Chapter 3

Polarization

3.1 General Aspects

An infinite plane wave in free space is transversally polarized which means that the electric field vector is always perpendicular to the direction of wave propagation z. The electric field is thus characterized by its components in the x-direction and the y-direction. For natural light and for the majority of lasers, the field vectors change their orientations randomly and in a short time interval compared to the detection time. Light sources with these properties are called unpolarized. In the following we discuss completely polarized light which means that the electric field vector either points in a fixed direction or changes its orientation periodically. The polarization of an electromagnetic field is completely characterized by the components of the electric field $\boldsymbol{E}$ in the x- and the y-direction. The field vector can be written such that common phase terms in the x- and the y-direction are extracted:

$$\boldsymbol{E} = \begin{pmatrix} E_{0x} \\ E_{0y} \exp[i\phi] \end{pmatrix} \exp[i(kz-\omega t+\psi)] \tag{3.1}$$

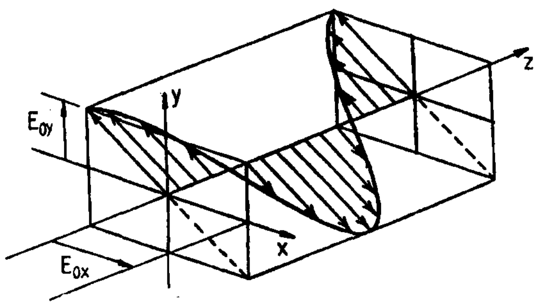

Fig. 3.1 The polarization of the electric field is characterized by the x-y-components field vector with the z-axis pointing into the direction of propagation. The polarization state is visualized in the x-y-plane by the projected curve traversed by the tip of the field vector. The field depicted is linearly polarized.

The field amplitudes E_{0x} and E_{0y} are both real numbers and the time averaged intensity of the wave is given by:

$$I = \frac{1}{2} c_0 \epsilon_0 \left(E_{0x}^2 + E_{0y}^2\right) \tag{3.2}$$

with: c_0: speed of light in vacuum
ϵ_0: vacuum permittivity

For the description of the polarization state the common time-dependent phase factor in (3.1) has no influence. In the following we will omit this term and deal only with the field vector:

$$\boldsymbol{E} = \begin{pmatrix} E_{0x} \\ E_{0y} \exp[i\phi] \end{pmatrix}$$

Special Polarization States

1) Linear Polarization, $\phi=0$ or $\phi=\pi$, (Fig. 3.2)
The two field components are either in phase or have opposite signs. The electric field vector oscillates at an angle α with respect to the x-axis with:

$$\alpha = \tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right) \qquad \textit{if} \ \phi = 0$$

$$\alpha = \pi - \tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right) \qquad \textit{if} \ \phi = \pi$$

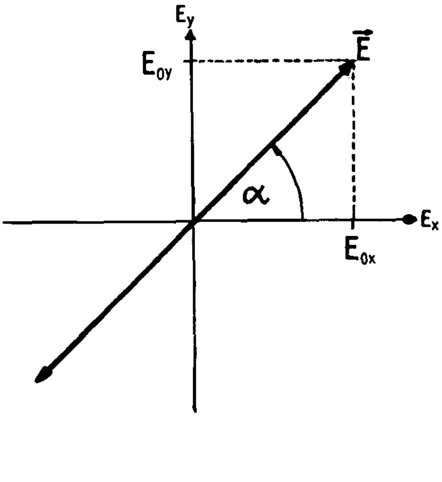

Fig. 3.2 For a linearly polarized electromagnetic wave the electric field vector oscillates only in one direction. The inclination is determined by the field amplitudes in the x- and y-direction.

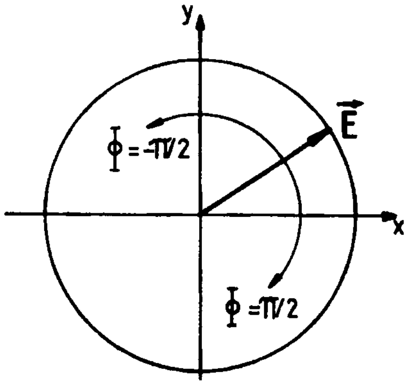

Fig. 3.3 For circularly polarized light, the tip of the field vector rotates on a circle.

2) Circular Polarization, $\phi=\pm\pi/2$, $E_{0x}=E_{0y}=E_0/\sqrt{2}$.
By using (3.1) we get the following field vectors for circularly polarized light:

$$E = \frac{E_0}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} \qquad if\ \phi = -\frac{\pi}{2}$$

$$E = \frac{E_0}{\sqrt{2}}\begin{pmatrix}1\\ +i\end{pmatrix} \qquad if\ \phi = +\frac{\pi}{2}$$

The temporal behavior of the field vector is depicted in Fig. 3.3. The two field components are oscillating with a phase difference of 90°. The electric field vector sweeps out a circle with radius E_0 as time evolves. If we look back at the source, the field vector rotates in clockwise direction for $\phi=+\pi/2$ (*right circularly polarized*) and in counterclockwise direction for $\phi=-\pi/2$ (*left circularly polarized*).

3) Elliptical Polarization, E_{0x},E_{0y}, and ϕ arbitrary.
If we assume that both field amplitudes are equal ($E_{0x}=E_{0y}=E_0$) the field vector is given by:

$$E = E_0\begin{pmatrix}1\\ \exp[i\phi]\end{pmatrix}$$

Projected onto the x-y plane, the tip of the field vector describes an ellipse whose semi-axes are rotated by 45° with respect to the x-axis. Linearly and circularly polarized light are special cases of elliptical polarization (Fig. 3.4).

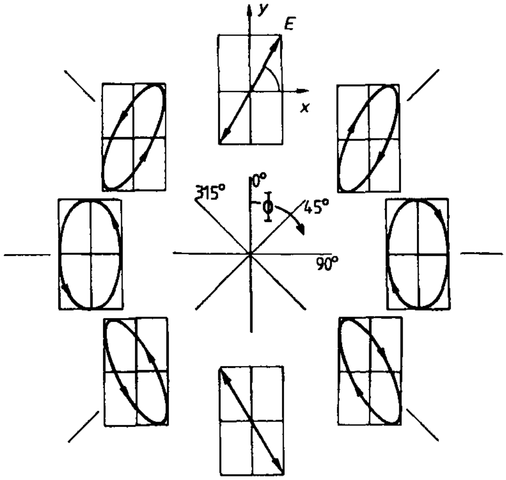

Fig. 3.4 Polarization states of light as a function of the phase difference ϕ between the electric field components in the x- and y-directions. In this graph the amplitudes of both components are different and the view is with the light propagation away from the observer.

Even if the two field amplitudes have different magnitudes, the field vector still traverses an ellipse with the semimajor axis rotated with respect to the x-axis by the angle α with:

$$\tan\alpha = \frac{E_{0y}^2 - E_{0x}^2 + \sqrt{4\cos^2\phi \; E_{0x}^2 E_{0y}^2 + (E_{0x}^2 - E_{0y}^2)^2}}{2\cos\phi \; E_{0x} E_{0y}}$$

Polarized light is called *right elliptically polarized* (clockwise rotation when looking back at the source) for $0<\phi<\pi$ and *left elliptically polarized* light is obtained for $\pi<\phi<2\pi$ (clockwise rotation when viewed in propagation direction).

3.2 Jones Matrices

3.2.1 Definition

Similarly to ray matrix theory in geometrical optics one can describe the changes in polarization generated by optical elements by means of a 2x2 matrix M^P. If E_0 denotes the field vector in front of the optical element, the new field vector at the output plane is given by:

$$\boldsymbol{E}_1 = \boldsymbol{M}^P \boldsymbol{E}_0 \tag{3.3}$$

The matrix M^P is referred to as the Jones matrix of the optical element [1.111,1.113]. In the following we present the Jones matrices for common polarizing optics.

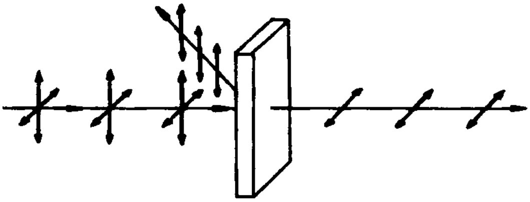

Fig. 3.5 Polarizer with a horizontal pass direction.

1) Polarizer

A polarizer transmits only electric fields oscillating in one direction called the pass direction. In case arbitrary polarized or unpolarized light is incident on the polarizer, only the field components in the pass direction will be transmitted. For an ideal polarizer, the light becomes linearly polarized. Thus, the Jones matrix of a polarizer can be written as:

$$M_P^P = t\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \qquad \textit{if the pass direction is the x-axis}$$

$$M_P^P = t\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \qquad \textit{if the pass direction is the y-axis}$$

The factor t with $0 \le t \le 1.0$ denotes the amplitude transmission of the polarizer. If the pass direction does not coincide with the x- or the y-direction, the Jones matrix can be found by rotation, as shall be discussed later.

