DAVE PRATT

HOW DO TEACHERS FOSTER STUDENTS' UNDERSTANDING OF PROBABILITY?

Chapter 7

> The goal of instruction should not be to exchange misconceptions for expert concepts but to provide the experiential basis for complex and gradual processes of conceptual change. (Smith, diSessa & Rochelle, 1993, p. 74)

1. INTRODUCTION: TWO KEY IDEAS IN PROBABILITY

Probability is unusual in many respects. As a knowledge domain, it straddles mathematics in its pure abstraction, and physics, economics and indeed most sciences and social sciences because of its wide range of applicability. Equally unusually for an aspect of mathematics, it explicitly pervades our everyday lives whereas most aspects of mathematics are hidden and, although they may have a fundamental impact on our lives, for the most part, we are unaware of their insidious effect (Noss, 1997). The language of probability pervades almost everything we do: sports commentators talk about a 50/50 ball, weather forecasters announce an 80% chance of rain; health is assessed in terms of risk factors based upon probabilistic calculations. Indeed it seems probability is one of the few areas of mathematics that informs explicitly the way in which we conduct our everyday lives.

In more recent years mathematics curricula have begun to recognize the significance of chance and probability, as illustrated below through the Australian, American and British curricula. In Australia, the *National Statement on Mathematics for Australian Schools* (AEC, 1991) informs the teaching that takes place in the various Australian states. Chance and Data is one of five mathematics content areas in that document and is seen as critical to the teaching of mathematics in a modern society.

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> A sound grasp of concepts in areas of chance, data handling and statistical inference is critical for the levels of numeracy appropriate for informed participation in society today. (AEC, p.163)

Similarly in the USA, the *Principles and Standards for School Mathematics* (NCTM, 2000) sets out 5 content areas, including one on Data Analysis and Probability. In England, the *National Curriculum: Mathematics* (DfEE, 1999a) inserts probability into one of four attainment targets, Handling Data.

Any one of these national curriculum documents could be used to illustrate the focus on probability in schools but I have chosen to draw upon the *National Numeracy Strategy for England and Wales* (DfEE, 1999b), which provides considerable detail. This document, often referred to as "the framework" for mathematics, sets out the teaching programmes, referenced against identified key objectives for ages 4 to 13 years. Nearly all teachers in state education in England and Wales follow this programme.

Probability becomes an explicit part of the curriculum from age 7 years on. Between ages 7 and 10, the curriculum focuses on the following ideas: (a) the language of probability with some emphasis on equally likely outcomes, (b) events that consist of two or more outcomes, (c) how the results from an experiment can vary and (d) the difference between theoretical and experimental probabilities. So the curriculum emphasizes at this stage the importance of variation, though it is limited in scope, and some elementary work on calculating probabilities.

In 2001, the corresponding teaching plans for ages 11 to 13 were published (DfEE, 2001). There is now an increased emphasis on calculating probabilities and the calculation of all possible combinations in various situations. There is also some emphasis on the estimation of probabilities from experiments. A key objective aimed only at the most able students at age 13 states, "Recognize that, with repeated trials, experimental probability tends to a limit…" (p. 283).

It is questionable whether sufficient emphasis is given to randomness in terms of time in the primary phase, and to the law of large numbers in either the middle school or secondary phase in terms of the range of ability for whom this is a key objective. In my view, the curriculum sends a clear signal that the ideas behind the law of large numbers are beyond the scope of all but the highest abilities.

Furthermore, there is a notable omission from the curriculum; indeed the central theme of this chapter is to address this omission. Mathematicians and statisticians would surely argue that the concept of distribution is central to their domain. The discussion about the comparison of theoretical and experimental probabilities, which is fostered by the curriculum, should find expression through the emergence of a notion of distribution. By limiting the experience of randomness and variation to situations for which children often already have an intuitive feel, they are not given in my view the opportunity to recognize the powerful connection between randomness, the law of large numbers and distribution.

Perhaps one reason for the limited extent to which these key concepts are addressed lies in their perceived difficulty. Teachers need to find ways of building on what children already know and to be aware of the limitations of that knowledge if they are to find pedagogic strategies that support the learning of these concepts.

