Chapter 1

Geometrical Optics

1.1 General Aspects

Light is an electromagnetic wave which differs from the waves known in radiofrequency technology only by a higher frequency ν and a shorter wavelength λ [1.1-1.5]. In both cases, the electromagnetic field is characterized by (Fig.1.1):

the electric field $\boldsymbol{E}$	...	(V/m)
the magnetic field $\boldsymbol{H}$	....	(A/m)
the wave vector $\boldsymbol{k}=2\pi/\lambda\, \boldsymbol{e}$	.	(1/m)

In homogeneous, isotropic, and unconfined media, all three vectors are perpendicular to one another and the wave vector $\boldsymbol{k}$ points into the direction of propagation $\boldsymbol{e}$ of the wave. The energy flow is characterized by the Poynting vector $\boldsymbol{S}=\boldsymbol{E}x\boldsymbol{H}$. The physical property that humans detect, with the eye or with a light-sensitive detector, is always the time averaged intensity:

$$I = \langle S \rangle \tag{1.1}$$

The power content of the electromagnetic wave is obtained by integrating the intensity over the area perpendicular to the propagation direction:

$$P = \int_A \langle S \rangle \, dA = \int_A \langle E \; x \; H \rangle \, dA \tag{1.2}$$

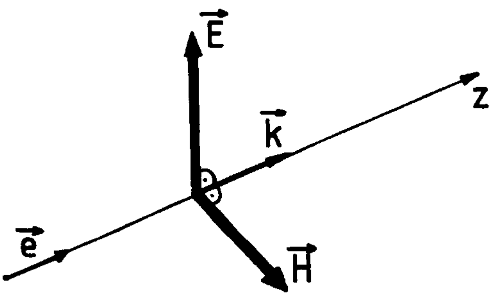


Fig. 1.1 The electromagnetic wave is completely determined by the field vectors $\boldsymbol{E}$, $\boldsymbol{H}$, and the wave vector $\boldsymbol{k}$ pointing into the direction of propagation.

Note that this expression contains both the electric and magnetic field. Within the scope of this book we are only interested in non-magnetic media with low absorption losses. In this case the magnetic field $\boldsymbol{H}$ is linked to the electric field vector $\boldsymbol{E}$ via:

$$\boldsymbol{H} = n\, c_0\, \epsilon_0\, \lfloor e \times \boldsymbol{E} \rfloor$$

The intensity then reads:

$$I = \frac{1}{2}\, n\, c_0\, \epsilon_0\, |E|^2 \tag{1.3}$$

with:	$c_o = 3\text{x}10^8$ m/s	speed of light in vacuum
	$\epsilon_o = 8.85\text{x}10^{-12}$ As/(Vm) ...	permittivity in vacuum
	n : dimensionless number =	refractive index of the medium
	$E=2E_o/(1+n)$	field inside medium, where E_o is the field in vacuum

Thus it is sufficient in most cases to use only the electric field to describe the properties of the electromagnetic wave. It can easily be shown that the magnetic and the electric field both contribute the same amount to the total power of (1.2).

In the following we consider only the electric field $\boldsymbol{E}$ which depends on the spatial coordinates x,y,z and is assumed to show a purely periodic oscillation with a frequency ν. It is common to chose the z-axis as the main direction of propagation. The general electromagnetic wave can then be expressed as:

$$\boldsymbol{E} = \boldsymbol{E_0}(x,y,z)\, \cos(\omega t - kz) \tag{1.4}$$

with $\omega = 2\pi\nu$. Since we consider only linear media, the frequency ν remains constant as the electromagnetic wave propagates through areas of different indices of refraction n. Linearity means that the index of refraction does not depend on the intensity I. This is in contrast to the propagation of high intensity beams through nonlinear materials such as KTP or BBO, which exhibit a change of the refractive index with the intensity, resulting in frequency conversion. The wavelength λ and the speed of light c, however, do change in linear media:

$$\lambda = \lambda_0/n \qquad c = c_0/n \qquad \nu = c/\lambda = c_0/\lambda_0 \tag{1.5}$$

with λ_0, c_0 being the wavelength and the speed of light both in vacuum.

Examples:

1) On a bright summer day in California the intensity of the sun light is about 500 W/m^2 for normal incidence. By using (1.1) we get an electric field of 614 V/m.

2) For a 1kW CO_2 laser with a beam diameter of 10mm the intensity is 12.7 MW/m^2. The corresponding electric field is 98,935 V/m.

1.2 Ray Transfer Matrices

In the following we will discuss the propagation of light geometrically by analyzing the propagation of rays [1.10]. In this geometrical optics approximation the spatial structure of the electromagnetic wave as well as diffraction effects caused by apertures or edges encountered by the wave are not taken into account. This approximate description of light propagation can be applied as long as one deals with a light beam whose characteristic parameter N, called the Fresnel number, is much greater than one:

$$N = \frac{a^2}{\lambda L} \gg 1 \tag{1.6}$$

with a: beam radius, λ: wavelength L: distance in propagation direction

The meaning of the Fresnel number will be discussed in later sections (see, for instance, Sec. 2.2). Light can only be described in terms of geometrical optics if we have a beam that is not too long or too thin. If the Fresnel number is close to one or lower, the beam propagation has to be calculated using diffraction theory. Unfortunately, there is no well defined Fresnel number which separates geometrical optics from diffraction theory. For Fresnel numbers N greater than twenty, however, geometrical optics can definitely be applied.

Example:
A light beam in the green spectral range (λ=500nm) with a diameter of $2a$=5mm can be described geometrically over a distance of about 1m (N=12.5).

1.2.1 One-Dimensional Optical Systems

In the geometrical approximation, light rays propagate along straight lines (in free space) and experience a declination if they pass through an optical element like a lens. If the ray starts in a plane perpendicular to the z-axis, it is completely determined by its starting points x,y and the inclination angles α in x-direction and β in y-direction. The z-axis is generally chosen such that it coincides with the optical axis. The optical axis is defined by the center of symmetry of the first optical element and is perpendicular to this elements' front surface. Thus, a ray can be expressed mathematically as a vector:

$$V = \begin{pmatrix} x \\ y \\ \alpha \\ \beta \end{pmatrix} \tag{1.7}$$

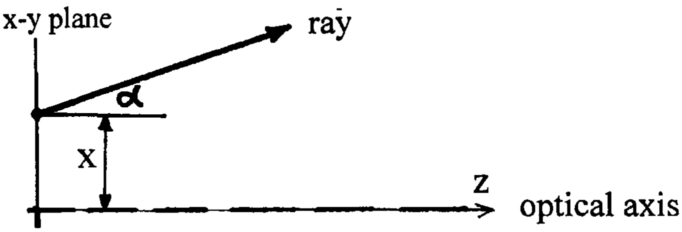


Fig. 1.2 In geometrical optics light is described by rays propagating along straight lines in free space.

If all optical elements in the propagation direction exhibit rotational symmetry (for example a spherical lens) the cartesian coordinate system can always be chosen such that the ray is completely determined by its starting point x and its inclination angle in x-direction α. This is due to the fact that the ray propagates only in the x-z-plane. In the following we will restrict our discussion to these one-dimensional optical systems. Ray propagation in two-dimensional optical systems such as cylinder lenses will be treated in Sec. 1.2.4.

The problem we have to solve is: How does this vector change as the ray propagates through an optical system? As far as the propagation in a medium with index of refraction n is concerned, the calculation of this problem is quite straightforward (Fig.1.3a). For the propagation over a distance L a ray starting at point x_1 under a small angle α_1 will end up at point x_2 with an inclination angle α_2, with:

$$x_2 = x_1 + L \cdot \alpha_1 \tag{1.8}$$

$$\alpha_2 = \alpha_1 \tag{1.9}$$

This can be written in form of a matrix equation:

$$\begin{pmatrix} x_2 \\ \alpha_2 \end{pmatrix} = M_{FS} \begin{pmatrix} x_1 \\ \alpha_1 \end{pmatrix} \qquad M_{FS} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \tag{1.10}$$

For every linear optical element such as a lens or a mirror one can find a matrix M that mathematically describes the transformation of a ray while going through this element [1.6,1.10]. This matrix M is called the ray transfer matrix. In order to find such a constant matrix - constant means that the matrix elements do not depend on the ray parameters - the approximation of small angles α has to be made (as we did in (1.8)). The ray transfer matrix theory is only applicable to the analysis of an optical system if the following relation holds for the maximum angle:

$$\sin\alpha \approx \tan\alpha \approx \alpha \tag{1.11}$$

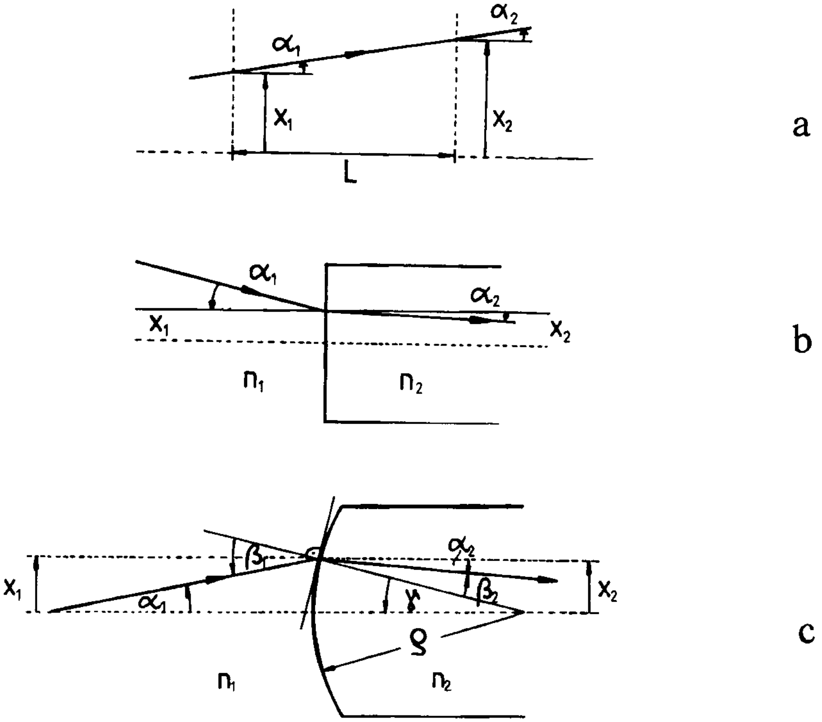


Fig. 1.3 Propagation of light rays. **a)** free space propagation over a distance L, **b)** refraction at the planar boundary surface between two media with different index of refraction, **c)** refraction at a spherical surface.

As a rule of thumb the maximum angle should not exceed 15° (α=0.262rad, sinα=0.259, tanα=0.268). This is not a serious limitation as far as optical resonators are concerned since the angle under which the light propagates inside a resonator is rarely greater than a couple of degrees. This limitation to small angles is called the paraxial approximation. It generates a linear transformation between ray vectors. If a light beam passes from one medium with index of refraction n_1 to another with index of refraction n_2, the rays are refracted at the interface (Fig.1.3b). In the paraxial approximation the refraction law, also known as Snell's law, reads:

$$\frac{\alpha_1}{\alpha_2} \cong \frac{\sin\alpha_1}{\sin\alpha_2} = \frac{n_2}{n_1} \qquad (1.12)$$

Thus, the ray transfer matrix M_R for refraction at a planar interface is fully determined since the position x of the ray at the surface remains unchanged:

$$M_R = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \tag{1.13}$$

A third important optical transition is the refraction at a spherical surface as depicted in Fig.1.3c. The surface geometry is characterized by the radius of curvature ρ. For a convex surface which means that the center of curvature is to the right of the surface and the ray is arriving from the left (as shown), the radius of curvature is positive. A negative radius of curvature defines a concave surface. Again the ray position at the surface remains fixed and Snell's law (1.12) now holds for the angles β_l and β_2. The angle α_2 can be calculated by using the geometrical relations $\gamma = x_1/\rho$, $\alpha_1 = \beta_1 - \gamma$, and $\alpha_2 = \beta_2 - \gamma$. The final result reads:

$$x_2 = x_1 \tag{1.14}$$

$$\alpha_2 = \frac{n_1 - n_2}{n_2 \rho} x_1 + \frac{n_1}{n_2} \alpha_1 \tag{1.15}$$

The ray transfer matrix for refraction at a spherical interface is given by:

$$M_{RS}(\rho) = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 \rho} & \frac{n_1}{n_2} \end{pmatrix} \tag{1.16}$$

As to be expected, this ray transfer matrix is transformed into matrix (1.13) if we set the radius of curvature ρ of the surface to infinity.

So far we have found two fundamental ray transfer matrices to describe ray propagation in free space $M_{FS}(L)$ and refraction at a spherical dielectric interface $M_{RS}(\rho)$. The knowledge of these two matrices is sufficient to describe arbitrary optical systems since ray transfer matrices for all optical elements can be derived by using these two fundamental matrices. Before we discuss this in more detail the reader should memorize the following rules to avoid confusion in later sections:

1) light rays always propagate from left to right.
2) convex dielectric surfaces have a positive radius of curvature, concave dielectric surfaces a negative radius of curvature. However, this is reversed for mirrors!
3) Angle orientations are defined mathematically. This means that in Fig.1.3c α_l is positive and α_2 is negative.

