time scale of turbulent energetic eddies is $T_t = \Lambda/q'$. As noted earlier, after a time T_t the eddies have lost their identity through nonlinear mechanisms and most of their energy has been transferred to smaller scales. A consequence is that for times much less than T_t, the dissipation rate can be considered constant. In the present formulation, Λ is related to the energetic eddies, so that the rate of dissipation can be defined by $\varepsilon = q'^3/\Lambda$ or $\varepsilon = q'^2/T_t$.

(ii) The distortion time is of the order of $T_d = 1/(\partial U_i/\partial x_j) = 1/S^*$. This is the linear response time of the turbulent stresses, which means that after a time T_d a significant evolution of k or the shear stress due to the distortion can be observed. For the cases where the distortion is applied over a time L_d/U that is larger than T_d (L_d is the length of the distorted zone and U an average value of the mean velocity), we must use L_d/U instead of T_d.

It is assumed here that in a rapid distortion the evolution of turbulent fluxes is controlled by the action of the mean distortion, and not by diffusive processes, turbulent or viscous. Only the case of high Reynolds number flow is considered. When the diffusion terms are neglected, the equation for the Reynolds stress $\sigma_{ij} = \widetilde{u_i' u_j'} = \overline{\rho u_i' u_j'}\big/\bar{\rho}$ can be written in the symbolic form:

$$\frac{D\sigma_{ij}}{Dt} = \quad \text{(production)} + \text{(pressure strain terms)} - \text{(dissipation)}.$$

The pressure strain terms cannot be neglected and need to be modeled. The production terms come in two parts: one from the interaction of the turbulence with mean velocity gradients, and one from the interaction of the turbulence with mean pressure gradients. The second part, in the Favre averaged equations, results from the interaction of turbulent mass fluxes with the mean pressure gradients, whereas for the Reynolds averaged equations it comes from the interaction of the turbulent mass flux with the mean acceleration. This part is sometimes called the *enthalpic production* term, but it is not discussed here because that would not contribute much to the physical understanding of the phenomena (for further details, see Dussauge and Gaviglio (1987)). Both parts of the production can be calculated directly from the mean field. For adiabatic, nonhypersonic Mach numbers, the two parts of the total production are comparable in magnitude and of order $q'^2 S$, and the dissipation terms are of order q'^3/Λ. If the evolution of σ_{ij} is to be independent of the dissipation, we first require that the dissipation be much less than the production and pressure strain terms. That is,

$$\frac{q'^3}{\Lambda} \ll q'^2 S \qquad \text{or} \qquad \frac{q'}{\Lambda S} \ll 1.$$

The result is identical to condition 4.21.

Second, we also require $D\sigma_{ij}/Dt \gg \varepsilon$ because $D\sigma_{ij}/Dt \sim q'^2 U/L$. This gives the same condition as condition 4.22:

$$\frac{q'}{U}\frac{L}{\Lambda} \ll 1.$$

The terms involving pressure fluctuations, that is, the pressure strain terms, need to be evaluated. A linearized equation for the pressure fluctuations can be derived by taking the divergence of the linearized momentum equation (neglecting diffusion), provided $\rho'/\rho \ll 1$ and $u'/U \ll 1$. This is in agreement with the requirements for RDT (in particular, conditions 4.20 and 4.24). The pressure strain terms can be written as the sum of gradients of pressure velocity correlations (the *pressure diffusion* term), and the product of pressure fluctuations and instantaneous velocity gradients (the *redistribution* term). When the initial turbulence is isotropic, exact expressions for the pressure strain terms can be found, but in the general case these terms need to be modeled. The pressure diffusion is simply neglected in RDA, although this may lead to inaccuracies. Hunt (1977) indicated that some rapid distortions may develop pressure fluctuations strong enough to produce significant pressure transport terms, but he suggested that an inhomogeneous version of the RDT could compute this effect. The redistribution term can also be split into two parts: the *return-to-isotropy* part and the *rapid* part. The return-to-isotropy part is of the order of the dissipation, and it is therefore neglected. Dussauge and Gaviglio (1987) developed a model for the rapid part of the pressure strain terms for solenoidal velocity fluctuations $\nabla \cdot \mathbf{u}' = 0$ and a compressible mean field. For such flows, the pressure fluctuation obeys a Poisson equation, and the mean dilatation does not contribute to p'. This result has two consequences.

First, if the mean distortion is isotropic, that is, a pure dilatation or compression with spherical symmetry, the pressure does not damp the effects of production. The only source term in the equation for σ_{ij} is then the production due to turbulence interacting with the mean velocity gradients. The Reynolds stress equation can be integrated, and as a result the tensor $T_{ij} = \rho^{-2/3}\sigma_{ij}$ is constant along a streamline, and the Reynolds stress σ_{ij} varies with density according to:

$$\sigma_{ij} \propto \rho^{2/3}.$$

This dependence was also proposed from dimensional considerations by Batchelor (1955).

Second, the compressible models for the rapid part of the pressure strain terms can be derived very simply from the subsonic formulations: the rule is that the velocity gradient $\partial U_i/\partial x_j$ should be replaced by its deviatoric $\partial U_i/\partial x_j - \frac{1}{3}\delta_{ij}\left(\partial U_k/\partial x_k\right)$. In particular, the rapid part can itself be split into two parts, one related to mean velocity gradients, $(\pi_{ij})_u$, and the other to mean pressure gradients, $(\pi_{ij})_p$. Because we assume $\nabla.\mathbf{u}' = \mathbf{0}$, the mean pressure contribution can be modelled according to the incompressible formulation by Lumley (1975, 1978). For the contribution due to mean velocity gradients, several different models are available (see, for example, Launder et al. (1975) and Shih and Lumley (1990)).

Finally, we obtain a Reynolds stress equation that can be solved once the mean field is known. As long as the pressure gradients are much larger than the turbulent stress gradients, the turbulence does not influence the mean field and the mean velocity and pressure gradients can be found using the Euler equation with appropriate boundary conditions. For a supersonic flow, the method of characteristics can be used.

