Chapter 3

Equations for Turbulent Flow

3.1 Definition of Averages

The complexity of the equations of motion is obvious. Even for incompressible flows, the fact that the velocity and pressure vary with time and cover a wide range of spatial scales precludes the prospect of a general analytical approach. When the flow is compressible, the temperature and density become additional variables, and the flow states become even more complex. The complete problem can only be studied by experiment, or by direct numerical simulations. Direct numerical simulations for compressible turbulent flows have made great strides in recent years, as has the work in large-eddy simulation (Lesieur et al., 1992), but because of practical limits on computer memory and processing speed these computations are currently only possible at Reynolds numbers typical of transitional flows. At Reynolds numbers corresponding to high-speed flight, analytical or numerical approaches have generally sought to reduce the amount of information contained in the solutions of the Navier-Stokes equations by considering only the statistical properties of the flowfield. The equations of motion are then written in terms of the magnitudes of mean quantities. This operation by itself does not present any major difficulties, but the resulting expressions contain more unknowns than there are equations. This is the well-known closure problem. To "close" the set of equations some empirical input is required, and the process of providing this input is called turbulence modeling.

The mean quantities appearing in the equations can be found by ensemble averaging (for flows where long-term variations occur in the flow state) or by time averaging (where such long-term variations are absent). The definition of the ensemble average $\hat{f}$ is

$$\hat{f} \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} n f_n,$$

where f is taken at a given position and time, and N is the rank of the ensemble (that is, the number of data points taken). In the case where the flow is statistically steady, which is the case for most of the practical examples considered here, it is possible to use the *ergodic hypothesis* to replace the ensemble averages by time averages (Hinze, 1975). If T is the mean integration time, where T is sufficiently long for the mean to be stationary, then the time average $\overline{f}$ is defined by:

$$\overline{f} \equiv \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f dt.$$

A number of simple rules govern this averaging procedure. For two arbitrary variables f and g we have:

$$\overline{(f+g)} = \overline{f} + \overline{g}, \qquad \overline{\overline{f}} = \overline{f}, \qquad \overline{\overline{f} \cdot \overline{g}} = \overline{f} \cdot \overline{g},$$

where the mean is represented by an overbar. In addition, we assume that the variables are sufficiently regular so that the operations of differentiation, integration, and averaging may be inverted in order. That is,

$$\overline{\frac{\partial f}{\partial x_i}} = \frac{\partial \overline{f}}{\partial x_i}, \qquad \overline{\int f dv} = \int \overline{f} dv,$$

$$\text{and} \qquad w = \overline{w} + w'' \qquad \text{with} \qquad \overline{w''} = 0,$$

where the double prime denotes a fluctuation from the (temporal) mean. This procedure is usually called Reynolds averaging, and the fluctuations are centered; that is, their mean is zero.

For compressible flows, Reynolds averaging can be used, but it is possible to simplify the resulting equations instead by using mass averaged variables in a procedure often called Favre averaging (Mieghem, 1949; Favre, 1965, 1976). Here we define:

$$w = \tilde{w} + w'$$

with

$$\bar{\rho}\tilde{w} = \overline{\rho w},$$

so that

$$\overline{\rho w'} = 0.$$

The tilde denotes the mass weighted average, and the single prime denotes a fluctuation from the mass averaged mean. In this case, $\overline{w'} \neq 0$. Because the fluctuations are no longer centered, certain statistical results become difficult to interpret. For example, the variances and covariances of two variables no longer obey Schwartz's inequality and therefore a correlation coefficient can take a value greater than one. However, this decomposition has a great advantage in that most of the resulting equations have much simpler forms than the corresponding Reynolds averaged equations.

3.1.1 Turbulent Averages

Because Favre averaging leads to a simpler notation, the mass averaged forms of the equations have been used almost exclusively for the computation of compressible flows. It is worth remembering that it is simply a convenient notation: no formal simplifications result. We show in Section 3.2 that in the continuity equation there is a physically important reason for preferring Favre averages. In the momentum and energy equations, however, terms such as the viscous terms and the dissipation terms are actually more complicated and less amenable to physical interpretation when expressed in mass averaged form.

Because we are concerned with the statistical analysis of turbulent flows, it is important to know which one of the decompositions is more physically correct in formulating models and closure hypotheses. For example, in using a gradient diffusion hypothesis do we use Reynolds averages or Favre averages? Which variables must be used in defining a mixing length? There are no clearcut answers to such questions, but closure relations are often the result of interpolation schemes: they are always approximations of some kind, and very often the uncertainties associated with the models are similar or possibly greater than the differences between the two decompositions. In the same spirit, it is often difficult to know what kind of averaging is performed by a measuring instrument. Is the velocity derived from Pitot tube measurements closer to a mass averaged value, or a Reynolds averaged value? Again, there is no obvious answer. Similarly, when a hot-wire is operated at high resistance ratio it is primarily sensitive to the fluctuating mass flux (see Section 1.7.1), but the assumptions required to obtain the instantaneous velocity or its mass weighted equivalent can lead to greater uncertainty in the data analysis than the differences that arise by using either of the two decompositions.

We can make some of these statements more precise by comparing the two types of averages and the relationships between them. We only discuss some of the simpler relationships, and more complete presentations are given by Favre (1976) and Cebeci and Smith (1974). As before, the single prime denotes a fluctuation weighted by the mass, and the double prime a fluctuation in the Reynolds decomposition. Consider the identity:

$$w = \tilde{w} + w' = \bar{w} + w''.$$

By multiplying by the density and taking the mean, we obtain:

$$\bar{\rho}\tilde{w} = \bar{\rho}\bar{w} + \overline{\rho w''} = \bar{\rho}\bar{w} + \overline{\rho' w''}. \tag{3.1}$$