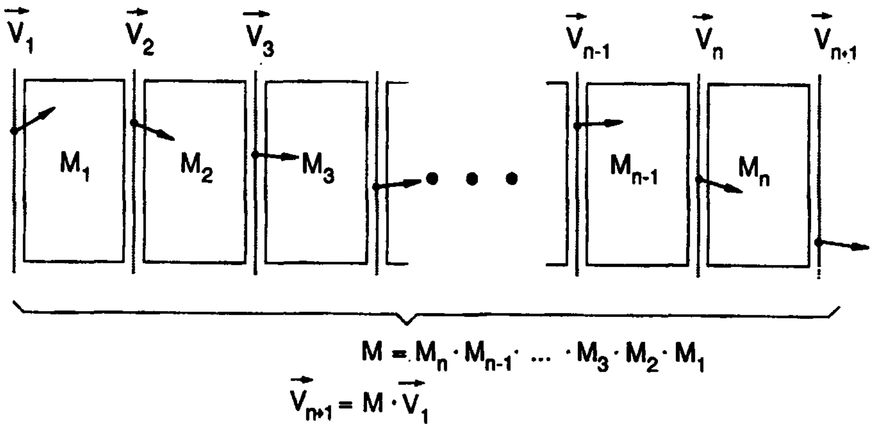


Fig. 1.4 Propagation of a light ray through a series of optical elements. The individual ray matrices can be combined to a resulting matrix *M*.

If the ray propagates through several optical elements their ray transfer matrices may be combined into a single one. This procedure is visualized in Fig. 1.4. The ray starts on the left plane having a ray vector v_1. This ray vector is transformed into the ray vector v_2 by the first optical element, the second element generates the ray vector v_3 on the third plane and so forth. If we have n optical elements we get n equations:

$$V_2 = M_1 \, V_1$$

$$V_3 = M_2 \, V_2$$

$$\vdots$$

$$V_{n+1} = M_n \, V_n$$

and therefore:

$$V_{n+1} = M_n M_{n-1} M_{n-2} \, \dots \, M_2 M_1 \, V_1 = M \, V_1 \quad . \tag{1.17}$$

Thus, the resulting ray transfer matrix *M* is obtained by multiplying all individual matrices in the opposite order (i.e. right to left) of the ray propagation. In other words, the first ray matrix 'seen' by the ray is on the right side of the matrix product, the last one on the left side. Now we have a powerful tool in hand to determine the ray transfer matrices of more complicated optical systems.

Examples (see also Fig.1.5)

1) Thin Lens

In general a spherical lens is determined by its index of refraction n_2, the thickness L and the curvatures of the front and the back surface. In the thin lens approximation, any change in ray position or angle inside the medium is neglected which means that we do not propagate the ray between the two surfaces. Thus the ray transfer matrix M_{TL} of a thin lens is the product of two transfer matrices for refraction at a spherical interface (1.16). If ρ_1, ρ_2 denote the radii of curvature of the front and back surface and the lens is surrounded by a medium with index of refraction n_1, the resulting ray transfer matrix reads:

$$M_{TL} = M_{RS}(\rho_2)\, M_{RS}(\rho_1)$$

$$= \begin{pmatrix} 1 & 0 \\ \dfrac{n_2 - n_1}{n_1 \rho_2} & \dfrac{n_2}{n_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \dfrac{n_1 - n_2}{n_2 \rho_1} & \dfrac{n_1}{n_2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix} \tag{1.18}$$

with a focal length f given by:

$$\frac{1}{f} = \frac{n_2 - n_1}{n_1} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \tag{1.19}$$

Note that the curvature ρ is positive for convex surfaces (center of curvature to the right of the interface) and negative for concave surfaces.

2) Plane Dielectric Slab

We consider a slab of length L with parallel surfaces and an index of refraction n_2. The ray transfer matrix is found by combining the two matrices M_R for refraction at the surfaces with the free space propagation matrix M_{FS}:

$$M_P = \begin{pmatrix} 1 & 0 \\ 0 & n_2/n_1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix} = \begin{pmatrix} 1 & n_1 L/n_2 \\ 0 & 1 \end{pmatrix} \tag{1.20}$$

Interestingly, this is exactly the matrix for free space propagation over an effective distance $n_1 L/n_2$. For a glass plate in air, for instance, this effective distance is smaller than the actual physical thickness L of the plate. This means that objects will appear closer to the eye if we look through the plate (this is, of course, only true for near objects which are viewed under an angle).

3) Spherical Mirror

A spherical mirror with radius of curvature ρ provides the same ray transformation properties as a lens except for the fact that the beam propagation direction is reversed. As shall be discussed later in more detail, beam reversal needs not to be incorporated into the ray matrices if the coordinate system is mirrored together with the beam. In other words, we always ride with the ray. Within this approach the ray transfer matrix of a mirror is identical to that of a thin lens (1.18) with focal length $f=\rho/2$:

$$M_M = \begin{pmatrix} 1 & 0 \\ -\frac{2}{\rho} & 1 \end{pmatrix} \tag{1.21}$$

The radius of curvature ρ is positive for a convex mirror surface which means the mirror opens to the left, towards the incoming ray (as shown in Fig. 1.5). It is most important for the reader to realize that this sign convention is contrary to the one used for dielectric interfaces. Keeping this in mind will save you a lot of calculation time.

A collection of commonly used ray transfer matrices is presented in Fig.1.5. It should be noted that from a mathematical point of view, ray transfer matrices can only be defined for optical elements that have parabolic index profiles (like the thermal lens in Fig.1.5) or parabolic surfaces (like an aberration free lens). Fortunately, near its center of symmetry a parabolic surface hardly differs from a spherical surface. Since only paraxial rays are considered we are limited to the central area of the optical elements. Thus, from a practical point of view, the ray transfer matrices can also be applied to optics with spherical, elliptical, or hyperbolic surfaces and index profiles without introducing noticeable errors.

The paraxial approximation also implies that the optical systems considered exhibit perfect imaging properties. For perfectly aligned optics a point will be imaged to a point because aberration is not incorporated into ray transfer matrix theory. For analyzing aberrations in optical systems numerical ray tracing algorithms are required (i.e. the commercially available optics design software like CODE V, ZEMAX, or OSLO). The impetus for computer solutions is that when the paraxial approximation (1.11) is no longer applicable, the time-efficient solution of the trigonometric functions for a great number of rays (on the order of 1,000) requires a numerical approach.

Astigmatism, however, which is a general feature of two-dimensional optics (cylindrical lenses) and is also induced in one-dimensional optical elements by tilt (i.e. a Brewster plate), can be analyzed by using 4x4 ray matrices (see Sec. 1.2.4).

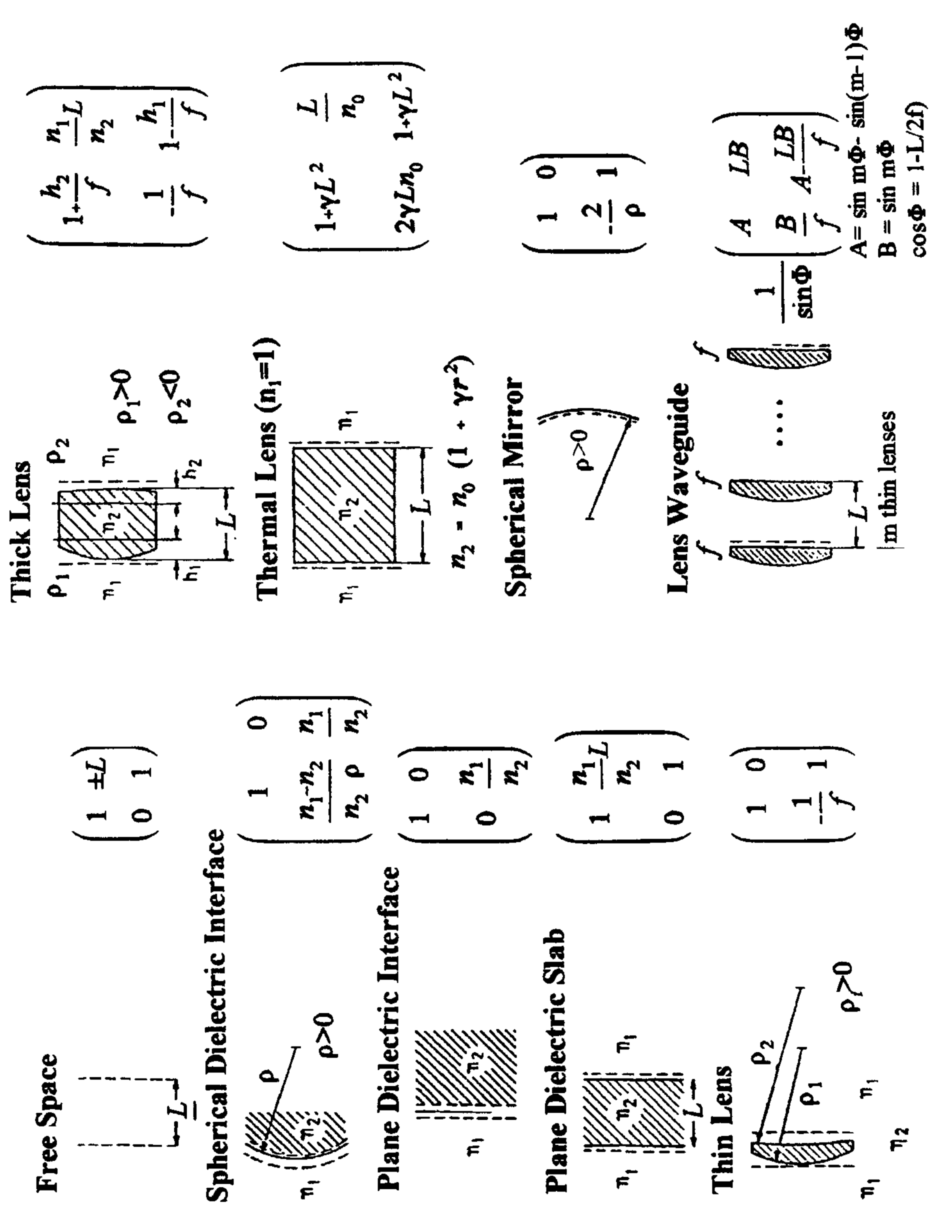


Fig. 1.5 Common optical elements and their ray transfer matrices. See (1.31) for the focal length and the principal planes of the thick lens.

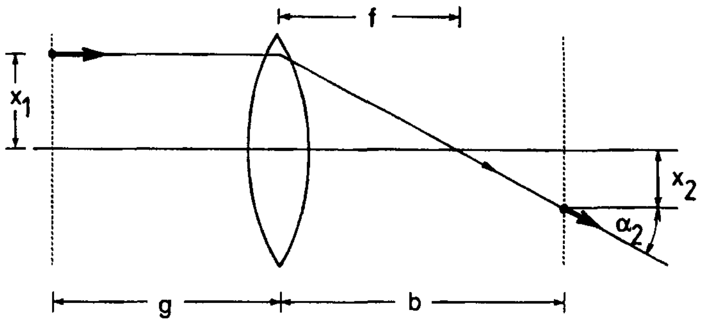


Fig. 1.6 Imaging with a biconvex focusing lens.

In order to get more experience and confidence in the utilization of ray transfer matrices the reader should first go through the two examples presented below. After this we will proceed with a more generalized presentation of ray matrix properties.

Imaging with a focusing lens

We are looking for the imaging condition for a focusing lens. The goal is to find the relation between the object distance g, the image distance b and the focal length f. Using (1.17) we first determine the resulting ray transfer matrix:

$$M_{IM} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{b}{f} & g+b-\frac{gb}{f} \\ -\frac{1}{f} & 1-\frac{g}{f} \end{pmatrix} \tag{1.22}$$

A ray starting on the left plane with parameters x_1, α_1 will intersect the image plane having the parameters:

$$x_2 = (1-\frac{b}{f})\, x_1 + (g+b-\frac{bg}{f})\, \alpha_1 \tag{1.23}$$

$$\alpha_2 = -\frac{1}{f}\, x_1 + (1-\frac{g}{f})\, \alpha_1 \tag{1.24}$$

This result can be easily verified geometrically by using a ray that propagates parallel to the optical axis ($\alpha_l=0$), as depicted in Fig. 1.6, and a second ray emerging from the same point but going through the center of the lens ($\alpha_l= -x_l/g$). Now we have to find a condition so that every point x_l is imaged onto one point x_2. This means that all rays starting at x_l have to end up at x_2 regardless of their angle α_1. A look at (1.23) indicates that this can only be accomplished if

$$g + b - \frac{bg}{f} = 0 \quad \longrightarrow \quad \frac{1}{b} + \frac{1}{g} = \frac{1}{f} \tag{1.25}$$

This is easily recognized as the imaging condition.

Simulation of a Spherical Mirror

While setting up an important optical experiment a student realizes that he does not have the spherical mirror needed. All he can find is a plane mirror. Is there a way to simulate a spherical mirror with a certain radius of curvature ρ just by placing a focusing lens at a distance L in front of the plane mirror (Fig. 1.7) ?

Here we are again confronted with the beam reversal problem and we should use this example to understand how the reflection of rays is dealt with. The basic idea is that we always ride with the ray. We can then mirror the optical system at the mirror plane and propagate through the whole system only in the forward direction. This method is shown in Fig. 1.7. The lens-mirror assembly with spacing L can be replaced by two lenses separated by twice the distance. In both presentations we get the same ray vector at the output plane. The ray transfer matrix for a round trip starting at the lens now reads:

$$M_{LM} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & 2L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1-2L/f & 2L \\ -2/f\,(1-L/f) & 1-2L/f \end{pmatrix}$$

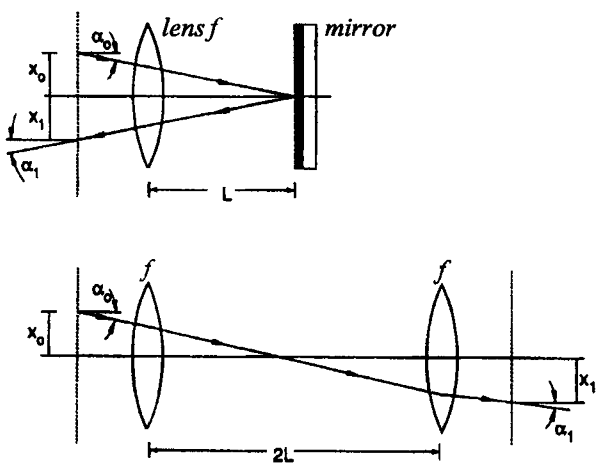


Fig. 1.7 Reflection at mirrors is dealt with by mirror imaging the optical system at the mirror plane. Thus, the lens-mirror combination is equivalent to the dual lens system.

