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Incidences

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Abstract

We survey recent progress related to the following general problem in combinatorial geometry: What is the maximum number of incidences between m *points and* n *members taken from a fixed family of curves or surfaces in* d*-space? Results of this kind have found numerous applications to geometric problems related to the distribution of distances among points, to questions in additive number theory, in analysis, and in computational geometry.*

1. Introduction

The problem and its relatives. Let P be a set of m distinct points, and let L be a set of n distinct lines in the plane. Let $I(P, L)$ denote the number of *incidences* between the points of P and the lines of L, i.e.,

$$I(P, L) = |\{(p, \ell)|\ p \in P, \ell \in L, p \in \ell\}|.$$

How large can $I(P, L)$ be? More precisely, determine or estimate $\max_{|P|=m,|L|=n} I(P, L)$.

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This simplest formulation of the incidence problem, due to Erdős and first settled by Szemerédi and Trotter, has been the starting point of extensive research that has picked up considerable momentum during the past two decades. It is the purpose of this survey to review the results obtained so far, describe the main techniques used in the analysis of this problem, and discuss many variations and extensions.

The problem can be generalized in many natural directions. One can ask the same question when the set L of lines is replaced by a set C of n curves of some other simple shape; the two cases involving respectively unit circles and arbitrary circles are of particular interest—see below.

A related problem involves the same kind of input—a set P of m points and a set C of n curves, but now we assume that no point of P lies on any curve of C. Let $\mathcal{A}(C)$ denote the *arrangement* of the curves of C, i.e., the decomposition of the plane into connected open cells of dimensions 0, 1, and 2 induced by drawing the elements of C. These cells are called *vertices, edges,* and *faces* of the arrangement, respectively. The total number of these cells is said to be the *combinatorial complexity* of the arrangement. See [21, 46] for details concerning arrangements. The combinatorial complexity of a single *face* is defined as the number of lower dimensional cells (i.e., vertices and edges) belonging to its boundary. The points of P then mark certain faces in the arrangement $\mathcal{A}(C)$ of the curves (assume for simplicity that there is at most one point of P in each face), and the goal is to establish an upper bound on $K(P, C)$, the combined combinatorial complexity of the marked faces. This problem is often referred to in the literature as the *Many-Faces Problem.*

One can extend the above questions to d-dimensional spaces, for $d > 2$. Here we can either continue to consider incidences between points and *curves,* or incidences between points and *surfaces* of any larger dimension k, $1 < k < d$. In the special case when $k = d-1$, we may also wish to study the natural generalization of the 'many-faces problem' described in the previous paragraph: to estimate the total combinatorial complexity of m marked (d-dimensional) cells in the arrangement of n given surfaces.

All of the above problems have many algorithmic variants. Perhaps the simplest question of this type is *Hopcroft's problem:* Given m points and n lines in the plane, how fast can one determine whether there exists any point that lies on any line? One can consider more general problems, like counting the number of incidences or reporting all of them, doing the same for a collection of curves rather than lines, computing m marked faces in an arrangement of n curves, and so on.

It turned out that two exciting *metric* problems (involving interpoint distances) proposed by Erdős in 1946 are strongly related to problems involving incidences.

1. *Repeated Distances Problem:* Given a set P of n points in the plane, what is the maximum number of pairs that are at distance exactly 1 from each other? To see the connection, let C be the set of unit circles centered at the points of P. Then two points $p, q \in P$ are at distance 1 apart if and only if the circle centered at

p passes through q and vice versa. Hence, $I(P, C)$ is twice the number of unit distances determined by P.

2. *Distinct Distances Problem:* Given a set P of n points in the plane, at least how many distinct distances must there always exist between its point pairs? Later we will show the connection between this problem and the problem of incidences between P and an appropriate set of circles of different radii.

Some other applications of the incidence problem and the many-faces problem will be reviewed at the end of this paper. They include the analysis of the maximum number of isosceles triangles, or triangles with a fixed area or perimeter, whose vertices belong to a planar point set; estimating the maximum number of mutually congruent simplices determined by a point set in higher dimensions; etc.

Historical perspective and overview. The first derivation of the tight upper bound

$$I(P, L) = \Theta(m^{2/3}n^{2/3} + m + n)$$

was given by Szemerédi and Trotter in their 1983 seminal paper [52]. They proved Erdős' conjecture, who found the matching lower bound (which was rediscovered many years later by Edelsbrunner and Welzl [25]). A slightly different lower bound construction was exhibited by Elekes [26] (see Section 2).

The original proof of Szemerédi and Trotter is rather involved, and yields a rather astronomical constant of proportionality hidden in the O-notation. A considerably simpler proof was found by Clarkson et al. [19] in 1990, using extremal graph theory combined with a geometric partitioning technique based on random sampling (see Section 3). Their paper contains many extensions and generalizations of the Szemerédi-Trotter theorem. Many further extensions can be found in subsequent papers by Edelsbrunner et al. [22, 23], by Agarwal and Aronov [1], by Aronov et al. [10], and by Pach and Sharir [41].

The next breakthrough occurred in 1997. In a surprising paper, Székely [51] gave an embarrassingly short proof (which we will review in Section 4) of the upper bound on $I(P, L)$ using a simple lower bound of Ajtai et al. [8] and of Leighton [34] on the *crossing number* of a graph G, i.e., the minimum number of edge crossings in the best drawing of G in the plane, where the edges are represented by Jordan arcs. In the literature this result is often referred to as the 'Crossing Lemma.' Székely's method can easily be extended to several other variants of the problem, but appears to be less general than the previous technique of Clarkson et al. [19].

Székely's paper has triggered an intensive re-examination of the problem. In particular, several attempts were made to improve the existing upper bound on the number of incidences between m points and n circles of arbitrary radii in the plane [42]. This was the simplest instance where Székely's proof technique failed. By combining Székely's method with a seemingly unrelated technique of Tamaki and Tokuyama [53] for cutting circles into 'pseudo-segments', Aronov and Sharir [13] managed to obtain an improved bound for this variant of the problem. Their work has then been followed by Agarwal et al. [2], who studied the complexity of many faces in arrangements of circles and pseudo-segments, and by Agarwal et al. [5], who extended this result to arrangements of pseudo-circles (see Section 5). Aronov et al. [11] generalized the problem to higher dimensions, while Sharir and Welzl [47] studied incidences between points and *lines* in three dimensions (see Section 6).

The related problems involving distances in a point set have had mixed success in recent studies. As for the Repeated Distances Problem in the plane, the best known upper bound on the number of times the same distance can occur among n points is $O(n^{4/3})$, which was obtained nearly 20 years ago by Spencer et al. [50]. This is far from the best known lower bound of Erdős, which is slightly super-linear (see [40]). The best known upper bound for the 3-dimensional case, due to Clarkson et al. [19], is roughly $O(n^{3/2})$, while the corresponding lower bound of Erdős is $\Omega(n^{4/3} \log \log n)$ (see [39]). Many variants of the problem have been studied; see, e.g., [28].