We see that in trying to solve the problem in terms of the Reynolds stresses, the basic concepts developed in RDT were used in RDA. However, in RDA the diffusion processes and the pressure transport are neglected, and the pressure strain terms need to be modeled. The current models were developed for incompressible turbulence and they use the incompressibility condition $\nabla.\mathbf{u}' = 0$ extensively. The discussion in the previous section also showed that the behavior of the pressure may depend strongly on the value of the gradient Mach number. Although these approximations may lead to inaccuracies, the resulting simplifications are attractive because they generally lead to tractable formulations.

4.4.4 Application to Shock-Free Flows

Computations using RDA were performed in expansions and compressions by Dussauge and Gaviglio (1987), Jayaram et al. (1989), Donovan (1989) and Smith et al. (1992). In these experiments, turbulent boundary layers were subjected to expansions and compressions produced by wall deflections, and they are described more fully in Chapter 9. Incoming Mach numbers ranged from 1.8 to 3.0. The pressure gradients were strong, so that the evolution of the supersonic part of the boundary layer could be computed using the Euler equations by the method of characteristics applied to the vortical flow of a perfect gas, together with the boundary conditions given by the experiment. The mean fields determined in this way were used as the input for the computation of the Reynolds stresses. This computation can be performed along streamlines because the diffusion terms were neglected. In all these flows, the gradient Mach number is small, and in any case less always than one. Favre averages were used, so that enthalpic production terms and their counterparts in the pressure strain models were retained.

The results for a 12° expansion at Mach 1.76 are shown in Figure 4.3. The general agreement between experiment and computation was excellent. The main effect on the velocity fluctuations appeared to be due to pure dilatation, but the rest of the mean distortion (due to mean normal strain and mean vorticity) also had a significant influence. The effect of dissipation was always negligible. Near the wall ($y/\delta < 0.4$), the RDA conditions were not fully met, and the predictions underestimate the turbulence damping. For the 20° expansion at Mach 2.89 studied by Smith et al. (1992), the computation also agreed well with the data, at least for $0.2 < y/\delta < 0.8$. In this region the dilatation alone accounted for 90% of the reduction in the streamwise component of the stress.

For the compressions, the task was more difficult. In each case, the compression was produced by a concave wall so that near the wall, where the turnover times are small, the relative length of the distortion increases. For this reason the compression was rapid in the outer flow (where the distortion length is short and the characteristic time scales are large) and slow near the wall. Nevertheless, in the outer layer of an 8° compression at Mach 2.87, Jayaram et al. (1989) found satisfactory agreement between the predictions of RDA and the measurements. Near the wall, the estimate of the nonlinear effects (associated with the neglected return-to-isotropy and dissipation terms) were compatible with the observed discrepancies between measurements and RDA. By including a crude estimate of the dissipation and the return-to-isotropy contributions, the predictions near the wall were improved considerably, as shown by Jayaram et al. (1989) and Donovan (1989). With these corrections, Donovan (1989) found very good agreement between the RDA predictions and the experimental data for the flow over a concavely curved wall with a turning angle of 16° at Mach 2.87, and for the flow over a flat wall with an externally

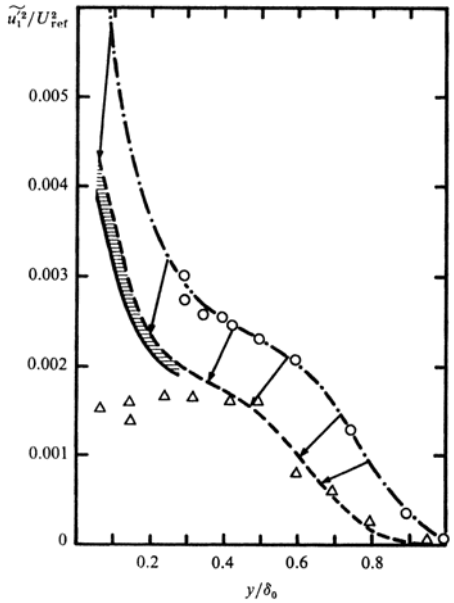

Figure 4.3. Velocity fluctuations in a 12° expansion corner at Mach 1.76. ○, upstream profile; △, along the last Mach wave in the expansion; - - - -, dilatation effect; shaded zone, dissipation effect. Arrows indicate streamline correspondence. (From Dussauge and Gaviglio (1987). Copyright 1987, Cambridge University Press. Reprinted with permission.)

imposed adverse pressure gradient of similar strength.

As we can see, RDA appears to be a useful tool for computing many continuous distortions of turbulent flows, provided the Reynolds number is high enough to satisfy the criteria of rapid distortion. It is also a useful tool to help understand more complex distortions. Smith and Smits (1996a), for example, used RDA to determine the relative importance of several competing distorting influences in the flow over a forward-facing step (see Chapter 9).

4.4.5 Shock Relations for the Turbulent Stresses

Another application of RDA is the case of a turbulent field passing through a shock wave. The main argument for using rapid distortion analysis is that the time of interaction between the shock and a turbulent eddy is very small so that the nonlinear energy cascade has no time to transfer energy. This leads to a linear problem if the strength of the shock does not depend on the incident turbulence. In particular, the shock is assumed to be steady. It was noted earlier that the nature of the pressure field may depend on the gradient Mach number $M_g = \Lambda S^*/a$. For large values of M_g, pressure fluctuations do not have enough time to propagate, so that there is no damping of the distortion by pressure. If this result still holds in vortical flows with entropy fluctuations, it suggests that the effect of pressure could be neglected for turbulent fluctuations passing through a shock, because the characteristic velocity gradient S^* takes very large values in a thin shock. The Reynolds stresses then experience the whole effect of the production terms. This hypothesis was used by Debiève (1983), who developed relationships for turbulence quantities across a shock (in effect, jump conditions for turbulence). The main difficulty in the analysis, the integration through the discontinuity, was resolved by defining a transport invariant relative to the Reynolds stresses, given by $n_i \sigma_{ij} n_j$, where $\mathbf{n}$ is a unit vector normal to the shock. This quantity is invariant for flows described by $D\sigma_{ij}/Dt =$ production. Turbulence source terms that modify the transport invariant were taken into account if they were either continuous through the shock, or if they varied like a Heaviside function.