A comparison with the matrix of a spherical mirror (1.21) indicates that we have to glue the lens right onto the plane mirror ($L=0$) to simulate a mirror curvature of $\rho=f$.

1.2.2 Matrix Elements and Liouville's Theorem

In order to get a better understanding of ray transfer matrices, we will now discuss the meaning of the matrix elements as far as ray transformation is concerned. This approach is very helpful for the initial layout of an optical system since each matrix element represents a characteristic property of the beam transformation. The most efficient way to visualize the meaning of the matrix elements is to set them to zero and analyze the propagation changes generated. In order to keep the discussion as general as possible we use the following form of the ray transfer matrix $\boldsymbol{M}$:

$$\boldsymbol{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

or

$$\begin{aligned} x_2 &= A\,x_1 + B\,\alpha_1 \\ \alpha_2 &= C\,x_1 + D\,\alpha_1 \end{aligned} \qquad (1.26)$$

<u>a) A=0</u>

The relationship between input and output ray parameters then reads:

$$x_2 = B\,\alpha_1 \qquad (1.27)$$

$$\alpha_2 = C\,x_1 + D\,\alpha_1 \qquad (1.28)$$

The position x_2 of the ray does not depend on the initial position x_1. All rays going through the system under an angle α_1 end up at the same coordinate x_2. This means that a parallel beam will be focused (Fig.1.9a).

Example: At which distance L are all parallel incoming rays focused by a lens with focal length f? By multiplying the ray transfer matrices for a thin lens and for free space propagation, the matrix elements A,B are found to be:

$$A = 1-\frac{L}{f} \qquad B = L$$

Matrix element A goes to zero if we go into the focal plane of the lens ($L=f$). In this case a parallel beam (see Fig. 1.9a), hitting the lens at an angle α_1 will be **focused** into the point $x_2=f\alpha_1$. This means that we obtain information on the angular ray distribution by looking at the intensity pattern in the focal plane of a focusing lens. We actually see the Fourier transform of the incoming beam (see Sec. 2.4).

b) B=0

The output ray parameters are now given by:

$$x_2 = A\, x_1 \qquad (1.29)$$

$$\alpha_2 = C\, x_1 + D\, \alpha_1 \qquad (1.30)$$

All rays starting at point x_1 under arbitrary angles will be reunified in point x_2. Thus setting B equal to zero creates an **imaging system** (Fig.1.9b), with the lateral magnification A.

Example: We have already derived the imaging condition for a thin lens. Let us now do the same for a thick lens (Fig. 1.8) since it is an important optical element in resonator physics. All laser media can be described in a first approach as a thick lens due to a refractive power induced by the pumping process. We utilize the ray matrix of a thick lens presented in Fig. 1.5 and first calculate the ray transfer matrix for the propagation from the input to the output plane as shown in Fig. 1.8. The resulting ray transfer matrix reads:

$$M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+h_2/f & Ln_1/n_2 \\ -1/f & 1-h_1/f \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \dfrac{h_2-b}{f} & L\dfrac{n_1}{n_2} + b(1 - \dfrac{h_1}{f}) + g(1 + \dfrac{h_2-b}{f}) \\ -\dfrac{1}{f} & 1 - \dfrac{h_1+g}{f} \end{pmatrix} . \qquad (1.31)$$

with $$\frac{1}{f} = \frac{n_2-n_1}{n_1}\left[\frac{1}{\rho_1} - \frac{1}{\rho_2} + \frac{n_2-n_1}{n_2}\frac{L}{\rho_1\rho_2}\right] ; \; h_i = \frac{f\,L}{\rho_j}\frac{n_1-n_2}{n_2} \quad i, j=1,2;\; i\neq j$$

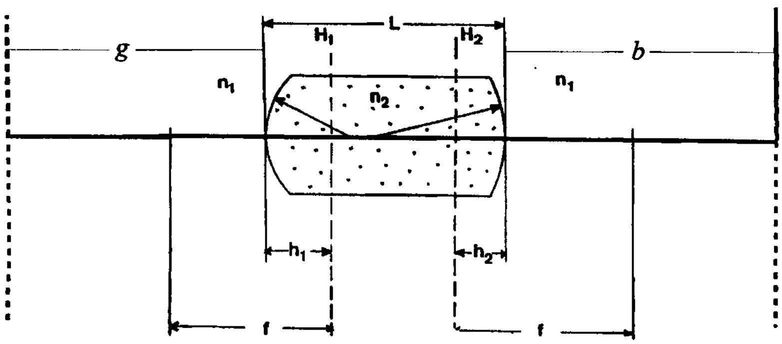


Fig. 1.8 Imaging by means of a thick lens. H_i denote the principal planes. The focal length f is measured from the principal planes. $h_i>0$, if the principal plane is located on the right hand side of the surface.

Setting matrix element B equal to zero yields:

$$b+g-\frac{gb}{f} = \frac{h_1 b - h_2 g}{f} - L\frac{n_1}{n_2}$$

which is equivalent to:

$$\frac{1}{g'} + \frac{1}{b'} = \frac{1}{f} \qquad with \quad g'=g+h_1 \quad b'=b-h_2 \tag{1.32}$$

We get the same imaging condition as compared to a thin lens if we measure the object and image distance from the principal planes H_i. Note that h_2 is negative (ρ_l is positive) !

c) C=0

At the output plane the ray parameters read:

$$x_2 = A\ x_1 + B\ \alpha_1 \tag{1.33}$$

$$\alpha_2 = D\ \alpha_1 \tag{1.34}$$

Parallel rays will still be parallel after passage through the optical system, but their inclination angle has changed by a factor of D. If we look through this optical system at a distant object, the size of the object is magnified by $/D/$. All telescopic systems such as the Galilean or the astronomical telescope have a zero C-component in their ray transfer matrix. D is the angle magnification.

d) D=0

At the output plane the ray parameters read:

$$x_2 = A\ x_1 + B\ \alpha_1 \tag{1.35}$$

$$\alpha_2 = C\ x_1 \tag{1.36}$$

We see that optical systems having $D=0$ generate a collimated beam from a divergent ray pattern. We can take the thin lens of Fig. 1.6 as an example. $D=0$ requires $g=f$. All rays starting in the point x_l at the front focal plane will be collimated to a parallel beam with an angle α_2 with respect to the z-axis.

Figure 1.9 gives an overview of the four cases discussed. The reader should keep the properties of the matrix elements in mind since this is very helpful in designing an optical system to provide a desired beam transformation. Generally, the optical properties of the components (distances, focal lengths, etc.) need to be varied until the corresponding matrix element goes to zero.

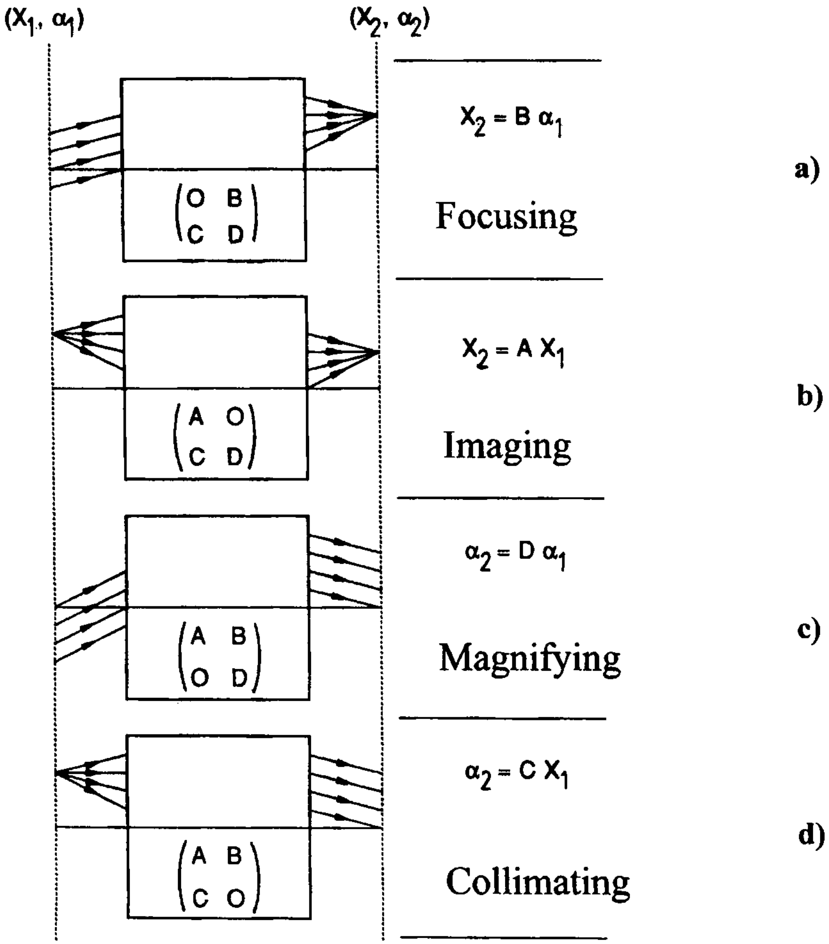


Fig. 1.9 Transformation of rays by optical systems having one vanishing element in their ray transfer matrix. The ray transfer matrix describes the ray propagation from left to right between the planes indicated by vertical lines.

Phase Space and Liouville's Theorem

So far we have discussed the propagation of single rays emerging from a light source through an optical system. In general the light source will emit over a spatial dimension Δx with an angular distribution at each point. An elegant way to visualize the effect of a beam propagation on the spatial and angular distribution of the rays is the presentation in phase space (Fig. 1.10). Each ray with starting point x and angle α can be represented by a point in phase space where the angular coordinate is plotted versus the spatial coordinate. As shown in Fig. 1.10 an extended light source of dimension Δx and angular width $\Delta\alpha$ is described by a rectangular area with height $\Delta\alpha$ and width Δx. Propagation through an optical system to a new plane now means that each point in this area will be imaged onto a new point in phase space. Therefore, the area will move as the light travels down the optical system.

It can be proven mathematically that the volume in a phase space cannot change as the physical system develops (Liouville's Theorem [1.7]). In our application Liouville's Theorem holds for light propagation in free space and through parabolic phase elements (e.g. aberration free lens or mirror). The area A in phase space must remain constant as long as the light propagates within the same medium. Let A_1 and A_2 be two areas in phase space with:

$$A_1 = \iint dx_1 d\alpha_1 \quad , \quad A_2 = \iint dx_2 d\alpha_2$$

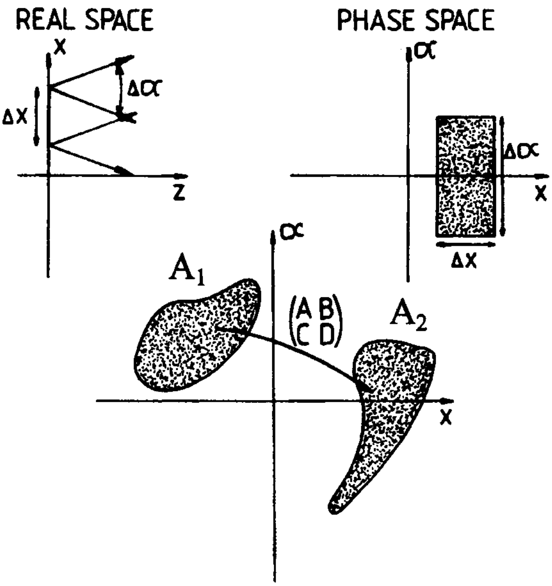


Fig. 1.10 Phase space presentation of a light source (upper graphs). The area moves in phase space as light propagates through optical systems.

The transformation of the differential is given by the determinant of the matrix $\boldsymbol{m}$:

$$\boldsymbol{m} = \begin{pmatrix} \frac{\delta x_2}{\delta x_1} & \frac{\delta x_2}{\delta \alpha_1} \\ \frac{\delta \alpha_2}{\delta x_1} & \frac{\delta \alpha_2}{\delta \alpha_1} \end{pmatrix} \tag{1.37}$$

By using the linear transformation (1.26) this results in:

$$dx_2 d\alpha_2 = (AD-BC)\ dx_1 d\alpha_1 \tag{1.38}$$

which means that the determinant of the ray transfer matrix must be equal to 1 to fulfill Liouville's Theorem $A_1=A_2$ (the refractive index is constant).