2. WHAT DO CHILDREN NOT KNOW AND WHAT *DO* THEY ALREADY KNOW?

The domain of probability and chance has been the focus of a great deal of research into the errors and irrational thinking that people, not just children, seem to exhibit when making judgments of chance. The failure of our intuitions has been so well documented that it is perhaps not surprising when teachers, confronted with the difficulties faced by their children, believe probability is simply counter-intuitive. A corollary to this view could be that our mental apparatus is hard-wired in such a way that it is beyond redress through any pedagogic strategy. It is certainly worth briefly summarizing that body of literature before evaluating whether the above perspective is the only defensible interpretation. Such a review has been completed in earlier chapters of this volume (e.g., Jones & Thornton; Langrall & Mooney; Watson) and so I refer the reader to the areas of most relevance to this chapter.

Research on What Children (and Adults) do not Know

The seminal work was carried out by Piaget and Inhelder (1951/ 1975). They noted that in order to accommodate probabilistic thinking the organism needs the capacity to recognize uncertainty and to be able to catalogue systematically all possible combinations. The latter requirement demands that probabilistic knowledge is a late development, well into the stage of formal operations. Probability theory can be seen as an invention by the organism to operationalize randomness.

Meanwhile, how are people to make judgments of chance in everyday life? Many researchers have offered descriptions of the sorts of heuristics that people use to make such judgments. The main body of literature has been provided by Kahneman and Tversky (e.g., Kahneman, Slovic, &

Tversky,1982) who catalogued during the 1960s and 1970s a long list of such heuristics. See Jones and Thornton, this volume, for a description of the *representativeness* and *availability* heuristics and also for an account of Konold's *outcome* approach (Konold, 1989). The outcome approach is one reason why, when teachers ask children to make a prediction about a chance situation, the children will respond that it is impossible to say, "it's just a matter of chance". For them perhaps, all that matters is what happens in practice. Lecoutre (1992) has reported a related phenomenon, named the *equiprobability bias*. (see Watson, this volume, for a full account of the equiprobability bias.) Lecoutre argued that the equiprobability bias was resistant to modification (even amongst individuals grounded in probability theory) but that a correct response could be induced by masking the chance element of the problem. She concluded that correct cognitive models are often available but are not spontaneously associated with the situations at hand.

In other words, children who were quite capable of identifying the possible combinations typically failed to use this information correctly when a random element was added to the task. They would tend to respond instead that it was just a matter of chance or "50/50." Lecoutre's work suggests that it is perfectly feasible to gain success by masking the random element in a task, and that our lack of comfort with randomness persists even beyond the point in our development when we are able to compute combinations.

The above account of human fallibility in making judgments of chance is depressing but it is not at all clear that the catalogue of failure necessarily implies that it is impossible to offer children productive learning experiences. Indeed the next section will strike an altogether more optimistic note.

Research on What Children (and Adults) do Know

Piaget's approach was to examine the epistemology of probabilistic knowledge from a genetic perspective and as such he was less interested in how the setting might shape such development. In contrast, teachers deal continuously with the partial knowledge of their children, and consequently teachers need guidance on how their actions, including the offering of certain types of resources, might shape children's knowledge development. Fischbein's work (1975, Fischbein & Schnarch,1997) on intuitions provides some constructs that relate to those aspects of our mental apparatus that are brought into play when we are making more immediate decisions and judgments. In one experiment by Fischbein, subjects were asked to predict the next event in a random sequence. Even very young children gradually tuned the proportions of their predictions to the relative frequencies of the outcomes, suggesting that they were able to intuit relative frequencies. Fischbein's thesis suggested that the weaknesses described above were the consequence of a pedagogy that emphasized the deterministic. According to Fischbein, children's early *primary* (by which he meant unschooled or untaught) intuitions fail to develop as effective *secondary* (scientifically learned or taught) intuitions because of a lack of support from the school system. However, his work does not quite reach the level of specificity that would help the teacher faced with the challenges of fostering probability learning.