While the Repeated Distances problem has been "stuck" for quite some time, more progress has been made on the companion problem of Distinct Distances. In the planar case, L. Moser [38], Chung [17], and Chung et al. [18] proved that the number of distinct distances determined by n points in the plane is $\Omega(n^{2/3})$, $\Omega(n^{5/7})$, and $\Omega(n^{4/5} /\text{polylog}(n))$, respectively. Székely [51] managed to get rid of the polylogarithmic factor, while Solymosi and Tóth [48] improved this bound to $\Omega(n^{6/7})$. This was a real breakthrough. Their analysis was subsequently refined by Tardos [54] and then by Katz and Tardos [33], who obtained the current record of $\Omega(n^{(48-14e)/(55-16e)-\varepsilon})$, for any $\varepsilon > 0$, which is $\Omega(n^{0.8641})$. In spite of this steady improvement, there is still a considerable gap to the best known upper bound of $O(n/\sqrt{\log n})$, due to Erdős [27] (see Section 7). In three dimensions, a recent result of Aronov et al. [12] yields a lower bound of $\Omega(n^{77/141-\varepsilon})$, for any $\varepsilon > 0$, which is $\Omega(n^{0.546})$. This is still far from the best known upper bound of $O(n^{2/3})$. A better lower bound of $\Omega(n^{0.5794})$ in a special case (involving "homogeneous" point sets) has recently been given by Solymosi and Vu [49]. Their analysis also applies to higher-dimensional homogeneous point sets, and yields the bound $\Omega(n^{2/d-1/d^2})$. In a sill unpublished manuscript, the same authors have tackled the general case, and obtained a lower bound of $\Omega(n^{2/d-1/d(d+2)})$.

For other surveys on related subjects, consult [15], Chapter 4 of [36], [39], and [40].

2. Lower Bounds

We describe a simple construction due to Elekes [26] of a set P of m points and a set L of n lines, so that $I(P, L) = \Omega(m^{2/3}n^{2/3} + m + n)$. We fix two integer parameters ξ, η. We take P to be the set of all lattice points in $\{1, 2, \ldots, \xi\} \times \{1, 2, \ldots, 2\xi\eta\}$. The set L consists of all lines of the form $y = ax + b$, where a is an integer in the range $1, \ldots, \eta$, and b is an integer in the range $1, \ldots, \xi\eta$. Clearly, each line in L passes through exactly ξ points of P. See Figure 1.

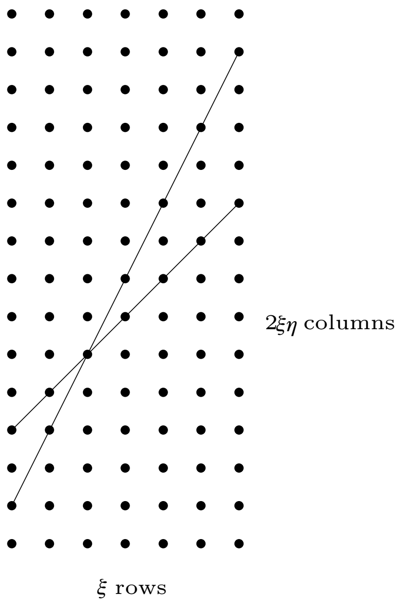

Figure 1. Elekes' construction.

We have $m = |P| = 2\xi^2\eta$, $n = |L| = \xi\eta^2$, and

$$I(P, L) = \xi|L| = \xi^2\eta^2 = \Omega(m^{2/3}n^{2/3}).$$

Given any sizes m, n so that $n^{1/2} \le m \le n^2$, we can find ξ, η that give rise to sets P, L whose sizes are within a constant factor of m and n, respectively. If m lies outside this range then $m^{2/3}n^{2/3}$ is dominated by $m + n$, and then it is trivial to construct sets P, L of respective sizes m, n so that $I(P, L) = \Omega(m + n)$. We have thus shown that

$$I(P, L) = \Omega(m^{2/3}n^{2/3} + m + n).$$

We note that this construction is easy to generalize to incidences involving other curves. For example, we can take P to be the grid $\{1, 2, \ldots, \xi\} \times \{1, 2, \ldots, 3\xi^2\eta\}$, and define C to be the set of all parabolas of the form $y = ax^2 + bx + c$, where $a \in \{1, \ldots, \eta\}$, $b \in \{1, \ldots, \xi\eta\}$, $c \in \{1, \ldots, \xi^2\eta\}$. Now we have $m = |P| = 3\xi^3\eta$, $n = |C| = \xi^3\eta^3$, and

$$I(P, C) = \xi|C| = \xi^4\eta^3 = \Omega(m^{1/2}n^{5/6}).$$

Note that in the construction we have $m = O(n)$. When m is larger, we use the preceding construction for points and lines, which can be easily transformed into a construction for points and parabolas, to obtain the overall lower bound for points and parabolas:

$$I(P, C) = \begin{cases} \Omega(m^{2/3}n^{2/3} + m), & \textit{if } m \ge n \\ \Omega(m^{1/2}n^{5/6} + n), & \textit{if } m \le n. \end{cases}$$

These constructions can be generalized to incidences involving graphs of polynomials of higher degrees.

From incidences to many faces. Let P be a set of m points and L a set of n lines in the plane, and put $I = I(P, L)$. Fix a sufficiently small parameter $\varepsilon > 0$, and replace each line $\ell \in L$ by two lines ℓ^+, ℓ^-, obtained by translating ℓ parallel to itself by distance ε in the two possible directions. We obtain a new collection L' of $2n$ lines. If ε is sufficiently small then each point $p \in P$ that is incident to $k \geq 2$ lines of L becomes a point that lies in a small face of $\mathcal{A}(L')$ that has $2k$ edges; note also that the circle of radius ε centered at p is tangent to all these edges. Moreover, these faces are distinct for different points p, when ε is sufficiently small.

We have thus shown that $K(P, L') \geq 2I(P, L) - 2m$ (where the last term accounts for points that lie on just one line of L). In particular, in view of the preceding construction, we have, for $|P| = m$, $|L| = n$,

$$K(P, L) = \Omega(m^{2/3}n^{2/3} + m + n).$$

An interesting consequence of this construction is as follows. Take $m = n$ and sets P, L that satisfy $I(P, L) = \Theta(n^{4/3})$. Let C be the collection of the $2n$ lines of L' and of the n circles of radius ε centered at the points of P. By applying an inversion,[1] we can turn all the curves in C into circles. We thus obtain a set C' of $3n$ circles with $\Theta(n^{4/3})$ tangent pairs. If we replace each of the circles centered at the points of P by circles with a slightly larger radius, we obtain a collection of $3n$ circles with $\Theta(\mathrm{n}^{4/3})$ *empty lenses,* namely faces of degree 2 in their arrangement. Empty lenses play an important role in the analysis of incidences between points and circles; see Section 5.

Lower bounds for incidences with unit circles. As noted, this problem is equivalent to the problem of Repeated Distances. Erdős [27] has shown that, for the vertices of an $n^{1/2} \times n^{1/2}$ grid, there exists a distance that occurs $\Omega(n^{1+\mathrm{c}/\log\log n})$ times, for an appropriate absolute constant $c > 0$. The details of this analysis, based on number-theoretic considerations, can be found in the monographs [36] and [40].

Lower bounds for incidences with arbitrary circles. As we will see later, we are still far from a sharp bound on the number of incidences between points and circles, especially when the number of points is small relative to the number of circles.

By taking sets P of m points and L of n lines with $I(P, L) = \Theta(m^{2/3}n^{2/3} + m + n)$, and by applying inversion to the plane, we obtain a set C of n circles and a set P' of m points with $I(P', C) = \Theta(m^{2/3}n^{2/3} + m + n)$. Hence the maximum number of incidences between m points and n circles is $\Omega(m^{2/3}n^{2/3} + m + n)$. However, we can slightly increase this lower bound, as follows.