The analysis leads to a particularly simple expression for the amplification of turbulence by a shock wave. The result depends on the orientation and strength of the shock wave, and it is given by

$$T_2 = K^* T_1 K \tag{4.25}$$

with $\mathbf{K} = \mathbf{I} - [\tilde{U}](\mathbf{n}^*/U_{2n})$, where T_1 and T_2 are the upstream and downstream Reynolds stress tensors, $\tilde{U}_n$ is the velocity normal to the shock, $[\tilde{U}]$ is the jump in velocity across the shock, and the asterisk denotes a transpose. Figure 4.4 gives a polar representation of the Reynolds stress upstream and downstream of the shock. The vector $\mathbf{OM}$ has a magnitude equal to the variance of the velocity fluctuation and a direction given by the unit vector $\mathbf{m}$. Note that $\mathbf{OM} = (\mathbf{u}' \cdot \mathbf{m})^2 \, \mathbf{m}$, and $\overline{u'v'} = OP - OQ$. The initial state corresponds to an isotropic tensor and it is represented by a circle. The diagram shows that the amplification through the shock is a maximum in the direction normal to the shock.

In this formulation the enthalpic production terms (which can probably be represented at least approximately as Dirac source terms) are neglected. Comparisons made with experiments in a 6° compression at Mach 1.8 showed good agreement, which suggests that the approximations based on gradient Mach number could also be extended to the rapid distortion of shear flows. However, Smits and Muck (1987) found that in stronger compression corner interactions at Mach 2.9, the agreement was not satisfactory, and for these flows the analysis explains only part of the observed turbulence amplification (see Chapter 10).

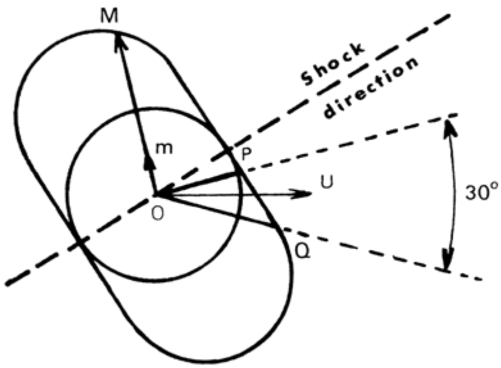

Figure 4.4. Evolution of the Reynolds stress tensor in a 6° compression ramp flow. (From Debiève (1983), with the author's permission.)

4.5 Mach Numbers for Turbulence

We have seen that a variety of Mach numbers has been used to quantify compressible turbulence. Turbulence models in their present form almost exclusively use the turbulence Mach number $M_t^2 = u'^2/a^2$, or $= \frac{1}{2}q'^2/a^2$, which suggests that strong turbulence fluctuations, of the magnitude of the speed of sound, will produce compressibility effects strong enough to change the global turbulence properties. If the velocity fluctuations are of the same order as the speed of sound, we can expect that shocks will be formed in the fluctuating motion. In some situations, however, this Mach number is not very useful. For example, acoustical phenomena are entirely dominated by compressibility, yet M_t is very small for this case. If the *rms* velocity is obtained from the isentropic relationship $p' = \rho a u'$, which can be rewritten as $p'/p = \gamma M_t$, we see that for typical acoustic pressure fluctuation levels it is difficult to obtain an acoustic wave with velocity fluctuations larger than the speed of sound. Another example can be found in direct numerical simulations. In simulating the decay of homogeneous turbulence, Blaisdell et al. (1993) found that the decay cannot be scaled by the turbulent Mach number alone, and that it depended strongly on initial conditions and on the relative importance of the solenoidal and nonsolenoidal parts of the velocity field.

Several other Mach numbers are also commonly used. In Chapter 3, it was shown that for thin shear layers in supersonic flow, the same approximations made in subsonic flows can be used, as long as $M_v \ll 1$, where M_v is the turbulence Mach number based on vertical velocity fluctuations. Some authors have introduced the fluctuating Mach number M', which takes into account the fluctuations in the speed of sound. In the analysis of boundary layers, and in Preston tube calibrations, a friction Mach number $M_\tau = u_\tau/a_w$ is often used. Finally, a parameter widely used in mixing layer studies is the convective Mach number briefly described in Chapter 1. Its main success is to collapse the data on the spreading rate of mixing layers formed between two different gases at high speed.

It is always puzzling, and certainly a little confusing, to introduce so many different parameters (even though they are all Mach numbers) to characterize situations that appear to be of the same type. Some questions need to be asked. Under what conditions are these Mach numbers defined? What is their physical meaning, and how they are related to each other? Is it possible to represent the properties of turbulence with some generality, as a function of any of these quantities?

We can start with the equations of motion. Dimensionless parameters generally define classes of problems, and they cannot be separated from the equations to which they are related. The momentum equation indicates that each time the magnitude of a turbulent stress is compared to the mean pressure, a Mach number is formed, for example, $M_v = v'/a$, as indicated earlier. This is similar to comparing the turbulent kinetic energy to the mean potential or internal energy, which yields $M_q^2 = \frac{1}{2}q'^2/a^2$. This is fundamentally the case found in the SRA relationship (see Chapter 3), where *rms* temperature fluctuations are proportional to M and M_t. In boundary layer studies, the friction Mach number $M_\tau = u_\tau/a_w$ can also be understood in the same way.