An example is shown in Fig. 1.11. Each point of the extended light source emits rays within an angular width $\Delta\alpha_1$. In phase space this source is represented by a rectangular area that has the width Δx_1, and height $\Delta\alpha_1$. The angular width remains constant as the rays propagate in free space, but the spatial extent of the beam increases from Δx_1 to Δx_i at the lens. In contrast to the light source the angular width at the lens is different for each point. The points P_1 and P_2 emit with zero divergence, whereas points in the center emit within the full angular width. In phase space the intermediate field at z_i is thus represented by a parallelogram. If the field is imaged by the lens, all beams starting from point Q_1 of the source are collimated into Q_2. Each point of the image now emits within the angular width $\Delta\alpha_2$. The image is again represented by a rectangular area in phase space with the same area as the source:

$$\Delta x_2 \Delta\alpha_2 = \Delta x_1 \Delta\alpha_1 \tag{1.39}$$

Note that this equation holds only for imaging! In the intermediate region z_i, the area in phase space is the same as that of the source or the image, but it cannot be written as a product of a spatial and an angular width since the area is not rectangular. The more general form of (1.39) is known as Abbe's sine law:

$$\Delta x_1\ n_1\ \sin\Delta\alpha_1 = \Delta x_2\ n_2\ \sin\Delta\alpha_2 \tag{1.40}$$

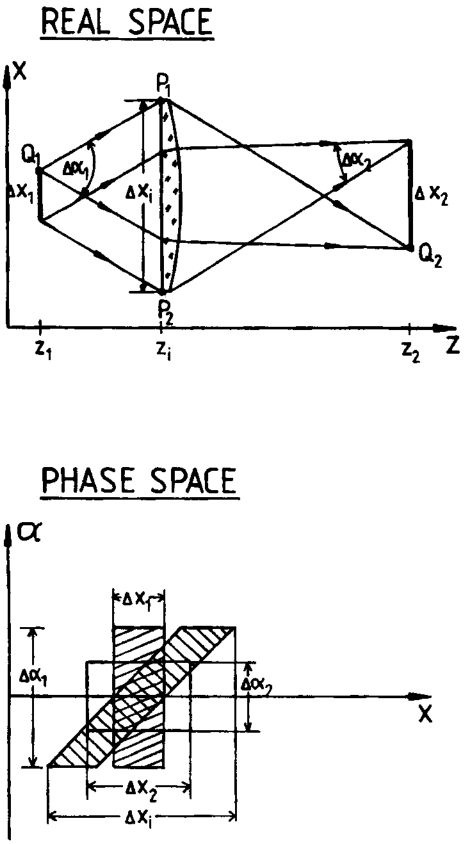


Fig. 1.11 Imaging by means of a focusing lens and the corresponding phase space presentation.

In the general case that light propagates from one plane (index n_1) to another (index n_2) and the ray propagation is described by the ray transfer matrix M, the following relation always holds:

$$\det M = AD - BC = \frac{n_1}{n_2} \tag{1.41}$$

The condition on the determinant also arises since energy conservative optical systems are reversible. An optical system can only exist if (1.41) holds for its ray transfer matrix. Thus it is not possible to build an optical funnel which transforms a large diameter divergent beam into a collimated thin ray . The product of beam size and beam divergence is a constant in imaging and is called beam parameter product $\Delta x \Delta \alpha / 4$ (product of the beam radius and the half angle of divergence). This product cannot be changed by any optical element without decreasing the power content of the beam (e.g. by using an aperture), if the light is completely incoherent. For coherent light, it is possible to convert any field distribution into a fundamental mode without loss of power by inserting suitable phase plates [1.18,1.20]. As will be discussed in detail in Sec.2.6.3, the beam parameter product is a direct measure for the beam quality, since a small focal spot size and a high depth of field can only be attained for small beam parameter products.

Example (Fig. 1.12):
A beam with diameter d and angle of divergence $\Delta\alpha$ is focused by means of a biconvex lens. Since the incoming rays are not all parallel to the optical axis they are not combined in one spot in the focal plane. We can easily calculate the resulting focus diameter by applying the phase space concept. In plane 1, the beam is characterized by a rectangle in phase space with area $d\Delta\alpha$. In the focal plane only rays with an angle α smaller than $d/2f$ can be detected (we assume that the beam diameter on the lens is also d). This means that the rectangular area in phase space has increased its height from $\Delta\alpha$ to d/f. According to (1.41) the area in phase space remains constant. This yields for the spot diameter s in the focal plane:

$$s\,\frac{d}{f} = d\,\Delta\alpha \quad \longrightarrow \quad s = f\,\Delta\alpha \tag{1.42}$$

Let us also look at the depth of field z_0 in the focal plane. We define the depth of field as the distance from the focal plane at which the beam diameter has increased by a factor of 2:

$$s + z_0\frac{d}{f} = 2s \quad \longrightarrow \quad z_0 = \frac{sf}{d} \tag{1.43}$$

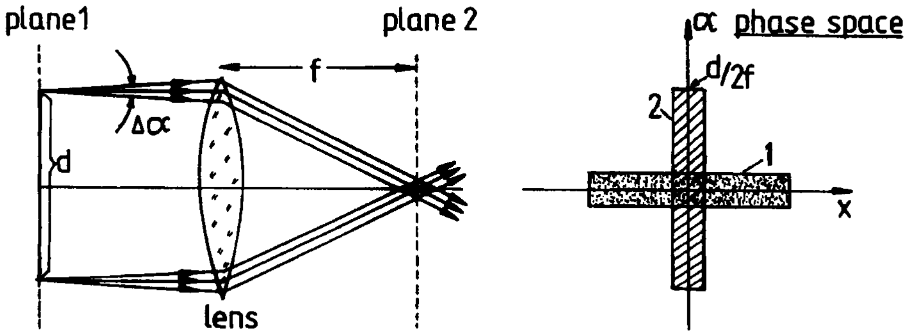


Fig. 1.12 Focusing of a beam with angular divergence Δα by a lens. The right hand graph shows the effect of the propagation from plane 1 to plane 2 in phase space.

The ratio of the focal spot area to the depth of field z_0 is a constant of the beam since it is proportional to the beam parameter product, or the phase space area:

$$\frac{\pi s^2}{4z_0} = \frac{\pi s d}{4f} = \pi \frac{d\,\Delta\alpha}{4} \tag{1.44}$$

Since the beam parameter product $d\Delta\alpha/4$ is constant, the depth of field decreases quadratically with the spot size. This fact clearly indicates that the beam parameter product $\Delta x\Delta\alpha/4$ needs to be as small as possible to attain good focusability. However, as we will see in Chapter 2, there is a general lower bound for the beam parameter product BPP or the area A in phase space, determined by the wavelength λ:

$$BPP = \frac{\Delta x \Delta\alpha}{4} \geq \frac{\lambda}{\pi} \tag{1.45}$$

In the green spectral range (λ=500nm), the minimum beam parameter product is 0.16mm mrad. The beam parameter products of lasers are close to this diffraction-limit. Conventional light sources, however, exhibit beam parameter products that are several orders of magnitude higher. Table 1.1 presents typical beam parameter products of different light sources.

Table 1.1 Beam parameter products of light sources.

light source	wavelength [nm]	beam parameter product [mm mrad]
200mW Nd:YVO4 laser (quadrupled)	266	0.09
20 W Nd:YVO4 laser (tripled)	355	0.13
20 W argon laser	488	0.16
100 mW laser diode	808	0.26
100 W cw Yb fiber laser	1,030	0.35
10 kW CO_2 laser	10,600	10
1 kW Yb:YAG disk laser	1,030	10
1 kW Nd:YAG rod laser	1,064	15
superbright LED (2 mW)	830-870	30
40 W fiber-coupled diode bar	808	50
conventional LED (2 mW)	630-670	200
flashlight (1 W)	400-800	1,000

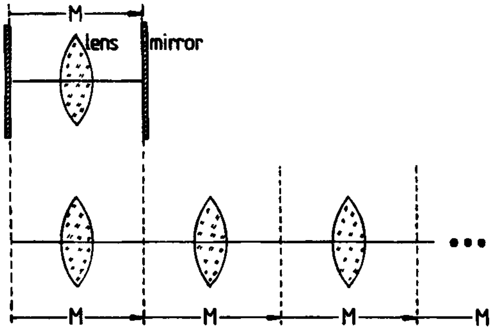


Fig. 1.13 Bouncing between two mirrors can be described as a straightforward propagation in a periodic lens guide. After propagating a distance containing N repetitions of the base unit cell described by the ray matrix M, the resulting matrix is M^N.

Sylvester's Theorem

A ray transfer matrix for periodic optical systems was derived by Tovar amd Casperson [1.17], based on Sylvester's formula [1.12]. This theorem is especially useful for the treatment of optical resonators since t he back and forth bouncing of a ray can be equivalently described as a straightforward propagation in a periodic lens guide (Fig. 1.13). For the symmetric resonator shown, the transit from one mirror to the other is the basic optical element, described by the ray matrix M. In order to determine the ray parameters after N subsequent travels, the N-th power of the fundamental ray matrix M has to be calculated. If we again take the general form of the ray transfer matrix M and restrict ourselves to matrices with determinant $AD\text{-}BC=1$, the N-th power of M reads:

$$M^N = \frac{1}{\sin\Phi}\begin{pmatrix} A\sin[N\Phi] - \sin[(N-1)\Phi] & B\sin[N\Phi] \\ C\sin[N\Phi] & D\sin[N\Phi] - \sin[(N-1)\Phi] \end{pmatrix} \tag{1.46}$$

with: $\cos\Phi = \dfrac{A+D}{2}$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

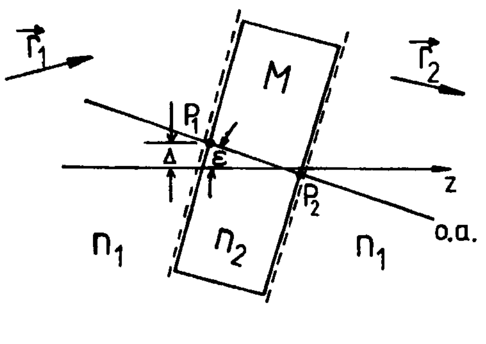


Fig. 1.14 An optical element with the optical axis, o.a., tilted by an angle ϵ about the front nodal vertex point P_1 and shifted by Δ perpendicularly to the z-direction ($\Delta>0$, $\epsilon<0$).

1.2.3 Misaligned Optical Elements

The matrices discussed so far hold for aligned optical elements. This means that the optical axis ***a*** of each element coincides with the reference axis z. The ray **a,** which represents the optical axis and intersects the surfaces of the optical element at the two vertex points P_1,P_2. This special ray is not affected by the element, it is neither shifted nor deflected. For a singlet, the spherical interface (see Fig. 1.5), the optical axis is degenerated. Each ray crossing the center of curvature, is optical axis; for the spherical lens the connecting line of the two centers of curvature is the optical axis. For a bifocal spherical interface the optical axis is defined by the ray, which is perpendicular to both center-lines of curvature. For a sequence of misaligned optical elements the axis depends on the misalignment parameters and will be calculated in this section. The optical axis of a resonator is defined by the ray, which is reproduced after one roundtrip.

Perfect alignment is impossible to achieve in reality, especially for a sequence of elements. In this section the influence of a shift and a tilt on the ray propagation will be briefly discussed. We assume an element which is tilted by an angle ϵ about the front vertex point P_1 and shifted by Δ, as shown in Fig. 1.14. The paraxial approximation holds with $|\epsilon| << 1$. For larger angles, the situation becomes more complicated because trigonometric functions are involved.

The misalignment of a single element is of no interest since the reference axis z can always be chosen along the optical axis of the element, and the ray transfer matrix of the aligned system still applies. However, for a sequence of optical elements with different alignments, the following relations become useful [1.13]. A ray $\boldsymbol{r}_1$ is incident on the optical system with:

$$\boldsymbol{r}_1 = \begin{pmatrix} x_1 \\ \alpha_1 \end{pmatrix} \qquad (1.47)$$

This vector transforms in the misaligned system to:

$$\boldsymbol{r}'_1 = \boldsymbol{r}_1 - \boldsymbol{s} \qquad (1.48)$$

with $\boldsymbol{s}$ being the misalignment vector:

$$\boldsymbol{s} = \begin{pmatrix} \Delta \\ \epsilon \end{pmatrix} \qquad (1.49)$$

The ray transfer matrix of the optical element transforms $\boldsymbol{r}'_1$ into $\boldsymbol{r}'_2$:

$$\boldsymbol{r}'_2 = \boldsymbol{M}\, \boldsymbol{r}'_1 \qquad (1.50)$$

This vector can be transformed back into the z-system by taking into account the propagation of vector $\boldsymbol{s}$ over the distance L between the reference planes:

$$\boldsymbol{r}_2 = \boldsymbol{r}'_2 + \boldsymbol{M}_{FS}\,\boldsymbol{s} = \boldsymbol{M}\boldsymbol{r}_1 - (\boldsymbol{M}-\boldsymbol{M}_{FS})\,\boldsymbol{s} \tag{1.51}$$

If the resulting misalignment vector is introduced:

$$\boldsymbol{s}_M = (\boldsymbol{M}-\boldsymbol{M}_{FS})\,\boldsymbol{s} \tag{1.52}$$

the ray vector behind a misaligned optical element reads:

$$\boldsymbol{r}_2 = M\,\boldsymbol{r}_1 - \boldsymbol{s}_M \tag{1.53}$$

The misalignment vector can be calculated for any element by using the matrices shown in Fig. 1.5:

<u>Thick lens with focal length f, thickness L, and principal plane distances h_i:</u>

$$\boldsymbol{M}_{FS} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \quad , \quad \boldsymbol{s}_M = \begin{pmatrix} \Delta\dfrac{h_2}{f} + \epsilon L(\dfrac{n_1}{n_2} - 1) \\ \dfrac{-\Delta}{f} - \epsilon\dfrac{h_1}{f} \end{pmatrix}$$

<u>Plane slab of length L:</u>

$$\boldsymbol{M}_{FS} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \quad , \quad \boldsymbol{s}_M = \begin{pmatrix} \epsilon L(\dfrac{n_1}{n_2} - 1) \\ 0 \end{pmatrix}$$

Example 1:
Let us consider a symmetric optical resonator with an internal lens as shown in Fig. 1.15. If all elements are aligned, the optical axis of the resonator, defined by the line going through both centers of curvature P_1,P_2 of the mirrors, coincides with the optical axis of the lens which is defined by the two focal points. Which ray represents the optical axis when the lens is shifted by Δ?

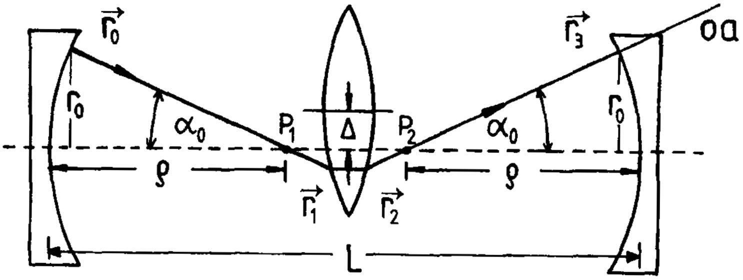


Fig. 1.15 A symmetric resonator with a shifted lens.