In reality, polarizers do not provide 100% linear polarization. To a certain degree, field components oscillating perpendicular to the pass direction are also transmitted. If $I_{\parallel}$ and $I_{\perp}$ denote the intensities of the transmitted light being linearly polarized parallel and perpendicular to the pass direction respectively, the quality of the polarizer is characterized by the degree of polarization P:

$$P = \frac{I_{\parallel} - I_{\perp}}{I_{\parallel} + I_{\perp}} \tag{3.4}$$

For the ideal polarizer $I_{\perp}=0$ holds, and the degree of polarization is $P=1$. The Jones matrix for a real polarizer with the pass direction along the y-axis reads:

$$M_P^P = t\begin{pmatrix}\dfrac{\sqrt{1-P}}{\sqrt{1+P}} & 0\\ 0 & 1\end{pmatrix} \tag{3.5}$$

High quality polarizers provide degrees of polarization of $P=0.999$ and greater.

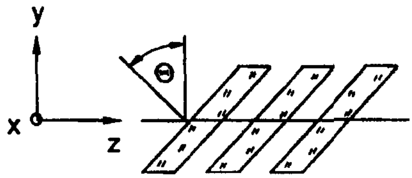

Fig. 3.6 Stack plate polarizer with three plates tilted by the Brewster angle Θ.

A simple way of generating a polarizer is by using an array of dielectric plates with refractive index n, arranged at the Brewster angle $\Theta = atan(n)$ (Brewster-plates), as shown in Fig. 3.6. This so called stack plate polarizer, used as early as the 19th century, has the advantage of transmitting one oscillation direction without reflection loss. Light polarized in the plane defined by the surface normal and the wave (propagation) vector of the light (p-polarized, along the y-axis in Fig. 3.6) passes a Brewster plate without loss. Field components oscillating in the perpendicular direction (s-polarized, along the x-axis) experience an amplitude transmission of:

$$t = \frac{2n}{n^2+1}$$

at each of the two interfaces. For a stack plate polarizer with N Brewster plates, as shown in Fig. 3.6, the Jones matrix reads:

$$M_{SPP}^{P} = \begin{pmatrix} \left[\dfrac{2n}{n^2+1}\right]^{2N} & 0 \\ 0 & 1 \end{pmatrix} \tag{3.6}$$

The degree of polarization is given by:

$$P = \frac{1 - t^{4N}}{1 + t^{4N}} \tag{3.7}$$

The intensity goes as the square of the field, and we can define the intensity transmission $T = t^{4N}$. For three glass plates ($n=1.5$) the intensity transmission for s-polarized light is 0.383 resulting in a degree of polarization P of 0.446. In low power gas lasers (HeNe, Ar^+) the discharge tubes are sealed off with Brewster glass plates ($N=2$). Note that the Brewster plates are inside the resonator - the resonator mirrors are typically located a few cm behind the plates. A resonator round trip will generate a loss ($=1-T^2$) of 0.72. Since the gain of these lasers is generally low and can therefore not compensate for combined losses of more than a couple of percent, the output beam is p-polarized.

2) Retardation Plates

Retardation plates are made of birefringent material that exhibit different indices of refraction along two perpendicular axes, denoted as the principal axes. If light incident on the retardation plate is oscillating along one of those principal axes, it experiences a phase shift of ϕ_1 or ϕ_2, depending on which principal axis lies along the oscillation direction. If the principal axes coincide with the axes of the coordinate frame, the Jones matrix of a retardation plate reads:

$$M_R^P = \begin{pmatrix} \exp[i\phi_1] & 0 \\ 0 & \exp[i\phi_2] \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \exp[i(\phi_2 - \phi_1)] \end{pmatrix} \exp[i\phi_1]$$

The phase term outside the matrix has no influence upon the polarization and therefore the Jones matrix can be written in the form:

$$M_R^P = \begin{pmatrix} 1 & 0 \\ 0 & \exp[i\delta] \end{pmatrix} \qquad \textit{with } \delta = \phi_2 - \phi_1 \tag{3.8}$$

Special Cases:

a) $\delta = \pm\pi/2$, 'Quarter Wave Plate'

The Jones matrix (3.8) now reads:

$$M_{\lambda/4}^P = \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix}$$

For linearly polarized light incident at an angle of 45° (field vector $E_0 = E_0(1,1)$), the output field vector becomes:

$$E_1 = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

Depending on the sign we get right circularly polarized or left circularly polarized light (Fig. 3.7).

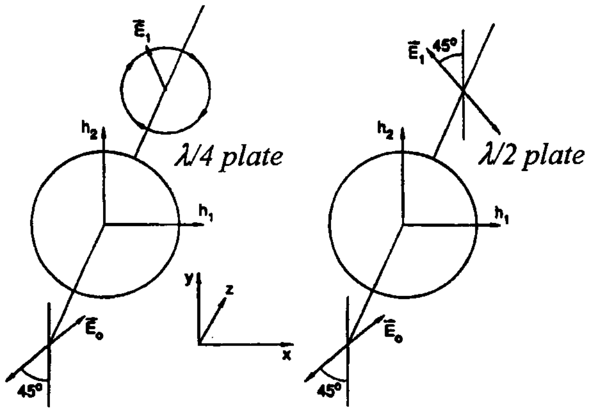

Fig. 3.7 Influence of a quarter wave plate and a half wave plate on the polarization of a linearly polarized beam oscillating at 45°. The principal axes of the plates are denoted by h_i.

b) $\delta=\pm\pi$, 'Half Wave Plate'

According to (3.8), the Jones matrix for the half wave plate is:

$$M^{P}_{\lambda/2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

After passage through the half wave plate, light linearly polarized under 45° will still be linearly polarized but the oscillation direction has rotated by 90°. The same effect can be generated by a series of two quarter wave plates.

3) Faraday Rotator

When inserted into the beam path, a Faraday rotator exhibits the unique property of rotating the plane of polarization for a linearly polarized wave regardless of the inclination angle at the entrance plane. This rotation, called the Faraday effect, can be generated in dielectric materials by applying a static magnetic field $\boldsymbol{B}$ along the propagation direction of the electromagnetic wave. The rotation angle β is proportional to the length L of Faraday rotator and to the magnetic field component in the direction of the wave vector $\boldsymbol{k}$:

$$\beta = V L \frac{\boldsymbol{k}\boldsymbol{B}}{|\boldsymbol{k}|} \tag{3.9}$$

The Verdet constant V determines the strength of the Faraday effect. Faraday rotators used in laser systems use doped glass or crystal rods with the magnetic field either generated electrically, as shown in Fig. 3.8, or by means of strong permanent magnets (Table 3.1).

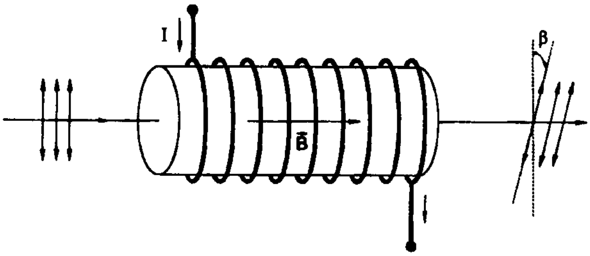

Fig. 3.8 The Faraday rotator rotates the plane of polarization of linearly polarized light.

Since the rotation of the polarization plane does not depend on the inclination of the input field vector, the Jones matrix of a Faraday rotator has the form of a rotation matrix (rotation in the counterclockwise direction when looking along the wave vector):

$$M_{FR}^{P} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \tag{3.10}$$

Table 3.1 Verdet constants of different materials.

Material	V[degree/(cm Tesla)]
water	2.2
phosphate glass	2.7
quartz	2.8
flint glass	5.3
Terbium doped glass	40.0
Terbium doped GGG	76.8

3.2.2 Matrices for Rotated Polarizing Optics

If the Jones matrix M^P for the aligned optics is known, we can calculate the Jones matrix $M^P(\alpha)$ for a rotation by an angle α by using (see Sec. 1.2.5):

$$M^{P}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} M^{P} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{3.11}$$

whereby the optics are rotated in the counterclockwise direction as we look towards the optics in the propagation direction of the electric field (Fig. 3.9).