[1] An inversion about, say, the unit circle centered at the origin, maps each point (x, y) to the point $(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. It maps lines to circles passing through the origin.

Let P be the set of vertices of the $m^{1/2} \times m^{1/2}$ integer lattice. As shown by Erdős [27], there are $t = \Theta(m/\sqrt{\log m})$ distinct distances between pairs of points of P. Draw a set C of mt circles, centered at the points of P and having as radii the t possible inter-point distances. Clearly, the number of incidences $I(P, C)$ is exactly $m(m-1)$. If the bound on $I(P, C)$ were $O(m^{2/3}n^{2/3} + m + n)$, then we would have

$$m(m-1) = I(P, C) = O(m^{2/3}(mt)^{2/3} + mt) = O(m^2/((\log m)^{1/3}),$$

a contradiction. This shows that, under the most optimistic conjecture, the maximum value of $I(P, C)$ should be larger than the corresponding bound for lines by at least some polylogarithmic factor.

3. Upper Bounds for Incidences via the Partition Technique

The approach presented in this section is due to Clarkson et al. [19]. It predated Székely's method, but it seems to be more flexible, and suitable for generalizations. It can also be used for the refinement of some proofs based on Székely's method.

We exemplify this technique by establishing an upper bound for the number of point-line incidences. Let P be a set of m points and L a set of n lines in the plane. First, we give a weaker bound on $I(P, L)$, as follows. Consider the bipartite graph $H \subseteq P \times L$ whose edges represent all incident pairs (p, ℓ), for $p \in P$, $\ell \in L$. Clearly, H does not contain $K_{2,2}$ as a subgraph. By the Kövari-Sós-Turán Theorem in extremal graph theory (see [40]), we have

$$I(P, L) = O(mn^{1/2} + n). \tag{1}$$

To improve this bound, we partition the plane into subregions, apply this bound within each subregion separately, and sum up the bounds. We fix a parameter r, $1 \le r \le n$, whose value will be determined shortly, and construct a so-called $(1/r)$-*cutting* of the arrangement $\mathcal{A}(L)$ of the lines of L. This is a decomposition of the plane into $O(r^2)$ vertical trapezoids with pairwise disjoint interiors, such that each trapezoid is crossed by at most n/r lines of L. The existence of such a cutting has been established by Chazelle and Friedman [16], following earlier and somewhat weaker results of Clarkson and Shor [20]. See [36] and [46] for more details.

For each cell τ of the cutting, let P_τ denote the set of points of P that lie in the interior of τ, and let L_τ denote the set of lines that cross τ. Put $m_\tau = |P_\tau|$ and $n_\tau = |L_\tau| \le n/r$. Using (1), we have

$$I(P_\tau, L_\tau) = O\left(m_\tau n_\tau^{1/2} + n_\tau\right) = O\left(m_\tau \left(\frac{n}{r}\right)^{1/2} + \frac{n}{r}\right).$$

Summing this over all $O(r^2)$ cells τ, we obtain a total of

$$\sum_{\tau} I(P_\tau, L_\tau) = O\left(m\left(\frac{n}{r}\right)^{1/2} + nr\right)$$

incidences. This does not quite complete the count, because we also need to consider points that lie on the boundary of the cells of the cutting. A point p that lies in the relative interior of an edge e of the cutting lies on the boundary of at most two cells, and any line that passes through p, with the possible exception of the single line that contains e, crosses both cells. Hence, we may simply assign p to one of these cells, and its incidences (except for at most one) will be counted within the subproblem associated with that cell. Consider then a point p which is a vertex of the cutting, and let l be a line incident to p. Then l either crosses or bounds some adjacent cell τ. Since a line can cross the boundary of a cell in at most two points, we can charge the incidence (p, l) to the pair (l, τ), use the fact that no cell is crossed by more than n/r lines, and conclude that the number of incidences involving vertices of the cutting is at most $O(nr)$.

We have thus shown that

$$I(P, L) = O\left(m\left(\frac{n}{r}\right)^{1/2} + nr\right).$$

Choose $r = m^{2/3}/n^{1/3}$. This choice makes sense provided that $1 \le r \le n$. If $r < 1$, then $m < n^{1/2}$ and (1) implies that $I(P, L) = O(n)$. Similarly, if $r > n$ then $m > n^2$ and (1) implies that $I(P, L) = O(m)$. If r lies in the desired range, we get $I(P, L) = O(m^{2/3}n^{2/3})$. Putting all these bounds together, we obtain the bound

$$I(P, L) = O(m^{2/3}n^{2/3} + m + n),$$

as required.

Remark. An equivalent statement is that, for a set P of m points in the plane, and for any integer $k \le m$, the number of lines that contain at least k points of P is at most

$$O\left(\frac{m^2}{k^3} + \frac{m}{k}\right).$$

Discussion. The cutting-based method is quite powerful, and can be extended in various ways. The crux of the technique is to derive somehow a weaker (but easier) bound on the number of incidences, construct a $(1/r)$-cutting of the set of curves, obtain the corresponding decomposition of the problem into $O(r^2)$ subproblems, apply the weaker bound within each subproblem, and sum up the bounds to obtain the overall bound. The work by Clarkson et al. [19] contains many such extensions.

Let us demonstrate this method to obtain an upper bound for the number of incidences between a set P of m points and a set C of n arbitrary circles in the plane.

Here the forbidden subgraph property is that the incidence graph $H \subseteq P \times C$ does not contain $K_{3,2}$ as a subgraph, and thus (see [40])

$$I(P, C) = O(mn^{2/3} + n).$$

We construct a $(1/r)$-cutting for C, apply this weak bound within each cell τ of the cutting, and handle incidences that occur on the cell boundaries exactly as above, to obtain

$$I(P, C) = \sum_{\tau} I(P_\tau, C_\tau) = O\left(m\left(\frac{n}{r}\right)^{2/3} + nr\right).$$

With an appropriate choice of $r = m^{3/5}/n^{1/5}$, this becomes

$$I(P, C) = O(m^{3/5}n^{4/5} + m + n).$$

However, as we shall see later, in Section 5, this bound can be considerably improved.

The case of a set C of n unit circles is handled similarly, observing that in this case the intersection graph H does not contain $K_{2,3}$. This yields the same upper bound $I(P, C) = O(mn^{1/2} + n)$, as in (1). The analysis then continues exactly as in the case of lines, and yields the bound

$$I(P, C) = O(m^{2/3}n^{2/3} + m + n).$$

We can apply this bound to the Repeated Distances Problem, recalling that the number of pairs of points in an n-element set of points in the plane that lie at distance exactly 1 from each other, is half the number of incidences between the points and the unit circles centered at them. Substituting $m = n$ in the above bound, we thus obtain that the number of repeated distances is at most $O(n^{4/3})$. This bound is far from the best known lower bound, mentioned in Section 2, and no improvement has been obtained since its original derivation in [50] in 1984.

As a matter of fact, this approach can be extended to any collection C of curves that have "d degrees of freedom", in the sense that any d points in the plane determine at most $t = O(1)$ curves from the family that pass through all of them, and any pair of curves intersect in only $O(1)$ points. The incidence graph does not contain $K_{d,t+1}$ as a subgraph, which implies that

$$I(P, C) = O(mn^{1-1/d} + n).$$

Combining this bound with a cutting-based decomposition yields the bound

$$I(P, C) = O(m^{d/(2d-1)}n^{(2d-2)/(2d-1)} + m + n).$$

Note that this bound extrapolates the previous bounds for the cases of lines ($d = 2$), unit circles ($d = 2$), and arbitrary circles ($d = 3$). See [42] for a slight generalization of this result, using Székely's method, outlined in the following section.