The optical axis is defined by the ray which travels back and forth along the same path, which means that the optical axis must be perpendicular to both mirror surfaces. The ray representing the optical axis on mirror 1 thus reads:

$$\boldsymbol{r}_0 = \begin{pmatrix} r_0 \\ \dfrac{-r_0}{\rho} \end{pmatrix}$$

The propagation of this ray to mirror 2 is given by:

$$\boldsymbol{r}_1 = \boldsymbol{M}_{FS}(L/2)\ \boldsymbol{r}_0$$

$$\boldsymbol{r}_2 = \boldsymbol{M}_{TL}\ \boldsymbol{r}_1 - \boldsymbol{s}_{TL}$$

$$\boldsymbol{r}_3 = \boldsymbol{M}_{FS}(L/2)\ \boldsymbol{r}_2$$

where $\boldsymbol{M}_{TL}$ is the ray transfer matrix for the thick lens. The combination of these three equations together with the fact that $\boldsymbol{r}_3$ must be equal to $\boldsymbol{r}_0$ (with a different sign of the angle α_0) yields the displacement and the tilt of the optical axis:

$$r_0 = \frac{\Delta}{1 - \dfrac{L}{2\rho} + \dfrac{2f}{\rho}}\ , \qquad \alpha_0 = \frac{\Delta}{\rho - \dfrac{L}{2} + 2f}$$

For plane mirrors with $\rho \to \infty$, r_0 is equal to Δ and α_0 equals zero, as it should be. The above relation indicates that the sensitivity of a resonator to lens shifting depends on the particular configuration.

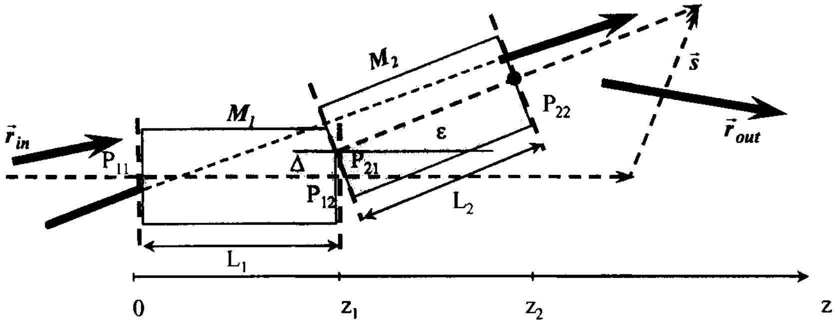


Fig. 1.16 The optical axis ā of a misaligned optical system. P_{ij} are the vertex points of the optical elements. The arrows indicate the direction of the vectors, not their moduli. $\bar{a}_i$ are the axes of the single elements.

The Optical Axis of Misaligned Systems

Two optical elements $\boldsymbol{M_1}$, $\boldsymbol{M_2}$ with geometrical lengths L_1, L_2 and optical axes $\boldsymbol{a_1}$, $\boldsymbol{a_2}$ are shifted and tilted against each other as shown in Fig. 1.16. The misalignment again is characterized by the vector $\boldsymbol{s}$ of (1.49), which now reads:

$$\boldsymbol{s} = \boldsymbol{a}_2 - \boldsymbol{a}_1 \tag{1.54}$$

The element $\boldsymbol{M_1}$ is aligned with respect to the z-axis, which determines the vector $\boldsymbol{a_1}$ to be:

$$\boldsymbol{a}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $\boldsymbol{a_2} = \boldsymbol{s}$. In the $\boldsymbol{a_1}$-reference system the output ray is related to the input ray according to (1.53):

$$\boldsymbol{r}_{out} = \boldsymbol{M}_2\boldsymbol{M}_1\,\boldsymbol{r}_\varepsilon - (\boldsymbol{M}_2 - \boldsymbol{M}_{FS2})\,\boldsymbol{s} \tag{1.55}$$

If $\boldsymbol{r_{in}}$ is the ray, which represents the optical axis $\boldsymbol{a}$ of the complete system, the following relations must hold:

$$\boldsymbol{r}_{in} = \boldsymbol{a}_{z=0} \qquad\qquad \boldsymbol{r}_{out} = \boldsymbol{a}_{z=z2} = \boldsymbol{M}_{FS2}\boldsymbol{M}_{FS1}\,\boldsymbol{a}_{z=0}$$

and (1.55) delivers an equation to determine the optical axis $\boldsymbol{a_{z=0}}$ of the misaligned system at the position z=0:

$$(M_2M_1 - M_{FS2}M_{FS1})\, a_{z=0} = (MM_2 - M_{FS2})\, s \tag{1.56}$$

If the ray vectors r_{in}, r_{out} are transformed into the a-system, according to

$$r'_{in} = r_{in} - a_{z=0} \qquad r'_{out} = r_{out} - a_{z=z2} = r_{out} - M_{FS2}M_{FS1}\, a_{z=0}$$

the simple matrix law can again be applied for the ray-vectors:

$$r'_{out} = M_2M_1\, r'_{in} \tag{1.57}$$

which can be proved easily from (1.55) and (1.56). This procedure can be continued for more elements. Any sequence of misaligned optical elements has a resulting optical axis. If the ray vectors are transformed into this reference system, the normal matrix laws hold.

Example 2:
As an example the resulting optical axis of a misaligned three-lens system as shown in Fig. 1.17 will be calculated. The matrix M_1 of the first optical element includes the two lenses on the left and the two distances L. The focal lengths of all three lenses are the same and equal to f. Then the two matrices read:

$$M_1 = \begin{pmatrix} (1-L/f)^2 - L/f & L(2-L/f) \\ (-2/f + L/f^2) & 1-L/f \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

The free-space matrices and the misalignment vector are

$$M_{FS1} = \begin{pmatrix} 1 & 2L \\ 0 & 1 \end{pmatrix} \qquad M_{FS2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad s = \begin{pmatrix} \Delta \\ 0 \end{pmatrix}$$

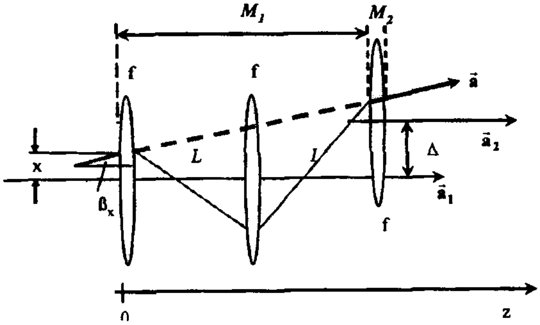


Fig. 1.17 The optical axis ā of a misaligned three-lens system.

With (1.56) the vector $\boldsymbol{a}$ of the optical axis $z = 0$ is obtained:

$$\boldsymbol{a} = \begin{pmatrix} x \\ \beta_x \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta/2}{L/f-3} \\ \Delta/(2L) \end{pmatrix}$$

Misalignment Matrices

The matrix of an optical element can be defined in such a way that the misalignment is included. Then 4x4-matrices are required, which is not very convenient. It becomes more difficult for two-dimensional rays, where 8x8 matrices now appear [1.19].

1.2.4 Two-Dimensional Optical Systems

In previous sections, we have restricted the calculation of ray propagation to one dimension, the x-z plane. This is sufficient as long as the radiation field and the optical elements exhibit rotational symmetry. In general, laser beams can be astigmatic (e.g. laser diodes) and parabolical optical elements can have different radii of curvature in x and y-directions as shown in Fig. 1.18. In this case, it is necessary to track all ray parameters. An arbitrary ray, starting at a x-y plane, is described by a four-dimensional ray vector (Fig. 1.18):

$$V_1 = \begin{pmatrix} x_1 \\ y_1 \\ \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{\gamma}_1 \end{pmatrix} \tag{1.58}$$

with $\boldsymbol{r}_1$, $\boldsymbol{\gamma}_1$ being two-dimensional vectors. Thus, we need a 4x4 matrix to describe the propagation from plane 1 to plane 2:

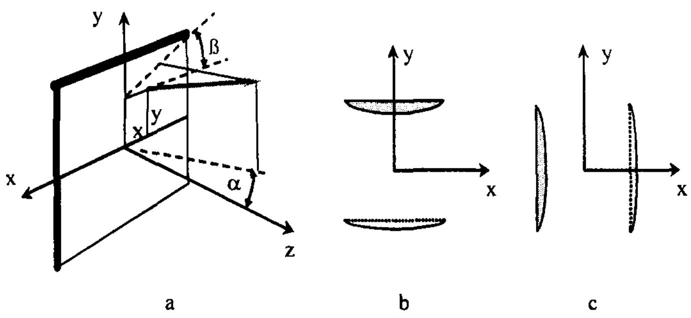


Fig. 1.18 A ray is generally described by four parameters (a). All four parameters have to be tracked when the ray passes through two-dimensional optics such as cylinder lenses (b,c).

$$\begin{pmatrix} x_2 \\ y_2 \\ \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} A_{xx} & A_{xy} & B_{xx} & B_{xy} \\ A_{yx} & A_{yy} & B_{yx} & B_{yy} \\ C_{xx} & C_{xy} & D_{xx} & D_{xy} \\ C_{yx} & C_{yy} & D_{yx} & D_{yy} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ \alpha_1 \\ \beta_1 \end{pmatrix} \tag{1.59}$$

By defining the sub-matrices $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$ this can be rewritten in an already familiar form:

$$\begin{pmatrix} \boldsymbol{r}_2 \\ \boldsymbol{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{\gamma}_1 \end{pmatrix} \tag{1.60}$$

The multiplication of matrices, whose elements are are again matrices is carried out the same way as for normal matrices. Relation (1.60) is a very convenient way to express ray transformations in two-dimensional optics since all relations that hold in one dimension can be generalized by using submatrices as matrix elements. An optical system is now described by a 4x4 matrix with 16 elements, but there are constraints to this matrix as was shown by Nemes [1.15,1.16]. The first is the determinant relation (1.41), which now reads:

$$\boldsymbol{A}\boldsymbol{D}^T - \boldsymbol{B}\boldsymbol{C}^T = \frac{n_1}{n_2}\boldsymbol{I} \tag{1.61}$$

with $\boldsymbol{I}$ being the identity matrix and $\boldsymbol{D}^T$ is the transposed matrix of $\boldsymbol{D}$ with:

$$\boldsymbol{D} = \begin{pmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{pmatrix} \qquad \boldsymbol{D}^T = \begin{pmatrix} D_{xx} & D_{yx} \\ D_{xy} & D_{yy} \end{pmatrix}$$

The two other constraints are:

$$\boldsymbol{A}\boldsymbol{B}^T = \boldsymbol{B}\boldsymbol{A}^T \tag{1.62}$$

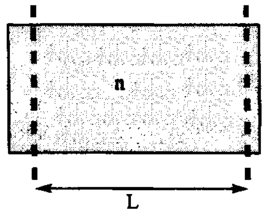
$$\boldsymbol{C}\boldsymbol{D}^T = \boldsymbol{D}\boldsymbol{C}^T \tag{1.63}$$

These three conditions reduce the number of independent elements to ten. This means that in the most general case of parabolic optical elements ten parameters are required for a complete description. The number of elements is further reduced if the optical system exhibits symmetry with respect to the coordinate system. For rotational symmetry, only four independent elements exist in general. If the refractive indices before and after the element are the same, only three independent elements remain.

For optical elements having mirror symmetry with respect to the x-axis or y-axis, the 4x4 ray transfer matrix can be easily derived by treating the x- and y-directions separately. In this case the sub-matrices ***A***,***B***,***C***,***D*** are diagonal and we thus have only six independent elements (for equal refractive indices). In the following we list the matrices for common optics:

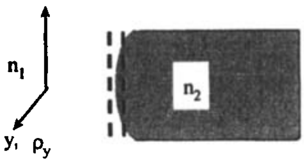
a) Propagation in isotropic, homogeneous medium over geometrical distance *L*

$$M_{FS} = \begin{pmatrix} 1 & 0 & L & 0 \\ 0 & 1 & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.64}$$



b) Aligned bifocal spherical interface (see Eq. 1.16)

$$M_{BSI} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{n_1-n_2}{n_2\rho_x} & 0 & \dfrac{n_1}{n_2} & 0 \\ 0 & \dfrac{n_1-n_2}{n_2\rho_y} & 0 & \dfrac{n_1}{n_2} \end{pmatrix} \tag{1.65}$$



c) Aligned bifocal thin lens

It has the refractive powers D_x in x-direction and D_y in y-direction. Special cases are the cylinder lenses with D_x or D_y equal to zero.

$$M_{BTL} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -D_x & 0 & 1 & 0 \\ 0 & -D_y & 0 & 1 \end{pmatrix} \tag{1.66}$$

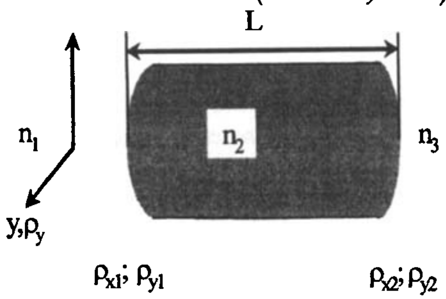


Fig.1.19 Aligned bifocal thick lens.

d) Aligned bifocal thick lens (Fig. 1.19)

The principal axes of both bifocal transitions are parallel to the x,y-axes. In the two-dimensional case, no principal planes exit. The system is characterized by seven independent parameters: n_1/n_2, n_2/n_3, ρ_{x1}, ρ_{y1}, ρ_{x2}, ρ_{y2}, and L. with $a=(n_1/n_2-1)$ and $b=(n_2/n_3-1)$.