These interpretations are simple, but not very informative. Models for compressible turbulence are more ambitious because they try to scale the ratio of the divergence-free part of the velocity field to the compressible part by a single parameter. A similar attempt was made in Section 4.4, when the order of magnitude of the ratio $(\nabla.\mathbf{u}')/(u'/\Lambda)$ was estimated in terms of M_t. Although the estimates were approximate, this approach is inadequate even in the simplest case because this ratio was a function of both M_t and T'/T. Therefore, the parameter M_t can be a useful parameter to classify phenomena in shear flows only when the flows have a high degree of similarity. For example, Blaisdell et al. (1993) showed in their simulations that the structure of homogeneous sheared turbulence was independent of the initial conditions and could be characterized by either M_t or M'.

Compressibility effects found in acoustic waves are certainly different. In shear flows, intense vortices produce pressure waves such as shock waves or sound waves, which in turn are a powerful mechanism to generate velocity divergence, and therefore "compressible" turbulence. The important point in this case is not so much the Mach number of the (small) velocity fluctuations produced by the waves, but the Mach number of the strong fluctuations generated at the origin of the wave. Because the velocity divergence depends on M_t and T'/T, a representation using M' is therefore probably more appropriate. Because we believe that only relative motions are important, the pressure fluctuations produced by vortices will be of a different nature if the velocity difference across the layer is subsonic or supersonic, so that the pressure term of Equation 4.14, and consequently $\nabla.\mathbf{u}'$ may depend in general on M' rather than M_t, whereas the entropy term is some function of M_t and T'/T.

These considerations are based on global statistics, rather than a physical view of the flow. More precise descriptions can be obtained by considering some particular cases, and it may be possible that this approach can be applied to other flows as well. For example, the fluctuating Mach number M' represents the magnitude of the local Mach number variation, and it may be the best indication of the effects of compressibility on the distortion of communication paths, and the possible appearance of shocklets. Similarly, the convective Mach number is based on the mean characteristics of the flow, and in simple cases it can be interpreted as the Mach number of the relative motion of the large eddies. This is consistent with the view that the scales of the energy-containing eddies are of the same order as the scales of the mean motion. If M_c is larger than one, compressibility affects these large eddies. The multiple scale aspects of turbulence complicate this approach considerably, and it is not clear that smaller scales, which are less energetic and which have smaller velocity scales, are also directly affected by compressibility. In the general situation, the problem is difficult. In self-similar flows, however, M_c could play the role of a Mach number for turbulence, because all properties will depend only on similarity parameters and only one Mach number is required to describe all compressibility effects. It then becomes possible to derive relationships among M_c, M_t, and M'. For example, in mixing layers it may be assumed that

$$u'/\Delta U = F(M_c),$$

where u' is the peak value of the *rms* velocity fluctuations at a given streamwise location, and F is the normalized spreading rate (see Chapter 5). Introducing the definition valid for mixing layers with the same gas on both sides $M_c = \Delta U/(a_1 + a_2)$, and remarking that $u'/(a_1 + a_2) \approx u'/2a$, we find that $M_t \approx M_c F(M_c)$. Because $F(M_c)$ varies from 1 to about 0.7 when M_c varies from 0 to 2, this relation shows that M_t and M_c are of the same order, and are nearly proportional to each other over a wide range of convective Mach numbers.

The picture would not be complete without referring again to the gradient Mach number $M_g = S\Lambda/a$. Here, $S\Lambda$ is the velocity difference across a distance Λ. If this relative velocity is supersonic, $M_g > 1$, and compressibility effects are expected to become important, in particular with respect to the pressure field. In the simple case of mixing layers, the scale Λ is of the order of δ_ω: $\Lambda = C\,(M_c)\,\delta_\omega$, where C can depend on M_c. In the middle of the layer, $\partial U/\partial y \approx \Delta U/\delta_\omega$, and therefore

$$M_g \approx \frac{\Delta U}{\delta_\omega}\frac{C\delta_\omega}{a} \approx C\,M_c\,.$$

Because C is of order 1, M_g, M_c, and M_t in these shear flows are all of the same order of magnitude, and they are nearly proportional to each other, although they can represent different physical aspects of the flow.

4.6 DNS and LES

Direct Numerical Simulations (DNS) based on the compressible Navier-Stokes equations require that a large number of scales need to be resolved. In many high Reynolds number variable-density flows produced in laboratory experiments, in industrial applications (aerodynamics) and in nature (atmospheric flows), the ratio of the large scales to the small scales is very often greater than 10^6, and a very high accuracy is therefore required. There is also one more variable, the density, and one more equation, the energy equation. Moreover, we have hyperbolic behavior and the equations allow shock solutions that the numerical algorithms must reproduce accurately. The methods also need to make compromises between conflicting requirements: they should have a low numerical viscosity and be able to capture discontinuities such as shock waves without oscillations. These problems remain challenging, and only the most refined numerical methods have these capabilities. At present, three-dimensional computations can have a resolution of 2048^3 (Kaneda and Ishihara, 2004), which means that the ratio of the smallest scales to the largest ones is two thousand. For isotropic flows, this may be enough to establish the high Reynolds number limit (Pearson et al., 2004), but for shear flows, especially wall-bounded flows at large Reynolds number, experiments still remain essential, and several new generations of computers (perhaps two or three) will need to be developed before this situation will change significantly.

There is also the question of which computational techniques can be used. Among the more accurate are spectral methods, usually in Fourier space, which naturally lend themselves to homogeneous or periodic flow conditions. This limits the simulation of compressible flow severely: as pointed out by Lele (1994), there exist only a few possible instances where compressible flows are homogeneous. For steady flow, there is only the case of temporally decaying, homogeneous, compressible turbulence, and the case of pure mean shear with uniform density. It is also possible to compute homogeneous compressions or expansions by balancing the dilatation by a time-varying density. There is obviously a great need to study numerically the behavior of inhomogeneous compressible turbulence, so that, at present, most of the simulations (DNS and LES) are very often performed with finite difference schemes of high accuracy. For reasons of computer power and capacity, it is also often preferred to simulate temporal problems, the longitudinal direction being considered as homogeneous. This indicates the difficulty found in computing fully inhomogeneous, spatially evolving flows. However, this is mostly a question of technical performance, which will probably find a cure in the next generation of computers, and which, it is hoped, does not alter significantly the understanding of the physics of such flows. Current indications are that this holds true, even for relatively low Reynolds number flows (see, for example, Martin (2004)).