Hence the difference between the mean quantities $\tilde{w}$ and $\bar{w}$ is given by:

$$\tilde{w} - \bar{w} = \frac{\overline{\rho' w''}}{\bar{\rho}}. \tag{3.2}$$

This difference is a mean quantity which in general depends on the strength of the mean flow gradients. The difference between the fluctuations w' and w'' is also a mean quantity:

$$w' - w'' = \frac{\overline{\rho' w''}}{\bar{\rho}}. \tag{3.3}$$

The distinction between the two decompositions therefore depends on the correlation of the variable w with the density, which in turn will depend on the particular flow under consideration. By way of example we can evaluate this difference for the velocity fluctuations in a boundary layer on an adiabatic flat plate. In Section 5.2, we show that in these flows the fluctuations in density and velocity are connected by the approximate relations:

$$\frac{\sqrt{\overline{\rho'^2}}}{\bar{\rho}} = (\gamma - 1)\, M^2 \frac{\sqrt{\overline{u'^2}}}{\overline{u}} \tag{3.4}$$

and

$$R_{\rho u} = \frac{-\overline{\rho' u'}}{\sqrt{\overline{\rho'^2}}\sqrt{\overline{u'^2}}} \approx 0.8, \tag{3.5}$$

where $R_{\rho u}$ is the correlation coefficient between the velocity and density fluctuations (Equations 3.4 and 3.5 represent one form of the Strong Reynolds Analogy). To simplify the notation we sometimes use the single prime in the statistical quantities, as we did in Equation 3.4. In that case the overbar denotes a Reynolds averaged quantity, and the tilde a mass averaged quantity, and there should be no confusion. Also, we generally adopt the convention $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, where u is the velocity in the streamwise direction, v is in the wall-normal direction and w is in the spanwise direction. Using Equation 3.2 we find:

$$\frac{\tilde{u} - \overline{u}}{\overline{u}} = R_{\rho u}\,(\gamma - 1)\, M_t^2,$$

where $M_t = \sqrt{\overline{u'^2}}\big/\bar{a}$ is the turbulence Mach number (see also Section 4.5). A typical maximum value for M_t in an adiabatic boundary layer with a freestream Mach number of 3 is about 0.2 (see Figure 7.1), which leads to a maximum difference between $\tilde{u}$ and $\overline{u}$ that is less than 1.5%. For a strongly cooled wall flow at Mach 7.2 (Owen and Horstman, 1972), the maximum value of M_t is about 0.4, which gives a maximum difference between $\tilde{u}$ and $\bar{u}$ of about 5%, if Equation 3.4 still holds for this case. It seems that for constant pressure adiabatic boundary layers the differences between the conventional means and the mass weighted means are small for boundary layer flows with Mach numbers less than about 5. In mixing layers or jets, where M_t can take large values even at relatively low Mach numbers, the differences between Favre averaged and Reynolds averaged variables may become important at much lower Mach numbers.

Finally, consider some of the terms in the equations that contain a mixture of the two types of variables. For example, terms such as $\overline{\rho' u_i'}$ and $\overline{p' u_i'}$ that appear in the turbulent kinetic energy equation are formed by combining ρ' and p', that are variables which have a zero mean, and u_i', that does not. As it turns out, these variables have the same value, regardless of the decomposition used. For example, if we begin with Equation 3.3 for the velocity fluctuation, multiply both sides by a centered fluctuation c'', and take the mean, we obtain:

$$\overline{c'' u_i'} - \overline{c'' u_i''} = \overline{c'' \frac{\overline{\rho' u_i''}}{\bar{\rho}}} = -\bar{c}'' \frac{\overline{\rho' u_i''}}{\bar{\rho}} = 0.$$

That is,

$$\overline{c'' u_i'} = \overline{c'' u_i''}.$$

Because these variables have values that are independent of the preferred decomposition, we can choose one decomposition to model the terms and then use the result in the other decomposition.

In what follows, we generally choose whichever approach leads to the simplest representation. As we have seen, the connections between the two representations are easily made, and in practical terms the differences between corresponding variables are usually small for boundary layers in nonhypersonic Mach numbers.

3.2 Equations for the Mean Flow

The full equations in Reynolds averaged and mass averaged form are given by Cebeci and Smith (1974). Here we consider the continuity and momentum equations, the energy equation, and the turbulence kinetic energy. Note that the mean density and pressure are always expressed in terms of a Reynolds average, and that ρ' and p' are used to denote fluctuations with respect to $\bar{\rho}$ and $\bar{p}$.

3.2.1 Continuity

The Reynolds averaged form is:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho}\overline{u_j} + \overline{\rho' u_j''}\right) = 0, \tag{3.6}$$

and the mass averaged form is:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho}\tilde{u}_j}{\partial x_j} = 0. \tag{3.7}$$

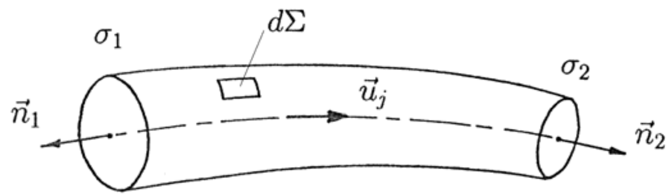

Figure 3.1. Control volume for mass conservation.

The equations show that $\tilde{u}_j$ is the mean velocity of mass transport, which is not true for $\overline{u_j}$. To illustrate the difference, consider the steady flow through the control volume shown in Figure 3.1. Here, $\tilde{u}_j$ is tangential to surface Σ and normal to the cross-sectional areas σ_1 and σ_2. Because the flow is steady, $\partial\bar{\rho}/\partial t = 0$, and Equation 3.6 can be integrated over the volume defined by these surfaces. Using the divergence theorem we obtain the Reynolds averaged form:

$$\int \overline{\rho' u_j''}\; n_j d\Sigma - \int \left(\bar{\rho}\overline{u_j} + \overline{\rho' u_j''}\right) n_1 d\sigma_1 + \int \left(\bar{\rho}\overline{u_j} + \overline{\rho' u_j''}\right) n_2 d\sigma_2 = 0.$$

The integrals over surfaces σ_1 and σ_2 represent the mass flow entering and leaving the control volume. Note that the surface Σ is not the surface of a stream tube because the mass flow across it, $\int \overline{\rho' u_j''}\; n_j d\Sigma$, is nonzero. This conceptual difficulty does not occur if the same calculation is performed using a control volume where Σ is tangential to $\tilde{u}_j$, and normal to the cross-sections σ_1 and σ_2. Then the mass averaged form is:

$$\int \bar{\rho}\tilde{u}_j\; n_1 d\sigma_1 - \int \bar{\rho}\tilde{u}_j\; n_2 d\sigma_2 = 0.$$

Because there is no mass flow across surface Σ it represents a stream tube, and therefore $\tilde{u}_j$ is the mean mass transport velocity.