$$M_{BFL} = \begin{pmatrix} 1+\frac{aL}{\rho_{x1}} & 0 & \frac{n_1}{n_2}L & 0 \\ 0 & 1+\frac{aL}{\rho_{y1}} & 0 & \frac{n_1}{n_2}L \\ \frac{b}{\rho_{x2}}+\frac{a}{\rho_{x1}}\left(\frac{n_2}{n_3}+\frac{bL}{\rho_{x2}}\right) & 0 & \frac{n_1}{n_3}+\frac{bL}{\rho_{x2}}\frac{n_1}{n_2} & 0 \\ 0 & \frac{b}{\rho_{y2}}+\frac{a}{\rho_{y1}}\left(\frac{n_2}{n_3}+\frac{bL}{\rho_{y2}}\right) & 0 & \frac{n_1}{n_3}+\frac{bL}{\rho_{y2}}\frac{n_1}{n_2} \end{pmatrix} \tag{1.67}$$

e) Free space propagation with different lengths in x and y

This special element consists of four aligned cylinder lenses to generate different propagation distances $L_x=L$ and $L_y=L+8f_x$ in x- and y-direction, respectively.

$$M_{FSA} = \begin{pmatrix} 1 & 0 & L_x & 0 \\ 0 & 1 & 0 & L_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.68}$$

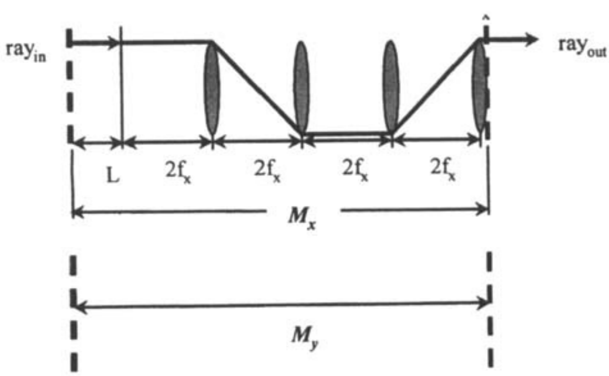


Fig. 1.20 Cylindrical telescope system with different propagation distances in x- and y-direction.

1.2.5 Rotation and Misalignment

If the optical element is rotated by an angle θ as shown for the cylinder lens in Fig. 1.21, the original ray transfer matrix M has to be transformed into an angle dependent transfer matrix $M(\theta)$ to describe the ray propagation through the element. In the following, the transformation rule for ray matrices under rotation is derived.

We know that the optical element can still be described by the initial ray matrix M if we use the rotated coordinate axes as the reference coordinate system (indicated by asterisks). Ray propagation can then be written as:

$$\begin{pmatrix} x_2^* \\ y_2^* \\ \alpha_2^* \\ \beta_2^* \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} x_1^* \\ y_1^* \\ \alpha_1^* \\ \beta_1^* \end{pmatrix} \tag{1.69}$$

The initial coordinate axes are related to the rotated ones (for ccw rotation) via:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.70}$$

This matrix equation also holds for the angles α,β since they are defined through x,y-coordinates. Thus the transformation rule for the ray vector reads:

$$\begin{pmatrix} x_1^* \\ y_1^* \\ \alpha_1^* \\ \beta_1^* \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ \alpha_1 \\ \beta_1 \end{pmatrix} := \boldsymbol{R} \begin{pmatrix} x_1 \\ y_1 \\ \alpha_1 \\ \beta_1 \end{pmatrix} \tag{1.71}$$

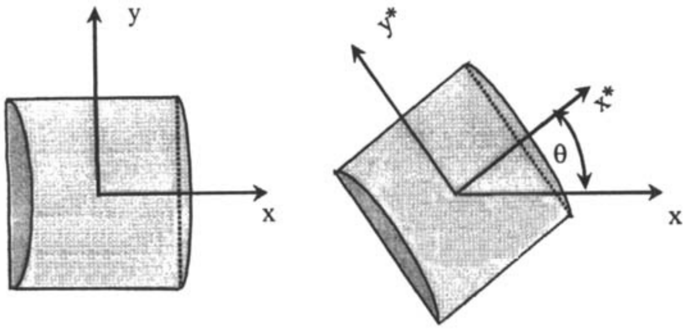


Fig. 1.21 Aligned cylinder lens (left) and rotated cylinder lens (right).

$$\begin{pmatrix} x_1 \\ y_1 \\ \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1^* \\ y_1^* \\ \alpha_1^* \\ \beta_1^* \end{pmatrix} := \boldsymbol{R}^{-1} \begin{pmatrix} x_1^* \\ y_1^* \\ \alpha_1^* \\ \beta_1^* \end{pmatrix} \tag{1.72}$$

By multiplying (1.69) with $\boldsymbol{R}^{-1}$ from the left and using the relation $\boldsymbol{RR}^{-1}=\boldsymbol{I}$ we obtain:

$$\begin{pmatrix} x_2 \\ y_2 \\ \alpha_2 \\ \beta_2 \end{pmatrix} = \boldsymbol{R}^{-1}\boldsymbol{MR} \begin{pmatrix} x_1 \\ y_1 \\ \alpha_1 \\ \beta_1 \end{pmatrix} \tag{1.73}$$

This is already what we are looking for! The ray transfer matrix $\boldsymbol{M}(\theta)$ for optics rotated in ccw direction thus reads:

$$\boldsymbol{M}(\theta) = \boldsymbol{R}^{-1}\boldsymbol{MR} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \boldsymbol{M} \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{1.74}$$

Example : Stokes Lens Pair (Fig. 1.22)

The Stokes Lens Pair consists of a negative and a positive cylinder lens with equal refractive power D [1.8]. The lenses can be rotated by an angle θ in opposite directions around the z-axis. We will show in the following that this element is equivalent to a combination of a spherical lens and a cylindrical lens under 45°, both with varying refractive power. Let us first consider the negative lens which is rotated counterclockwise. By using (1.66) and (1.74) we obtain:

$$\boldsymbol{M}_{-CYL} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ D & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

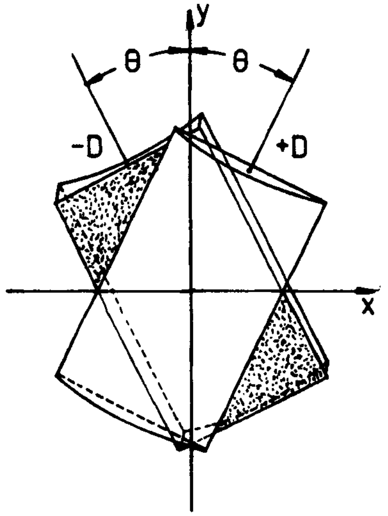


Fig. 1.22 A Stokes lens pair as an example of a rotated optics. This lens pair is used in ophthalmic devices (phoropters, refractors) to measure cylinder errors (astigmatism) of human eyes.

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ D\cos^2\theta & D\sin\theta\cos\theta & 1 & 0 \\ D\sin\theta\cos\theta & D\sin^2\theta & 0 & 1 \end{pmatrix} \qquad D > 0 \qquad (1.75)$$

The matrix M_{+CYL} for the positive cylinder lens can be found by replacing D with $-D$ and θ with $-\theta$ in M_{-CYL}. The resulting matrix for the combined lens pair is given by:

$$M_{STOKES} = M_{-CYL}\, M_{+CYL} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & D\sin2\theta & 1 & 0 \\ D\sin2\theta & 0 & 0 & 1 \end{pmatrix} \qquad D>0 \qquad (1.76)$$

Now we compare the Stokes Lens Pair with a system consisting of a negative cylinder lens rotated by 45° with refractive power $-D^*$ $(D^*>0)$ combined with a positive spherical lens with refractive power $D^*/2$. Matrix (1.71) can be used to determine the ray transfer matrix of the cylinder lens $M_{-CYL}(45°)$. The ray transfer matrix of the equivalent system is:

$$M_{EQU.} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{D^*}{2} & 0 & 1 & 0 \\ 0 & -\frac{D^*}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{D^*}{2} & \frac{D^*}{2} & 1 & 0 \\ \frac{D^*}{2} & \frac{D^*}{2} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{D^*}{2} & 1 & 0 \\ \frac{D^*}{2} & 0 & 0 & 1 \end{pmatrix} \tag{1.77}$$

This is equivalent to (1.76) if D^* equals $2D\sin(2\theta)$. A Stokes lens pair can thus be made equivalent to 45°-cylinder lenses with refractive powers D^* between $-2D$ and $+2D$. By adding a second lens pair rotated at 45° with respect to the first one, both pairs together can be made equivalent to any angle cylinder lens with any power between $\pm 2D$.

Example: The Phase Space Beam Analyzer

The Phase Space Beam Analyzer is a fascinating optical system which enables one to record the phase space presentation of a one-dimensional light source (Fig. 1.23) [1.11,1.14,1.21]. A small slit in plane 1 generates a line source so that only rays with $y=0$ can enter the system. The second slit, which is preferably located close to the spherical lens ($c=0$), selects only rays with zero inclination in y-direction. This means that the whole setup is only sensitive to rays having ray vectors $v=(x, \alpha, 0, 0)$. The basic principle of the device is that on plane 4 the y-coordinate depends only on the angle α whereas in the x-direction we see the image of the first slit. Imaging of the slit is accomplished by the spherical lens and the quadrupole lens (a Stokes lens pair with $\theta=-45°$) translates the angle α into a shift in the y-direction.

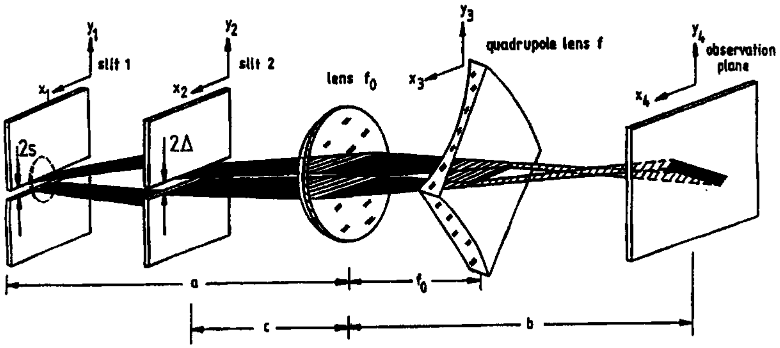


Fig. 1.23 The Phase Space Beam Analyzer.

By multiplying all ray transfer matrices between plane 1 and 4, the resulting matrix reads:

$$M_{PSBA} = \begin{pmatrix} 1 & 0 & b-f_0 & 0 \\ 0 & 1 & 0 & b-f_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1/f & 1 & 0 \\ -1/f & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & f_0 & 0 \\ 0 & 1 & 0 & f_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/f_0 & 0 & 1 & 0 \\ 0 & -1/f_0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= -\begin{pmatrix} P & 0 & 0 & S \\ 0 & P & S & 0 \\ 1/f_0 & 0 & 1/P & f_0/f \\ 0 & 1/f_0 & f_0/f & 1/P \end{pmatrix} \quad with \;\; P = \frac{b-f_0}{f_0} \;\; ; \;\; S = \frac{f_0}{f}(b-f_0) \qquad (1.78)$$

Since only rays with $v=(x,0,\alpha,0)$ can reach the observation plane, (1.78) yields for the ray coordinates in plane 4:

$$x_4 = -P\,x_1 \;\; , \quad y_4 = -S\,\alpha_1 \qquad (1.79)$$

In plane 4 we observe the phase space presentation of the horizontal line source in plane 1, scaled by the factor P and S. In order to visualize the phase space presentation for different parts of the light source, the phase space analyzer has to be rotated around the z-axis.

Meridional Rays and Skew Rays

If the ray is propagating in a plane that contains the z-axis, it is called a meridional ray. By rotating the reference frame, the four-dimensional meridional ray can be transformed into a two-dimensional ray. With $y_2=0$ and $\beta_2=0$, the transformation (1.71) then yields:

$$\begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \begin{pmatrix} \alpha_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$$

This can be accomplished by chosing the angle of rotation θ as:

$$\tan\theta = \frac{y_1}{x_1} = \frac{\alpha_1}{\beta_1}$$

If this condition holds the ray is a meridional ray, if not (i.e. $y_1/x_1 \neq \alpha_1/\beta_1$) it is a skew ray as shown in Fig.1.24.

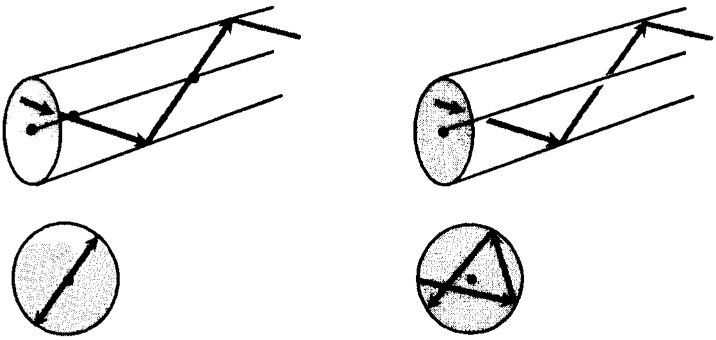


Fig.1.24 Meridional (left) and skew rays (right) in a step-index fiber. The meridional ray propagates in a plane that contains the symmetry axis of the fiber. The skew ray is propagating around the axis.