4. Incidences via Crossing Numbers—Székely's Method

A graph G is said to be *drawn* in the plane if its vertices are mapped to distinct points in the plane, and each of its edges is represented by a Jordan are connecting the corresponding pair of points. It is assumed that no edge passes through any vertex other than its endpoints, and that when two edges meet at a common interior point, they properly *cross* each other there, i.e., each curve passes from one side of the other curve to the other side. Such a point is called a *crossing*. In the literature, a graph drawn in the plane with the above properties is often called a *topological graph*. If, in addition, the edges are represented by straight-line segments, then the drawing is said to be a *geometric graph*.

As we have indicated before, Székely discovered that the analysis outlined in the previous section can be substantially simplified, applying the following so-called Crossing Lemma for graphs drawn in the plane.

Lemma 4.1 (Leighton [34], Ajtai et al. [8]) *Let G be a simple graph drawn in the plane with V vertices and E edges. If* $E > 4V$ *then there are* $\Omega(E^3/V^2)$ *crossing pairs of edges.*

To establish the lemma, denote by $\text{cr}(G)$ the minimum number of crossing pairs of edges in any 'legal' drawing of G. Since G contains too many edges, it is not planar, and therefore $\text{cr}(G) \geq 1$. In fact, using Euler's formula, a simple counting argument shows that $\text{cr(G)} \geq E - 3V + 6 > E - 3V$. We next apply this inequality to a random sample G' of G, which is an induced subgraph obtained by choosing each vertex of G independently with some probability p. By applying expectations, we obtain $\mathbf{E}[\text{cr}(G')] \geq \mathbf{E}[E'] - 3\mathbf{E}[V']$, where E', V' are the numbers of edges and vertices in G', respectively. This can be rewritten as $\text{cr}(G)p^4 \geq Ep^2 - 3Vp$, and choosing $p = 4V/E$ completes the proof of Lemma 4.1.

We remark that the constant of proportionality in the asserted bound, as yielded by the preceding proof, is 1/64, but it has been improved by Pach and Tóth [44]. They proved that $\text{cr}(G) \geq E^3/(33.75\text{V}^2)$ whenever $E \geq 7.5V$. In fact, the slightly weaker inequality $\text{cr}(G) \geq E^3/(33.75V^2) - 0.9V$ holds without any extra assumption. We also note that it is crucial that the graph G be *simple* (i.e., any two vertices be connected by at most one edge), for otherwise no crossing can be guaranteed, regardless of how large E is.

Let P be a set of m points and L a set of n lines in the plane. We associate with P and L the following plane drawing of a graph G. The vertices of (this drawing of) G are the points of P. For each line $\ell \in L$, we connect each pair of points of $P \cap \ell$ that are

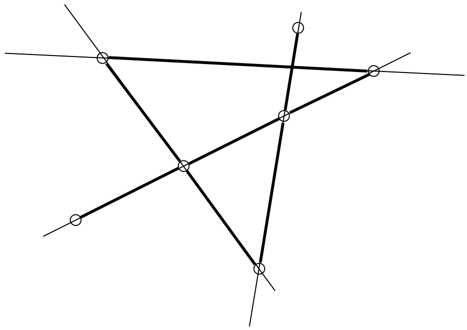

Figure 2. Székely's graph for points and lines in the plane.

consecutive along ℓ by an edge of G, drawn as the straight segment between these points (which is contained in ℓ). See Figure 2 for an illustration. Clearly, G is a simple graph, and, assuming that each line of L contains at least one point of P, we have $V = m$ and $E = I(P, L) - n$ (the number of edges along a line is smaller by 1 than the number of incidences with that line). Hence, either $E < 4V$, and then $I(P, L) < 4m + n$, or $\text{cr}(G) \geq E^3/(cV^2) = (I(P, L) - n)^3/(cm^2)$. However, we have, trivially, $\text{cr}(G) \leq \binom{n}{2}$, implying that $I(P, L) \leq (c/2)^{1/3} m^{2/3} n^{2/3} + n \leq 2.57 m^{2/3} n^{2/3} + n$.

Extensions: Many faces and unit circles. The simple idea behind Székely's proof is quite powerful, and can be applied to many variants of the problem, as long as the corresponding graph G is simple, or, alternatively, has a bounded edge multiplicity. For example, consider the case of incidences between a set P of m points and a set C of n unit circles. Draw the graph G exactly as in the case of lines, but only along circles that contain more than two points of P, to avoid loops and multiple edges along the same circle. We have $V = m$ and $E \geq I(P, C) - 2n$. In this case, G need not be simple, but the maximum edge multiplicity is at most two; see Figure 3. Hence, by

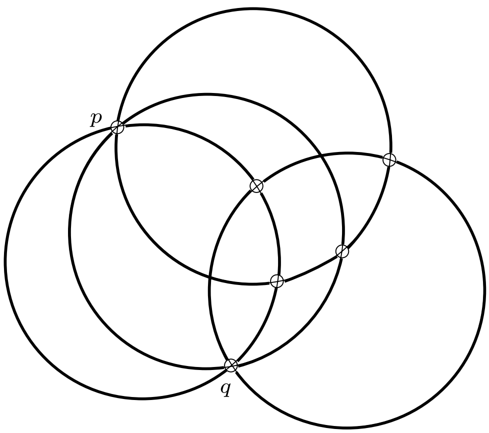

Figure 3. Székely's graph for points and unit circles in the plane: The maximum edge multiplicity is two—see the edges connecting p and q.

deleting at most half of the edges of G we make it into a simple graph. Moreover, $\mathrm{cr}(G) \le n(n-1)$, so we get $I(P, C) = O(m^{2/3}n^{2/3} + m + n)$, again with a rather small constant of proportionality.

We can also apply this technique to obtain an upper bound on the complexity of many faces in an arrangement of lines. Let P be a set of m points and L a set of n lines in the plane, so that no point lies on any line and each point lies in a distinct face of $\mathcal{A}(L)$. The graph G is now constructed in the following slightly different manner. Its vertices are the points of P. For each $\ell \in L$, we consider all faces of $\mathcal{A}(L)$ that are marked by points of P, are bounded by ℓ and lie on a fixed side of ℓ. For each pair f_1, f_2 of such faces that are consecutive along ℓ (the portion of ℓ between ∂f_1 and ∂f_2 does not meet any other marked face on the same side), we connect the corresponding marking points p_1, p_2 by an edge, and draw it as a polygonal path $p_1q_1q_2p_2$, where $q_1 \in \ell \cap \partial f_1$ and $q_2 \in \ell \cap \partial f_2$. We actually shift the edge slightly away from ℓ so as to avoid its overlapping with edges drawn for faces on the other side of ℓ. The points q_1, q_2 can be chosen in such a way that a pair of edges meet each other only at intersection points of pairs of lines of L. See Figure 4. Here we have $V = m$, $E \ge K(P, L) - 2n$, and $\mathrm{cr}(G) \le 2n(n-1)$ (each pair of lines can give rise to at most four pairs of crossing edges, near the same intersection point). Again, G is not simple, but the maximum edge multiplicity is at most two, because, if two faces f_1, f_2 are connected along a line ℓ, then ℓ is a common external tangent to both faces. Since f_1 and f_2 are disjoint convex sets, they can have at most two external common tangents. Hence, arguing as above, we obtain $K(P, L) = O(m^{2/3}n^{2/3} + m + n)$. We remark that the same upper bound can also be obtained via the partition technique, as shown by Clarkson et al. [19]. Moreover, in view of the discussion in Section 2, this bound is tight.