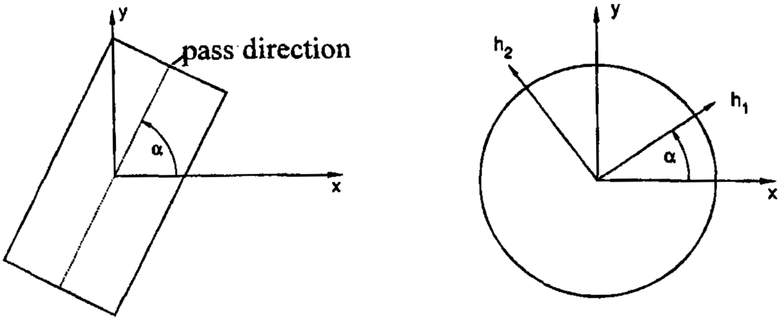

Fig. 3.9 Rotated polarizer and rotated retardation plate (viewed along the propagation direction of the field).

a) Rotated Polarizer (Fig. 3.9)

We assume that the pass direction of the aligned polarizer (with $t=1$) is along the x-axis. Application of (3.11) yields:

$$M_P^P(\alpha) = \begin{pmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix} \tag{3.12}$$

b) Rotated Retardation Plate (Fig. 3.9)

By inserting the Jones matrix of the aligned retarder (3.8) into the transformation law (3.11) one gets:

$$M_{RP}^P(\alpha) = \begin{pmatrix} \cos^2\alpha + \exp[i\delta]\sin^2\alpha & \sin\alpha\cos\alpha(1-\exp[i\delta]) \\ \sin\alpha\cos\alpha(1-\exp[i\delta]) & \sin^2\alpha + \exp[i\delta]\cos^2\alpha \end{pmatrix} \tag{3.13}$$

For a $\lambda/2$-plate ($\delta=\pi$), the matrix (3.13) reads:

$$M_{RP}^P(\alpha) = \begin{pmatrix} \cos^2\alpha-\sin^2\alpha & 2\sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & \sin^2\alpha-\cos^2\alpha \end{pmatrix} = \begin{pmatrix} \cos2\alpha & \sin2\alpha \\ \sin2\alpha & -\cos2\alpha \end{pmatrix} \tag{3.14}$$

Light that is polarized linearly in x- or in y-direction will be rotated by an angle 2α.

c) Rotated Faraday Rotator

The Jones matrix of the Faraday rotator is invariant under rotation:

$$M^{P}_{FR}(\alpha) = M^{P}_{FR}(0) \tag{3.15}$$

This is easy to understand since the Faraday effect does not depend on the inclination of the polarization plane at the entrance face of the rotator.

3.2.3 Combination of Several Polarizing Optics

When several polarizing optics are located in the beam path, the Jones matrices of the individual components have to be multiplied to find the resulting Jones matrix of the total optical system. If M_i^P denotes the Jones matrix of the i-th optical element passed by the beam, the resulting Jones matrix for N elements is given by:

$$M^{P} = M^{P}_{N}\, M^{P}_{N-1} \,\dots\, M^{P}_{2}\, M^{P}_{1} \tag{3.16}$$

Similar to the ray matrices in geometrical optics, the optical element passed first stands at the right hand side of the matrix product. Note that (3.16) can only be used if no more than one Jones-matrix depends on spatial coordinates (such as for radially and radial-azimuthally birefringent materials). If the optical system comprises more than one of these elements, the ray propagation between the elements has to be taken into account.

When the light is reflected off a mirror and travels through the same polarizing optics again, but in the opposite direction, the same Jones matrices for these elements are used. As already discussed in geometrical optics, the observer always rides with the beam which means that the coordinate frame is reflected by the mirror such that the z-axis points again in the propagation direction. There is, however, one exception to this rule: For polarization rotators based on optical activity (such as crystalline quartz, sugar solution, or milk acid), the negative angle $-\beta$ has to be used for the reverse direction in the Jones matrix (3.10). This is the mathematical description of the fact that for an optically active medium, the polarization is completely unaffected if the beam propagates through the medium twice, once in the forward and once in the reverse direction.

A high-reflecting mirror generally does not influence the polarization properties of the beam since both field components experience the same phase shift of π and the same reflectivity. Partially reflecting mirrors with non-normal incidence and all reflecting devices based on total internal reflection (porro prisms, corner cubes), however, affect the polarization state since the reflectivities and the phase shifts are different for p-polarized and s-polarized light. The general Jones matrix for a mirror thus reads:

$$M_M^P = \begin{pmatrix} r_x & 0 \\ 0 & r_y \exp[i\delta] \end{pmatrix} \tag{3.17}$$

with r_x,r_y are the amplitude reflectivities in the x- and the y-direction, respectively. For a 90° roof top prism with index of refraction n, the phase shift δ is given by:

$$\delta = 4\tan^{-1}\sqrt{1-2/n^2} - \pi \qquad \textit{roof edge along y-axis} \tag{3.18}$$

$$\delta = \pi - 4\tan^{-1}\sqrt{1-2/n^2} \qquad \textit{roof edge along x-axis} \tag{3.19}$$

A roof top prism acts like a $\lambda/4$ plate for an index of refraction of n=*1.5538*.

Examples:

1) The Optical Diode

A combination of a polarizer and a quarter wave plate with an angle of 45° between the pass direction of the polarizer and the principal axes of the plate is called an optical diode (Fig. 3.10). Light reflected off a mirror or a target (e.g. work piece in laser material processing) cannot pass the polarizer in backward direction. The reflected light is prevented from entering the laser system which can cause unwanted feedback effects. The optical diode is also used to physically separate the signal beam from the source beam path in laser measurement systems that make use of light scattered or reflected back into the source (remote sensing, LIDAR).

The principle of operation of an optical diode is based on the fact that the combination of forward and backward propagation through the quarter wave plate will rotate the polarization plane by 90°. Any back propagating portion of the beam will be completely absorbed or reflected by the polarizer. This is, of course, only true if the mirror or the target does not change the polarization state.

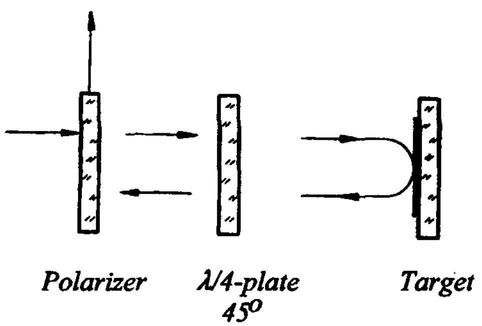

Fig. 3.10 The optical diode. The quarter wave plate rotates the polarization plane by 90°. For best results, the target must not affect the polarization.

By calculating the resulting Jones matrix with (3.16) we find that the resulting Jones matrix has indeed only zero components:

$$M_{OD}^{P} = M_{P}^{P}\, M_{\lambda/4}^{P}(45^{\circ})\, M_{\lambda/4}^{P}(45^{\circ})\, M_{P}^{P} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.20}$$

If the target affects the polarization of the beam, the quarter wave plate has to be replaced by a Faraday rotator with rotation angle β=45° and a polarizer attached to its end face (such that the rotated beam will be completely transmitted). If the pass direction of the first polarizer is in the y-direction, the resulting Jones matrix for this system reads:

$$M_{OD}^{P} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Pol.(y)}} \underbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{\text{FR}(45^{\circ})} \underbrace{\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}}_{\text{Pol.}(45^{\circ})} \underbrace{\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}}_{\text{Target}} \underbrace{\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}}_{\text{Pol}(45^{\circ})} \underbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{\text{FR}(45^{\circ})} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Pol.(y)}}$$

$$M_{OD}^{P} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} \tag{3.21}$$

Equation (3.21) clearly shows that this system always blocks the light in the backwards direction, no matter what the Jones matrix elements m_{ij} of the target are.

2) Rotatable Retardation Plate between Crossed Polarizers

Two crossed polarizers cannot transmit light. Only if the polarization is changed between the polarizers can field components in the pass direction of the second polarizer be generated. This effect can be used to visualize stress in transparent materials. Since stress induces birefringence, the intensity distribution behind the second polarizer contains information on the area and the magnitude of stress in materials. The retardation plate is a model for such a stressed material. The resulting Jones matrix for the system depicted in Fig. 3.11 reads:

$$M^{P} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos^2\alpha + \exp[i\delta]\sin^2\alpha & \sin\alpha\cos\alpha(1-\exp[i\delta]) \\ \sin\alpha\cos\alpha(1-\exp[i\delta]) & \sin^2\alpha + \exp[i\delta]\cos^2\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ \sin\alpha\cos\alpha(1-\exp[i\delta]) & 0 \end{pmatrix} \tag{3.22}$$

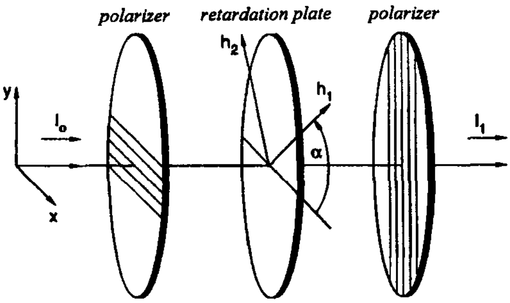

Fig. 3.11 Rotatable retardation plate between crossed polarizers

An unpolarized input beam with intensity I_0 results in an output intensity I of:

$$I = I_0\, 2\sin^2\alpha\cos^2\alpha(1-\cos\delta) \tag{3.23}$$

We get zero transmission when the pass directions of the polarizers coincide with the principal axes of the retardation plate (α=0°,90°,180°,270°), and maximum transmission is attained for α=45°,135°, 225°, and 315°. The phase shift δ can be determined from two intensity measurements, performed at different angles.