3.2.2 Momentum

The Reynolds averaged form of the momentum equation is:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\bar{\rho}\overline{u_i} + \overline{\rho' u_i''}\right) &+ \frac{\partial}{\partial x_j}\left(\bar{\rho}\overline{u_i}\,\overline{u_j} + \overline{u_i}\,\overline{\rho' u_j''}\right) \\ &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \overline{u_j}\,\overline{\rho' u_i''} - \bar{\rho}\overline{u_i'' u_j''} - \overline{\rho' u_i'' u_j''}\right),\end{aligned} \tag{3.8}$$

and the mass averaged form is:

$$\frac{\partial \bar{\rho}\tilde{u}_i}{\partial t} + \frac{\partial \bar{\rho}\tilde{u}_i\tilde{u}_j}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \overline{\rho u_i' u_j'}\right). \tag{3.9}$$

Similar remarks to those made regarding the continuity equation can be made here. Because mass conservation is inherent in the use of $\tilde{u}_j$, the mass averaged equation is simpler. Moreover, the term $\overline{u_j}\overline{\rho' u_i''}$ in the friction term of Equation 3.8 does not appear in the mass averaged form and as a result Equation 3.9 is very similar in form to the momentum equation for incompressible flows. The Reynolds stress in mass averaged variables is simply $\sigma_{ij} = \widetilde{u_i' u_j'} = \overline{\rho u_i' u_j'} \big/ \overline{\rho}$. The difference between this term and $\overline{\rho}\overline{u_i'' u_j''}$ is mainly due to the term $\overline{u_j}\overline{\rho' u_i''}$, because the triple velocity correlation is expected to be an order of magnitude smaller than the other stress terms in Equation 3.8. The term $\overline{u_j}\overline{\rho' u_i''}$ is of the same order as the other terms, and cannot be neglected (Spina et al., 1994). However, the corresponding production term in the turbulent kinetic energy equation is at least two orders of magnitude smaller than the production due to $\overline{\rho}\overline{u_i'' u_j''}$, and it is not important for the energy flow in a compressible boundary layer.

At hypersonic Mach numbers, it is possible that the triple correlation $-\overline{\rho' u_i'' v_j''}$ may become comparable to the "incompressible" Reynolds shear stress, $\overline{\rho}\overline{u_i'' u_j''}$, because $\rho'/\overline{\rho} \sim M^2 u''/U$. Owen (1990) evaluated the various contributions to the "compressible" Reynolds shear stress at Mach 6 through simultaneous use of two-component LDV and a normal hot-wire. His results indicate that $-\overline{\rho' u_i'' v_j''}$ is negligible compared to $\overline{\rho}\overline{u_i'' u_j''}$. Even though density fluctuations increase with the square of the Mach number, it should be remembered that the main contribution to the Reynolds shear stress occurs in the region where the local Mach number is small compared to the freestream value, so this "hypersonic effect" may only be important at very high freestream Mach numbers.

3.2.3 Energy

The total enthalpy is given by:

$$h_0 = h + \tfrac{1}{2} u_i u_i. \tag{3.10}$$

Using the definitions given earlier, we have:

$$\tilde{h}_0 = \tilde{h} + \tfrac{1}{2} \frac{\overline{\rho u_i u_i}}{\overline{\rho}}.$$

That is,

$$\tilde{h}_0 = \tilde{h} + \tfrac{1}{2} \tilde{u}_i \tilde{u}_i + \tfrac{1}{2} \frac{\overline{\rho u_i' u_i'}}{\overline{\rho}}, \tag{3.11}$$

where $\tilde{h}_0 = \overline{\rho h_0} \big/ \overline{\rho}$, and $\tilde{h} = \overline{\rho h} \big/ \overline{\rho}$. Using these definitions in Equation 2.19, we obtain:

$$\frac{\partial \overline{\rho} \tilde{h}_0}{\partial t} + \frac{\partial \overline{\rho} \tilde{u}_j \tilde{h}_0}{\partial x_j} = \frac{\partial \overline{p}}{\partial t} + \frac{\partial}{\partial x_j} \left(-\overline{\rho u_j' h_0'} + \overline{u_i d_{ij}} - \overline{q_j} \right).$$

It is often useful to expand the stagnation enthalpy in terms of the velocity and temperature. Hence:

$$h_0' = h' + \tilde{u}_i u_i' + \tfrac{1}{2}\left(u_i' u_i' - \frac{\overline{\rho u_i' u_i'}}{\bar{\rho}}\right) \tag{3.12}$$

and finally:

$$\begin{aligned}\frac{\partial \bar{\rho}\tilde{h}_0}{\partial t} + \frac{\partial \bar{\rho}\tilde{u}_j \tilde{h}_0}{\partial x_j} = \frac{\partial \bar{p}}{\partial t} &+ \frac{\partial}{\partial x_j}\left(\tilde{u}_i \overline{d_{ij}} + \overline{u_i' d_{ij}} - \tilde{u}_i \bar{\rho} u_i' u_j' - \tfrac{1}{2}\overline{\rho u_i' u_i' u_j'}\right) \\ &- \frac{\partial}{\partial x_j}\left(\overline{q_j} + \overline{\rho u_j' h'}\right).\end{aligned} \tag{3.13}$$