These computations are very appealing, despite their limitations, because they can produce results that are very difficult or even impossible to obtain experimentally. First, they can give field properties, because they produce an entire three-dimensional, time-dependent data set, and quantities such as vorticity or velocity divergence can be determined, and the existence of shock waves can be observed or derived. Such data are difficult to obtain experimentally under three-dimensional unsteady conditions such as found in turbulent flows. Second, they provide data on thermodynamic properties such as pressure, density, and temperature. Direct measurements of these quantities in supersonic turbulent flows are very difficult in many respects, and information from simulations, even if it is imperfect or incomplete, can prove to be very useful. Third, some particularly interesting experiments are very difficult to perform at high speed because of experimental difficulties. For example, it is very difficult to produce a homogeneous, decaying turbulence field in a flow of good quality, or to generate grid turbulence subjected to homogeneous shear, or to control the motion of the foot of a shock wave in an experiment. Similarly, because of the small physical size of the flows produced in typical research wind tunnels, and because the Reynolds number is usually high, it is generally more difficult in high-speed flows to take measurements in the near-wall region, or to measure the slope of the spectrum at high frequencies. For such problems, the simulations can play an essential role. The status of the simulations for homogeneous flows and simple shear flows is reviewed in the next section, and the case of inhomogeneous shear flows such as mixing layers, boundary layers, and channel flows is examined in Section 4.6.4. The particular example of the numerical simulation of the interaction of a shock wave with an incoming turbulence field is discussed in Section 4.7. We should also mention the recent simulations of channel flows by Coleman et al. (1995) and Huang et al. (1995). Their results, in particular regarding the pressure and density fluctuations, and the turbulent time scales, are in good agreement with what is known about supersonic wall layers, as described in the following chapters.

4.6.1 Homogeneous Decaying Turbulence

For the case of homogeneous turbulence in a uniform stream with constant mean density, the first simulations were two-dimensional (Passot and Pouquet, 1987). Shock waves were observed, and at the intersection of shock waves vortices were seen forming. In the three-dimensional case, it was more difficult to observe these shock waves, partly because of the problems of spatial resolution. In fact, the formation of shocks depends critically on the amount of compressible turbulence put in as part of the initial conditions. A useful parameter is χ, the ratio of the kinetic energy of the part of the fluctuations with nonzero divergence to the total kinetic energy. When χ is $O(1)$, nonlinear effects are observed in the acoustic mode and shock waves are produced by wave steepening. However, even with an initial χ of zero, Lee et al. (1991) report the occurrence of shocklets for high enough values of the turbulence Mach number M_t. One of the difficulties in interpreting these simulations is that the observed decay depends on the amount of dilatation dissipation ε_d present at any time, where ε_d itself depends on the initial values of χ, M_t, and the density fluctuations. It also seems that differences occur if the acoustic mode is prescribed independently of the vortical mode (see Sarkar et al. (1991) and Blaisdell et al. (1993)). When the initial conditions of these two modes are specified separately, their superposition is obtained: from Kovasznay's mode theory it is expected that their interaction is weak. Such behavior was confirmed by the work of Kida and Orszag (1990). Moreover, the resolved large scales had a limited statistical ensemble, and the oscillations in the solution obscured the observed trends (Lele, 1994).

Nevertheless, the simulations can give some guidance on the trends that can be expected, and they have been used to calibrate compressible turbulence closure models. In their present state, they are probably not reliable enough because of the ambiguities listed in the previous paragraph, and there is some question as to whether the simulations compute turbulence and the pressure field generated by the eddies, or whether they compute eddies and acoustic waves with weak coupling. In addition, because of insufficient dynamics, the resolved scales are often entirely dominated by viscous effects (Lele, 1994). Even if the functional form of the closure model is correct, therefore, the values of the modeling constants probably have to be treated somewhat skeptically.

4.6.2 Turbulence Subjected to Constant Shear

The situation is brighter for homogeneous turbulence subjected to uniform shear. The simulations are limited to flows with constant mean density, and they can only describe the shear flows found in real life incompletely because they give information on the effect of shear alone. This flow is fundamentally different from an inhomogeneous shear flow, because the latter is bounded and can lose energy by acoustic radiation. In the homogeneous case, the initial conditions are rather quickly forgotten, and after some time the ratio of the solenoidal and dilatational parts of dissipation is nearly constant (Blaisdell et al., 1993). As expected, the amplification of fluctuations is less than at low speeds. Long and thin structures with strong negative velocity divergence have been identified, and they have been interpreted as shocklets developing between zones where the vorticity has concentrated. This supports the notion that shock waves can exist within the turbulent motions present in shear flows.

The simulations also suggest that in practical flows a part of density is related to pressure through nearly isentropic processes, and another part is connected to the turbulent transport of heat, which, at moderate Mach numbers, is nearly isobaric but produces large temperature fluctuations.

Sarkar (1995) has performed simulations for various values of the turbulent Mach number M_t and gradient Mach number M_g. The gradient Mach number was based on the velocity difference occurring over a distance equal to an integral scale (Section 4.4.2), and it was shown that the structure of the pressure field in rapid distortion problems depends critically on this parameter, for both irrotational mean distortions and for shear flows. In these simulations, Sarkar attempted to make M_t and M_g vary independently. The data show that the observed low amplification rate of fluctuations is primarily due to a low level of turbulent friction and to a lesser extent to dilatation dissipation and pressure divergence terms. The amplification rate depends strongly on M_g and it is practically insensitive to variations in M_t. The results suggest that the first signs of compressibility effects are to be found in changes in the anisotropy of the Reynolds stresses. This aspect has been widely ignored in simple compressible turbulence models, and if confirmed, it implies that we are probably just starting to discover the properties of turbulence in high-speed shear flows where compressibility effects are significant.