3) Two half-wave plates offset by 45°

The first half wave plate has its principal axes rotated by an angle α, the second plate is rotated by an additional 45° (angle: $\alpha+\pi/4$). By using (3.14), the resulting Jones-matrix can be easily calculated:

$$M^P_{2\bullet\pi}(\alpha) = \begin{pmatrix} \cos2(\alpha+\pi/4) & \sin2(\alpha+\pi/4) \\ \sin2(\alpha+\pi/4) & -\cos2(\alpha+\pi/4) \end{pmatrix} \begin{pmatrix} \cos2\alpha & \sin2\alpha \\ \sin2\alpha & -\cos2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} -\sin2\alpha & \cos2\alpha \\ \cos2\alpha & \sin2\alpha \end{pmatrix} \begin{pmatrix} \cos2\alpha & \sin2\alpha \\ \sin2\alpha & -\cos2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

A comparison with the matrix of a Faraday rotator (3.10) indicates that this combination of two half-wave plates is equivalent to a 90° rotator with counter- clockwise rotation (viewed along the propagation direction). If the second wave plate were rotated by an angle $\alpha+\pi/4$, a 90° rotation in the clockwise direction would result.

3.3 Eigenstates of Polarization

For any combination of polarizing optics we can find states of polarization which remain unaffected after passage. These polarization states are called the eigenstates of the optical system and they play an important role in resonator physics since they represent the steady state solutions. If $\boldsymbol{M}^P$ denotes the Jones matrix of the optical system, the field vector $\boldsymbol{E}_I$ must be proportional to the input field vector $\boldsymbol{E}_0$, with:

$$\boldsymbol{E}_1 = \mu^P \boldsymbol{E}_0 = \boldsymbol{M}^P \boldsymbol{E}_0 \tag{3.24}$$

The eigenstates of polarization are given by the two eigenvectors of the Jones matrix $\boldsymbol{M}^P$. The physical meaning of the eigenvalues μ^P is revealed if we compare the intensities at the input plane I_0 and at the output plane I_I:

$$I_1 = \frac{1}{2} c\epsilon_0 \boldsymbol{E}_1 \boldsymbol{E}_1^* = \mu^P \mu^{P*} \frac{1}{2} c\epsilon_0 \boldsymbol{E}_0 \boldsymbol{E}_0^* = |\mu^P|^2 I_0 \tag{3.25}$$

The factor $|\mu^P|^2$ represents the intensity fraction remaining in the beam after passage through the polarizing optics. This factor is called the loss factor V. The power fraction getting lost due to reflection or absorption is given by $\Delta V = 1 - V$. The two eigenvalues for the general Jones matrix given by:

$$\boldsymbol{M}^P = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

can be calculated by using:

$$\mu^P_{1,2} = \frac{m_{11}+m_{22}}{2} \pm \sqrt{\left(\frac{m_{11} - m_{22}}{2}\right)^2 + m_{12} m_{21}} \tag{3.26}$$

The corresponding eigenvectors read:

$$\boldsymbol{E}_i^P = \begin{pmatrix} 1 \\ \dfrac{\mu_i^P - m_{11}}{m_{12}} \end{pmatrix} \; ; \quad i=1,2 \qquad \text{if } m_{12} \neq 0 \tag{3.27}$$

$$\boldsymbol{E}_1^P = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \; \boldsymbol{E}_2^P = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; \qquad \text{if } m_{12} = m_{21} = 0 \tag{3.28}$$

Examples:

1) Retardation Plate

The Jones matrix of the rotated retardation plate is:

$$M_{RP}^{P}(\alpha) = \begin{pmatrix} \cos^2\alpha + \exp[i\delta]\sin^2\alpha & \sin\alpha\cos\alpha(1-\exp[i\delta]) \\ \sin\alpha\cos\alpha(1-\exp[i\delta]) & \sin^2\alpha + \exp[i\delta]\cos^2\alpha \end{pmatrix}$$

By using (3.26) and (3.27) we get the following eigenvectors and eigenvalues:

$$E_1^P = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \quad , \quad \mu_1^P = 1$$

$$E_2^P = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} \quad , \quad \mu_2^P = \exp[i\delta]$$

The eigenstates are linear polarization along the two principal axes. Since the retardation plate does not induce any losses (surface reflections are not accounted for in the Jones matrix!), both loss factors V are equal to 1.

2) Brewster Plate

The Jones matrix for a Brewster plate with index of refraction n and the surface normal in the y-z plane reads:

$$M_{BR}^{P} = \begin{pmatrix} \left[\dfrac{2n}{n^2+1}\right]^2 & 0 \\ 0 & 1 \end{pmatrix}$$

The eigenvectors and eigenvalues are:

$$E_1^P = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \mu_1^P = \left[\frac{2n}{n^2-1}\right]^2$$

$$E_2^P = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad \mu_1^P = 1$$

The s-polarized wave E_1^P exhibits a loss of $1-|\mu_1^P|^2$ due to reflection at the two interfaces.

3) Faraday Rotator

It can be easily shown that left circular and right circular polarizations are the eigenstates of a Faraday rotator, with both loss factors being equal to 1.

3.4 Polarization in Optical Resonators

3.4.1 Eigenstates of the Round trip Jones Matrix

The Jones matrix formalism enables us to determine the polarization of the electric field in laser resonators [1.112]. If polarizing optical elements are inserted into the resonator, the polarization will reproduce itself after every round trip as soon as a steady state electric field has been established. The steady state polarization is thus determined by the polarization eigenstates of the round trip Jones matrix. The reference plane from which the round trip is started can be chosen arbitrarily since the steady state condition must hold everywhere inside the resonator.

A polarized laser beam, however, can only be generated if the loss factor for one polarization eigenstate is lower than that for the other one. In this case, the polarization with the lowest loss is preferred. This is due to the fact that the threshold condition is reached first for the lowest loss polarization. After the onset of laser oscillation in this polarization eigenstate, gain saturation will prevent the second polarization eigenstate from reaching the threshold. In case both loss factors are equal, the electric field will be unpolarized. A laser resonator with an internal Brewster plate (Fig.3.12) will therefore emit a linearly polarized beam. The round trip Jones matrix starting on mirror 1 reads:

$$M^{P} = \begin{pmatrix} \left[\dfrac{2n}{n^2+1}\right]^4 & 0 \\ 0 & 1 \end{pmatrix}$$

The polarization eigenstates are linear polarizations along the x- and y- axis, respectively. The field will, however, be polarized along the y-axis (p-polarization), since this oscillation mode will not generate any reflection losses at the Brewster plate. The second eigenvector representing the s-polarized beam exhibits a round trip loss of:

$$\Delta V = 1 - \left[\frac{2n}{n^2+1}\right]^8$$

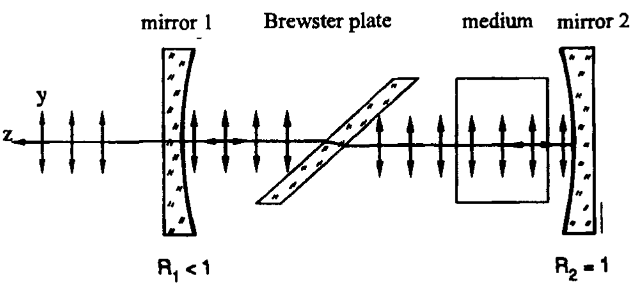

Fig. 3.12 A laser resonator with an internal Brewster plate generates a linearly polarized beam (p-polarized).

3.4.2 Polarization and Diffraction Integrals

In Sec. 2.8 we presented the integral equations for the calculation of the electric field distributions at the resonator mirrors and at any other plane inside the resonator. For the derivation of the Collins integral it was assumed that the electric field is a scalar quantity, neglecting the polarization. Fortunately, as far as optical resonators are concerned, it is not necessary to repeat a similar derivation of the diffraction integrals when the polarization is taken into account. Since the scalar wave equation holds for both the x- and the y-component of the field vector, we can apply the Collins integral in the x- and y-directions separately. If $\boldsymbol{K}E$ denotes the Collins integral for a round trip inside the resonator applied to the scalar field E, we can write the integral equation for the vector field in the form:

$$\begin{pmatrix} \gamma_x E_x \\ \gamma_y E_y \end{pmatrix} = \begin{pmatrix} \boldsymbol{K} & 0 \\ 0 & \boldsymbol{K} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{3.29}$$

The two field components do not mix since we have no polarizing element inside the resonator. We are dealing with two scalar field integral equations that can be solved separately.