3.2.4 Turbulent Kinetic Energy

The Reynolds averaged form is:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\overline{\rho u_i'' u_i''}\right) + \frac{\partial}{\partial x_i}\left(\overline{u_j}\tfrac{1}{2}\overline{\rho u_i'' u_i''}\right) &= -\bar{\rho}\overline{u_i'' u_j''}\frac{\partial \overline{u_i}}{\partial x_j} \\ &- \overline{u_j}\overline{\rho' u_i''}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{p'\frac{\partial u_j''}{\partial x_j}} - \frac{\partial}{\partial x_j}\left(\overline{p' u_j''} + \tfrac{1}{2}\overline{\rho u_i'' u_i'' u_j''}\right) \\ &- \text{viscous diffusion} - \text{dissipation},\end{aligned} \tag{3.14}$$

and the mass averaged form is:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\overline{\rho u_i' u_i'}\right) + \frac{\partial}{\partial x_i}\left(\overline{u_j}\tfrac{1}{2}\overline{\rho u_i' u_i'}\right) &= -\overline{\rho u_i' u_j'}\frac{\partial \tilde{u}_i}{\partial x_j} \\ &- \frac{\overline{\rho' u_i'}}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x_i} + \overline{p'\frac{\partial u_j'}{\partial x_j}} - \frac{\partial}{\partial x_j}\left(\overline{p' u_j'} + \tfrac{1}{2}\overline{\rho u_i' u_i' u_j'}\right) \\ &- \text{viscous diffusion} - \text{dissipation}.\end{aligned} \tag{3.15}$$

The turbulent kinetic energy equations appear very similar, if one excludes the extra convection terms introduced by the Reynolds decomposition. However, the Reynolds averaged form contains the relative acceleration $\overline{u_j}\overline{\rho' u_i''}\,(\partial \overline{u_i}/\partial x_j)$ which represents the work per unit time required to accelerate a fluid particle of a given mass. This term does not appear in the mass averaged form, but its place is taken by $\overline{\rho' u_i'}\,(\partial \bar{p}/\partial x_i)$. That is, in the absence of friction it is the pressure gradient that produces the force to accelerate the flow. The Favre averaged form is simpler in its interpretation, in that the pressure gradient term disappears in constant pressure flows, whereas the relative acceleration term in the Reynolds averaged equation does not.

The closure problem is also evident from the above equations: the equations for the mean flow contain second-order mean products of fluctuating quantities, and the equations for the second-order quantities contain third-order products. Equations for the third-order products can also be derived from the Navier-Stokes equations but these will contain fourth-order products, and so on. If the Reynolds stress equations are to be useful, then at some point the equations need to be closed; that is, the highest-order products need to be expressed in terms of lower-order products ($\overline{\rho u_i' u_i' u_j'}$ in terms of $\overline{\rho u_i' u_i'}$, for example) so that the number of equations equals the number of unknowns. In this process of turbulence modeling, most attention has been focused on the turbulence kinetic energy $\frac{1}{2}\overline{\rho q^2}\left(= \overline{\rho u_i' u_i'}\right)$, despite the fact that the turbulent kinetic energy does not appear in any of the mean momentum equations. There are two main reasons for this: there are more data available on the behavior of the terms appearing in the turbulent kinetic energy equation, compared to the equations for the components of the Reynolds stress tensor, and the redistribution term $\overline{p' \left(\partial u_j' / \partial x_j\right)}$ vanishes in an incompressible flow.

3.3 Thin Shear Layer Equations

Thin shear layers are flows where the characteristic scale in the cross-stream direction is much smaller than the characteristic scale in the streamwise direction. As a consequence, derivatives of mean quantities taken in the direction across the flow are always much larger than similar derivatives taken in the freestream direction. Typical examples include mixing layers, jets, wakes and boundary layers where the pressure gradients are not too large. For this class of flows, we can derive a set of approximate equations that are useful for the understanding of compressible turbulent shear layers.

From a mathematical viewpoint, it is always a risky procedure to simplify equations before solving them: any simplification will mean that the general solution cannot be obtained. However, the complexity of the original equations is so extensive that it is not possible to find the general solution and then simplify the result for a particular case. This is true even when we consider only the mean equations together with a closure hypothesis, and therefore it is very attractive to try to use some empirical observations to derive a simpler set of equations. Unfortunately, even with very simple closure schemes the thin shear layer equations themselves cannot be solved analytically. It is also widely recognized now that the thin shear layer equations are not a good starting point for a calculation method. In complex flows where, for example, streamline curvature and pressure gradients are present, the approximations used in deriving the thin shear layer equations can lead to errors that are of the same order as the errors introduced by the turbulence model (for a discussion relevant to subsonic flow, see Hunt and Joubert (1979)). Current calculation methods for turbulent shear layers now often use the full equations for the mean flow, and the approximations are made in the turbulence model, not in the equations themselves. Any prediction method based on the thin shear layer equations will lack generality, and will require various levels of additional modeling to produce a reasonable level of agreement with experiment (see, for example, Bradshaw (1973, 1974)).

Despite these limitations, the thin shear layer equations still play an important role. The equations are derived using empirical input regarding the characteristic scales, which are then used in an order-of-magnitude argument to identify the dominant terms in the original equation. This process helps to provide some physical insight into the behavior of compressible turbulent shear layers by establishing a basis of comparison with the incompressible case, and by identifying the characteristic scales that govern the shear layer behavior.

In what follows, we use order-of-magnitude arguments to derive the thin shear layer equations for compressible turbulent flows. It is useful to derive the equations for the special case of the boundary layer, although the equations also describe the behavior of other thin shear layers such as mixing layers, jets, and wakes. Only zero pressure gradient flows are examined.