Nevertheless, Fischbein offers a more positive outlook in the sense that his work promotes the notion that new pedagogies might support the development of "better" intuitions, rather than leaving our children to develop in a state of epistemological anxiety (Wilensky, 1997).

Research has reported children at age 10 or 11 years with well-established intuitions for randomness. Young children seem to recognize random experiments as involving the following characteristics: unpredictability, irregularity, unsteerability, and fairness (Pratt, 1998a).

1. *Unpredictability*: If the next outcome is not predictable, a child might regard the experiment as random,
2. *Irregularity*: If there is evidently no patterned sequence in prior results, a child might refer to the experiment as random,
3. *Unsteerability*: If the child is unable to exert physical control over the outcome of the phenomenon, the experiment might be seen as random, and
4. *Fairness*: If there seems to be a rough symmetry in the experiment, a child may think of the experiment as random.

I would claim that these intuitions for randomness are not so different from the expert perspective, though, whereas these four intuitions might be about as much as a 10 year old child knows, the expert's knowledge will connect these intuitions to a rich and extensive concept image (Tall & Vinner, 1981). An expert recognizes the differences between fair and random. In particular, random can be biased and so perhaps might be regarded as unfair. My perspective is that we should not dismiss the child's knowledge as a misconception to be eradicated (see Smith, diSessa, & Rochelle, 1993 for a brilliant articulation of this perspective). Instead, we should *accept* the pedagogic challenge of how to build on the child's impoverished view of randomness so that it is connected to, but not identical with, that of fairness.

From an expert perspective, the four intuitions for randomness sometimes appear self-contradictory. To a young child, a spinner, whose equal-sized sectors read 1, 2, 3, 4, 5 and 6 might be seen as fair and so random. Now consider a spinner also numbered 1 to 6 but in such a way that the 6 sector is twice the size of the others. The same child might well regard this spinner as unfair and so non-random. Both these spinners are in fact largely unpredictable and unsteerable and will both generate irregular results, and so in these respects the experiment with the non-uniform spinner might have been regarded as random too. In my research, children often appeared unconcerned by such inconsistencies; they adopted whichever stance was cued by the most obvious characteristics of the situation in question. The teacher's role might be to find a way to raise these inconsistencies in the hope that the cognitive conflict somehow helps the child to begin to distinguish between fairness and randomness. Of course, the problem for the teacher might be that the child continues to ignore the conflict, which might appear to be more of a conflict for the teacher than it is for the child!

There is one final point that I wish to raise about the differences between the limited, but useful, intuitions of the 10-year-old child and the powerful expert understanding, and this is in my view the most significant issue. The four intuitions for randomness focus entirely on immediately observable aspects of the experiment. The children in my research did not in the initial interviews exhibit any awareness of the longer term aggregated properties of randomness, Yet, the crucial understanding that an expert has, and one of the key objectives identified in the first section, is that set out in the law of large numbers. The mathematically exciting property of random experiments is not so much their unpredictability in the short term but their predictability over a large number of trials, in the sense that the relative frequency of an outcome tends towards its probability.

The law of large numbers, regarded by the Numeracy Strategy for England and Wales as accessible to only the most able children, appears to be the principal aspect of randomness that distinguishes an expert understanding of randomness from that of some 10-year-old children. How might teachers support the development of intuitions for this idea (and indeed for distribution)? Some educators have suggested that technology could have a particularly significant role to play. In the next section, a short review of the research on technology and probability will be given before continuing with a more detailed summary of a research study that focuses on technology-supported probability learning. I will use this latter study to infer some pedagogic principles to guide probability teaching.

3. THE ROLE OF TECHNOLOGY IN FOSTERING STUDENTS' UNDERSTANDING OF PROBABILITY

A literature describing the use of technology in the teaching and learning of probability has slowly emerged over the past two decades. The following review provides a background against which the subsequent study can be better understood. Below the main issues are listed.

There has been some speculation that the use of computers in stochastic work might be hindered by learners' concerns about the nature of computer-based randomness. It has been recommended that the pseudo-random nature of randomness on the computer may need to be made transparent as part of the activity (Borovcnik & Peard, 1996). Although the complex algorithms used to generate pseudo-random numbers are likely to be hidden from students, it appears that top-level engagement with the model may provide reassurance or clarification.