4.6.3 Spectra for Compressible Turbulence

The form of the spectrum in compressible turbulence is still unknown. A number of theoretical proposals have been made and they were summarized by Passot and Pouquet (1987) and Lian and Aubry (1993). If acoustic waves are superimposed on incompressible turbulence, or if shock waves are present in one-dimensional or two-dimensional turbulence, the shape of the spectrum may be altered. In fact, there is some evidence to suggest that this is possible. In most numerical simulations the number of scales is too small to develop an inertial subrange. Typically, the spectra contain only two decades of wave numbers. However, the first simulations of two-dimensional compressible turbulence showed that there was a dramatic increase in the spectra at high wave numbers when shock waves were produced, suggesting that the slope of the spectrum for small scales could be significantly altered by compressibility. The more recent simulations by Porter et al. (1994, 1995) of forced and decaying turbulence contradict these results. For decaying turbulence, the initial *rms* Mach number was unity, and they reported the existence of a small spectral range with a $-5/3$ slope. Their simulations for turbulence subjected to a shear showed a larger range. When the solenoidal part and the compressible part of the velocity are considered separately, a $-5/3$ range was found for each part. This implies that the part of the motion which is directly controlled by the shocklets can also follow a $-5/3$ law. In a hypersonic boundary layer, Laderman and Demetriades (1974) found experimentally a slope close to $-5/3$ in the spectrum at high wave numbers, but these measurements are very difficult: the $-5/3$ slope is found in a frequency range where the frequency compensation of the hot-wire probe is difficult, so that the accuracy of the measurement may be in question.

More recently, Fauchet (1998) and Fauchet and Bertoglio (1998) have explored the shape of the spectra provided by the DIA model adapted for compressible isotropic turbulence. They also found a very robust persistence of the $-5/3$ power law for the solenoidal spectrum, whatever the value of M_t. Their results suggest, for low M_t, a $-3/2$ power law for the compressible part of the motion, and a -3 power law at large M_t. These results are obtained in the high wave number ranges of isotropic turbulence. Existing measurements are restricted to low energetic wave numbers in shear flow turbulence. In this case a distortion of the spectra is observed, the energetic scales being about half the size of their subsonic counterparts. These results are discussed in greater detail in Chapter 8. In summary, although the shape of the spectra appear to be modified, the asymptotic power laws are still observed.

4.6.4 Shear Flows

The computation of simple shear flows has been very successfully developed during the last few years, and an overview of these results in canonical flows such as mixing layers, boundary layers, and channel flows is given in this section. Simulations have been performed with (LES) or without a turbulence model (DNS). Obviously, the LES aim at simulating flows at larger Reynolds numbers. Many models have been used in LES, from the simple MILES (Boris et al., 1992), in which the filtering produced by the truncature at the level of the mesh acts as a sink for the fluctuations of larger scale, to more elaborate subgrid closures such as the dynamic mixed scale model (Sagaut, 2002). The adequacy of these models for high-speed flows is not discussed here, because at the present stage, no specific two-point closure is routinely used in the simulations. However, the discussion presented in Section 4.2 suggests that in flows where M_t is small, the fluctuating field is almost solenoidal. It may therefore be expected that in such flows, typically in channel flows and boundary layers with $M < 4$ or 5, no compressible events are expected. That is, it is probable that no significant singularities such as shocklets occur in the flow, and DNS may be easier. If no compressibility modeling is needed, low-speed formulations can probably be used with success.

Mixing layer simulations are more difficult, in that compressibility effects are important, even for relatively low values of M_c. In Section 6.10, we compare the predicted anisotropy of the Reynolds stresses to the experimental values. The behavior of, for example, $-\overline{u'v'}/\overline{u'^2}$ reveals more or less similar trends (see figure 6.16), but the most striking observations are the discrepancies shown at low Mach numbers, indicating some serious shortcomings in the computations.

Despite these difficulties, some important results have been uncovered from the computations. There is uniform agreement among the computations quoted here (Vreman et al., 1996; Freund et al., 2000; Pantano and Sarkar, 2002). that the dilatational dissipation and the pressure divergence terms are weak, and that the decrease in the spatial growth rate reflects a change in the structure of the fluctuating pressure. So the computations have already proved their worth by uncovering important physical mechanisms, even if their accuracy remains insufficient to calibrate models.

The predictions seem to be much better for boundary layer and channel flows. Many authors have performed DNS and LES simulations for compressible channel flows, including Coleman et al. (1995), Huang et al. (1995), and Lechner et al. (2001) among others. Their results, in particular regarding the pressure and density fluctuations, and the turbulent time scales, are in good agreement with what is known about supersonic wall layers, as described in Chapter 8. As for flat plate boundary layers, many computations are now available that demonstrate good results, including Guarini et al. (2000), Maeder et al. (2001), Stolz and Adams (2003), Pirozzoli et al. (2004), Sagaut et al. (2004), and Martin (2004), among others. The agreement with measurements in boundary layers at moderate Reynolds numbers is often excellent. An example of LES results obtained by Sagaut et al. (2004) is given in Figures 4.5 and 4.6. The result of the computation is in close agreement with the measurements. Particular hypotheses, such as the assumption of small pressure fluctuations, are well verified.

Recent DNS results obtained by Martin (2003) and (2004) are particularly interesting. The computations covered a Mach number range from 3 to 8, for a constant Reynolds number of $\delta^+ = 400$ (corresponding to a value of $Re_\theta = 2390$ at the lowest Mach number, and $Re_\theta = 13,060$ at the highest). A complementary run at $M = 2.23$ and $Re_\theta = 2390$ allowed comparisons with the experiments of Debiève (1983), at the same Mach and Reynolds number. This body of work is discussed more extensively in Chapter 8, but the initial evaluations show remarkable agreement with the known behavior of high-speed boundary layers, and further analysis of the DNS database will undoubtedly lead to a greatly expanded insight into the underlying physics. What is of equal interest are the DNS results on shock wave-boundary layer interactions obtained by Adams (2000) and Wu and Martin (2004). A preliminary result is shown in Figure 10.9, highlighting the wrinkled character of the shock as it interacts with the incoming turbulent boundary layer. Further consideration of this work is deferred to Chapter 10.