3.3.1 Characteristic Scales

To begin the derivation of the thin shear layer equations, we need to define the characteristic scales for the order-of-magnitude analysis. Here we use U and V as the scales for the streamwise and normal velocities, q' for the turbulent fluctuations, ρ^* for the density, and L and δ for the distances in the streamwise direction and the wall-normal direction, respectively. The velocity and density scales are not defined very precisely, but if chosen properly they should be of the same order of magnitude as the primary variables. For example, in the outer part of the layer, we can assume that U could be taken as the freestream velocity. The length scales are equally ill-defined, but again they should be chosen so that the nondimensionalized derivatives are of order unity. Usually, L is taken to be the distance from the origin of the boundary layer (in terms of order-of-magnitude arguments, the difference between the virtual origin of the layer and the beginning of the turbulent flow is not important). For the outer flow, δ is taken as the local boundary layer thickness and the velocity gradients in the direction normal to the wall will, by definition of ΔU, be of order $\Delta U/\delta$, where ΔU, which does not depend on viscosity, will be specified for each particular case. At reasonable Reynolds numbers it is a matter of observation that $\delta/L \ll 1$. Near the wall, where the velocity gradients are large, a new set of scales may be needed. The effects of viscosity dominate, and the appropriate length scale is probably the thickness of the viscous sublayer δ_v. The velocity scale for the mean flow should still be the freestream velocity, or some fraction of it, except perhaps very near the wall where the velocity scale should be defined using the wall stress. Thus there exist two distinct regions that scale with different similarity variables. In the inner layer, the viscous stress is important, whereas in the outer layer the turbulent frictional stress dominates. Depending on the region of the flow under consideration, one or the other form of friction can become important, and in scaling the turbulence quantities we must look carefully at the contributions of the turbulence and molecular stresses to the total stress. Here, we consider the outer region first. The arguments developed here are similar to the used by Tennekes and Lumley (1972) and Cousteix (1989).

3.3.2 Continuity

For incompressible flow, the continuity equation may be written with the order of magnitude of each term underneath, as follows,

$$\frac{\partial \bar{\rho}\tilde{u}}{\partial x} + \frac{\partial \bar{\rho}\tilde{v}}{\partial y} = 0 \tag{3.16}$$

$$\frac{\Delta U}{L} \qquad \frac{V}{\delta}.$$

We see that the two terms are of the same order of magnitude, and therefore:

$$\frac{V}{\Delta U} \sim \frac{\delta}{L} \ll 1.$$

In compressible flow, two additional questions arise: the effect of the density gradient on the order-of-magnitude argument, and the proper estimate of the velocity gradient $\partial\tilde{u}/\partial x$. The continuity equation (Equation 3.16) can be written in the form

$$\frac{\partial \bar{\rho}\tilde{v}}{\partial y} = -\bar{\rho}\frac{\partial \tilde{u}}{\partial x} - \tilde{u}\frac{\partial \bar{\rho}}{\partial x}.$$

To relate the density and velocity gradients, we need information from the energy equation. Here, we consider the case of adiabatic layers in a perfect gas, where the total temperature gradient is small enough to be neglected for the purpose of order-of-magnitude arguments. That is,

$$\frac{\partial \tilde{T}_0}{\partial x} = \frac{\partial \tilde{T}}{\partial x} + \frac{\tilde{u}}{C_p}\frac{\partial \tilde{u}}{\partial x} = 0.$$

If the pressure is constant, it follows that

$$\frac{\partial \bar{\rho}}{\partial x} = (\gamma - 1)\, M^2 \frac{\bar{\rho}}{\tilde{u}}\frac{\partial \tilde{u}}{\partial x}$$

and

$$\frac{\partial \bar{\rho}\tilde{v}}{\partial y} \approx \bar{\rho}\frac{\partial \tilde{u}}{\partial x}\left(1 + (\gamma - 1)\, M^2\right).$$

Integrating from the outer edge of the layer yields:

$$\bar{\rho}_e \tilde{v}_e - \overline{\rho v} = \int_{\delta}^{y} \bar{\rho} \frac{\partial \tilde{u}}{\partial x} \left(1 + (\gamma - 1) M^2\right) dy,$$

where $\bar{\rho}_e \tilde{v}_e$ is the wall-normal mass flux at the edge of the layer. For the outer part of the layer, the orders of magnitude of both sides of this expression are

$$\rho^* V \sim \rho^* \left(1 + (\gamma - 1) M^2\right) \delta \left(\frac{\partial \tilde{u}}{\partial x}\right).$$

To find the order of magnitude of $\partial \tilde{u}/\partial x$, it is assumed that in the outer layer that the time scales of the mean and turbulent motions, $(\partial \tilde{u}/\partial y)^{-1}$ and δ / q', respectively, are of the same order, as in subsonic flows (Cousteix, 1989). That is,

$$\frac{\partial \tilde{u}}{\partial y} \sim \frac{q'}{\delta},$$

which leads to the conclusion

$$\tilde{u}_e - \tilde{u} \sim q',$$

so that

$$O\left(\frac{\partial \tilde{u}}{\partial x}\right) = \frac{\Delta U}{L} = \frac{q'}{L}.$$

We need to find the magnitude of q'. It is shown in Chapter 5 that Morkovin's hypothesis applies to nonhypersonic boundary layers. As a consequence, we find that the characteristic scale for the velocity fluctuations is $(\rho_w/\rho)^{1/2}\, u_\tau$, where u_τ is the friction velocity. For an adiabatic plate, the ratio ρ_w/ρ has a magnitude given by $1 + \frac{1}{2}(\gamma - 1) M^2$.

Finally, we obtain an estimate for the order of magnitude of the wall-normal velocity:

$$V \sim u_\tau \frac{\delta}{L} \frac{1 + (\gamma - 1) M^2}{\left(1 + \frac{1}{2}(\gamma - 1) M^2\right)^{1/2}}.$$

So a correction to the incompressible estimate appears that depends on Mach number. However, for Mach numbers less than 5, the incompressible estimate $V/\Delta U \sim \delta / L$ remains valid, and it is used in the rest of the analysis.

3.3.3 Momentum

For the streamwise momentum equation we have:

$$\bar{\rho}\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \bar{\rho}\tilde{v}\frac{\partial \tilde{u}}{\partial y} = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x}\left(-\bar{\rho}\widetilde{u'^2} + \tau_{11}\right) + \frac{\partial}{\partial y}\left(-\bar{\rho}\widetilde{u'v'} + \tau_{12}\right), \qquad (3.17)$$

where $\widetilde{u_i'u_j'} = \overline{\rho u_i' u_j'} \big/ \bar{\rho}$, as before. First, we can show that the two terms in the convective acceleration are of the same order. That is,

$$\bar{\rho}\tilde{u}\frac{\partial \tilde{u}}{\partial x} \sim \rho^* U \frac{\Delta U}{L},$$

and

$$\bar{\rho}\tilde{v}\frac{\partial \tilde{u}}{\partial y} \sim \rho^* V \frac{\Delta U}{\delta} \sim \rho^* \frac{(\Delta U)^2}{L},$$

for moderate Mach numbers. As in the outer part of low-speed boundary layers, we find that the second term in the acceleration is smaller than the first one. Increasing the Mach number will increase the relative importance of the second term, and the analysis suggests that in hypersonic boundary layers both terms are of comparable magnitude. In mixing layers and jets, $\Delta U \sim U$, so that no inertial term can be neglected. In the general case, therefore, both terms on the left-hand side must be retained.