If a polarizing optical element with Jones matrix M^P is located inside the resonator, as depicted in Fig. 3.13, the two components will interact. We can still calculate the four propagations between the mirrors and the polarizing elements using diagonal integral matrices as in (3.29), but we now have to mix the x- and y-components of the field at the plane of the polarizing element according to its Jones matrix. The integral equation for a resonator round trip will then read:

$$\begin{pmatrix} \gamma_x E_x \\ \gamma_y E_y \end{pmatrix} = \begin{pmatrix} \boldsymbol{K}_4 & 0 \\ 0 & \boldsymbol{K}_4 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_3 & 0 \\ 0 & \boldsymbol{K}_3 \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_2 & 0 \\ 0 & \boldsymbol{K}_2 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_1 & 0 \\ 0 & \boldsymbol{K}_1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{3.30}$$

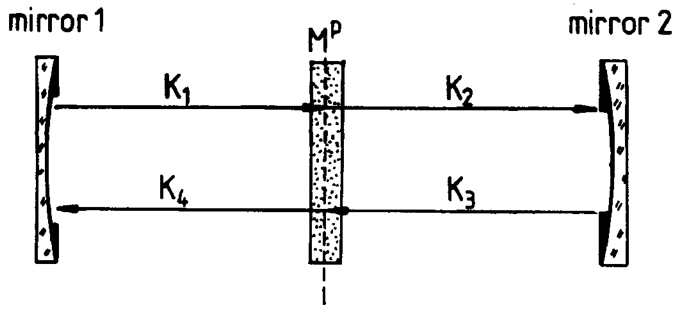

Fig. 3.13 Round trip in an optical resonator with an internal polarizing element. It is assumed that the length of this element is small compared to the resonator length. The field is propagated to and from the dotted plane by means of Collins integrals .

which can be written as:

$$\gamma_x E_x = [K_4 m_{11} K_3 K_2 m_{11} K_1 + K_4 m_{12} K_3 K_2 m_{21} K_1]\ E_x + [K_4 m_{11} K_3 K_2 m_{12} K_1 + K_4 m_{12} K_3 K_2 m_{22} K_1]\ E_y$$

$$\gamma_y E_y = [K_4 m_{12} K_3 K_2 m_{11} K_1 + K_4 m_{22} K_3 K_2 m_{21} K_1]\ E_x + [K_4 m_{21} K_3 K_2 m_{12} K_1 + K_4 m_{22} K_3 K_2 m_{22} K_1]\ E_y$$

If the Jones matrix has non-vanishing elements m_{12}, m_{21}, the two field components become coupled and we have to solve both integral equations simultaneously. Note that the swapping of the integral operators and the Jones matrix elements in the above equations is prohibited if the Jones matrix elements depend on the spatial coordinates. Once the solutions of these coupled integral equations are found (in most cases numerically), the field distribution E and the loss factor V are given by:

$$E = \begin{pmatrix} E_x(x,y) \\ E_y(x,y) \end{pmatrix}, \quad V = \frac{|\gamma_x E_x|^2 + |\gamma_y E_y|^2}{|E_x|^2 + |E_y|^2} \tag{3.31}$$

3.5 Depolarizers

In the preceding sections we have only discussed polarizing optics that either change the polarization state of polarized light, like the retardation plate, or generate a well-defined polarization from unpolarized light, like the polarizer. In addition to this class of polarizing optics, optical elements that transform polarized light into unpolarized light also exist. The latter type of polarizing optics are called *depolarizers*. Whereas the generation of polarized light out of unpolarized light cannot be accomplished without loss of energy, the depolarization can be achieved in a lossless way. The depolarization of polarized laser beams is most often applied to decrease measurement errors in power and intensity measurements since the polarization interacts with the measurement apparatus. Furthermore, as far as material processing is concerned, an unpolarized beam provides a more flexible tool since the interaction with the work piece is less affected by changes in the processing geometry (e.g. angle of incidence, direction of focal spot movement). Depolarization can be attained by reflection off or transmission through a scattering screen. This technique will, however, generate considerable loss and will also spoil the beam quality. A more suitable technique is to generate a continuum of different polarization states across the beam. The beam will then behave like an unpolarized beam. One commonly used depolarizer working on this principle is the Cornu depolarizer, as shown in Fig. 3.14.

This depolarizer for monochromatic light consists of two crystalline quartz prisms attached such that they form a cube. The first quartz prism (left handed quartz) acts like a retardation plate whereby the induced phase shift δ_l is proportional to the distance over which the beam has to propagate within the prism. At the interface, the phase shift of a collimated beam is a function of the entrance height y of the beam:

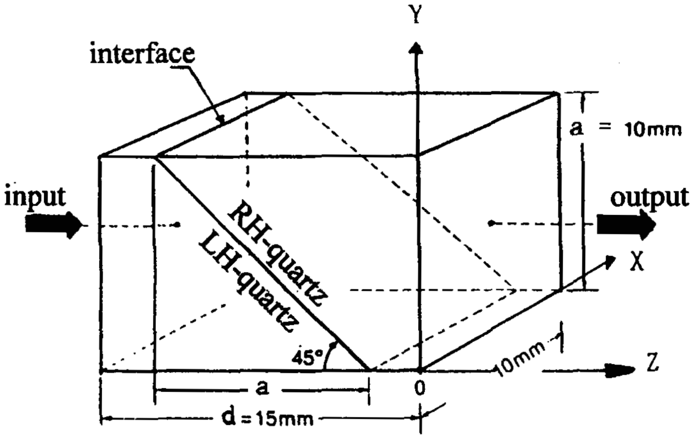

Fig. 3.14 Cornu Depolarizer.

$$\delta_1(y) = \frac{2\pi}{\lambda}[n_2 - n_1]\left(\frac{d-a}{2} + a - y\right) \tag{3.32}$$

Without the second prism, the beam would be deflected from the optical axis due to refraction. The second prism must be arranged such that the phase shift has a negative sign compared to the first one, otherwise the total phase shift would not depend on the y-coordinate. This is accomplished by switching the principal axes (right handed quartz). The phase shift induced by the second prism is given by:

$$\delta_2(y) = \frac{2\pi}{\lambda}[n_1 - n_2]\left(y + \frac{d-a}{2}\right) \tag{3.33}$$

Addition of the two phase shifts results in the total phase shift:

$$\delta(y) = \frac{2\pi}{\lambda}[n_1 - n_2](2y - a) \tag{3.34}$$

The beam emerging from the Cornu depolarizer exhibits different polarization states at different heights. Hence, the beam is not truly unpolarized, but by averaging over the total beam cross section it will simulate the behavior of unpolarized radiation. It is advantageous to focus the exit beam to a smaller spot size to more closely mimic unpolarized light.

3.6 Momentum and Angular Momentum of a Beam

The electromagnetic field carries momentum and angular momentum, which depend on the Poynting vector, the structure of the beam and its polarization. This is easy to understand, if we remember that in Quantum Optics the beam can be represented by a stream of photons. Each of them with a momentum $\hbar\boldsymbol{k}$ and an angular momentum $\hbar$, which is related to the polarisation. If these photons are absorbed by a target (a small piece of matter), momentum and angular momentum are transferred and the target will be pushed away and starts to rotate. However, this is not a quantum effect, and it has been already well known in classical electrodynamics [1.116]. The interaction between field and matter occurs by the Coulomb/Lorentz-force $\boldsymbol{F}$ acting on the electrons [1.117]. This force depends on the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$. For dielectric media with $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ the Coulomb/Lorentz force reads:

$$\boldsymbol{F} = e\,\boldsymbol{E} + \frac{e}{\epsilon_0 c_0^2}\,\boldsymbol{v} \times \boldsymbol{H} \tag{3.35}$$

with e : charge of the electron ($1.6021 \cdot 10^{-19}$ As)
v : velocity of the electron

The Coulomb force is parallel to the electric field, the Lorentz force perpendicular to the magnetic field and the velocity of the electron. The oscillating electric field forces the electron to oscillate in the same direction as the electric field vector. The oscillating magnetic field interacts with the oscillating electron and generates a force in direction $\boldsymbol{k}$ of the propagating field, as shown in Fig. 3.15. Momentum is transferred to the target, but only if the field is partly absorbed or reflected. This force can be used to deflect atoms [1.118] or to balance small glass beads in the gravitational field [1.119, 1.120].

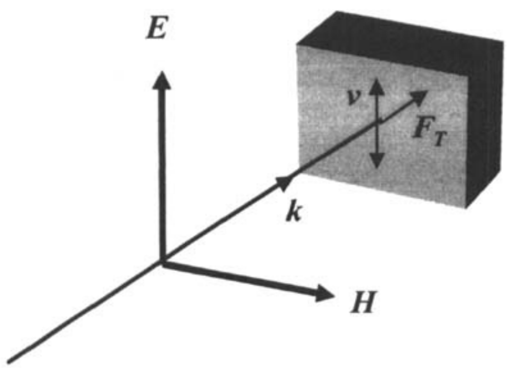

Fig. 3.15 A plane wave is transmitting a dielectric target and partly reflected or absorbed. The interplay of Coulomb and Lorentz force generates a force $\boldsymbol{F}_T$ on the target in the direction $\boldsymbol{k}$ of beam propagation.