Misalignment of Two-Dimensional Optical Elements

Misalignment of two-dimensional elements can be described in the same way as for one-dimensional elements in Sec. 1.2.3. Equations (1.50)-(1.53) also hold for four-dimensional vectors and matrices. The misalignment vector now reads:

$$s = \begin{pmatrix} \Delta_x \\ \Delta_y \\ \epsilon_x \\ \epsilon_y \end{pmatrix}$$

The tilt introduced in Sec. 1.2.3 was assumed to be very small. For large tilt angles the trigonometric functions have to be considered. We define a tilt as a rotation by an angle θ around the x-axis as shown for the lens in Fig. 1.25. The projection of the lens onto the x-y plane now is an ellipse with the smaller axis along the y-axis. The plane defined by the small ellipse axis and the z-axis is called the *tangential plane*, the plane defined by the z-axis and the large ellipse axis is the *sagittal plane*. In the following we restrict the discussion to optical elements that exhibit mirror symmetry in x- and y-direction. We are thus dealing with the ray transfer matrix:

$$\begin{pmatrix} A_{xx} & 0 & B_{xx} & 0 \\ 0 & A_{yy} & 0 & B_{yy} \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix}$$

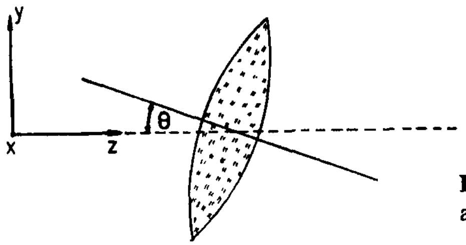


Fig. 1.25 A tilted focusing lens (rotation around x-axis) as seen in the tangential plane.

Since all submatrices are diagonal we can treat ray propagation separately for the two planes. The ray matrices of the most common tilted optics can be derived by using the matrix of a tilted spherical dielectric interface (Fig. 1.26). If θ_l denotes the angle of incidence of the optical axis with respect to the surface normal (this is also the tilt angle) and θ_2 is the corresponding angle of refraction, the ray transfer matrix for the tangential plane (y-z plane) reads:

$$M^T_{SI,TILT} = \begin{pmatrix} A_{yy} & B_{yy} \\ C_{yy} & D_{yy} \end{pmatrix} = \begin{pmatrix} \dfrac{\cos\theta_2}{\cos\theta_1} & 0 \\ \dfrac{n_2\cos\theta_2 - n_1\cos\theta_1}{n_2\rho\ \cos\theta_1\cos\theta_2} & \dfrac{n_1\cos\theta_1}{n_2\cos\theta_2} \end{pmatrix} \tag{1.80}$$

and for the sagittal plane (x-z plane) we get:

$$M^S_{SI,TILT} = \begin{pmatrix} A_{xx} & B_{xx} \\ C_{xx} & D_{xx} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_2\rho} & \dfrac{n_1}{n_2} \end{pmatrix} \tag{1.81}$$

with $$\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}$$

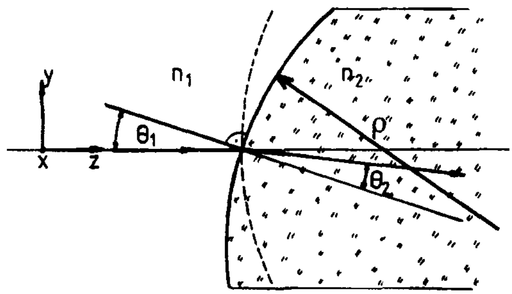


Fig. 1.26 The spherical dielectric interface, tilted by an angle θ_l around the x-axis.

The 4x4 ray transfer matrix for the spherical interface tilted around the x-axis is given by:

$$M_{SI,TILT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\cos\theta_2}{\cos\theta_1} & 0 & 0 \\ \dfrac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_2\,\rho} & 0 & \dfrac{n_1}{n_2} & 0 \\ 0 & \dfrac{n_2\cos\theta_2 - n_1\cos\theta_1}{n_2\rho\,\cos\theta_1\cos\theta_2} & 0 & \dfrac{n_1\cos\theta_1}{n_2\cos\theta_2} \end{pmatrix} \tag{1.82}$$

The knowledge of this ray matrix enables us to derive the 4x4 ray transfer matrix for two important tilted optical elements used in laser resonators: the tilted thin lens, and the tilted slab (Fig. 1.27).

Tilted Thin Lens

The lens with index of refraction n_2 is surrounded by a medium with index n_1 and tilted by an angle θ_1 around the x-axis as shown in Fig. 1.27. In order to simplify the discussion we assume that both interfaces have the same radii of curvature $|\rho_1|,|\rho_2|$. The ray transfer matrix in the y-direction (tangential plane) is obtained by multiplying two matrices (1.80) for a tilted spherical interface:

$$\mathcal{M}^T_{TL,TILT} = \begin{pmatrix} \dfrac{\cos\theta_4}{\cos\theta_3} & 0 \\ \dfrac{n_1\cos\theta_4 - n_2\cos\theta_3}{n_1\rho\,\cos\theta_3\cos\theta_4} & \dfrac{n_2\cos\theta_3}{n_1\cos\theta_4} \end{pmatrix}\begin{pmatrix} \dfrac{\cos\theta_2}{\cos\theta_1} & 0 \\ \dfrac{n_2\cos\theta_2 - n_1\cos\theta_1}{n_2\rho\,\cos\theta_1\cos\theta_2} & \dfrac{n_1\cos\theta_1}{n_2\cos\theta_2} \end{pmatrix} \tag{1.83}$$

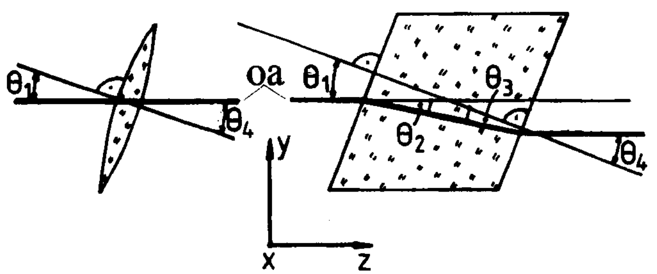


Fig. 1.27 Tilted thin lens and tilted slab (oa: optical axis).

with θ_1, θ_3 being the angles of incidence and θ_2, θ_4 the angles of refraction of the optical axis. Since both curvatures are equal and the lens has no thickness, the relations $\theta_1 = \theta_4$ and $\theta_2 = \theta_3$ hold. Insertion into (1.83) yields:

$$M^T_{TL,TILT} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\cos\theta_1}\left[\frac{n_2\cos\theta_2}{n_1\cos\theta_1} - 1\right]\left[\frac{1}{\rho_1} - \frac{1}{\rho_2}\right] & 1 \end{pmatrix} \tag{1.84}$$

The ray transfer matrix for the sagittal plane (x-z plane) can be found in a similar way by using the product of two ray transfer matrices (1.81):

$$M^S_{TL,TILT} = \begin{pmatrix} 1 & 0 \\ \cos\theta_1\left[\frac{n_2\cos\theta_2}{n_1\cos\theta_1} - 1\right]\left[\frac{1}{\rho_1} - \frac{1}{\rho_2}\right] & 1 \end{pmatrix} \tag{1.85}$$

Tilted Slab

We can use the ray matrices for the spherical interface with infinite radius of curvature and combine two of those with the ray transfer matrix for free space propagation. In the tangential and the sagittal plane we get:

$$M^T_{SLAB,TILT} = \begin{pmatrix} 1 & \frac{Ln_1}{n_2}\left[\frac{\cos\theta_1}{\cos\theta_2}\right]^2 \\ 0 & 1 \end{pmatrix} \tag{1.86}$$

$$M^S_{SLAB,TILT} = \begin{pmatrix} 1 & \frac{Ln_1}{n_2} \\ 0 & 1 \end{pmatrix} \tag{1.87}$$

In the sagittal plane, we obtain the ray transfer matrix of the aligned slab, whereas the tilt decreases the effective length of the slab in the tangential plane. For a slab inserted at the Brewster angle, the 4x4 ray transfer matrix is given by:

$$M_{Brewster\ Slab} = \begin{pmatrix} 1 & 0 & Ln_1/n_2 & 0 \\ 0 & 1 & 0 & L(n_1/n_2)^3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.88}$$

1.2.6 The ABCD Law for the Radius of Curvature

The knowledge about ray matrices we have acquired enables us to send rays into optical systems and calculate the coordinates and the angles at which they will emerge. In some applications, however, it is more interesting to find a relation among radii of curvature of spherical waves. In geometrical optics a spherical wave is defined by rays having a virtual source at the center of curvature (Fig.1.28). After passage through the optics the wavefront will still be spherical, but with a changed radius of curvature R. This occurs since all optical elements are assumed to have parabolic surfaces or index profiles. The relationship between the two curvatures is called the *ABCD law*. For the derivation we first consider the case of one-dimensional optics and express the radius of curvature R_l of the incident wave in terms of the ray parameters of the surface normals (for small angles α):

$$R_1 = x_1/\alpha_1 \tag{1.89}$$

Accordingly, at the other side of the optical system with ray transfer matrix M the radius of curvature is $R_2 = x_2/\alpha_2$. We find the relationship between the two curvatures by dividing the two equations for the ray parameters:

$$\frac{x_2}{\alpha_2} = \frac{Ax_1 + B\,\alpha_1}{Cx_1 + D\,\alpha_1}$$

which yields the ABCD law:

$$R_2 = \frac{A\,R_1 + B}{C\,R_1 + D} \tag{1.90}$$

Note that the radius of curvature is positive for a divergent wave (center of curvature is to the left of the wavefront) and negative for a convergent wave.

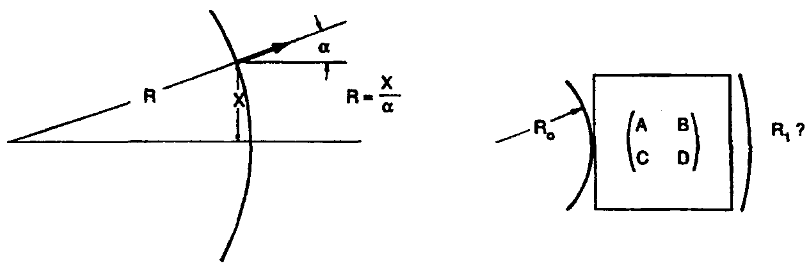


Fig. 1.28 Definition of a spherical wavefront. The radius of curvature is changed after passage through the optical system.

Example: Using the ray matrix for a thin lens (1.18), the ABCD law (1.90) yields

$$\frac{1}{R_1} - \frac{1}{R_2} = \frac{1}{f}$$

This is equivalent to the imaging condition since the object distance is R_1 and the image distance is $-R_2$.

Similar to the determinant relation (1.41), the ABCD law can be extended to the case of arbitrary two-dimensional optical systems. The two-dimensional ABCD law can be written as (the superscript *-1* denotes the inverse matrix):

$$\boldsymbol{R}_2 = (\boldsymbol{A}\,\boldsymbol{R}_1 + \boldsymbol{B})(\boldsymbol{C}\,\boldsymbol{R}_1 + \boldsymbol{D})^{-1} \tag{1.91}$$

with $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$ being the 2x2 submatrices according to (1.60), and the curvature matrix is given by:

$$\boldsymbol{R} = \begin{pmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{pmatrix} \tag{1.92}$$

If the wavefront exhibits symmetry along the x- and y-direction, the curvature matrix is diagonal and R_{xx}, R_{yy} denote the radii of curvature along the x-axis and y-axis respectively. If (1.92) is not diagonal, the coordinate frame has to be rotated around the z-axis to find the new x,y-frame in which the wavefront becomes symmetric. This procedure is equivalent to finding the eigenvalues and eigenvectors of the curvature matrix (1.92). The eigenvalues give the radii of curvature R^*_{xx} and R^*_{yy} along the x*- and y*-axis of the new symmetric coordinate system with:

$$R^*_{xx} = \frac{R_{xx} + R_{yy}}{2} + \sqrt{\left(\frac{R_{xx} - R_{yy}}{2}\right)^2 + R_{xy}R_{yx}} \tag{1.93}$$

$$R^*_{yy} = \frac{R_{xx} + R_{yy}}{2} - \sqrt{\left(\frac{R_{xx} - R_{yy}}{2}\right)^2 + R_{xy}R_{yx}} \tag{1.94}$$

and the eigenvectors $\boldsymbol{e}^*_x$, $\boldsymbol{e}^*_y$ represent the new coordinate axes expressed in the initial coordinate frame:

$$e_x^* = \begin{pmatrix} 1 \\ \dfrac{R_{xx}^* - R_{xx}}{R_{xy}} \end{pmatrix} \qquad e_y^* = \begin{pmatrix} 1 \\ \dfrac{R_{yy}^* - R_{xx}}{R_{xy}} \end{pmatrix} \qquad \begin{matrix} (1.95) \\ (1.96) \end{matrix}$$

Example: We want to calculate the change in spherical wave curvature that is induced by a cylindrical lens rotated clockwise by an angle of 45° with respect to the y-axis. Without rotation the lens focuses in the x-direction with a refractive power *D (D>0)* (see the positive lens in Fig. 1.21). We assume that the incident spherical wave has the same curvature with respect to x- and y-axis and its curvature matrix is thus given by:

$$\boldsymbol{R}_1 = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

The submatrices for the rotated cylinder lens are:

$$\boldsymbol{A} = \boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \boldsymbol{C} = \begin{pmatrix} -D\cos^2\theta & \dfrac{D}{2}\sin(2\theta) \\ \dfrac{D}{2}\sin(2\theta) & -D\sin^2\theta \end{pmatrix}$$

Insertion into the ABCD law (1.91) yields for the new curvature matrix (with θ=45°):

$$\boldsymbol{R}_2 = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} -RD/2 + 1 & RD/2 \\ RD/2 & -RD/2 + 1 \end{pmatrix}^{-1} .$$

$$= \frac{R}{1-RD} \begin{pmatrix} 1 - RD/2 & -RD/2 \\ -RD/2 & 1 - RD/2 \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{pmatrix}$$

With (1.93) - (1.96) we get:

$$R_{xx}^* = \frac{R}{1-RD} \,,\; e_x^* = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad R_{yy}^* = R \,,\; e_y^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

As expected the new coordinate frame is rotated by -45° with respect to the old one and we obtain the imaging condition $1/R - 1/R_{xx}^* = D$ in the x*-direction.

