Chapter 11

A DIFFERENTIAL GAME OF ADVERTISING FOR NATIONAL AND STORE BRANDS

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Abstract We consider a differential game model for a marketing channel formed by one manufacturer and one retailer. The latter sells the manufacturer's product and may also introduce a private label at a lower price than the manufacturer's brand. The aim of this paper is twofold. We first assess in a dynamic context the impact of a private label introduction on the players' payoffs. If this is beneficial for the retailer to propose his brand to consumers and detrimental to the manufacturer, we wish then to investigate if a cooperative advertising program could help the manufacturer to mitigate the negative impact of the private label.

1. Introduction

Private labels (or store brand) are taking increasing shares in the retail market in Europe and North America. National manufacturers are threatened by such private labels that can cannibalize their market shares and steal their consumers, but they can also benefit from the store traffic generated by their presence. In any event, the store brand introduction in a product category affects both retailers and manufacturers marketing decisions and profits. This impact has been studied using static game models with prices as sole decision variables. Mills (1995, 1999) and Narasimhan and Wilcox (1998) showed that for a bilateral monopoly, the presence of a private label gives a bigger bargaining power to the retailer and increases her profit, while the manufacturer gets lower profit. Adding competition at the manufacturing level, Raju et al. (1995) identified favorable factors to the introduction of a private label for the retailer. They showed in a static context that price competition between the store and the national brands, and between national brands has considerable impact on the profitability of the private label introduction.

Although price competition is important to understand the competitive interactions between national and private labels, the retailer's promotional decisions do also affect the sales of both product (Dhar and Hoch 1997). Many retailers do indeed accompany the introduction of a private label by heavy store promotions and invest more funds to promote their own brand than to promote the national ones in some product categories (Chintagunta et al. 2002).

In this paper, we present a dynamic model for a marketing channel formed by one manufacturer and one retailer. The latter sells the manufacturer's product (the national brand) and may also introduce a private brand which would be offered to consumers at a lower price than the manufacturer's brand. The aim of this paper is twofold. We first assess in a dynamic context the impact of a private label introduction on the players' profits. If we find the same results obtained from static models, i.e., that it is beneficial for the retailer to propose his brand to consumers and detrimental to the manufacturer, we wish then to investigate if a cooperative advertising program could help the manufacturer to mitigate, at least partially, the negative impact of the private label.

A *cooperative advertising* (or promotion) program is a cost sharing mechanism where a manufacturer pays part of the cost incurred by a retailer to promote the manufacturer's brand. One of the first attempts to study cooperative advertising, using a (static) game model, is Berger (1972). He studied a case where the manufacturer gives an advertising allowance to his retailer as a fixed discount per item purchased and showed that the use of quantitative analysis is a powerful tool to maximize the profits in the channel. Dant and Berger (1996) used a Stackelberg game to demonstrate that advertising allowance increases retailer's level of local advertising and total channel profits. Bergen and John (1997) examined a static game where they considered two channel structures: A manufacturer with two competing retailers and two manufacturers with two competing retailers. They showed that the participation of the manufacturers in the advertising expenses of their dealers increases with the degree of competition between these dealers, with advertising spillover and with consumer's willingness to pay. Kim and Staelin (1999) also explored the two-manufacturers, two-retailers channel, where the cooperative strategy is based on advertising allowances.

Studies of cooperative advertising as a coordinating mechanism in a dynamic context are of recent vintages (see, e.g., Jørgensen et al. (2000, 2001), Jørgensen and Zaccour (2003), Jørgensen et al. (2003)).

Jørgensen et al. (2000) examine a case where both channel members make both long and short term advertising efforts, to stimulate current sales and build up goodwill. The authors suggest a cooperative advertising program that can take different forms, i.e., a full-support program where the manufacturer contributes to both types of the retailer's advertising expenditures (long and short term) or a partial-support program where the manufacturer supports only one of the two types of retailer advertising. The authors show that all three cooperative advertising programs are Pareto-improving (profit-wise) and that both players prefer the full support program. The conclusion is thus that a coop advertising program is a coordinating mechanism in also a dynamic setting. Due to the special structure of the game, long term advertising strategies are constant over time. This is less realistic in a dynamic game with an infinite time horizon. A more intuitive strategy is obtained in Jørgensen et al. (2001). This paper reconsiders the issue of cooperative advertising in a two-member channel in which there is, however, only one type of advertising of each player. The manufacturer advertises in national media while the retailer promotes the brand locally. The sales response function is linear in promotion and concave in goodwill. The dynamics are a Nerlove-Arrow-type goodwill evolution equation, depending only on the manufacturer's national advertising. In this case, one obtains a nondegenerate Markovian advertising strategy, being linearly decreasing in goodwill.