An application is the optical tweezer [1.128] a tool for handling microscopic particles. The correct calculation of this force is given in text books [1.117,1.121, 1.130]. A laser pulse with an energy E_p consists of $N = E_p/\hbar\omega$ photons. Each photon carries the momentum $\hbar\boldsymbol{k}$ which results in the total momentum $\boldsymbol{P}$ in direction of propagation:

$$\boldsymbol{P} = \frac{E_p}{c_0}\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \tag{3.36}$$

The momentum flux is the momentum per time and is equivalent to a force. It reads for a continuous field of power $P=dE_p/dt$:

$$\frac{d\boldsymbol{P}}{dt} = \frac{P}{c_0}\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \tag{3.37}$$

This equation holds for plane waves. A more precise expression for arbitrary in non-magnetic media reads [1.121]:

$$\boldsymbol{P} = \frac{1}{c_0^2}\iiint \boldsymbol{E} \times \boldsymbol{H}\ dxdydz = \frac{1}{c_0^2}\iiint \boldsymbol{S}\ dxdydz \tag{3.38}$$

and the momentum flux in z -direction is given by:

$$\frac{d\boldsymbol{P}}{dt} = \frac{1}{c_0}\iint \boldsymbol{S}_z\ dxdy = \frac{P}{c_0} \tag{3.39}$$

This momentum flux can transfer momentum to a target only if power is absorbed. A loss free crystal interacts with the radiation field, demonstrated by the lower speed of light inside the medium. But in average, the Lorentz force is zero and no momentum is transferred, because the velocity of the electron and the magnetic field have a phase shift of 90^0. In order to transfer momentum, absorption is necessary, which results in a slight change the phase shift. If reflection occurs, the incident and the reflected wave generate a standing wave. In this case the electron velocity and the magnetic field are in phase and momentum transfer occurs. The momentum flux absorbed or reflected is equal to the force on the target. Therefore, the force on the target is given by:

$$\boldsymbol{F}_T = \frac{P}{c_0}(1+R-T)\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \tag{3.40}$$

where P is the incident power, R is the reflectivity and T the transmission of the target.

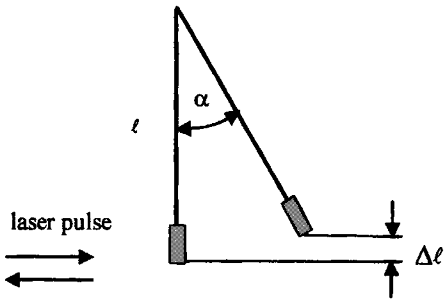

Fig. 3.16 Deflection of a mirror by reflection of a laser pulse.

Example:

A tiny mirror with a mass of $m = 2\times10^{-5}$ kg is suspended by a string with a length of ℓ=0.1 m, as shown in Fig. 3.16. A short laser pulse with energy E_p=1 Joule is completely reflected by the mirror. The momentum of the beam is changed by Δp=$2E_p/c_0$. This momentum has to be absorbed by the mirror $p = mv = 2E_p/c_0$ leading to a recoil. The kinetic energy of the mirror becomes:

$$E_{kin} = \frac{1}{2}mv^2 = \frac{2}{m}\left[\frac{E_p}{c_0}\right]^2$$

The mirror is deflected and lifted by $\Delta\ell$ in the gravitational field, at which point the potential energy equals the initial kinematic energy:

$$E_{pot} = mg\Delta\ell = \frac{2}{m}\left[\frac{E_p}{c_0}\right]^2$$

with g = 9.81 m/s^2. The maximum deflection angle $\Delta\alpha$ is thus given by:

$$\Delta\alpha = \sqrt{\frac{\Delta\ell}{\ell}} = \frac{2}{m}\left|\frac{E_p}{c_0}\right|\frac{1}{\sqrt{g\ell}}$$

For the parameters given above, the resulting angle of 0.3 mrad can be easily observed.

Using the momentum transfer due to reflection, it is even possible to stabilize a small particle in the center of a Gaussian beam [1.121, 1.122], as is shown in Fig.3.17. This can be easily explained by examining the resulting momentum. If the particle is decentred, the conservation of momentum results in a momentum ΔP, which pushes the particle back to the center.

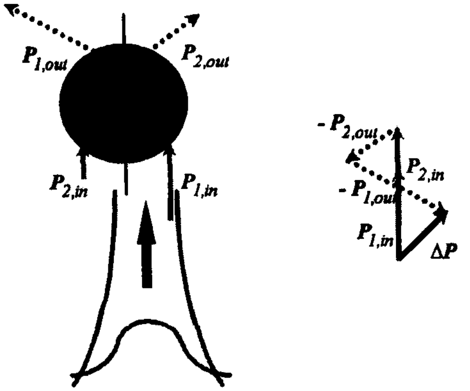

Fig. 3.17 A small particle can be stabilized in the centre of a Gaussian beam (left). If the particle is decentered, the resulting transferred momentum pushes it back to the center (right).

3.6.2 The Poynting Vector of Structured Beams

Before dealing with the angular momentum let us briefly discuss the pointing vector of structured fields. The field of a quasi-plane wave propagating in z-direction, slightly distorted by diffraction, can be described by the slowly varying envelope approximation:

$$E(x,y,z,t) = \boldsymbol{E}_0 \; u(x,y,z) \; \exp[i(\omega t - kz)] \; , \qquad \boldsymbol{E}_0 = \begin{pmatrix} \boldsymbol{E}_T \\ \boldsymbol{E}_z \end{pmatrix} \tag{3.41}$$

where $\boldsymbol{E}_T$ and $\boldsymbol{E}_z$ are the transverse and longitudinal components, respectively. Slowly varying envelope approximation means that

$$\frac{d|u|}{dz} \ll k|u|$$

In the case of elliptically polarized light, the transverse part of the electric field reads:

$$\boldsymbol{E}_z = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 1 \\ \exp[i\Phi] \end{pmatrix} \tag{3.42}$$

The Poynting vector for such a field is again given by Eq.(1.1). It is a bit troublesome, but nonetheless straight forward, to derive the magnetic field by applying Maxwell's equations. Neglecting higher order derivations of u, one obtains [1.123, 1.136] for the Poynting vector:

$$S = \begin{pmatrix} S_T \\ S_z \end{pmatrix}$$

Now a transverse flow of power S_T appears. The width of the beam is increasing when propagating in free space, a result of diffraction and energy flowing radially. The transverse part of the Poynting vector is given by:

$$S_T = \frac{1}{4k}\epsilon_0 c_0 |E_0|^2 \begin{pmatrix} i\left[u^*\frac{\partial u}{\partial x} - u\frac{\partial u^*}{\partial x}\right] & - \sin\Phi\left[u\frac{\partial u^*}{\partial y} - u^*\frac{\partial u}{\partial y}\right] \\ i\left[u^*\frac{\partial u}{\partial y} - u\frac{\partial u^*}{\partial y}\right] & + \sin\Phi\left[u\frac{\partial u^*}{\partial x} - u^*\frac{\partial u}{\partial x}\right] \end{pmatrix} \tag{3.43}$$

The first part is related to the structure of the beam and is called the structural term, the second part depends on the polarisation and is called the polarization term. The z-component of the Poynting vector is the well known plane wave energy flow:

$$S_z = \frac{1}{2}\epsilon_0 c_0 \; |E_0 u|^2 \tag{3.44}$$

Example:

In order to gain a better understanding let us calculate the Poynting vector for a circularly polarized Gaussian beam with $\Phi=\pi/2$. As discussed in detail in Sec. 2.5, the electric field at the distance z from the waist position is given by:

$$E = E_0 \; u = \frac{E_0}{1-iz/z_0} \exp\left[-ik\,\frac{x^2+y^2}{2q}\right]$$

If the transverse Poynting vector is normalized with respect to the z-component, the two contributions are:

$$\frac{S_{T,struct.}}{|S_z|} = \begin{pmatrix} x/R \\ y/R \end{pmatrix} \quad , \quad \frac{S_{T,pol.}}{|S_z|} = \begin{pmatrix} -y/R \\ x/R \end{pmatrix}$$

with the radius of curvature $R= z(z/z_0+z_0/z)$ (see Sec. 2.5.1). The meaning of these two expressions is easier to understand if cylindrical coordinates are introduced:

$$S = \begin{pmatrix} S_\rho \\ S_\Phi \end{pmatrix}$$

Then the two parts of the transverse Poynting vector read:

$$\frac{S_{T,struct.}}{|S_z|} = \frac{\rho}{R}e_\rho \quad , \qquad \frac{S_{T,pol.}}{|S_z|} = \frac{\rho}{R}e_\phi$$

with $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ being the radial and azimuthal unit vectors, respectively. The structural part represents a radial energy flow as discussed in Sec. 2.9, which means that the resulting Poynting vector S is always perpendicular to the phase fronts, as shown in Fig. 2.42. The polarization part has an azimuthal component only. The energy is circulating around the z-axis, a consequence of the circular polarization.

3.6.3 Angular Momentum

The angular momentum L is defined the same way as in classical mechanics as the cross product of the position vector r and the momentum P :

$$L = \frac{1}{c_0^2} \iiint r \times E \times H \; dxdydz = \frac{1}{c_0^2} \iiint r \times S \; dxdydz \tag{3.45}$$

In most cases only the angular momentum flow in propagation direction is of interest. This flow is obtained from (3.45) with $dz/dt = c_0$:

$$J_z = \frac{\partial}{\partial t}L_z = \frac{1}{c_0} \iint (r \times S)_z \; dxdy \tag{3.46}$$

The vector of the cross product is perpendicular to both vectors r and S. If S has only a z-component, the cross product has no z-component and the angular momentum flow is zero. Due to diffraction, S exhibits x,y-components, as was discussed in the previous section. Inserting (3.43) into (3.46) delivers two terms for the angular momentum [1.124,1.125]:

$$J_{z,struct.} = \frac{1}{4\omega}\epsilon_0 c_0 |E_0|^2 \iint i\left[yu\frac{\partial u^*}{\partial x} - xu\frac{\partial u^*}{\partial y}\right] dxdy \; + \; cc \tag{3.47}$$

$$J_{z,pol.} = \frac{1}{4\omega}\epsilon_0 c_0 |E_0|^2 \sin\Phi \iint \left[xu\frac{\partial u^*}{\partial x} + yu\frac{\partial u^*}{\partial y}\right] dxdy \; + \; cc \tag{3.48}$$

where cc denotes the complex conjugate. Note that momentum flux is equivalent to a torque.