However, Székely's technique does not always apply. The simplest example where it fails is when we want to establish an upper bound on the number of incidences

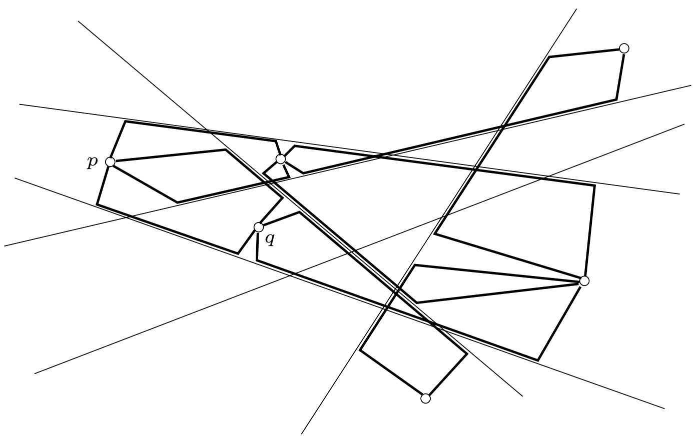

Figure 4. Székely's graph for face-marking points and lines in the plane. The maximum edge multiplicity is two—see, e.g., the edges connecting p and q.

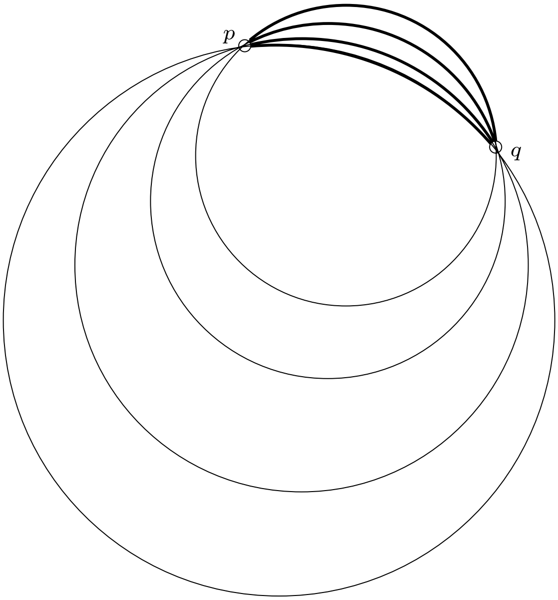

Figure 5. Székely's graph need not be simple for points and arbitrary circles in the plane.

between points and circles of arbitrary radii. If we follow the same approach as for equal circles, and construct a graph analogously, we may now create edges with arbitrarily large multiplicities, as is illustrated in Figure 5. We will tackle this problem in the next section.

Another case where the technique fails is when we wish to bound the total complexity of many faces in an arrangement of line *segments*. If we try to construct the graph in the same way as we did for full lines, the faces may not be convex any more, and we can create edges of high multiplicity; see Figure 6.

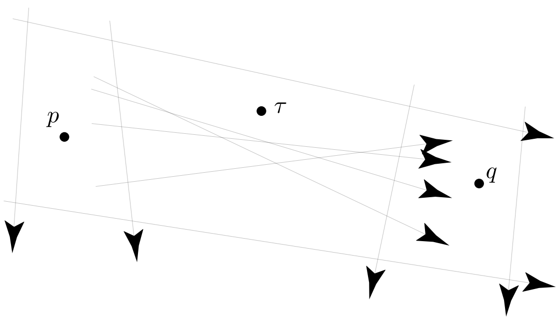

Figure 6. Székely's graph need not be simple for marked faces and segments in the plane: An arbitrarily large number of segments bounds all three faces marked by the points $p, q, r,$ so the edges (p, r) and (r, q) in Székely's graph have arbitrarily large multiplicity.

5. Improvements by Cutting into Pseudo-segments

Consider the case of incidences between points and circles of arbitrary radii. One way to overcome the technical problem in applying Székely's technique in this case is to cut the given circles into arcs so that any two of them intersect at most once. We refer to such a collection of arcs as a collection of *pseudo-segments.*

The first step in this direction has been taken by Tamaki and Tokuyama [53], who have shown that any collection C of n *pseudo-circles,* namely, closed Jordan curves, each pair of which intersect at most twice, can be cut into $O(n^{5/3})$ arcs that form a family of pseudosegments. The union of two arcs that belong to distinct pseudo-circles and connect the same pair of points is called a *lens.* Let $\chi(C)$ denote the minimum number of points that can be removed from the curves of C, so that any two members of the resulting family of arcs have at most one point in common. Clearly, every lens must contain at least one of these cutting points, so Tamaki and Tokuyama's problem asks in fact for an upper bound on the number of points needed to "stab" all lenses. Equivalently, this problem can be reformulated, as follows.

Consider a hypergraph H whose vertex set consists of the edges of the arrangement $\mathcal{A}(C)$, i.e., the arcs between two consecutive crossings. Assign to each lens a *hyperedge* consisting of all arcs that belong to the lens. We are interested in finding the *transversal number* (or the size of the smallest "hitting set") of H, i.e., the smallest number of vertices of H that can be picked with the property that every hyperedge contains at least one of them. Based on Lovász' analysis [35] (see also [40]) of the greedy algorithm for bounding the transversal number from above (i.e., for constructing a hitting set), this quantity is not much bigger than the size of the largest *matching* in H, i.e., the maximum number of pairwise disjoint hyperedges. This is the same as the largest number of pairwise non-overlapping lenses, that is, the largest number of lenses, no two of which share a common edge of the arrangement $\mathcal{A}(C)$ (see Figure 7). Viewing

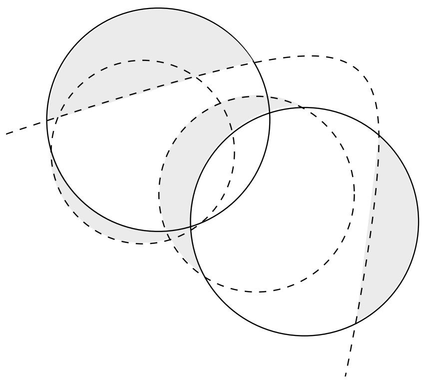

Figure 7. The boundaries of the shaded regions are nonoverlapping lenses in an arrangement of pseudo-circles. (Observe that the *regions* bounded by nonoverlapping lenses can overlap, as is illustrated here.)

such a family as a graph G, whose edges connect pairs of curves that form a lens in the family, Tamaki and Tokuyama proved that G does not contain $K_{3,3}$ as a subgraph, and this leads to the asserted bound on the number of cuts.

In order to establish an upper bound on the number of incidences between a set of m points P and a set of n circles (or pseudo-circles) C, let us construct a modified version G' of Székely's graph: its vertices are the points of P, and its edges connect adjacent pairs of points along the new pseudo-segment arcs. That is, we do not connect a pair of points that are adjacent along an original curve, if the arc that connects them has been cut by some point of the hitting set. Moreover, as in the original analysis of Székely, we do not connect points along pseudo-circles that are incident to only one or two points of P, to avoid loops and trivial multiplicities.