In Jørgensen et al. (2000, 2001), it is an assumption that the retailer's promotion affects positively the brand image (goodwill stock). Jørgensen, et al. (2003) study the case where promotions damage the brand image and ask the question whether a cooperative advertising program is meaningful in such context. The answer is yes if the initial brand image is "weak" or if the initial brand image is at an "intermediate" level and retailer promotions are not "too" damaging to the brand image.

Jørgensen and Zaccour (2003) suggest an extension of the setup in Jørgensen et al. (2003). The idea now is that excessive promotions, and not instantaneous action, is harmful to the brand image.

To achieve our objective, we shall consider three scenarios or games:

1. Game N: the retailer carries only the National brand and no cooperative advertising program is available. The manufacturer and the retailers play a noncooperative game and a feedback Nash equilibrium is found.

2. Game S: the retailer offers a Store brand along with the manufacturer's product and there is no cooperative advertising program.

The mode of play is noncooperative and a feedback Nash equilibrium is the solution concept.

3. Game C: the retailer still offers both brands and the manufacturer proposes to the retailer a Cooperative advertising program. The game is played à la Stackelberg with the manufacturer as leader. As in the two other games, we adopt a feedback information structure.

Comparing players' payoffs of the first two games allows to measure the impact of the private label introduction by the retailer. Comparing the players' payoffs of the last two games permits to see if a cooperative advertising program reduces the harm of the private label for the manufacturer. A necessary condition for the coop plan to be attractive is that it also improves the retailer's profit, otherwise the will not accept to implement it.

The remaining of this paper is organized as follows: In Section 2 we introduce the differential game model and define rigorously the three above games. In Section 3 we derive the equilibria for the three games and compare the results in Section 4. In Section 5 we conclude.

2. Model

Let the marketing channel be formed of a manufacturer (player M) and a retailer (player R). The manufacturer controls the rate of national advertising for his brand $A(t), t \in [0, \infty)$. Denote by $G(t)$ the goodwill of the manufacturer's brand, which dynamics evolve à la Nerlove and Arrow (1962):

$$\dot{G}(t) = \lambda A(t) - \delta G(t), \quad G(0) = G_0 \geq 0, \tag{11.1}$$

where λ is a positive scaling parameter and $\delta > 0$ is the decay rate.

The retailer controls the promotion efforts for the national brand, denoted by $p_1(t)$, and for the store brand, denoted by $p_2(t)$.
We consider that promotions have an immediate impact on sales and do not affect the goodwill of the brand. The demand functions for the national brand (Q_1) and for the store brand (Q_2) are as follows:

$$\begin{aligned} Q_1(p_1, p_2, G) &= \alpha p_1(t) - \beta p_2(t) + \theta G(t) - \mu G^2(t), & (11.2) \\ Q_2(p_1, p_2, G) &= \alpha p_2(t) - \psi p_1(t) - \gamma G(t), & (11.3) \end{aligned}$$

where $\alpha, \beta, \theta, \mu, \psi$ andγ are positive parameters.

Thus, the demand for each brand depends on the retailer's promotions for both brands and on the goodwill of the national brand. Both demands are linear in promotions.

We have assumed for simplicity that the sensitivity of demand to own promotion is the same for both brands considering that the retailer is using usually the same media and methods to promote both brands. However, the cross effect is different allowing for asymmetry in brand substitution. We assume that own brand promotion has a greater impact on sales, in absolute value, than competitive brand promotion, i.e., $\alpha > \beta$ and $\alpha > \psi$. This assumption mirrors the one usually made on prices in oligopoly theory. We further suppose that the marginal effect of promoting the national brand on the sales of the store brand is higher than the marginal effect of promoting the store brand on the sales of the national brand, i.e., $\psi > \beta$. This actually means that the manufacturer's brand enjoys a priori a stronger consumer preference than the retailer's one. Putting together these inequalities leads to the following assumption

$$\mathrm{A1} : \alpha > \psi > \beta > 0.$$

Finally, the demand for the national brand is concave increasing in its goodwill (i.e., $\frac{\partial Q_1}{\partial G} = \theta - 2\mu G > 0, \forall G > 0$) and the demand for the store brand is decreasing in that goodwill.

Denote by $D(t), 0 \leq D(t) \leq 1$, the coop participation rate of the manufacturer in the retailer's promotion cost of the national brand. We assume as in, e.g., Jørgensen et al. (2000, 2003), that the players face quadratic advertising and promotion costs. The net cost incurred by the manufacturer and the retailer are as follows

$$\begin{aligned} C_M(A) &= \frac{1}{2}u_M A^2(t) + \frac{1}{2}u_R D\left(t\right) p_1^2(t), \\ C_R\left(p_1, p_2\right) &= \frac{1}{2}u_R \Big\{ \left[1 - D\left(t\right)\right] p_1^2(t) + p_2^2(t) \Big\}, \end{aligned}$$

where $u_R, u_M > 0$.