Cliff Konold has been one of the pioneers in exploring the use of computers in the teaching and learning of probability. Reflecting on an experiment (1995) in which he placed bets against a student with respect to the outcomes of a series of coin tosses, Konold demonstrated how (a) we are more likely to find stories and explanations for the vicissitudes of the data than to regard the data as forceful in its explanatory power, (b) technology itself is not necessarily engaging but rather the task design is fundamentally important, (c) there is a tendency for students to underestimate just how much data is needed to draw reasonably sound conclusions, (d) variation tends to be ignored by designers who often fail to exploit the ability of technology to repeat trials and experiments, and (e) the focus of software design should be on sense-making and the enrichment of intuitions.

More recent work (Stohl & Tarr, 2002) has focused on how notions of inference can be fostered in sixth grade students using a software tool, Probability Explorer, to formulate and evaluate inferences. This study revealed that students were able to understand the interplay between empirical and theoretical probability, recognize the importance of using larger samples to make inferences, and justify their claims with data-based evidence.

At about the same time as Konold was reporting on his work, Wilensky (1993) was completing research that was set in the context of a *connected* mathematics project and focused on university students' use of StarLogo microworlds. Wilensky described how students worked through various epistemological anxieties to begin to see randomness in a connected way, neither representing complete ignorance, nor just a mathematical formalism.

To reach this position, the students struggled against a range of obstacles that reflect the relative infancy of probability and the lack of feedback from everyday experience to point up the inadequacy of their probability judgments.

A central notion in Wilensky's use of StarLogo is that, unlike the more conventional uses of computers, the child interacts with the formalisms themselves to build new products, a process which brings the learner into a closer intimacy with fundamental epistemological and conceptual barriers embedded in the stochastic. As an example, consider the following classic problem:

> A chord is drawn randomly across a circle of diameter 10cm. What is the probability that this chord is longer than the radius of the circle?

Wilensky's students were confused in that it was possible to establish different answers to this problem and yet impossible to disprove any of those answers. When the students tried to program the problem into the computer, they recognized that the solution would depend on how they programmed the random generation. According to Wilensky, this was a critical breakthrough in their beginning to address epistemological difficulties with the notion of randomness.

Other researchers have built on Wilensky's Logo-based approach. Abrahamson & Wilensky (2002) describe the design of a NetLogo microworld that enables the observation of the incremental growth of a bell-shaped curve. Reflecting on this process, Abrahamson and Wilensky recognize a range of epistemological perspectives that constitute the sort of complexity identified in Wilensky's original work.

Paparistodemou, Noss and Pratt (2002) built a probability game into Imagine, a powerful version of Logo. In studying young children's understanding of random mixture, portrayed as a dynamically shifting set of bouncing balls, they showed that even children as young as six years of age were able to make sense of random mixtures represented in such quasi-concrete ways.

The original observations by Konold (in particular), together with the early microworld studies, found subsequent expression in the formulation of the Chance-Maker microworld that is described in the next section. The study associated with the Chance-Maker microworld provides a number of insights, which may, in the mind of the reader at least, inform our thinking about how teachers might foster children's understanding of probability.

4. THE CHANCE-MAKER STUDY

In providing this synopsis of part of my own research I will illustrate what I see as a number of important pedagogical guidelines that hopefully flesh out Fischbein's precept: greater emphasis on stochastics would provide better support for children's early intuitions (1975, p.73).

My study was in fact a piece of design research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) in which one aim was to build a microworld, which eventually became known as *Chance-Maker*. At the same time, I aimed to gain fresh insights into how children's stochastic thinking evolved through the use of the developing computer-based tools. The findings about these children's initial intuitions for randomness have already been discussed. In this section, I wish to emphasize the emergence of new knowledge.