Second, we note that when the friction (that is, the force due to shear stress gradients) is small, the acceleration terms on the left-hand side are counterbalanced by the pressure gradient. In general, the pressure gradient term can be of the same order of magnitude as the acceleration terms and it must be retained.

As a first step in considering the other terms on the right-hand side of Equation 3.17, we can show that the viscous terms τ_{11} and τ_{12} are small in the outer region. With Stokes's hypothesis, we obtain:

$$\tau_{11} = \overline{\mu \left[2\frac{\partial u}{\partial x} - \tfrac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right]} \tag{3.18}$$

$$\tau_{12} = \overline{\mu \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]}. \tag{3.19}$$

For boundary layers on an adiabatic flat plate, we can use the approximation

$$\overline{\mu \frac{\partial u_i}{\partial x_j}} \approx \overline{\mu} \frac{\partial \tilde{u}_i}{\partial x_j},$$

where we have neglected terms of the type

$$\overline{\mu \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)}.$$

As justification, consider, for example, $\overline{\mu \, (\partial u' / \partial y)}$. Here,

$$\begin{aligned} \overline{\mu \frac{\partial u}{\partial y}} &= \overline{(\overline{\mu} + \mu') \frac{\partial}{\partial y} (\tilde{u} + u')} \\ &= \overline{\mu}\frac{\partial \tilde{u}}{\partial y} + \overline{\mu'}\frac{\partial \tilde{u}}{\partial y} + \overline{\mu}\frac{\partial \overline{u'}}{\partial y} + \overline{\mu' \frac{\partial u'}{\partial y}}. \end{aligned} \tag{3.20}$$

The second term in Equation 3.20 is zero because $\overline{\mu'} = 0$. If the gradients of $\overline{u'}$ and $\tilde{u}$ have comparable length scales (which requires that the density and velocity fluctuations have similar length scales), then the ratio $(\partial\tilde{u}/\partial y)\Big/\left(\partial\overline{u'}\big/\partial y\right)$ has the same order of magnitude as $\tilde{u}\big/\overline{u'}$. More precisely:

$$\left(\overline{\mu}\frac{\partial\overline{u'}}{\partial y}\right)\Bigg/\left(\overline{\mu}\frac{\partial\tilde{u}}{\partial y}\right) = \frac{\overline{u'}}{\tilde{u}}.$$

By using the Strong Reynolds Analogy (see Equations 3.4, 3.5 and Section 5.2):

$$\frac{\overline{u'}}{\tilde{u}} = \frac{-\overline{\rho' u'}}{\bar{\rho}\tilde{u}} = -R_{\rho u}\left(\gamma - 1\right)\frac{\overline{u'^2}}{a^2} = -R_{\rho u}\left(\gamma - 1\right) {M_t}^2.$$

The term containing u' is therefore negligible if the square of the Mach number of the fluctuating velocity is small; that is, $M_t^2 \ll 1$.

To estimate the order of the last term in Equation 3.20, we use the fact that the viscosity varies with temperature according to $(\mu/\mu_0) = (T/T_0)^\omega$, which implies that $\mu'/\overline{\mu} = \omega\left(T'\big/\overline{T}\right)$ for small fluctuations. If pressure fluctuations are small, then $\mu'/\overline{\mu} = -\omega\left(\rho'/\bar{\rho}\right)$, and by using the same approximations adopted for evaluating the second term in Equation 3.20 we find:

$$\begin{aligned}
\left(\overline{\mu'\frac{\partial u'}{\partial y}}\Bigg/\overline{\mu}\frac{\partial\tilde{u}}{\partial y}\right) &\approx \omega R_{\rho u}\left(\gamma - 1\right) M^2\left(\frac{\overline{u'^2}}{\overline{\overline{u^2}}}\right)\\
\left(\overline{\mu'\frac{\partial u'}{\partial y}}\Bigg/\overline{\mu}\frac{\partial\tilde{u}}{\partial y}\right) &\approx \omega R_{\rho u}\left(\gamma - 1\right) M_t^2.
\end{aligned}$$

So this term will also be negligible if the velocity fluctuations are subsonic ($\omega = 0.76$ according to Equation 2.11). In any case, because $\partial\tilde{u}/\partial y$ is $O(\Delta U/\delta)$, and $\partial\tilde{v}/\partial x$ is $O(V/L)$, we can now write $\tau_{12} \sim \overline{\mu}\left(\partial\tilde{u}/\partial y\right)$. From the continuity equation, we know that $\partial\tilde{u}/\partial x$ and $\partial\tilde{v}/\partial y$ are the same order of magnitude, and therefore $\tau_{11} \sim \overline{\mu}\left(\partial\tilde{u}/\partial x\right)$. If the length scales fulfill the conditions noted earlier, then $(\partial\tau_{12}/\partial y) \gg (\partial\tau_{11}/\partial x)$, and

$$\frac{\partial\tau_{12}}{\partial y} \sim \mu^*\frac{\Delta U}{\delta^2} = \rho^*\frac{U\Delta U}{L}\frac{L}{\delta}\frac{\mu^*}{\rho^* U\delta},$$

where μ^* is the characteristic scale for the viscosity.

We see that if the Reynolds number $\rho^* U\delta/\mu^*$ is large enough then the viscous terms are negligible. Such flows are sometimes called "fully turbulent". This means more than simply requiring that the flow must have developed to the point where there are no intermittent periods of laminar flow, as it does in the region where the flow is still transitional. That is, we require the outer flow to be independent of Reynolds number. Specifically, a wake parameter that depends on Reynolds number is not permitted (see Chapter 7).