Denote by m_0 the manufacturer's margin, by m_1 the retailer's margin on the national brand and by m_2 her margin on the store brand. Based on empirical observations, we suppose that the retailer has a higher margin on the private label than on the national brand, i.e., $m_2 > m_1$. Ailawadi and Harlam (2004) found indeed that for product categories where national brands are heavily advertised, the percent retail margins are significantly higher for store brands than for national brands.

We denote by r the common discount rate and we assume that each player maximizes her stream of discounted profit over an infinite horizon. Omitting the time argument when no ambiguity may arise, the optimization problems of players M and R in the different games are as follows:

- Game C: Both brands are offered and a coop program is available.

$$\max_{A,D} J_M^C = \int_0^{+\infty} e^{-rt}\Big[m_0\big(\alpha p_1 - \beta p_2 + \theta G - \mu G^2\big) - \frac{u_M}{2}A^2 - \frac{u_R}{2}Dp_1^2\Big]dt,$$

$$\max_{p_1,p_2} J_R^C = \int_0^{+\infty} e^{-rt}\Big[m_1\big(\alpha p_1 - \beta p_2 + \theta G - \mu G^2\big) + m_2\big(\alpha p_2 - \psi p_1 - \gamma G\big) - \frac{1}{2}u_R\Big[\big(1-D\big)p_1^2 + p_2^2\Big]\Big]dt.$$

- Game S: Both brands are available and there is no coop program.

$$\max_{A} J_M^S = \int_0^{+\infty} e^{-rt}\Big[m_0\big(\alpha p_1 - \beta p_2 + \theta G - \mu G^2\big) - \frac{u_M}{2}A^2\Big]dt,$$

$$\max_{p_1,p_2} J_R^S = \int_0^{+\infty} e^{-rt}\Big[m_1\big(\alpha p_1 - \beta p_2 + \theta G - \mu G^2\big) + m_2\big(\alpha p_2 - \psi p_1 - \gamma G\big) - \frac{u_R}{2}\Big(p_1^2 + p_2^2\Big)\Big]dt.$$

- Game N: Only manufacturer's brand is offered and there is no coop program.

$$\max_{A} J_M^N = \int_0^{+\infty} e^{-rt}\Big[m_0\big(\alpha p_1 + \theta G - \mu G^2\big) - \frac{u_M}{2}A^2\Big]dt,$$

$$\max_{p_1} J_R^N = \int_0^{+\infty} e^{-rt}\Big[m_1\big(\alpha p_1 + \theta G - \mu G^2\big) - \frac{u_R}{2}p_1^2\Big]dt.$$

3. Equilibria

We characterize in this section the equilibria of the three games. In all cases, we assume that the players adopt stationary Markovian strategies, which is rather standard in infinite-horizon differential games. The following proposition gives the result for Game N.

PROPOSITION 11.1 *When the retailer does not sell a store brand and the manufacturer does not provide any coop support to the retailer, stationary feedback Nash advertising and promotional strategies are given by*

$$p_1^N = \frac{\alpha m_1}{u_R},$$

$$A^N(G) = X + YG,$$

where

$$X = \frac{2m_0\theta\lambda}{\left(r + 2\sqrt{\Delta_1}\right)u_M}, \quad Y = \frac{r + 2\delta - 2\sqrt{\Delta_1}}{2\lambda},$$
$$\Delta_1 = \left(\delta + \frac{r}{2}\right)^2 + \frac{2\mu m_0 \lambda^2}{u_M}.$$

Proof. A sufficient condition for a stationary feedback Nash equilibrium is the following: Suppose there exists a unique and absolutely continuous solution $G(t)$ to the initial value problem and there exist bounded and continuously differentiable functions $V_i : \Re_+ \to \Re, i \in \{M, R\}$, such that the Hamilton-Jacobi-Bellman (HJB) equations are satisfied for all $G \geq 0$:

$$rV_M(G) = \max_A \Big\{ m_0\left(\alpha p_1 + \theta G - \mu G^2\right) - \frac{1}{2}u_M A^2 + V_M'(G)\left(\lambda A - \delta G\right) \mid A \geq 0 \Big\}, \tag{11.4}$$

$$rV_R(G) = \max_{p_1} \Big\{ m_1\left(\alpha p_1 + \theta G - \mu G^2\right) - \frac{1}{2}u_R p_1^2 + V_R'(G)\left(\lambda A - \delta G\right) \mid p_1 \geq 0 \Big\}. \tag{11.5}$$