The Chance-Maker Microworld

In the final iteration of the research, the children were given a series of *gadgets* (Figure 1), mini-computational devices that simulate everyday random generators (a coin, a spinner, a dice and so on). The design of these devices was based on the assumption that the children would regard the normative state for such gadgets as one of being fair. In order to appreciate their understanding of chance, I needed to challenge this perspective. Hence, the gadgets were in some cases intentionally broken, in the sense that some sort of bias had been inserted into their operation. The children were asked to identify which gadgets were not working properly. The gadgets also contained a variety of tools. The children were challenged to use these tools to mend the broken gadgets. My assumption was that they would aim to make the gadgets fair but the research showed that fairness has many ways of manifesting itself.

Each of the gadgets shown in Figure 1 has a strength control. This allows the child to control how hard the coin, spinner or dice is thrown or tossed. Higher strengths make the simulation continue for a longer time period though in fact strength has no effect on outcome. Alternatively, the child can click with the mouse directly on the gadget, in which case it is triggered with the same strength. This allows replications of experiments which do not necessarily generate the same outcome.

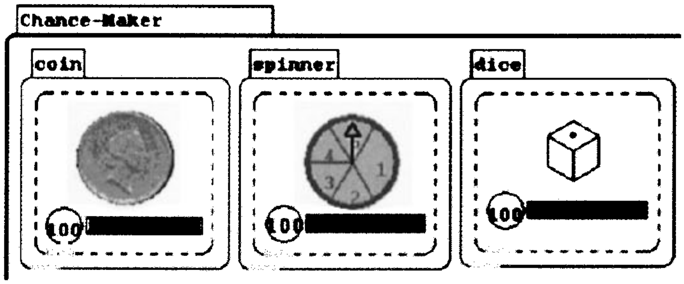

Figure 1. Three of the gadgets in the Chance-Maker microworld

When a child wishes to mend any of the gadgets, she opens up the gadget to gain access to the mending tools. In Figure 2 the tools for the dice gadget are shown. The results are listed (in the *Results* box) and can be displayed as a pictogram (*Pict* button) or as a pie chart. (*Pie* button). Trials of an experiment can be repeated many times (in Figure 2, the *Repeat* tool is prepared for an experiment of 100 trials), usually by turning the graphics off to save time (*On/Off* button). Results will accumulate until a new experiment is begun (*New* button).

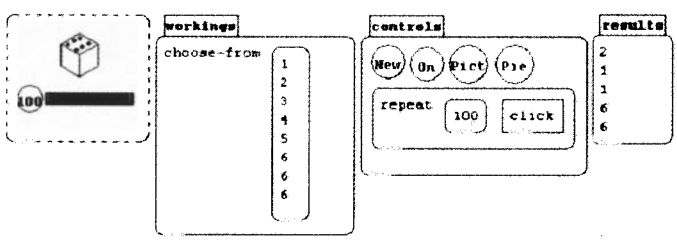

Figure 2. The main tools in the dice gadget

The workings box shows the computational core of the gadget. In this particular case, the dice "chooses" from the list 1, 2, 3, 4, 5, 6, 6, 6. The workings box can be edited by the child to change the way it works.

Emergent Knowledge

In order to extract the pedagogic issues, I summarize below a typical evolution of knowledge, though, of course, there were variations in the ways that different children interacted with the software. The qualitative methodology adopted for this study does not allow claims of statistical generality, though hopefully the reader may find some resonance, which in a sense imbues the work with a degree of generality (for more detail, you may wish to refer to Pratt, 1998b, 2000).

The children began by simply triggering the gadgets into action, usually through the strength control. Their challenge was to ascertain which gadgets seemed in their view not to be working properly. There was much evidence to support the intuitions for randomness identified in the pre-interviews.

By default, the coin gadget was unbiased. Nevertheless the children thought that they identified effects due to the strength control. However, the children discovered that any apparent pattern was not maintained; conjectures about the effect of the strength control upon the results was not supported over more extended periods of time.

They were confused that the pie chart for the coin did not appear uniform. They experimented again with the strength control and various other features to see if they could make the pie chart display as they felt it should. They would consistently use small numbers of trials and so the pie chart was never satisfactory.