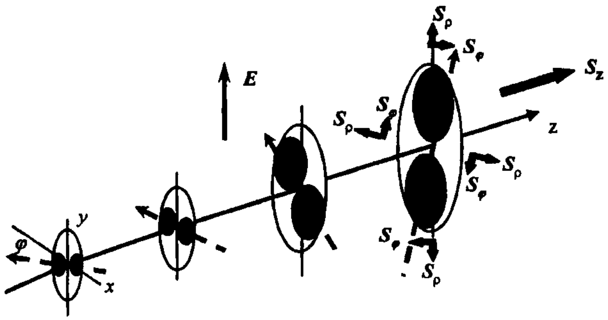

Fig. 3.18 A rotating intensity distribution carries an orbital angular momentum.

The structural (orbital) angular momentum

The structural term is also referred to as the orbital angular momentum. In Sec. 2.6.2, the second intensity moments of abeam were discussed. From the definition of the mixed moments $<w_x\theta_y>$ and $<w_y\theta_x>$, a relation between these twist parameters and the orbital angular momentum is obtained:

$$J_{z,struct.} = \frac{P}{4c_0}\left[<w_x\theta_y> - <w_y\theta_x>\right] \tag{3.49}$$

where P is the total power of the beam. The existence of a twist or orbital angular momentum means that the beam is rotating around the z-axis with constant direction of the electric field vector as depicted in Fig. 3.18. An experimental example of such a field are the Gauss-Laguerre eigenmodes, which are discussed in detail in Sec. 5.2.1. The field and the intensity of such an eigenmode of order ℓ for a rotating field read: (see Eq. (5.6)):

$$\begin{aligned} E(\rho,\phi) &= E_0\, f(\rho)\, \exp[i(\omega t - kz) \pm i\ell\phi] \\ I(\rho) &= I_0\, |f(\rho)|^2 \end{aligned} \tag{3.50}$$

For a standing wave, which can be described as the superposition of two counter-rotating modes with equal amplitude, the corresponding expressions read:

$$\begin{aligned} E(\rho,\phi) &= E_0\, f(\rho)\, \exp[i(\omega t - kz)]\, [\exp(-i\ell\phi) + \exp(+i\ell\phi)] \\ I(\rho) &= 4I_0\, |f(\rho)|^2 \cos^2(\ell\phi) \end{aligned} \tag{3.51}$$

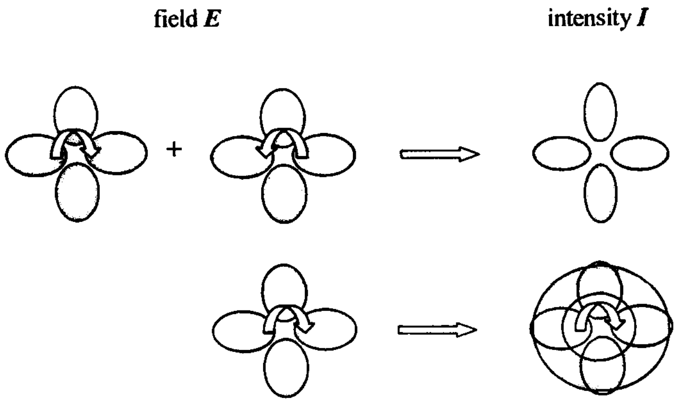

Fig. 3.19 The superposition of two counter-rotating Gaus-Laguerre modes with mode order p=0 and ℓ=2 generate an intensity distribution with an azimuthal structure (top). A single mode exhibits a ring-like structure (bottom).

The standing wave produces an azimuthal intensity pattern, whereas the rotating field intensity (3.50) exhibits a ring-like structure as shown in Fig. 3.19. Therefore, only the ring mode has an angular orbital momentum. Equations (3.50) and (3.47) deliver for the flux:

$$J_{z,struct.} = \pm \ell \frac{P}{\omega} \tag{3.52}$$

which means that each photon of the beam has an orbital angular momentum of $\ell\hbar$.

Generation of modes with orbital angular momentum (twist)
Gauss-Laguerre modes can be generated in laser systems of perfect circular symmetry, which is difficult to realize. It is much easier to transform Gauss-Hermite modes into Gauss-Laguerre modes by using astigmatic optical systems [1.125-1.127]. An experimental setup is shown in Fig. 3.20. At $z = 0$ is the waist of a Gauss-Hermite mode with order $(m,0)$. This mode is transformed by a system of three cylinder lenses with focal lengths $f/2$, f and $f/2$ equally spaced by a distance $f/2$. The lens system is rotated by 45^0 with respect to the mode axis and the focal length is equal to the Rayleigh range z_0. If the Rayleigh ranges are different in the x- and the y- direction, they have to be equalized by a suitable bifocal lens system [1.134]. In the distance f/2 behind the third lens, the waist of a Gauss-Laguerre mode will appear. Let us first evaluate the variance matrices of the beam.

If the beam has its waist in the plane z=0, the second order moment or variance matrix reads in the x_1,y_1 reference system:

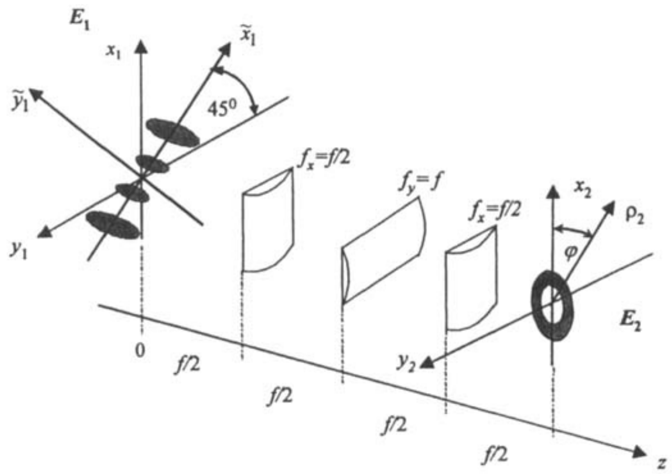

Fig. 3.20 Transformation of a Gauss-Hermite mode with order (m,0) into a Gauss-Laguerre mode of order p=0 and ℓ=2.

$$\boldsymbol{P}_1 = \begin{pmatrix} w^2_{x,1} & 0 & 0 & 0 \\ 0 & w^2_{y,1} & 0 & 0 \\ 0 & 0 & \theta^2_{x,1} & 0 \\ 0 & 0 & 0 & \theta^2_{y,1} \end{pmatrix} \tag{3.53}$$

It is a simple astigmatic beam with different waist sizes and divergence angles in the x_1-and the y_1-directions. For a Gauss-Hermite mode of order (m,0) the following relations hold:

$$\begin{aligned} w_{x,1} &= w_0\sqrt{2m+1} \ , \qquad \theta_{x,1} = \theta_0\sqrt{2m+1} \\ w_{y,1} &= w_0 \ , \qquad \theta_{y,1} = \theta_0 = \frac{\lambda}{\pi w_0} \ , \qquad z_0 = kw_0^2/2 \end{aligned} \tag{3.54}$$

The variance matrix $\boldsymbol{P}_2$ in the x_2,y_2 plane is obtained by applying Eq. (2.126):

$$\boldsymbol{P}_2 = \boldsymbol{M}\boldsymbol{P}_1\boldsymbol{M}^T$$

The matrix M is the four-dimensional ray transfer matrix of the complete system and includes all distances and elements as well as the 45° rotation:

$$M = R^{-}(45^{\circ})M_{lens}(f/2)M_{distance}(f/2)M_{lens}(f)M_{distance}(f/2)M_{lens}(f/2)M_{distance}(f/2)R^{+}(45^{\circ})$$

The different matrices are compiled in Sec. 1.2.4. Instead of going straight ahead with the matrix multiplication, we use an easier method to obtain the resulting matrix. First we calculate the 2x2 matrices of the system in the x- and in the y-direction without taking the rotation into account:

$$M_x = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & f \\ -1/f & 0 \end{pmatrix}$$

$$M_y = \begin{pmatrix} 1 & f/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/f & 1 \end{pmatrix}\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/f & 1 \end{pmatrix}\begin{pmatrix} 1 & f/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Combination of both matrices results in the four-dimensional ray transfer matrix of the non-rotated system:

$$M_{non-rotated} = \begin{pmatrix} 0 & 0 & f & 0 \\ 0 & -1 & 0 & 0 \\ -1/f & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