The maximization of the right-hand-side of equations (11.4) and (11.5) yields the following advertising and promotional rates:

$$A(G) = \frac{\lambda}{u_M}V_m'(G), \quad p_1 = \frac{\alpha m_1}{u_R}.$$

Substituting the above in (11.4) and (11.5) leads to the following expressions

$$rV_M(G) = m_0\left(\frac{\alpha^2 m_1}{u_R} + \theta G - \mu G^2\right) + \frac{\lambda^2}{2u_M}\left[V_M'(G)\right]^2 - \delta G V_M'(G), \tag{11.6}$$

$$rV_R(G) = m_1\left(\frac{\alpha^2 m_1}{2u_R} + \theta G - \mu G^2\right) + V_R'(G)\left[\frac{\lambda^2}{u_M}V_M'(G) - \delta G\right]. \tag{11.7}$$

It is easy to show that the following quadratic value functions solve the HJB equations;

$$V_M(G) = a_1 + a_2 G + \frac{1}{2} a_3 G^2, \quad V_R(G) = b_1 + b_2 G + \frac{1}{2} b_3 G^2,$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are constants. Substitute $V_M(G)$, $V_R(G)$ and their derivatives into equations (11.6) and (11.7) to obtain:

$$r\left(a_1 + a_2 G + \frac{a_3}{2} G^2\right) = \frac{m_0 \alpha^2 m_1}{u_R} + \frac{\lambda^2 a_2^2}{2u_M}$$
$$+\left(m_0\theta - \delta a_2 + \frac{\lambda^2 a_2 a_3}{u_M}\right) G - \left(\mu m_0 + \delta a_3 - \frac{\lambda^2 a_3^2}{2u_M}\right) G^2,$$

$$r\left(b_1 + b_2 G + \frac{1}{2} b_3 G^2\right) = \frac{\alpha^2 m_1^2}{2u_R} + \frac{\lambda^2}{u_M} a_2 b_2$$
$$+\left(m_1\theta - \delta b_2 + \frac{\lambda^2}{u_M}(a_2 b_3 + a_3 b_2)\right) G - \left(m_1\mu + \delta b_3 - \frac{\lambda^2}{u_M} b_3 a_3\right) G^2.$$

By identification, we obtain the following values for the coefficients of the value functions:

$$a_3 = \frac{\left(\delta + \frac{r}{2}\right) \pm \sqrt{\Delta_1}}{\lambda^2 / u_M}, \qquad b_3 = -\frac{m_1 \mu}{\frac{r}{2} + \delta - \frac{\lambda^2}{u_M} a_3}$$

$$a_2 = \frac{m_0 \theta}{r + \delta - \frac{\lambda^2}{u_M} a_3}, \qquad b_2 = \frac{m_1\theta + \frac{\lambda^2}{u_M} b_3 a_2}{r + \delta - \frac{\lambda^2}{u_M} a_3}$$

$$a_1 = \frac{m_0 \alpha^2 m_1}{r u_R} + \frac{\lambda^2 a_2^2}{2 r u_M}, \qquad b_1 = \frac{\alpha^2 m_1^2}{2 r u_R} + \frac{\lambda^2 a_2 b_2}{r u_M}$$

where

$$\Delta_1 = \left(\delta + \frac{r}{2}\right)^2 + \frac{2\mu m_0 \lambda^2}{u_M}.$$

To obtain an asymptotically stable steady state, choose the negative solution for a_3. Note that the identified solution must satisfy the constraint $A(G) > 0$. Since $\frac{\lambda}{u_M} V_M'(G) = A(G)$, this assumption is true for $G \in \left[0, \bar{G}^N\right]$, where

$$\bar{G}^N = -\frac{a_2}{a_3}, \quad A(G) = \frac{\lambda}{u_M} V_M'(G) = \frac{2 m_0 \theta \lambda}{\left(r + 2\sqrt{\Delta_1}\right) u_M} + \frac{r + 2\delta - 2\sqrt{\Delta_1}}{2\lambda} G.$$

$\square$

The above proposition shows that the retailer promotes always the manufacturer's brand at a positive constant rate and that the advertising strategy is decreasing in the goodwill. The next proposition characterizes the feedback Nash equilibrium in Game S.

PROPOSITION 11.2 *When the retailer does sell a store brand and the manufacturer does not provide any coop support to the retailer, assuming an interior solution, stationary feedback Nash advertising and promotional strategies are given by*

$$p_1^S = \frac{\alpha m_1 - \psi m_2}{u_R}, \quad p_2^S = \frac{\alpha m_2 - \beta m_1}{u_R}, \quad A^S(G) = A^N(G).$$

Proof. The proof proceeds exactly as the previous one and we therefore print only important steps. The HJB equations are given by:

$$rV_M(G) = \max_A \Big\{ m_0\big(\alpha p_1 - \beta p_2 + \theta G - \mu G^2\big) - \frac{u_M}{2} A^2 + V_M'(G)\big(\lambda A - \delta G\big) \mid A \geq 0 \Big\},$$

$$rV_R(G) = \max_{p_1,p_2} \Big\{ m_1\big(\alpha p_1 - \beta p_2 + \theta G - \mu G^2\big) + m_2\big(\alpha p_2 - \psi p_1 - \gamma G\big) - \frac{u_R}{2}\big(p_1^2 + p_2^2\big) + V_R'(G)\big(\lambda A - \delta G\big) \mid \big(p_1, p_2\big) \geq 0 \Big\}.$$