Sometimes by accident (the software accumulates results unless the *New* button is pressed), and sometimes after a researcher prompt, the children tried increasing the number of trials and found that the pie chart would then appear to be more even. Thus, after some time working with the coin, the children would articulate thoughts such as: "the more the number of times we throw the coin, the more even is its pie chart". This expression is, in my view, a good example of what Noss and Hoyles (1996) call a *situated abstraction*. The children have abstracted a rule for how the phenomenon behaves but the abstraction is apparently tied to its setting in so far as one can ascertain from their language.

The initial impression of the spinner gadget is that it looks unfair (the sectors are not uniform) and so the children quickly suspected it was not working properly (they felt it should be unbiased). This was confirmed by the appearance of uneven pie charts. Their attention was drawn by this unfairness to the unfairness of the workings box. However, editing the workings box so that each outcome appeared only once did not seem to solve the problem. The pie chart for example still appeared "unfair", in the sense that the sectors were unequal.

Instead of reusing their situated abstraction from the coin gadget, they continued to use small numbers of trials and tried many configurations of the workings box, adjusting the values to compensate for discrepancies in the previous pie chart. Nevertheless, the pie was inconsistent in its appearance.

This activity seemed to demonstrate the deep situatedness of the knowledge gained from working with the coin gadget.

Eventually, perhaps out of desperation, they recalled quite explicitly what they had learned about the coin gadget. They tried a greater number of trials and found at last that the pie chart now appeared to them to be fair.

Along the way they articulated the situated abstraction that "the spinner's workings box makes the pie chart fair". When challenged by the researcher, some of the children realized that this would only work when the number of trials was high.

The children began working on the dice exactly as they had with the spinner. Again they ignored their previous learning. However, on this occasion they were much quicker to turn to their previous situated abstractions to explore what would happen with a uniform workings box and a high number of trials. It seemed that, by the third gadget, the new knowledge was now sufficiently reliable to be called up in preference to other ideas that they might have for how it worked.

This brief synopsis only covers some aspects of the research but nevertheless it allows me to draw out some pedagogic guidelines, which the teacher might find resonates with their own experiences in the classroom.

5. IMPLICATIONS FOR PEDAGOGY

Purpose and Utility

Teachers are confronted with what Ainley, Pratt, and Hansen (in press) have called the *planning paradox*. Like the Numeracy Strategy in England and Wales, most curricula are set out in terms of teaching objectives, based directly on mathematical concepts or skills. We claim that if teachers plan from teaching objectives, the tasks are likely to be unrewarding for the children and mathematically impoverished. However, if teachers plan from tasks, the activity is likely to be unfocussed and unassessable.

We propose two constructs to discuss task design. The first construct we call *purpose*. We define a purposeful task as one that has a meaningful outcome for the learner, in terms of an actual or virtual product, or the solution of an engaging problem. The second construct is *utility*. We have found that it is possible to plan for opportunities for learners to appreciate the utility of mathematical concepts and techniques in the sense that they learn how and why that idea is useful by applying it in a purposeful context. We claim this dualistic approach stands in contrast to the conventional emphasis on how to carry out a technique.

The difficulty in planning lies in linking purpose to utility in such a way that there is a high probability that the learner will stumble across the utility of the mathematical concept as they engage in the purposeful activity. I see this notion as critically important in the successful design of the Chance-Maker task. The children enjoyed the idea of working on the concrete and meaningful task of figuring out which gadgets were not working properly and then attempting to mend them. However, purpose is insufficient. The design of the Chance-Maker tools made it almost impossible for the children to avoid ideas that lie behind the two principal concepts: (i) the law of large numbers (through the repeat tool and the graphing facilities), and (ii) distribution (through the workings box).

No doubt there are many ways of engaging children in work on these two key ideas and the Chance-Maker case is just one such method. Perhaps the most critical implication that can be drawn for our pedagogy is not limited to probability but applies to task design in the general practice of teaching mathematics.