Clearly, the graph G' is simple, and the number E' of its edges is at least $I(P, C) - \chi(C) - 2n$. The crossing number of G' is, as before, at most the number of crossings between the original curves in C, which is at most $n(n-1)$. Using the Crossing Lemma (Lemma 4.1), we thus obtain

$$I(P, C) = O(m^{2/3}n^{2/3} + \chi(C) + m + n).$$

Hence, applying the Tamaki-Tokuyama bound on $\chi(C)$, we can conclude that

$$I(P, C) = O(m^{2/3}n^{2/3} + n^{5/3} + m).$$

An interesting property of this bound is that it is tight when $m \geq n^{3/2}$. In this case, the bound becomes $I(P, C) = O(m^{2/3}n^{2/3} + m)$, matching the lower bound for incidences between points and lines, which also serves as a lower bound for the number of incidences between points and circles or parabolas. However, for smaller values of m, the term $O(n^{5/3})$ dominates, and the dependence on m disappears. This can be rectified by combining this bound with a cutting-based problem decomposition, similar to the one used in the preceding section, and we shall do so shortly.

Before proceeding, though, we note that Tamaki and Tokuyama's bound is not tight. The best known lower bound is $\Omega(n^{4/3})$, which follows from the lower bound construction for incidences between points and lines. (That is, we have already seen that this construction can be modified so as to yield a collection C of n circles with $\Theta(n^{4/3})$ empty lenses. Clearly, each such lens requires a separate cut, so $\chi(C) = \Omega(n^{4/3})$.) Recent work by Alon et al. [9], Aronov and Sharir [13], and Agarwal et al. [5] has led to improved bounds. Specifically, it was shown in [5] that $\chi(C) = O(n^{8/5})$, for families C of *pseudo-parabolas* (graphs of continuous everywhere defined functions, each pair of which intersect at most twice), and, more generally, for families of x-*monotone* pseudo-circles (closed Jordan curves with the same property, so that the two portions of their boundaries connecting their leftmost and rightmost points are graphs of two continuous functions, defined on a common interval).

In certain special cases, including the cases of circles and of vertical parabolas (i.e., parabolas of the form $y = ax^2 + bx + c$), one can do better, and show that

$$\chi(C) = O(n^{3/2}k(n)),$$

where

$$\kappa(n) = (\log n)^{O(\alpha^2(n))},$$

and where $\alpha(n)$ is the extremely slowly growing inverse Ackermann's function. This bound was established in [5], and it improves a slightly weaker bound obtained by Aronov et al. [13]. The technique used for deriving this result is interesting in its own right, and raises several deep open problems, which we omit in this survey.

With the aid of this improved bound on $\chi(C)$, the modification of Székely's method reviewed above yields, for a set C of n circles and a set P of m points,

$$I(P, C) = O(m^{2/3}n^{2/3} + n^{3/2}\kappa(n) + m).$$

As already noted, this bound is tight when it is dominated by the first or last terms, which happens when m is roughly larger than $n^{5/4}$. For smaller values of m, we decompose the problem into subproblems, using the following so-called "dual" partitioning technique. We map each circle $(x - a)^2 + (y - b)^2 = \rho^2$ in C to the "dual" point $(a, b, \rho^2 - a^2 - b^2)$ in 3-space, and map each point (ξ, η) of P to the "dual" plane $z = -2\xi x - 2\eta y + (\xi^2 + \eta^2)$. As is easily verified, each incidence between a point of P and a circle of C is mapped to an incidence between the dual plane and point. We now fix a parameter r, and construct a $(1/r)$-cutting of the arrangement of the dual planes, which partitions $\mathbb{R}^3$ into $O(r^3)$ cells (which is a tight bound in the case of planes), each crossed by at most m/r dual planes and containing at most n/r^3 dual points (the latter property, which is not an intrinsic property of the cutting, can be enforced by further partitioning cells that contain more than n/r^3 points). We apply, for each cell τ of the cutting, the preceding bound for the set P_τ of points of P whose dual planes cross τ, and for the set C_τ of circles whose dual points lie in τ. (Some special handling of circles whose dual points lie on boundaries of cells of the cutting is needed, as in Section 3, but we omit the treatment of this special case.) This yields the bound

$$\begin{aligned} I(P, C) &= O(r^3) \cdot O\left(\left(\frac{m}{r}\right)^{2/3}\left(\frac{n}{r^3}\right)^{2/3} + \left(\frac{n}{r^3}\right)^{3/2}\kappa\left(\frac{n}{r^3}\right) + \frac{m}{r}\right) \\ &= O\left(m^{2/3}n^{2/3}r^{1/3} + \frac{n^{3/2}}{r^{3/2}}\kappa\left(\frac{n}{r^3}\right) + mr^2\right). \end{aligned}$$

Assume that m lies between $n^{1/3}$ and $n^{5/4}$, and choose $r = n^{5/11}/m^{4/11}$ in the last bound, to obtain

$$I(P, C) = O(m^{2/3}n^{2/3} + m^{6/11}n^{9/11}\kappa(m^3/n) + m + n).$$

It is not hard to see that this bound also holds for the complementary ranges of m.

6. Incidences in Higher Dimensions

It is natural to extend the study of incidences to instances involving points and curves or surfaces in higher dimensions. The case of incidences between points and (hyper)surfaces (mainly hyperplanes) has been studied earlier. Edelsbrunner et al. [23] considered incidences between points and planes in three dimensions. It is important to note that, without imposing some restrictions either on the set P of points or on the set H of planes, one can easily obtain $|P| \cdot |H|$ incidences, simply by placing all the points of P on a line, and making all the planes of H pass through that line. Some natural restrictions are to require that no three points be collinear, or that no three planes be collinear, or that the points be vertices of the arrangement $\mathcal{A}(H)$, and so on. Different assumptions lead to different bounds. For example, Agarwal and Aronov [1] proved an asymptotically tight bound $\Theta(m^{2/3}n^{d/3} + n^{d-1})$ for the number of incidences between n hyperplanes in d dimensions and $m > n^{d-2}$ vertices of their arrangement (see also [23]), as well as for the number of facets bounding m distinct cells in such an arrangement. Edelsbrunner and Sharir [24] considered the problem of incidences between points and hyperplanes in four dimensions, under the assumption that all points lie on the upper envelope of the hyperplanes. They obtained the bound $O(m^{2/3}n^{2/3} + m + n)$ for the number of such incidences, and applied the result to obtain the same upper bound on the number of bichromatic minimal distance pairs between a set of m blue points and a set of n red points in three dimensions. Another set of bounds and related results are obtained by Brass and Knauer [14], for incidences between m points and n planes in 3-space, and also for incidences in higher dimensions.

The case of incidences between points and *curves* in higher dimensions has been studied only recently. There are only two papers that address this problem. One of them, by Sharir and Welzl [47], studies incidences between points and lines in 3-space. The other, by Aronov et al. [11], is concerned with incidences between points and circles in higher dimensions. Both works were motivated by problems asked by Elekes. We briefly review these results in the following two subsections.

6.1. Points and Lines in Three Dimensions

Let P be a set of m points and L a set of n lines in 3-space. Without making some assumptions on P and L, the problem is trivial, for the following reason. Project P and L onto some generic plane. Incidences between points of P and lines of L are bijectively mapped to incidences between the projected points and lines, so we have $I(P, L) = O(m^{2/3}n^{2/3} + m + n)$. Moreover, this bound is tight, as is shown by the planar lower bound construction. (As a matter of fact, this reduction holds in any dimension $d \geq 3$.)