The maximization of the right-hand-side of the above equations yields the following advertising and promotional rates:

$$A(G) = \frac{\lambda}{u_M} V_m'(G), \quad p_1 = \frac{\alpha m_1 - \psi m_2}{u_R}, \quad p_2 = \frac{\alpha m_2 - \beta m_1}{u_R}.$$

We next insert the values of $A(G)$, p_1 and p_2 from above in the HJB equations and assume that the resulting equations are solved by the following quadratic functions:

$$V_M(G) = s_1 + s_2 G + \frac{1}{2} s_3 G^2, \quad V_R(G) = k_1 + k_2 G + \frac{1}{2} k_3 G^2,$$

where $k_1, k_2, k_3, s_1, s_2, s_3$ are constants. Following the same procedure as in the proof of the previous proposition, we obtain

$$s_3 = \frac{\big(\delta + \frac{r}{2}\big) \pm \sqrt{\Delta_2}}{\lambda^2 / u_M}, \qquad k_3 = -\frac{m_1 \mu}{\frac{r}{2} + \delta - \frac{\lambda^2}{u_M} s_3},$$

$$s_2 = \frac{m_0\theta}{r+\delta-\frac{\lambda^2}{u_M}s_3}, \qquad k_2 = \frac{m_1\theta - m_2\gamma + \frac{\lambda^2}{u_M}k_3 s_2}{r+\delta-\frac{\lambda^2}{u_M}s_3},$$

$$s_1 = \frac{m_0}{ru_R}\Big(\alpha\,(m_1\alpha - m_2\psi) - \beta\,(m_2\alpha - m_1\beta)\Big) + \frac{\lambda^2}{2ru_M}s_2^2,$$

$$k_1 = \frac{1}{2ru_R}\Big((m_1\alpha - m_2\psi)^2 + (m_2\alpha - m_1\beta)^2\Big) + \frac{\lambda^2}{ru_M}k_2 s_2,$$

where

$$\Delta_2 = \Delta_1 = \left(\delta + \frac{r}{2}\right)^2 + \frac{2\mu m_0\lambda^2}{u_M}.$$

In order to obtain an asymptotically stable steady state, we choose for s_3 the negative solution. The assumption $A(G) > 0$ holds for $G \in \left[0, \bar{G}^S\right]$, where $\bar{G}^S = -\frac{s_2}{s_3}$. Note also that $s_3 = a_3$, $s_2 = a_2$ and $b_3 = k_3$. Thus $A^S(G) = A^N(G)$ and $\bar{G}^S = \bar{G}^N$. □

REMARK 11.1 Under A1 $(\alpha > \psi > \beta > 0)$ and the assumption that $m_2 > m_1$, the retailer will always promote his brand, i.e., $p_2^S = \frac{\alpha m_2 - \beta m_1}{u_R} > 0$. For $p_1^S = \frac{\alpha m_1 - \psi m_2}{u_R}$ to be positive and thus the solution to be interior, it is necessary that $(\alpha m_1 - \psi m_2) > 0$. This means that the retailer will promote the national brand if the marginal revenue from doing so exceeds the marginal loss on the store brand. This condition has thus an important impact on the results and we shall come back to it in the conclusion.

In the last game, the manufacturer offers a coop promotion program to her retailer and acts as leader in a Stackelberg game. The results are summarized in the following proposition.

PROPOSITION 11.3 *When the retailer does sell a store brand and the manufacturer provides a coop support to the retailer, assuming an interior solution, stationary feedback Stackelberg advertising and promotional strategies are given by*

$$p_1^C = \frac{2\alpha m_0 + (\alpha m_1 - \psi m_2)}{2u_R}, \quad p_2^C = \frac{\alpha m_2 - \beta m_1}{u_R},$$

$$A^C(G) = A^S(G), \quad D = \frac{2\alpha m_0 - (\alpha m_1 - \psi m_2)}{2\alpha m_0 + (\alpha m_1 - \psi m_2)}.$$

Proof. We first obtain the reaction functions of the follower (retailer) to the leader's announcement of an advertising strategy and a coop support rate. The later HJB equation is the following

$$rV_R(G) = \max_{p_1,p_2} \Big\{ m_1\big(\alpha p_1 - \beta p_2 + \theta G - \mu G^2\big) + m_2\big(\alpha p_2 - \psi p_1 - \gamma G\big) \tag{11.8}$$

$$- \frac{u_R}{2}\big((1-D)\, p_1^2 + p_2^2\big) + V_R'(G)\big(\lambda A - \delta G\big) \mid \big(p_1, p_2\big) \geq 0 \Big\}.$$