Now we return to Equation 3.17. As the next step in approximating the right-hand side, consider the turbulent friction, that is, the turbulent shear stress. Experimental results taken in turbulent shear flows have shown that all turbulent stresses have a similar order of magnitude, and the magnitude of $\widetilde{u'^2}$, $\widetilde{v'^2}$, and $\widetilde{w'^2}$ may be represented by q'^2. A typical value of $-\widetilde{u'v'}\big/\widetilde{q'^2}$ is about 0.15, and therefore we can assume that $-\widetilde{u'v'}$ is $O(q'^2)$ for an order-of-magnitude argument. Hence the term $\partial\left(\bar{\rho}\widetilde{u'v'}\right)\big/\partial y$ will be of order $\rho^* q'^2/\delta$, and because $\partial\left(\bar{\rho}\widetilde{u'^2}\right)\big/\partial x$ will be of order $\rho^* q'^2/L$, the shear stress gradient will be the dominant term, as in subsonic flows (note that we need $\delta/L \ll 1$ to be sure of this approximation). Also, the viscous part is of order $\mu^* \tilde{u}/\delta^2$, and the turbulent friction is larger than the viscous term by a factor $(\rho^* U\delta/\mu^*)\left(\widetilde{u^2}\big/\widetilde{u'^2}\right)$, which is always very large in the outer flow.

Finally, in Equation 3.17, because the term involving $-\bar{\rho}\widetilde{u'^2}$ and the first term in the convective acceleration are of the same order, we have

$$\frac{q'^2}{U\Delta U} \sim \frac{\delta}{L}$$

in the outer layer. If $\Delta U \sim q'$, as in boundary layers and wakes, then $q'/U \sim \delta/L$, and if $\Delta U \sim U$, as in mixing layers and jets, $q'/\Delta U \sim (\delta/L)^{1/2}$ as at low speed. It is shown in Chapter 6 that the experimental evidence corroborates this second result.

Subject to the approximations made so far, the boundary layer form of the mean momentum equation for the streamwise direction is given by:

$$\bar{\rho}\tilde{u}\frac{\partial \tilde{v}}{\partial x} + \bar{\rho}\tilde{v}\frac{\partial \tilde{v}}{\partial y} = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y}\left(-\bar{\rho}\widetilde{uv} + \overline{\mu}\frac{\partial \tilde{u}}{\partial y}\right). \tag{3.21}$$

This equation has the same form as for subsonic flows, but of course $\bar{\rho}$ and $\overline{\mu}$ are not constant in supersonic flows.

Consider now the mean momentum equation for the direction normal to the wall, that is, the y-component momentum equation. The pressure gradient across the layer $\partial\bar{p}/\partial y$ is of special interest because all pressure disturbances in a compressible flow propagate only along characteristic directions. If we restrict ourselves to the outer region where the viscous terms can be neglected:

$$\bar{\rho}\tilde{u}\frac{\partial \tilde{v}}{\partial x} + \bar{\rho}\tilde{v}\frac{\partial \tilde{v}}{\partial y} = -\frac{\partial \bar{p}}{\partial y} - \frac{\partial}{\partial x}\bar{\rho}\widetilde{u'v'} - \frac{\partial}{\partial y}\bar{\rho}\widetilde{v'^2} \tag{3.22}$$

$$\frac{\rho^* UV}{L} \qquad \frac{\rho^* V^2}{\delta} \qquad\qquad \frac{\rho^* q'^2}{L} \qquad \frac{\rho^* q'^2}{\delta}.$$

The terms on the left-hand side are all of the same order, and they are smaller than the corresponding terms in the x-component momentum equation by a factor V/U (or δ/L). Also, the first stress gradient is an order of magnitude smaller than the second, and by considering the estimate for $q'/\Delta U$, the turbulent friction term can be shown to be much larger than the convection term in the y-momentum equation. The reduced equation reads:

$$\frac{\partial}{\partial y}\left(\bar{p}+\bar{\rho}\widetilde{v'^2}\right)=0,$$

and because

$$\frac{\bar{\rho}\widetilde{v'^2}}{\bar{p}}=\gamma M^2\frac{\widetilde{v'^2}}{\widetilde{u}^2}=\gamma M_v^2,$$

the pressure is constant across the boundary layer if the Mach number of the normal velocity fluctuations is small (the approximation breaks down at high Mach numbers, as first pointed out by Finley (1977)). This result may be compared to the result for subsonic flows where the pressure is constant if the dynamic pressure associated with the normal velocity fluctuations is small. When $M_t^2 \ll 1$, $M_v^2 \ll 1$, and then the pressure is only a function of streamwise distance: the pressure gradient term in the x-component momentum equation is set by the conditions in the external flow, and, as in subsonic flow, we speak of the pressure gradient being "imposed" on the boundary layer.

Within the viscous sublayer, the viscous terms need to be taken into account. In high-speed flows, we do not have many detailed measurements in this region, but all indications are that for moderate Mach numbers the behavior is similar to that occurring at low speeds: the convection terms are small, the anisotropy of the stresses is modified but the dominant terms are still $\partial\left(\bar{\rho}\widetilde{u'v'}\right)\big/\partial y$ and $\overline{\mu}\partial\tilde{u}/\partial y$. Above all the total stress is constant and equal to the wall stress, and because it may be shown by similar arguments to those given earlier that the viscous terms in the y-momentum equation are an order of magnitude smaller than the wall stress, we arrive at the same conclusion. That is, the pressure is constant across the boundary layer when $M_t^2 \ll 1$.