By using the rotation matrix (1.68):

$$R^{\pm}(45^{\circ}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \pm 1 & 0 & 0 \\ \mp 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \pm 1 \\ 0 & 0 & \mp 1 & 1 \end{pmatrix}$$

the complete matrix of the system is finally obtained:

$$M = R^{-}M_{non\text{-}rotated}R^{+} = \frac{1}{2}\begin{pmatrix} -1 & 1 & f & f \\ 1 & -1 & f & f \\ -1/f & -1/f & -1 & 1 \\ -1/f & -1/f & 1 & -1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.55}$$

Equations (3.53) and (3.55) deliver the new variance matrix P_2 of the field in the x_2,y_2-plane:

$$P_2 = \begin{pmatrix} w^2_{x,1}+w^2_{y,1} & 0 & 0 & \dfrac{(w^2_{x,1}-w^2_{y,1})}{z_0} \\ 0 & w^2_{x,1}+w^2_{y,1} & \dfrac{(w^2_{x,1}-w^2_{y,1})}{z_0} & 0 \\ 0 & -\dfrac{(w^2_{x,1}-w^2_{y,1})}{z_0} & \theta^2_{x,1}+\theta^2_{y,1} & 0 \\ \dfrac{(w^2_{x,1}-w^2_{y,1})}{z_0} & 0 & 0 & \theta^2_{x,1}+\theta^2_{y,1} \end{pmatrix} \tag{3.56}$$

This is a field of rotational symmetry with a waist in this plane. Waist radii and angles of divergence, respectively, are equal in the two directions:

$$w^2_{x,2} = w^2_{y,2} = \frac{w^2_{x,1}+w^2_{y,1}}{2} \quad , \qquad \theta^2_{x,2} = \theta^2_{y,2} = \frac{\theta^2_{x,1}+\theta^2_{y,1}}{2}$$

The astigmatic beam has been transformed into a beam of rotational symmetry. However, it is not a stigmatic beam since new non-diagonal term appear in the variance matrix. The can be identified as a twist:

$$<w_x\theta_y> = -<w_y\theta_x> = \frac{w^2_{x,1}-w^2_{y,1}}{2z_0}$$

or as an orbital angular momentum:

$$J_{z,struct.} = \frac{P}{4c_0}\,\frac{w^2_{x,1}-w^2_{y,1}}{z_0} \tag{3.57}$$

This equation holds for any astigmatic beam with variance matric P_1, which is transformed into a beam of rotational symmetry by a suitable optical system. By using the parameters (3.54) of the Gauss-Hermite modes of order (m,0), the angular momentum reads:

$$J_{z,struct.} = m \frac{P}{\omega} \tag{3.58}$$

n order to calculate the field distribution in the x_2,y_2-plane, the two-dimensional Collins-ntegral (Sec.2.3.2) has to be solved using the ABCD-parameters of the system matrix (3.55). In the x_1,y_1-plane, the field of a Gauss-Hermite mode of order (m,2) is given by:

$$E_1(x_1,y_1) = E_0 \; H_m(\sqrt{2}x_1/w_0) \; \exp\left[-\frac{x_1^2+y_1^2}{w_0^2}\right]$$

After propagation through the optical system, the output field in the x_2,y_2-plane reads [1.125-1.127]:

$$E_2(x_2,y_2) = E_0 \; i^m \sqrt{-i} \sqrt{2^{m-1}} \left(\frac{\sqrt{2}\rho_2}{w_0}\right)^m \exp\left[-\frac{\rho_2^2}{w_0^2}\right] \exp[im\phi]$$

with $\rho_2^2=x_2^2+y_2^2$. An experimental example is shown in Fig. 3.21. Other combinations of spherical and cylindrical lenses, as proposed by several authors [1.128-1.130], can also be used to transform astigmatic beams into beams with rotational symmetry. This is of particular interest for fiber coupling of diode lasers.

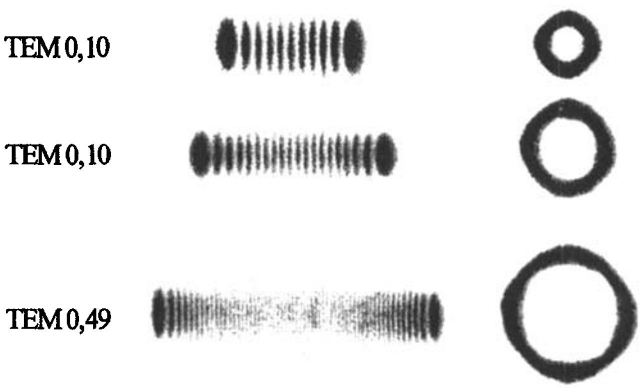

Fig. 3.21 Transformation of Gauss-Hermite modes into Gauss-Laguerre modes using the optical system depicted in the previous figure. A diode-pumped 1064nm Nd:YAG laser was used to generate the Gauss-Hermite modes. The recorded intensity distributions at the input plane (left) and the output plane (right) are shown [1.125,1.127].

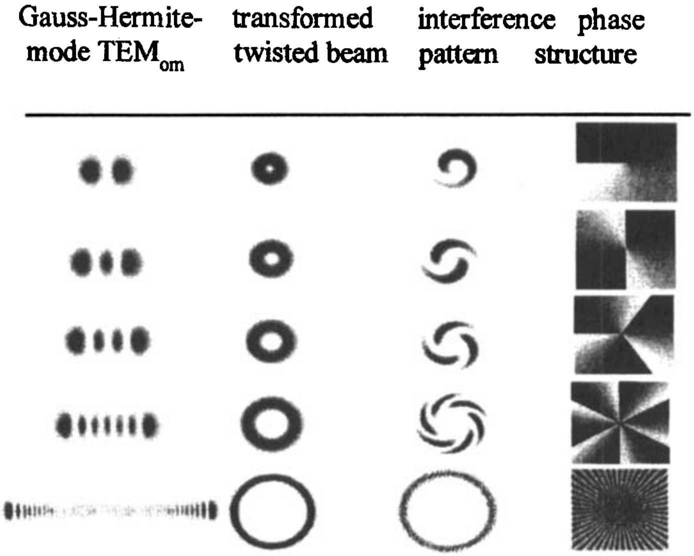

Fig. 3.22 Interference pattern of a plane wave with a twisted Gauss-Laguerre mode and its phase structure [1.125].

Example:
A Nd:YAG laser emits a Gauss-Hermite mode of order (m,n)=(10,0) with an output power of 10W at 1064nm. After transformation into a Gauss-Laguerre mode, the beam, according to (3.58), exhibits torque of $J_{z,struct.} = 5.4 \cdot 10^{-15}$ Nm. This is certainly a low value, but it is sufficient to make microscopic particles rotate.

The polarization angular momentum
The angular momentum flux due to the polarization term given by Eq. (3.48) can be written as:

$$J_{z,pol.} = \frac{\epsilon_0 c_0}{4\omega}|E_0|^2 \sin\Phi \iint \left| x\frac{\partial |u|^2}{\partial x} + y\frac{\partial |u|^2}{\partial y} \right| dxdy$$

Integration of this equation yields a very simple result:

$$J_{z,pol.} = -\sin\Phi \; \frac{P}{\omega}$$

This means, that for counterclockwise or clockwise circular polarization ($\Phi=\mp\pi$), each photon carries an angular momentum of $\pm\hbar$ [1.132]. The relation between circular polarization and spin of the photon was proven experimentally by Beth in 1936 [1.134]. The set-up he used is depicted in Fig. 3.23. The radiation of a conventional light source was collimated, left circularly polarized with a combination of a polarizer and a retarder plate and chopped with a shutter. The light was incident on a $\lambda/2$–plate, which was suspended on a thin quartz fibre. Each photon that goes through the plate, changes its angular momentum by $2\hbar$. This angular momentum is transferred to the plate, which results in an oscillation with the shutter frequency. To enhance the sensitivity the shutter frequency is equal the resonance frequency of the suspended plate.

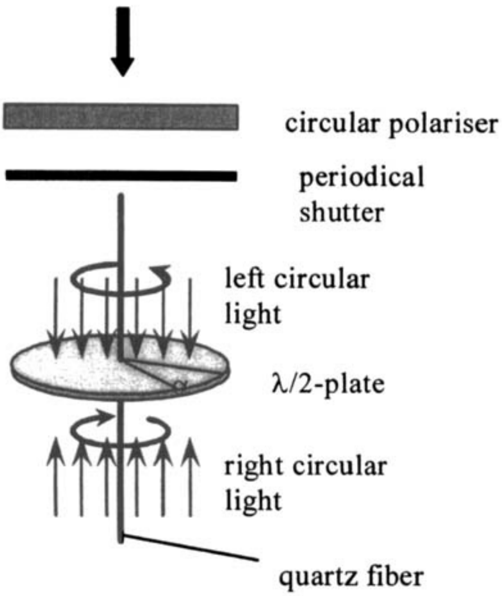

Fig. 3.23 Set-up of Beth's experiment to measure the intrinsic angular momentum of circularly polarized light [1.134].

Part II

Basic Properties of Optical Resonators