Maximization of the right-hand-side of (11.8) yields

$$p_1 = \frac{\alpha m_1 - \psi m_2}{u_R(1-D)}, \quad p_2 = \frac{\alpha m_2 - \beta m_1}{u_R}. \tag{11.9}$$

The manufacturer's HJB equation is:

$$rV_M(G) = \max_{A,D} \Big\{ m_0\big(\alpha p_1 - \beta p_2 + \theta G - \mu G^2\big) - \frac{u_M}{2} A^2 - \frac{1}{2} u_R D p_1^2$$

$$+ V_M'(G)\big(\lambda A - \delta G\big) \mid A \geq 0, \quad 0 \leq D \leq 1 \Big\}.$$

Substituting for promotion rates from (11.9) into manufacturer's HJB equation yields

$$rV_M(G) = \quad \max_{A,D} \Big\{ m_0\big(\alpha \frac{\alpha m_1 - \psi m_2}{u_R(1-D)} - \beta \frac{\alpha m_2 - \beta m_1}{u_R} + \theta G - \mu G^2\big)$$

$$- \frac{u_M}{2} A^2 - \frac{u_R}{2} D\Big(\frac{\alpha m_1 - \psi m_2}{u_R(1-D)}\Big)^2 + V_M'(G)\big(\lambda A - \delta G\big) \Big\}$$

Maximizing the right-hand-side leads to

$$A(G) = \frac{\lambda}{u_M} V_M'(G), \quad D = \frac{2\alpha m_0 - (\alpha m_1 - \psi m_2)}{2\alpha m_0 + (\alpha m_1 - \psi m_2)}. \tag{11.10}$$

Using (11.9) and (11.10) provides the retailer's promotional strategies

$$p_1 = \frac{2\alpha m_0 + (\alpha m_1 - \psi m_2)}{2u_R}, \quad p_2 = \frac{\alpha m_2 - \beta m_1}{u_R}.$$

Following a similar procedure to the one in the proof of Proposition 11.1, it is easy to check that following quadratic value functions provide unique solutions for the HJB equations,

$$V_M(G) = n_1 + n_2 G + \frac{1}{2} n_3 G^2, \quad V_R(G) = l_1 + l_2 G + \frac{1}{2} l_3 G^2,$$

where $n_1, n_2, n_3, l_1, l_2, l_3$ are constants given by:

$$n_3 = \frac{\big(\delta + \frac{r}{2}\big) \pm \sqrt{\Delta_3}}{\lambda^2 / u_M}, \qquad l_3 = -\frac{m_1 \mu}{\frac{r}{2} + \delta - \frac{\lambda^2}{u_M} n_3},$$

$$n_2 = \frac{m_0\theta}{r+\delta-\frac{\lambda^2}{u_M}n_3}, \qquad l_2 = \frac{m_1\theta - m_2\gamma + \frac{\lambda^2}{u_M}l_3 n_2}{r+\delta-\frac{\lambda^2}{u_M}n_3},$$

$$\begin{aligned} n_1 &= \frac{m_0}{ru_R}\Big(\alpha\big(\alpha m_0 + \frac{1}{2}(m_1\alpha - m_2\psi)\big) - \beta\big(m_2\alpha - m_1\beta\big)\Big) \\ &\quad + \frac{\lambda^2}{2ru_M}n_2^2 - \frac{1}{2ru_R}\Big(\alpha^2{m_0}^2 - \frac{1}{4}\big(m_1\alpha - m_2\psi\big)^2\Big), \\ l_1 &= \frac{(m_1\alpha - m_2\psi)}{2ru_R}\Big(\alpha m_0 + \frac{1}{2}\big(m_1\alpha - m_2\psi\big)\Big) \\ &\quad + \frac{(m_2\alpha - m_1\beta)^2}{2ru_R} + \frac{\lambda^2 l_2 n_2}{ru_M}, \end{aligned}$$

where $\Delta_3 = \Delta_2 = \Delta_1 = \big(\delta + \frac{r}{2}\big)^2 + 2\mu m_0 \frac{\lambda^2}{u_M}$.

To obtain an asymptotically stable steady state, we choose the negative solution for n_3. Note that $n_3 = s_3 = a_3$, $n_2 = s_2 = a_2$, $l_3 = k_3 = b_3$ and $l_2 = k_2$. Thus $A^C(G) = A^S(G) = A^N(G)$. □

REMARK 11.2 As in Game S, the retailer will always promote her brand at a positive constant rate. The condition for promoting the manufacturer's brand is $(2\alpha m_0 + \alpha m_1 - \psi m_2) > 0$ (the numerator of p_1^C has to be positive). The condition for an interior solution in Game S was that $(\alpha m_1 - \psi m_2) > 0$. Thus if p_1^S is positive, then p_1^C is also positive.