The condition that $M_t^2 \ll 1$ has now appeared twice. From the results given in Chapter 1, we know that $\bar{\rho}\widetilde{u'^2}\big/\tau_w$ appears to be a nearly universal function of the nondimensional distance y/δ where this function f is apparently independent of Mach number and only weakly dependent on Reynolds number (see Chapter 8 for further discussion). For a constant pressure boundary layer, the maximum value of $\bar{\rho}\widetilde{u'^2}\big/\tau_w$ occurs near the wall, and it is about 8 for an adiabatic wall, as in subsonic flows. That is,

$$\bar{\rho}\widetilde{u'^2}=\tau_w f\left(\frac{y}{\delta}\right)$$

implying that:

$$M_t^2=\frac{\widetilde{u'^2}}{a^2}=\tfrac{1}{2}C_f M_e^2\, f\left(\frac{y}{\delta}\right),$$

which gives a maximum value for M_t^2 of about $4C_f M_e^2$. For a boundary layer in a Mach 3 flow with $C_f = 0.001$, we have $M_t^2 = 0.036$, which is very much less than one. At Mach 5 with the same value of C_f, we get $M_t^2 = 0.10$ and $\gamma M_v^2 = 0.047$ (near the wall $\widetilde{u'^2}/\widetilde{v'^2}$ is about 6). This is still small, but at Mach 10, we have $M_t^2 = 0.40$ and $\gamma M_v^2 = 0.18$, which indicates that under these conditions the pressure increases across the layer by about 20% if the anisotropy of the turbulence remains about the same as at lower Mach numbers. This result is interesting in that it implies that by measuring the mean pressure distribution across the layer, we can estimate the Mach number of the fluctuating velocity field and the extent to which compressibility is influencing the turbulence.

3.3.4 Total Enthalpy

Because we know that gradients of a given quantity in the direction of the mean flow are always much less than gradients in the direction normal to the wall, the total enthalpy equation for a steady, two-dimensional boundary layer becomes:

$$\frac{\partial}{\partial x}\left(\bar{\rho}\tilde{u}\widetilde{h_0}\right) + \frac{\partial}{\partial y}\left(\bar{\rho}\tilde{v}\tilde{h}_0\right) = \frac{\partial}{\partial y}\left(-\overline{\rho v' h_0'} + \overline{u\mu\frac{\partial u}{\partial y}} + \overline{k\frac{\partial T}{\partial y}}\right)$$
$$= \frac{\partial}{\partial y}\left(\tfrac{1}{2}\overline{\mu\frac{\partial u^2}{\partial y}} - \tilde{u}\overline{\rho u'v'} - \tfrac{1}{2}\overline{\rho u'u'v'}\right) + \frac{\partial}{\partial y}\left(\overline{k\frac{\partial T}{\partial y}} - \overline{\rho h'v'}\right). \quad (3.23)$$

Clearly, the terms on the right-hand side of the energy equation are considerably more complicated than the corresponding terms in the momentum equation. In the viscous term, we find terms that are quadratic in fluctuating velocity, and the diffusion of kinetic energy contains terms that are cubic in fluctuating velocity. It is possible to neglect certain terms in some regions of the boundary layer, but only at some values of the Reynolds number, and in general all terms must be retained. However, the following approximations appear to be reasonable:

$$\overline{\mu\frac{\partial u^2}{\partial y}} \approx \overline{\mu}\frac{\partial \tilde{u}^2}{\partial y} \quad \text{and} \quad \overline{k\frac{\partial T}{\partial y}} \approx \overline{k}\frac{\partial \tilde{T}}{\partial y}.$$

We can estimate the errors introduced by these approximations for the case of a boundary layer on an adiabatic wall, using the procedure given in the previous section, if we assume that the fluctuations in the total temperature and the pressure are small. This would show that the approximations are valid if $u'/\tilde{u}$, MM_t and M_t^2 are all small compared to one. Thus the condition that $M_t \ll 1$ is necessary to obtain the usual form of the energy equation for a boundary layer (see below).

3.4 Summary

In summary, the equations of motion for a turbulent boundary layer in a steady, two-dimensional, adiabatic supersonic flow are given by:

$$\frac{\partial \bar{\rho}\tilde{u}}{\partial x} + \frac{\partial \bar{\rho}\tilde{v}}{\partial y} = 0, \tag{3.24}$$

$$\bar{\rho}\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \bar{\rho}\tilde{v}\frac{\partial \tilde{u}}{\partial y} = -\frac{d\bar{p}}{dx} + \frac{\partial}{\partial y}\left(-\bar{\rho}\widetilde{u'v'} + \bar{\mu}\frac{\partial \tilde{u}}{\partial y}\right), \tag{3.25}$$

$$\begin{aligned} \bar{\rho}\tilde{u}\frac{\partial \tilde{h}_0}{\partial x} + \bar{\rho}\tilde{v}\frac{\partial \tilde{h}_0}{\partial y} &= \frac{\partial}{\partial y}\left(-\bar{\rho}\widetilde{h_0'v'} + \bar{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial y} + \bar{k}\frac{\partial \tilde{T}}{\partial y}\right) \\ &= \frac{\partial}{\partial y}\left(-\bar{\rho}\widetilde{h'v'} - \bar{\rho}\tilde{u}\widetilde{u'v'} + \bar{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial y} + \bar{k}\frac{\partial \tilde{T}}{\partial y}\right), \end{aligned} \tag{3.26}$$

where we have neglected the term $\overline{\rho u'u'v'}$ compared to $\bar{\rho}\tilde{u}\widetilde{u'v'}$.

As we have noted, the condition $M_t^2 \ll 1$ is necessary to write the usual form of the boundary layer equations for supersonic flow. For incompressible flows, the boundary layer equations are often called the thin shear layer equations in that they also apply to free shear layers, mixing layers, jets and wakes, although it should be understood that the approximations involved become less satisfactory for flows that are not bounded by a wall, mainly because the velocity fluctuations are generally larger and the layers grow relatively faster. Similar considerations will apply for compressible flows with the additional constraint that M_t is typically larger in free shear layers than in boundary layers.

It was shown that the continuity and momentum equations for thin shear layers in supersonic flows are identical to those in subsonic flow, as long as the fluctuating Mach number is small compared to one. This formal analogy does not prove that the turbulent fluxes are of the same nature at low and high speeds. In particular, the validity of using turbulence models developed for subsonic flows in calculations of supersonic flows must be verified. Throughout this chapter, the fluctuations in velocity were linked to the fluctuations in mass flux, pressure. and temperature, and therefore it is important to study the behavior of these parameters in compressible flows, as we show in later chapters.