1.2.7 Eigensolutions and Eigenvalues

As mentioned in the introductory remarks on optical resonators we are mainly interested in light intensity distributions on the resonator mirrors that reproduce themselves after each round trip in the resonator. In terms of ray propagation we are looking for spherical wavefronts whose radii of curvature are not affected by the resonator round trip.

If M denotes the general ray transfer matrix of an optical system, we can always find self-reproducing spherical waves with radius of curvature R by applying the ABCD law (1.90) with $R_2=R_1=R$:

$$R = \frac{A R + B}{C R + D} \tag{1.97}$$

There are generally two solutions to this equation and these two self-reproducing wave curvatures R_a,R_b are linked to the two eigenvectors v_a, v_b and eigenvalues μ_a,μ_b of the ray transfer matrix M defined by:

$$\mu_{a,b} v_{a,b} = M\, v_{a,b} \tag{1.98}$$

The effect of the optical system on rays defined by eigenvectors is a multiplication of both the coordinate x and the angle α by a factor of μ. The absolute value of the eigenvalue is therefore called the magnification of the optical system. All rays defining a self-reproducing spherical wavefront according to (1.97) are automatically eigenvectors of the ray transfer matrix, since:

$$R_2 = \frac{x_2}{\alpha_2} = \frac{\mu x_1}{\mu \alpha_1} = \frac{x_1}{\alpha_1} = R_1 \tag{1.99}$$

For every 2x2 ray transfer matrix M *with* $C \neq 0$ the eigenvalues and corresponding reproducing radii of curvature are given by:

$$\mu_a = \frac{A+D}{2} + \sqrt{\left(\frac{A+D}{2}\right)^2 - 1} \qquad R_a = \frac{A-D + \sqrt{(A+D)^2 - 4}}{2C} \qquad (1.100)\ (1.101)$$

$$\mu_b = \frac{A+D}{2} - \sqrt{\left(\frac{A+D}{2}\right)^2 - 1} \qquad R_b = \frac{A-D - \sqrt{(A+D)^2 - 4}}{2C} \qquad (1.102)\ (1.103)$$

An incident beam with diameter d and radius of curvature $R_{a,b}$ will emerge from the optical system without a change in curvature but with a diameter $|\mu_{a,b}|d$.

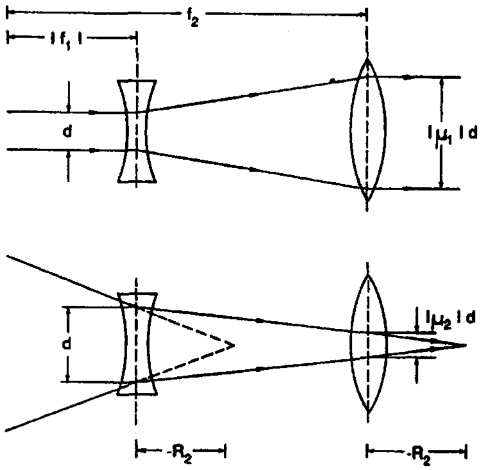


Fig. 1.29 Self-reproducing spherical waves in a Galilean Telescope.

Example: Galilean Telescope

The Galilean Telescope comprises a focusing lens with focal length f_2 and a negative lens with focal length $-f_1$ $(f_1>0)$ which are spaced such that they have a common focal point (Fig. 1.29). The ray transfer matrix for the whole system reads:

$$M_{GT} = \begin{pmatrix} \frac{f_2}{f_1} & f_2 - f_1 \\ 0 & \frac{f_1}{f_2} \end{pmatrix}$$

Equations (1.100-1.103) yield for the eigensolutions:

$$\mu_a = \frac{f_2}{f_1} \qquad R_a = \infty$$

$$\mu_b = \frac{f_1}{f_2} \qquad R_b = -\frac{(f_1 f_2)^2}{f_2 - f_1} < 0$$

The eigensolution given by the first row is probably familiar to all readers: a plane wave will reemerge with a plane wavefront, but the beam diameter has increased by f_2/f_1. There is, however, a second spherical wavefront that reproduces its radius of curvature. This eigensolution of the telescope is often referred to as the convergent wave and its beam diameter has decreased by f_1/f_2.

1.3 Optical Resonators and Ray Transfer Matrices

An optical resonator usually consists of two spherical mirrors with radius of curvatures ρ_1, ρ_2 separated by a distance L (Fig.1.30). It is customary to replace each mirror by two lenses with focal length $f_i = \rho_i$ and locate the reference planes in between. This technique generally simplifies the ray transfer matrix $(A=D)$ and allows for an easier memorizing of the matrices. The reference planes are the mirror surfaces, which means a plane wave in this representation is actually a spherical wave having the curvature of the mirror in real space. The ray transfer matrix for the round trip inside an arbitrary optical resonator starting at mirror 1 is given by:

$$M_{RES} = \begin{pmatrix} 2g_1g_2-1 & 2Lg_2 \\ \dfrac{(2g_1g_2-1)^2-1}{2Lg_2} & 2g_1g_2-1 \end{pmatrix} \tag{1.104}$$

with $g_i = 1 - \dfrac{L}{\rho_i}$; $i=1,2$

The parameters g_1 and g_2 are called the g-parameters of the optical resonator. Note that the radius of curvature is positive for a concave mirror (focusing mirror) and negative for a convex mirror! According to (1.104), the imaging properties of two-mirror resonators are fully defined by the g-parameters and the mirror spacing L. We can further simplify the ray transfer matrix M_{RES} by introducing the *equivalent G-Parameter* $G=2g_1g_2-1$:

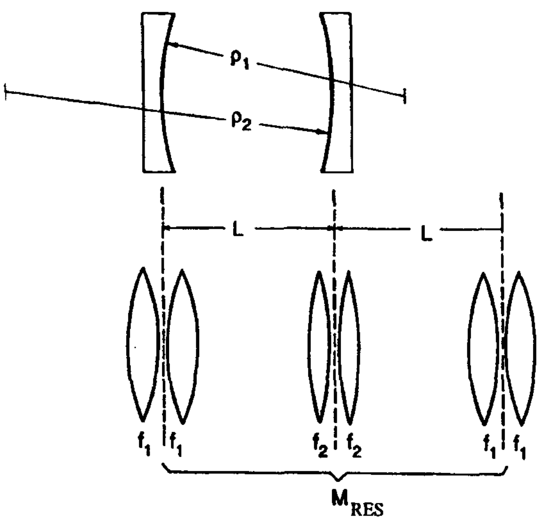


Fig. 1.30 A round trip in an optical resonator can be described as a transit in an equivalent lens waveguide. Each mirror has been replaced by a lens pair with focal length $f_i = \rho_i$.

$$M_{RES} = \begin{pmatrix} G & 2Lg_2 \\ \frac{G^2-1}{2Lg_2} & G \end{pmatrix} \tag{1.105}$$

We can now use this matrix to find eigensolutions of the resonator. We are looking for spherical wavefronts starting on mirror 1 which reproduce themselves after one round trip. Application of (1.100)-(1.103) yields for the eigenvalues and the corresponding self-reproducing wave curvatures:

$$\mu_a = G + \sqrt{G^2 - 1} \qquad R_a = \frac{+2Lg_2}{\sqrt{G^2 - 1}} \qquad (1.106)\ (1.107)$$

$$\mu_b = G - \sqrt{G^2 - 1} \qquad R_b = \frac{-2Lg_2}{\sqrt{G^2 - 1}} \qquad (1.108)\ (1.109)$$

Note that R_a and R_b refer to the dotted planes in Fig. 1.30. According to these eigensolutions we can distinguish between three different types of optical resonators:

1) |G|>1, equivalent to $|g_1g_2|>1$
We can find two spherical waves with real values for the radii of curvature $R_{a,b}$ reproducing themselves inside the resonator. If the beam diameter on mirror 1 is d, the spherical wave with radius of curvature R_a increases its diameter by $|\mu_a|$ every round trip (Fig. 1.31a). This eigensolution is called the divergent wave. The second eigensolution, the convergent wave leads to a decrease of the beam diameter by $|\mu_b|$ per round trip. Resonators having these ray propagation properties are called **unstable resonators**.

2) |G|=1, equivalent to $|g_1g_2|=1$
Both radii of curvature R_a and R_b are infinite and both eigenvalues are equal to 1. This means that a plane wavefront is coming back planar after the round trip without change in diameter. These resonators are referred to as the **resonators on the stability boundaries**. The plane-plane resonator in Fig. 1.31b is one example of such a resonator.

3) |G|<1, equivalent to $|g_1g_2|<1$
The radii of curvature and the eigenvalues are all complex numbers. This is a very puzzling result since we do not know what to make of complex curvatures. We can only interpret it in such a way that in geometrical optics no eigensolutions can be found in this type of resonator. We shall see in Section 2 that eigensolutions exist, but they can only be calculated by applying diffraction theory. These resonators are called **stable resonators**.

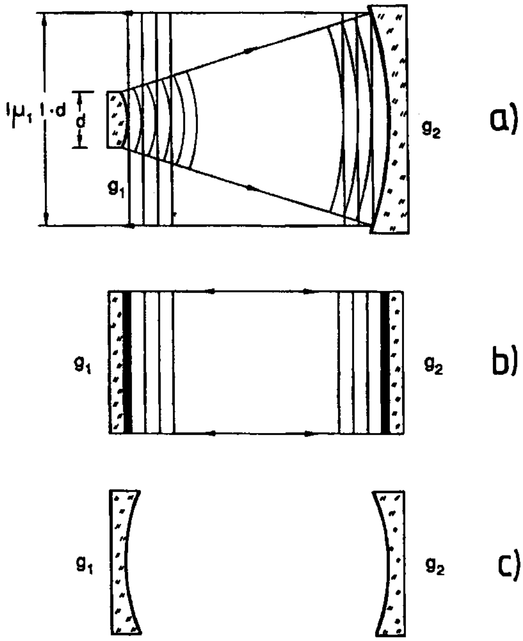


Fig. 1.31 The three types of optical resonators with spherical mirrors. a) unstable resonator, b) plane-plane resonator as an example of a resonator on the stability boundaries, c) stable resonator having no eigensolutions in geometrical optics.

There is a different and more commonly used approach to define the three resonator types. It is the tracking of rays launched into the resonator parallel to the optical axis as a function of the number of round trips. Using Sylvester's Theorem (1.46) the ray coordinate after N round trips reads:

$$x_{N+1} = (G\ \sin[N\Phi] - \sin[(N-1)\Phi])\ x_1 \qquad (1.110)$$

with $\cos\Phi = G$

The ray stays confined within the resonator if $|G| \leq 1$ holds. For $|G| > 1$, $\cos\Phi$ has to be replaced by the hyperbolic function, which means that x_N increases exponentially with the number of round trips. The ray leaves the system. This makes it more understandable why resonators with $|G| < 1$ are called stable.

Optical resonators with two mirrors can be visualized in a diagram where the g-parameters represent the coordinate axes (Fig. 1.32). This diagram is referred to as the g-diagram or, more often, as the stability diagram. A resonator defined by the g-parameters g_1 and g_2 is represented by a point in the stability diagram. Unfortunately, this representation is not unambiguous because the mirror spacing is not included.

A special class of resonators are the confocal resonators with both mirrors having a common focal point. The confocal condition reads:

$$g_1 + g_2 = 2g_1g_2 \tag{1.111}$$

Examples:

ρ_1=1m, ρ_2=1m,	L=1m	⇒	g_1=0.0, g_2=0.0	confocal
ρ_1=2m, ρ_2=∞,	L=0.5m	⇒	g_1=0.5, g_2=1.0	stable
ρ_1=-0.5m, ρ_2=1.5m,	L=0.5m	⇒	g_1=2.0, g_2=0.66	unstable, confocal

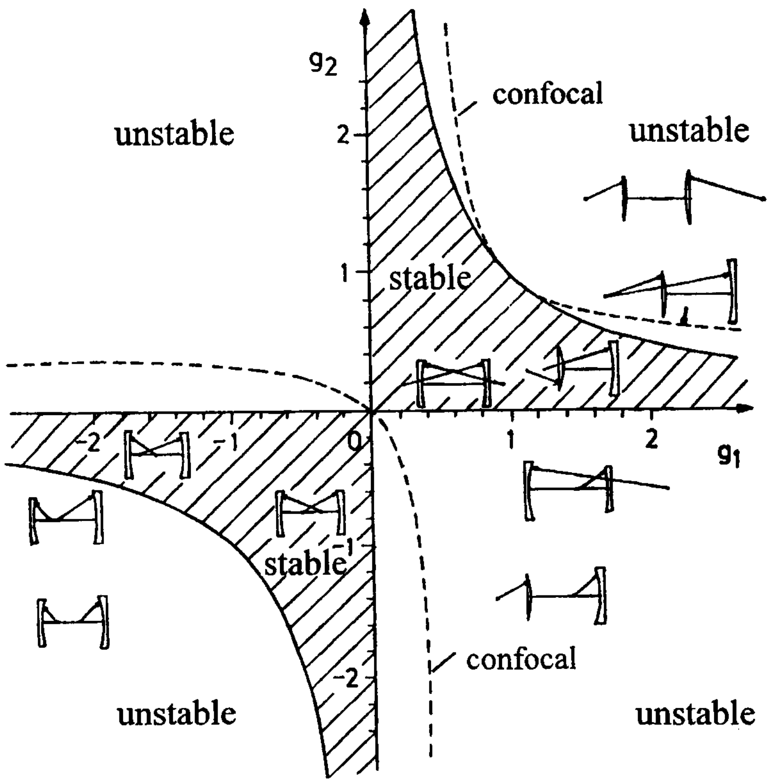


Fig. 1.32 The stability diagram of optical resonators with two spherical mirrors.