To conclude this short review, it seems that for homogeneous flows and simple wall flows (and possibly even for shock wave-boundary layer interactions), DNS and LES may be accurate enough to predict global properties in a very reliable way. If valid, this conclusion will open up a new era in high-speed

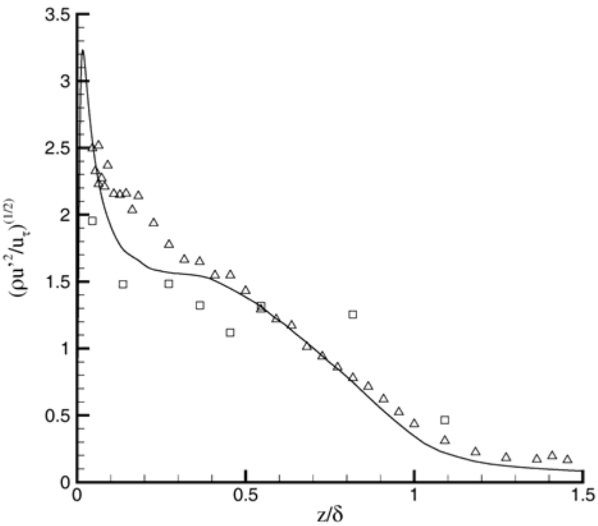

Figure 4.5. Longitudinal velocity variance in a supersonic boundary layer, $M = 2.3$, $Re_\theta = 4500$. □, hot-wire measurements, Debiève (1983); △, LDV measurements, Eléna and Lacharme (1988); ——, LES. (From Sagaut et al. (2004), with permission.)

boundary layer research, where many old questions can be addressed in an entirely new way. For example, numerical results can explore the immediate vicinity of the wall, a zone that is out of reach of most experimental methods. Moreover, time-dependent phenomena can be examined in great detail, such as the three-dimensional deformation of a single large-scale boundary layer motion as it encounters a shock wave. Finally, the computations have already been used for calibrating turbulence models, and they can help significantly in their development. It is not clear, however, that all the pertinent parameters have been systematically examined, and the available results are very often dominated by viscous effects. Whatever the present imperfections, it is expected that future progress in this area will help to improve our understanding of at least some compressibility effects, and it is likely to have a crucial impact on the development of compressible turbulence models.

4.7 Modeling Issues

In developing turbulence models for high-speed shear flows, Morkovin's (1962) hypothesis has often been a starting point. In essence, Morkovin suggested that for moderate Mach numbers compressibility effects did not influence the

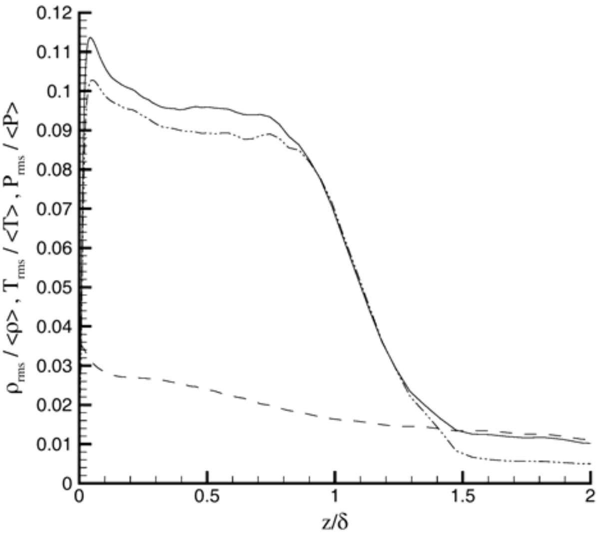

Figure 4.6. Density, pressure, and temperature fluctuations in a supersonic boundary layer, $M = 2.3$, $Re_\theta = 4500$, obtained using LES. ——- , *rms* density; - - - - , *rms* pressure; —- · ·, *rms* temperature. (From Sagaut et al. (2004), with permission.)

dynamic behavior of turbulence directly, and the principal effect of high speeds was felt through the change in fluid properties. The next chapter is concerned with the range of validity of this hypothesis, among other topics, and we show that it will be possible to describe some flows, such as nonhypersonic boundary layers with zero pressure gradient, using relatively straightforward extensions of simple low-speed models. In other flows, such as in mixing layers (see, for example, Barre et al. (1994)), or in boundary layers with $M_e > 5$ or with nonzero pressure gradients, there is a need for models taking compressibility effects into account. Here, we discuss some of the difficulties encountered when trying to model compressible turbulence using second-order closures. As we show, this is not a simple task. In contrast to the situation in low-speed flows, no satisfactory model has been found that is general enough to give accurate results over a reasonably extended range of applications. The models for compressible flow are rapidly evolving in as much as this field is relatively new, and we run the risk of making comments that will become obsolete at the same rate. Nevertheless, it is probably a risk worth taking, in order to check that the current proposals are sensible.

It was seen in Chapter 3 that the equation for turbulent kinetic energy does not have the same form in low- and high-speed flows. For compressible flows, there is a pressure velocity divergence term, which represents the power produced (or lost) by pressure in volume changes. The problem is to determine whether we can keep the modeling of the other terms unchanged (that is, use the models derived for incompressible flows) and add one or more extra terms, appropriately modeled, or whether compressibility modifies several or all of the terms, or even if the pressure divergence term is actually the most important one. Another question of interest is to determine if in models of the k–ε variety, the dissipation equation needs to be modified, and how this should be done.