Testing Personal Conjectures

We know from the literature that children (and adults) have many idiosyncratic ways of thinking about chance situations. Some have been outlined above. How are we to regard these ways of thinking? My theoretical framework (Pratt & Noss, 2002), built as a synthesis of the work by Noss and Hoyles (1996) and diSessa (1993), asserts that old pieces of knowledge coexist with newer pieces of knowledge, either in a connected way or perhaps isolated from each other. This framework stands in opposition to much of the misconceptions literature, which often presents misconceptions as ideas to be eradicated and replaced with normative views. In fact, misconceptions are typically naïve ideas that nevertheless contain some element of the normative view. Thus, like experts, children see randomness as unpredictable but need to learn that there is also a long-term sense in which randomness is in fact predictable. Misconceptions can be useful platforms for further learning in the sense that they become connected to new knowledge. Thus, knowledge about the unpredictability of short-term randomness was connected to the behavior of Chance-Maker's gadgets in the longer term. As a result it becomes possible to abstract limitations on both the predictable and unpredictable faces of randomness. When misconceptions are so wrong-headed that they seem to have no pedagogic potential, a strategy can be developed by which children recognize the lack of explanatory power of that idea compared to an alternative view. Hence the strength control was designed to show that changing the force with which the gadget was thrown had rather less explanatory power than the role of either the workings box or the number of trials.

Feedback is crucial here. If children are to be in a position to refute long standing beliefs, or, as I would prefer to say, if they are to have less reliance upon those beliefs, they need feedback that gives them good evidence of the weakness of their current ideas. The children need to be able to test out their personal conjectures and evaluate them.

In the Chance-Maker study above, the children were able to test out the notion that the strength affected the result of throwing the gadgets. I refer to the strength bar as a *redundant control* in the sense that, mathematically the children do not need it. However, psychologically it is crucial that the children are able to test out their personal conjectures.

In the context of probability, this is even more important than elsewhere in mathematics. At this level, probability theory is essentially a model for describing certain types of phenomena. When we experience those phenomena, we make judgments about them, which rarely receive feedback that might cause us to reflect on whether our judgments were correct. When we play games, we attend to the excitement of the game; we do not usually reflect on our strategy (cf. the outcome approach). In any case, we are not usually in a position to try out the sort of long-term experiments that might give us helpful feedback.

Whatever tasks we design to help children understand key objectives like the law of large numbers and distribution, those tasks must provide a mechanism for the children to appreciate the power of these ideas compared to their own intuitions.

Large Scale Experiments

An appreciation of the law of large numbers cannot be realized without the facility to carry out long-term experiments. It seems, from my research, that some ten-year-old children may well not have appreciated how randomness behaves in the long term. Their tendency, it seems, is to carry out a small number of trials of an experiment when given the choice. Why would you do otherwise unless you had good reason? Indeed children seem to follow a Law of Small Numbers (Kahneman et al., 1982).

My research suggests that the tasks that teachers give children should encourage them to decide for themselves how many trials to use. If they are simply told to use a large number of trials then how are they to realize the problems in using small numbers of trials, which they may well have done given a free choice? Even so, the task needs to encourage them to try increasingly high numbers of trials, and the available tools need to facilitate such large-scale experiments.

Systematic Variation of the Context

The situated cognition movement (see for example Lave, 1988,1991) argues persuasively that knowledge is deeply contingent upon the setting. The children in my research did however eventually reuse their situated abstractions. Why do I think they did this? It seems that, once the children had seen the lack of explanatory power of their own ideas, they would reconsider recently learned knowledge. Furthermore, as those ideas proved reliable across different contexts, the ideas took on a higher priority (here I lean heavily on the notion of phenomenological primitives or p-prims: diSessa, 1988)[1] and were more easily cued as sense-making devices.

I believe that the children were eventually able to connect across the gadgets because there were huge structural similarities. In a sense the only difference between them was the outward appearance, which to the child was highly significant but to the mathematician is irrelevant. The underlying tools were identical in each case and, of course, the gadgets were wrapped up inside the same microworld.

Although a difficult challenge, new pedagogies that provide different contexts for the same mathematical idea and offer similar tools within each context may prove more effective in helping children to appreciate the wide applicability of the two key concepts, the law of large numbers and distribution.