REMARK 11.3 The support rate is constrained to be between 0 and 1. It is easy to verify that if $p_1^C > 0$, then a necessary condition for $D < 1$ is that $(\alpha m_1 - \psi m_2) > 0$, i.e., $p_1^S > 0$. Assuming $p_1^C > 0$, otherwise there is no reason for the manufacturer to provide a support, the necessary condition for having $D > 0$ is $(2\alpha m_0 - \alpha m_1 + \psi m_2) > 0$.

4. Comparison

In making the comparisons, we assume that the solutions in the three games are interior. The following table collects the equilibrium strategies and value functions obtained in the three games.

In terms of strategies, it is readily seen that the manufacturer's advertising strategy $(A(G))$ is the same in all three games. This is probably a by-product of the structure of the model. Indeed, advertising does not affect sales directly but do it through the goodwill. Although the later has an impact on the sales of the store brand, this does not affect the profits earned by the manufacturer. The retailer adopts the same promotional strategy for the private label in the games where such brand is available, i.e., whether a coop program is offered or not. This is also due to the simple structure of our model.

Table 11.1. Summary of Results

	Game N	Game S	Game C
p_1	$\frac{\alpha m_1}{u_R}$	$\frac{\alpha m_1 - \psi m_2}{u_R}$	$\frac{2\alpha m_0 + (\alpha m_1 - \psi m_2)}{2u_R}$
p_2		$\frac{\alpha m_2 - \beta m_1}{u_R}$	$\frac{\alpha m_2 - \beta m_1}{u_R}$
$A(G)$	$A^N(G)$	$A^N(G)$	$A^N(G)$
D			$\frac{2\alpha m_0 - (\alpha m_1 - \psi m_2)}{2\alpha m_0 + (\alpha m_1 - \psi m_2)}$
$V_M(G)$	$a_1 + a_2 G + \frac{a_3}{2} G^2$	$s_1 + a_2 G + \frac{a_3}{2} G^2$	$n_1 + a_2 G + \frac{a_3}{2} G^2$
$V_R(G)$	$b_1 + b_2 G + \frac{b_3}{2} G^2$	$k_1 + k_2 G + \frac{b_3}{2} G^2$	$l_1 + k_2 G + \frac{b_3}{2} G^2$

The remaining and most interesting item is how the retailer promotes the manufacturer's brand in the different games. The introduction of the store brand leads to a reduction in the promotional effort of the manufacturer's brand ($p_1^N - p_1^S = \frac{\psi m_2}{u_R} > 0$). The coop program can however reverse the course of action and increases the promotional effort for the manufacturer's brand $\left(p_1^C - p_1^S = \frac{2\alpha m_0 - \alpha m_1 + \psi m_2}{2u_R} > 0\right)$. This result is expected and has also been obtained in the literature cited in the introduction. What is not clear cut is whether the level of promotion could reach back the one in the game without the store brand. Indeed, $\left(p_1^N - p_1^C\right)$ is positive if the condition that $(\alpha m_1 + \psi m_2 > 2\alpha m_0)$ is satisfied.

We now compare the players' payoffs in the different games and thus answer the questions raised in this paper.

Proposition 11.4 *The store brand introduction is harmful for the manufacturer for all values of the parameters.*

Proof. From the results of Propositions 11.1 and 11.2, we have:

$$V_M^S(G_0) - V_M^N(G_0) = s_1 - a_1 = -\frac{m_0}{r u_R}\left[m_2 \psi \alpha + \beta\left(m_2 \alpha - m_1 \beta\right)\right] < 0.$$

□

For the retailer, we cannot state a clear-cut result. Compute,

$$\begin{aligned} V_R^S(G_0) - V_R^N(G_0) &= k_1 - b_1 + (k_2 - b_2) G_0 \\ &= \frac{1}{2 r u_R}\left[(m_1 \alpha - m_2 \psi)^2 + (m_2 \alpha - m_1 \beta)^2 - \alpha^2 m_1^2\right] \\ &\quad + \frac{4 \lambda^2 m_0 m_2 \theta \gamma}{r u_M \left(r + \sqrt{\Delta_2}\right)^2} + \frac{2 m_2 \gamma}{r\left(r + \sqrt{\Delta_2}\right)} G_0. \end{aligned}$$

Thus for the retailer to benefit from the introduction of a store brand, the following condition must be satisfied

$$V_R^S(G_0) - V_R^N(G_0) > 0 \Leftrightarrow G_0 > \frac{\left(r+\sqrt{\Delta_2}\right)}{4m_2\gamma u_R}\alpha^2 m_1^2 - \frac{2\lambda^2 m_0\theta}{u_M\left(r+\sqrt{\Delta_2}\right)} - \frac{\left(r+\sqrt{\Delta_2}\right)}{4m_2\gamma u_R}\left[(m_1\alpha - m_2\psi)^2 + (m_2\alpha - m_1\beta)^2\right].$$

The above inequality says that the retailer will benefit from the introduction of a store brand unless the initial goodwill of the national one is "too low". One conjecture is that in such case the two brands would be too close and no benefit is generated for the retailer from the product variety. The result that the introduction of a private label is not always in the best interest of a retailer has also been obtained by Raju et al. (1995) who considered price competition between two national brands and a private label.