A useful starting point is the case of subsonic flows with density gradients. These flows are very different from supersonic and hypersonic flows, but they can be considered as a benchmark for defining the action of compressibility as distinct from the departures due to high Mach numbers. At low speed, the degenerate form of a compressible turbulence model should match the variable density form. Somewhat surprisingly, work has been performed in this area only recently. A model accounting for the modification of the turbulent kinetic-energy diffusion was proposed by Shih et al. (1987), and Huang et al. (1994) showed that standard k–ε models have enormous difficulties in computing the equilibrium layer of a subsonic boundary layer with variable density (k–ω models are more successful in this respect).

One of the difficulties comes from the following reason. In the equilibrium, constant stress zone, we have:

$$\rho_w u_\tau^2 = -\overline{\rho u'v'} = \text{constant} \qquad \text{and} \qquad -\overline{u'v'} \propto k.$$

Therefore $\bar{\rho}k$ is constant, but k is not. In the equation for k, if the diffusion term is kept proportional to $\partial k/\partial y$, there exists obviously a diffusion term in the constant stress zone. This term is a function of the density gradient, which alters the balance between production and dissipation. As the constants of these models are linked together, there is a possibility to compensate partially this imbalance by adjusting one of the constants, for example the turbulent Prandtl number for the diffusion of the dissipation rate. Guézengar et al. (1999) and (2000) have used this possibility to compute boundary layers and mixing layers. Catris (2000) and Aupoix (2002, 2004) have explored another way by remarking that the diffusion term in the equation for k should be based on the gradient of $\bar{\rho}k$ to recover equilibrium conditions. Although this modeling brings a clear improvement, modifications to the equation for ϵ are necessary to recover the influence of density gradient on the spreading rate of the mixing layer. They do not reproduce the effect of compressibility on free shear flows.

These adjustments are necessary to cure some pathological aspects of the k-equation in models for incompressible flows. They should not hide more precise compressibility problems, related to the existence of nonzero velocity divergence, that result in two consequences for the k-equation: the existence of a dilatation dissipation and of a pressure divergence term. These terms represent the explicit effects of compressibility, because new terms appear in the equation. This behavior does not exclude the possibility of implicit effects: compressibility may modify the solenoidal dissipation and the structure of the fluctuating pressure field, even in the pseudo-sound approximation.

The first attempts to model compressible turbulence (Zeman, 1990; Sarkar et al., 1991) concentrated on these explicit aspects. In order to proceed, the source of the divergence terms needs to be identified. Dussauge et al. (1989); Zeman (1990), and Blaisdell et al. (1993) used order-of-magnitude estimates, analysis, and numerical simulations, and their conclusions supported the idea that the strongest sources of velocity divergence are shocklets, not the heating due to viscous dissipation in solenoidal fluctuations (see also Section 4.2). More recent work, however, driven by DNS (Vreman et al., 1996; Freund et al., 2000; Pantano and Sarkar, 2002; Section 4.6), has shown that in many supersonic shear flows, dilatation dissipation and pressure divergence are negligible, and the explicit effects cannot explain the anomalous spatial growth rate of mixing layers. The arguments for a fourth-order scaling in M_t presented in Section 4.2 seem well confirmed, and high-compressibility regimes should be considered to obtain a significant level of $\partial u'_k/\partial x_k$. Moreover, the analysis proposed by Ristorcelli (1997) suggests that pressure divergence scales as $M_t^2(P/\epsilon_s - 1)$. This scaling implies that pressure divergence is important if M_t is not small and if the flow is strongly out of equilibrium ($P/\epsilon_s \neq 1$). This explains probably why subsonic models can often provide reasonable predictions of supersonic boundary layers: in such flows M_t remains small and production balances dissipation over most of the layer, so that dilatation dissipation and pressure-divergence are small. The scaling suggests also that pressure divergence can be important when production and dissipation are very different, that is, in distorted or perturbed flows. This remains a domain that has not been fully explored.

The implicit effects are more puzzling. The work by Aupoix (2004) shows that, before taking compressibility into account, it is necessary to tune the models to reproduce density effects. Equally, it shows that some improvements are still needed to predict accurately the spreading rate of the subsonic, variable density mixing layer. Vreman et al. (1996) have clearly shown that the effect of compressibility in free shear flows is to change the structure of pressure fluctuations and as a consequence, to reduce $-\overline{u'v'}$. This means that taking pressure divergence and dilatation dissipation into account does not solve the central issue, and implies that the modeling of $-\overline{u'v'}$ needs to be revised. In two-equation models such as k–ϵ or k–ω, an eddy viscosity is used, and the formulation of ν_t should probably be modified. One possibility is to make ν_t a function of some local Mach number. Because the evolution of $-\overline{u'v'}$ is, to a large extent, related to the rapid part of pressure, the choice of a gradient Mach number appears as a likely candidate. If more detailed models such as Reynolds stress models, are considered, it is clear that the focus should be put on the modeling of the pressure strain terms. Their rapid part is again related to the gradient Mach numbers, and the nonlinear part (the return-to-isotropy in incompressible turbulence) depends on the turbulent Mach number. Note that rapid distortion studies for compressible turbulence have already proposed models for the dilatational part of the motion (Durbin and Zeman, 1992; Jacquin et al., 1993; Simone et al., 1997). These models, however, need to be extensively tested.

The previous considerations have dealt mainly with simple shear flows. The problems that were raised are found equally in flows strongly out of equilibrium, as in shock/boundary layer interactions. Such interactions may constitute new sources of vorticity, and of velocity divergence and of sound. They may alter significantly the balance between these various elements, and enhance compressibility effects. The same points as discussed with respect to simple shear flows will need to be addressed again, and the specific problems related to the presence of a shock (production of vorticity, temperature, and pressure fluctuations through the shock, flow separation, and shock unsteadiness) will require extensive further studies. The review paper by Knight et al. 2003 gives a good overview of the ability of the current turbulence models to predict such interactions. We must note that LES may also predict some of these interactions very well, as shown at moderate Mach numbers by Garnier et al. (2002), and that a great deal of new information is coming out of DNS (Adams, 2000; Li and Coleman, 2003; Wu and Martin, 2004).