There are several ways in which the problem can be made interesting. First, suppose that the points of P are *joints* in the arrangement $\mathcal{A}(L)$, namely, each point is incident to at least three non-coplanar lines of L. In this case, one has $I(P, L) = O(n^{5/3})$ [47]. Note that this bound is independent of m. In fact, it is known that the number of joints is at most $O(n^{23/14} \log^{31/14} n)$, which is $O(n^{1.643})$ [45] (the best lower bound, based on lines forming a cube grid, is only $\Omega(n^{3/2})$).

For general point sets P, one can use a new measure of incidences, which aims to ignore incidences between a point and many incident coplanar lines. Specifically, we define the *plane cover* $\pi_L(p)$ of a point p to be the minimum number of planes that pass through p so that their union contains all lines of L incident to p, and define $I_c(P, L) = \sum_{p \in P} \pi_L(p)$. It is shown in [47] that

$$I_c(P, L) = O(m^{4/7} m^{5/7} + m + n),$$

which is smaller than the planar bound of Szemerédi and Trotter.

Another way in which we can make the problem "truly 3-dimensional" is to require that all lines in L be *equally inclined,* meaning that each of them forms a fixed angle (say, 45°) with the z-direction. In this case, every point of P that is incident to at least three lines of L is a joint, but this special case admits better upper bounds. Specifically, we have

$$I(P, L) = O(\min\left\{m^{3/4} n^{1/2} \kappa(m), m^{4/7} n^{5/7}\right\} + m + n).$$

The best known lower bound is

$$I(P, L) = \Omega(m^{2/3} n^{1/2}).$$

Let us briefly sketch the proof of the upper bound $O(m^{3/4} n^{1/2} \kappa(m))$. For each $p \in P$ let C_p denote the (double) cone whose apex is p, whose symmetry axis is the vertical line through p, and whose opening angle is 45°. Fix some generic horizontal plane π_0, and map each $p \in P$ to the circle $C_p \cap \pi_0$. Each line $\ell \in L$ is mapped to the point $\ell \cap \pi_0$, coupled with the projection ℓ^* of ℓ onto π_0. Note that an incidence between a point $p \in P$ and a line $\ell \in L$ is mapped to the configuration in which the circle dual to p is incident to the point dual to ℓ and the projection of ℓ passes through the center of the circle; see Figure 8. Hence, if a line ℓ is incident to several points $p_1, \ldots, p_k \in P$, then the dual circles $p_1^*, \ldots, p_k^*$ are all tangent to each other at the common point $\ell \cap \pi_0$. Viewing these tangencies as a collection of degenerate lenses, we can bound the overall number of these tangencies, which is equal to $I(P, L)$, by $O(n^{3/2} \kappa(n))$. By a slightly more careful analysis, again based on cutting, one can obtain the bound $O(m^{3/4} n^{1/2} \kappa(m))$.

6.2. *Points and Circles in Three and Higher Dimensions*

Let C be a set of n circles and P a set of m points in 3-space. Unlike in the case of lines, there is no obvious reduction of the problem to a planar one, because the projection of C onto some generic plane yields a collection of ellipses, rather than circles, which can cross each other at four points per pair. However, using a more refined analysis, Aronov et al. [11] have obtained the same asymptotic bound of

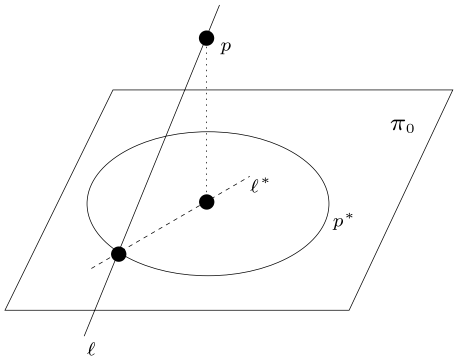

Figure 8. Transforming incidences between points and equally inclined lines to tangencies between circles in the plane.

$I(P, C) = O(m^{2/3}n^{2/3} + m^{6/11}n^{9/11}\kappa(m^3/n) + m + n)$ for $I(P, C)$. The same bound applies in any dimension $d \geq 3$.

7. Applications

The problem of bounding the number of incidences between various geometric objects is elegant and fascinating, and it has been mostly studied for its own sake. However, it is closely related to a variety of questions in combinatorial and computational geometry. In this section, we briefly review some of these connections and applications.

7.1. Algorithmic Issues

There are two types of algorithmic problems related to incidences. The first group includes problems where we wish to actually determine the number of incidences between certain objects, e.g., between given sets of points and curves, or we wish to compute (describe) a collection of marked faces in an arrangement of curves or surfaces. The second group contains completely different questions whose solution requires tools and techniques developed for the analysis of incidence problems.

In the simplest problem of the first kind, known as Hopcroft's problem, we are given a set P of m points and a set L of n lines in the plane, and we ask whether there exists at least one incidence between P and L. The best running time known for this problem is $O(m^{2/3}n^{2/3} \cdot 2^{O(\log^*(m+n))})$ [37] (see [31] for a matching lower bound). Similar running time bounds hold for the problems of counting or reporting all the incidences in $I(P, L)$. The solutions are based on constructing cuttings of an appropriate size and thereby obtaining a decomposition of the problem into subproblems, each of which can be solved by a more brute-force approach. In other words, the solution can be viewed as an implementation of the cutting-based analysis of the combinatorial bound for $I(P, L)$, as presented in Section 3.

The case of incidences between a set P of m points and a set C of n circles in the plane is more interesting, because the analysis that leads to the current best upper bound on $I(P, C)$ is not easy to implement. In particular, suppose that we have already cut the circles of C into roughly $O(n^{3/2})$ pseudo-segments (an interesting and non-trivial algorithmic task in itself), and we now wish to compute the incidences between these pseudo-segments and the points of P. Székely's technique is non-algorithmic, so instead we would like to apply the cutting-based approach to these pseudo-segments and points. However, this approach, for the case of lines, after decomposing the problem into subproblems, proceeds by duality. Specifically, it maps the points in a subproblem to dual lines, constructs the arrangement of these dual lines, and locates in the arrangement the points dual to the lines in the subproblem. When dealing with the case of pseudo-segments, there is no obvious incidence-preserving duality that maps them to points and maps the points to pseudo-lines. Nevertheless, such a duality has been recently defined by Agarwal and Sharir [7] (refining an older and less efficient duality given by Goodman [32]), which can be implemented efficiently and thus yields an efficient algorithm for computing $I(P, C)$, whose running time is comparable with the bound on $I(P, C)$ given above. A similar approach can be used to compute many faces in arrangements of pseudo-circles; see [2] and [7]. Algorithmic aspects of incidence problems have also been studied in higher dimensions; see, e.g., Brass and Knauer [14].

The cutting-based approach has by now become a standard tool in the design of efficient geometric algorithms in a variety of applications in range searching, geometric optimization, ray shooting, and many others. It is beyond the scope of this survey to discuss these applications, and the reader is referred, e.g., to the survey of Agarwal and Erickson [3] and to the references therein.