Turning now to the question whether a coop advertising program can mitigate, at least partially, the losses for the manufacturer, we have the following result.

PROPOSITION 11.5 *The cooperative advertising program is profit Pareto-improving for both players.*

Proof. Recall that $k_2 = l_2$, $k_3 = l_3 = n_3$ and $n_2 = s_2$. Thus for the manufacturer, we have

$$V_M^3(G_0) - V_M^2(G_0) = n_1 - s_1 = \frac{1}{8ru_R}\left[2\alpha m_0 - (\alpha m_1 - \psi m_2)\right]^2 > 0.$$

For the retailer

$$V_R^C(G_0) - V_R^S(G_0) = l_1 - k_1 = \frac{1}{4ru_R}(m_1\alpha - m_2\psi)(2\alpha m_0 - m_1\alpha + m_2\psi)$$

which is positive. Indeed, $(m_1\alpha - m_2\psi) = u_r p_1^S$ which is positive by the assumption of interior solution and $(2\alpha m_0 - m_1\alpha + m_2\psi)$ which is also positive (it is the numerator of D). □

The above proposition shows that the answer to our question is indeed yes and, importantly, the retailer would be willing to accept a coop program when suggested by the manufacturer.

5. Concluding Remarks

The results so far obtained rely heavily on the assumption that the solution of Game S is interior. Indeed, we have assumed that the retailer will promote the manufacturer's brand in that game. A natural question is that what would happen if it were not the case? Recall that we required that

$$p_1^S = \frac{\alpha m_1 - \psi m_2}{u_R} > 0 \Leftrightarrow \alpha m_1 > \psi m_2.$$

If $\alpha m_1 > \psi m_2$ is not satisfied, then $p_1^S = 0$ and the players' payoffs should be adjusted accordingly. The crucial point however is that in such event, the constraint on the participation rate in Game C would be impossible to satisfy. Indeed, recall that

$$D = \frac{2\alpha m_0 - (\alpha m_1 - \psi m_2)}{2\alpha m_0 + \alpha m_1 - \psi m_2},$$

and compute

$$1 - D = \frac{2\,(\alpha m_1 - \psi m_2)}{2\alpha m_0 + \alpha m_1 - \psi m_2}.$$

Hence, under the condition that $(\alpha m_1 - \psi m_2 < 0)$, the retailer does not invest in any promotions for the national brand after introducing the private label $\left(p_1^S = 0\right)$. In this case, the cooperative advertising program can be implemented only if the retailer does promote the national brand and the manufacturer offers the cooperative advertising program i.e., a positive coop participation rate, which is possible only if $(2\alpha m_0 + \alpha m_1 - \psi m_2) > 0$.

Now, suppose that we are in a situation where the following conditions are true

$$\alpha m_1 - \psi m_2 < 0 \quad \text{and} \quad 2\alpha m_0 + \alpha m_1 - \psi m_2 > 0 \tag{11.11}$$

In this case, the retailer does promote the manufacturer's product $(p_1^C > 0)$, however we obtain $D > 1$. This means that the manufacturer has to pay more than the actual cost to get her brand promoted by the retailer in Game C and the constraint $D < 1$ has to be removed.

For $p_1^S = 0$ and when the conditions in (11.11) are satisfied, it is easy to show that the effect of the cooperative advertising program on the profits of retailer and the manufacturer are given by

$$V_R^C\,(G_0) - V_R^S\,(G_0) = \frac{(\alpha m_1 - \psi m_2)}{4u_r}\,(2\alpha m_0 + m_1\alpha - m_2\psi) < 0$$

$$V_M^C\,(G_0) - V_M^S\,(G_0) = \frac{1}{8u_r}\,(2m_0\alpha + \alpha m_1 - \psi m_2)^2 > 0$$

In this case, even if the manufacturer is willing to pay the retailer more then the costs incurred by advertising the national brand, the retailer will not implement the cooperative program.

To wrap up, the message is that the implementability of a coop promotion program depends on the type of competition one assumes between the two brands and the revenues generated from their sales to the retailer. The model we used here is rather simple and some extensions are desirable such as, e.g., letting the margins or prices be endogenous.

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