6. FINAL REMARKS

The four pedagogic implications, purpose and utility, testing personal conjectures, large-scale experiments and systematic variation of the context, have been abstracted from research that depended fundamentally on technology. Critics of technology-based research in this domain refer to how

[1] DiSessa's work provides a detailed model of conceptual change in which knowledge is seen as fragmented - at least in its initial stages. Small pieces of knowledge, p-prims, are abstracted directly from experience. One example of such a p-prim could be characterised as "I push – it moves". P-prims have priorities attached to them, which determine how likely any particular p-prim is to be cued, and this cueing priority is in turn modified according to how consistent and helpful the p-prim turned out to be in practice. Gradually, through "tuning towards expertise", p-prims may become connected to each other, forming what we might think of as concepts. Although situated abstractions are at a much higher grain level than p-prims, I have found his model useful as a way of thinking about the coexistence of different, possibly contradictory, situated abstractions and the process of tuning that might increase the likelihood of activation of normative abstractions.

children might not believe in the randomness of the computer, which is after all only pseudorandom anyway. I found that it was important that the children were able to persuade themselves that they could not predict or control the outcome from the computer, nor that they could find patterns in the results. Under these circumstances, children began to believe that the computer was indeed generating random results. The idea that the numbers from the computer are pseudorandom seems far less worrisome. From a modeling perspective, the use of a stochastic model to describe the results of a computer random generator are no different from using such a model to describe the results from a dice or any other physical random generator. It is worth reflecting though on the special nature of technology in relation to each of the four pedagogic guidelines listed above.

Purpose and utility are ideas that have arisen naturally from the *constructionist* ideas of Papert and others (Harel & Papert, 1991). Papert has argued that building concrete or virtual objects is a particularly appropriate way of learning. Building seems to provide a concrete focus that lends greater meaning to the activity. The mending task in the Chance-Maker study fits easily into this paradigm and illustrates how it is possible to link purpose and utility. Task design is central since it is the task that imbues initial purpose and drives the activity thereafter. Equally it is the task design that leads the learner to the utilities associated with the planted mathematical concepts. Elsewhere (Ainley & Pratt, 2002), I have discussed other fruitful areas to explore in order to link purpose and utility. Thus, although computer environments used in a constructionist way facilitate the resolution of the planning paradox, they are not unique.

The testing of personal conjectures is an especially difficult aim to achieve without the use of technology. Hard-pressed teachers are unable to provide sufficient feedback to satisfy knowledge hungry children, given permission to explore. Without access to technology, teachers would have to employ other techniques such as group work and whole class sessions. The difficulty then is that, on the one hand without computers the feedback is not neutral, it is less personal and it may be incorrect. On the other hand, computers are not very good at handling the range of possible idiosyncratic ideas that a child may hold.

Large-scale experiments are also difficult to handle without technology. To avoid tedium, teachers tend to collate the work of whole classes. Unfortunately there is then some loss of individuality which means you may find yourself exploring someone else's way of thinking about the situation rather than your own. In many schools computers are still relatively rare resources. To relieve this problem and to give experience with conventional random generators, teachers often begin with group work away from the computer. This process is about making explicit a range of personal conjectures. In order that these conjectures can be tested in a large-scale way, individual or small group exploration on the computer is nevertheless likely to be needed as a follow-up to the initial group work.

Setting up different contexts might on the face of it look easier to do away from the computer, and there is some truth to this. However, it is important, and this is where the computer has salience, that the contexts have the same mathematical structure and probably some similar surface features to cue the reuse of recently learned knowledge.

One huge dilemma remains. When children use computers, will what they learn about probability "transfer" to conventional settings? Said in another way: "Will children reuse the knowledge in the new setting?" The research in probability suggests no, at least not in any simplistic way. If we think of a task set in the physical world as just another gadget, we see the problem. This gadget is so very different from those on the computer. The affordances or attributes of the physical world are in many respects insufficient for most people to gain rich intuitions for the key ideas. In the end, this is why the literature is full of reported failures. However, I hope that the suggestions above point towards the redesign of pedagogies that might bring virtual and physical settings closer together, and enable children to reuse in conventional settings ideas initially constructed in the virtual world.

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