7.2. Distinct Distances

The above techniques can be applied to obtain some nontrivial results concerning the Distinct Distances problem of Erdős [27] formulated in the Introduction: what is the minimum number of distinct distances determined by n points in the plane? As we have indicated after presenting the proof of the Crossing Lemma (Lemma 4.1), Székely's idea can also be applied in several situations where the underlying graph is not *simple,* i.e., two vertices can be connected by more than one edge. However, for the method to work it is important to have an upper bound for the multiplicity of the edges. Székely [51] formulated the following natural generalization of Lemma 4.1.

Lemma. *Let G be a multigraph drawn in the plane with V vertices, E edges, and with maximal edge-multiplicity M. Then there are* $\Omega\left(\frac{E^3}{MV^2}\right) - O(M^2 V)$ *crossing pairs of edges.*

Székely applied this statement to the Distinct Distances problem, and improved by a polylogarithmic factor the best previously known lower bound of Chung et al. [18] on the minimum number of distinct distances determined by n points in the plane. His new bound was $\Omega(n^{4/5})$. However, Solymosi and Tóth [48] have realized that, combining Székely's analysis of distinct distances with the Szemerédi-Trotter theorem for the number of incidences between m points and n lines in the plane, this lower bound can be substantially improved. They managed to raise the bound to $\Omega(n^{6/7})$. Later, Tardos and Katz have further improved this result, using the same general approach, but improving upon a key algebraic step of the analysis. In their latest paper [33], they combined their methods to prove that the minimum number of distinct distances determined by n points in the plane is $\Omega(n^{(48-14e)/(55-16e)-\varepsilon})$, for any $\varepsilon > 0$, which is $\Omega(n^{0.8641})$. This is the best known result so far. A close inspection of the general Solymosi-Tóth approach shows that, without any additional geometric idea, it can never lead to a lower bound better than $\Omega(n^{8/9})$.

7.3. Equal-area, Equal-perimeter, and Isoceles Triangles

Let P be a set of n points in the plane. We wish to bound the number of triangles spanned by the points of P that have a *given area,* say 1. To do so, we note that if we fix two points $a, b \in P$, any third point $p \in P$ for which Area $(\Delta abp) = 1$ lies on the union of two fixed lines parallel to ab. Pairs (a, b) for which such a line ℓ_{ab} contains fewer than $n^{1/3}$ points of P generate at most $O(n^{7/3})$ unit area triangles. For the other pairs, we observe that the number of lines containing more than $n^{1/3}$ points of P is at most $O(n^2/(n^{1/3})^3) = O(n)$, which, as already mentioned, is an immediate consequence of the Szemerédi-Trotter theorem. The number of incidences between these lines and the points of P is at most $O(n^{4/3})$. We next observe that any line ℓ can be equal to one of the two lines ℓ_{ab} for at most n pairs a, b, because, given ℓ and a, there can be at most two points b for which $\ell = \ell_{ab}$. It follows that the lines containing more than $n^{1/3}$ points of P can be associated with at most $O(n \cdot n^{4/3}) = O(n^{7/3})$ unit area triangles. Hence, overall, P determines at most $O(n^{7/3})$ unit area triangles. The best known lower bound is $\Omega(n^2 \log\log n)$ (see [15]).

Next, consider the problem of estimating the number of *unit perimeter* triangles determined by P. Here we note that if we fix $a, b \in P$, with $|ab| < 1$, any third point $p \in P$ for which Perimeter$(\Delta abp) = 1$ lies on an ellipse whose foci are a and b and whose major axis is $1 - |ab|$. Clearly, any two distinct pairs of points of P give rise to distinct ellipses, and the number of unit perimeter triangles determined by P is equal to one third of the number of incidences between these $O(n^2)$ ellipses and the points of P. The set of these ellipses has four degrees of freedom, in the sense of Pach and Sharir [42] (see also Section 3), and hence the number of incidences between them and the points of P, and consequently the number of unit perimeter triangles determined by P, is at most

$$O(n^{4/7}(n^2)^{6/7}) = O(n^{16/7}).$$

Here the best known lower bound is very weak—only $\Omega(ne^{c\frac{\log n}{\log\log n}})$ [15].

Finally, consider the problem of estimating the number of *isosceles* triangles determined by P. Recently, Pach and Tardos [43] proved that the number of isosceles triangles induced by triples of an n-element point set in the plane is $O(n^{(11-3\alpha)/(5-\alpha)})$ (where the constant of proportionality depends on α), provided that $0 < \alpha < \frac{10-3e}{24-7e}$. In particular, the number of isoceles triangles is $O(n^{2.136})$. The best known lower bound is $\Omega(n^2 \log n)$ [15]. The proof proceeds through two steps, interesting in their own right.

(i) Let P be a set of n distinct points and let C be a set of ℓ distinct circles in the plane, with $m \le \ell$ distinct centers. Then, for any $0 < \alpha < 1/e$, the number I of incidences between the points in P and the circles of C is

$$O\left(n + \ell + n^{\frac{2}{3}}\ell^{\frac{2}{3}} + n^{\frac{4}{7}}m^{\frac{1+\alpha}{7}}\ell^{\frac{5-\alpha}{7}} + n^{\frac{12+14\alpha}{21+3\alpha}}m^{\frac{3+5\alpha}{21+3\alpha}}\ell^{\frac{15-3\alpha}{21+3\alpha}} + n^{\frac{8+2\alpha}{14+\alpha}}m^{\frac{2+2\alpha}{14+\alpha}}\ell^{\frac{10-2\alpha}{14+\alpha}}\right),$$

where the constant of proportionality depends on α.

(ii) As a corollary, we obtain the following statement. Let P be a set of n distinct points and let C be a set of ℓ distinct circles in the plane such that they have at most n distinct centers. Then, for any $0 < \alpha < 1/e$, the number of incidences between the points in P and the circles in C is

$$O\left(n^{\frac{5+3\alpha}{7+\alpha}}\ell^{\frac{5-\alpha}{7+\alpha}} + n\right).$$

In view of a recent result of Katz and Tardos [33], both statements extend to all $0 < \alpha < \frac{10-3e}{24-7e}$, which easily implies the above bound on the number of isosceles triangles.

7.4. Congruent Simplices

Bounding the number of incidences between points and circles in higher dimensions can be applied to the following interesting question asked by Erdős and Purdy [29, 30] and discussed by Agarwal and Sharir [6]. Determine the largest number of simplices congruent to a fixed simplex σ, which can be spanned by an n-element point set $P \subset \mathbb{R}^k$?

Here we consider only the case when $P \subset \mathbb{R}^4$ and $\sigma = abcd$ is a 3-simplex. Fix three points $p, q, r \in P$ such that the triangle pqr is congruent to the face abc of σ. Then any fourth point $v \in P$ for which $pqrv$ is congruent to σ must lie on a circle whose plane is orthogonal to the triangle pqr, whose radius is equal to the height of σ from d, and whose center is at the foot of that height. Hence, bounding the number of congruent simplices can be reduced to the problem of bounding the number of incidences between circles and points in 4-space. (The actual reduction is slightly more involved, because the same circle can arise for more than one triangle pqr; see [6] for details.) Using the bound of [11], mentioned in Section 6, one can deduce that the number of congruent 3-simplices determined by n points in 4-space is $O(n^{20/9+\varepsilon})$, for any $\varepsilon > 0$.

This is just one instance of a collection of bounds obtained in [6] for the number of congruent copies of a k-simplex in an n-element point set in $\mathbb{R}^d$, whose review is beyond the scope of this survey.

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