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COMBINATORICS AND THE DEVELOPMENT OF CHILDREN'S COMBINATORIAL REASONING

Chapter 5

Simple combinatorics is the backbone of elementary probability and our teaching of probability should take account of this fact. (Freudenthal, 1973, p. 596)

1. INTRODUCTION

Combinatorics is one of the oldest branches of discrete mathematics, dating back to the 16^{th} century when games of chance played a key role in society life (Abramovich & Pieper, 1996). To provide a theory for these games, specific counting techniques and mathematical ideas were created. In particular, the work of Pascal and Fermat, who studied the theory of combinatorial problems, laid a foundation for the theory of probability and provided approaches to the development of "enumerative combinatorics" (Abramovich & Pieper, 1996).

Combinatorics may be defined as a principle of calculation involving the selection and arrangement of objects in a finite set. Combinatorics is a significant component of the mathematics curriculum, comprising a rich structure of powerful principles that underlie several other areas such as counting, computation, and probability (Borovcnik & Peard, 1996; English, 1993).

Recommendations to incorporate combinatorics in the school mathematics curriculum date back to the early 1970s (e.g., Kapur, 1970; Kenny & Hirsh, 1991; National Council of Teachers of Mathematics [NCTM], 1989). The Working Group (K-4) of the *Commission on Standards for School Mathematics* (NCTM, 1986) highlighted combinatorics as an area of exploration within two of its themes for curriculum development; these themes were "Ways of building models of representations" and "Ways of counting/computing." Not long after this recommendation came the 1991 NCTM Yearbook titled, *Discrete Mathematics across the Curriculum, K-12*

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(Kenny & Hirsch), in which several chapters were devoted to the teaching of combinatorics especially in the middle and secondary school years.

Despite its importance in the mathematics curriculum, combinatorics continues to remain neglected, particularly at the elementary school level. Yet, as Kapur (1970) pointed out over 30 years ago, the real-world nature of the domain makes it suitable for study at all grade levels. Indeed, combinatorics provides the basis for meaningful problems to be solved in a variety of ways and with a variety of representational tools (including manipulative materials).

Combinatorial problems also facilitate the development of enumeration processes, as well as conjectures, generalisations, and systematic thinking. For example, to determine all the possible outfits from a set of differently colored shirts and pants, one needs to systematically match one colored shirt with each pair of pants and then repeat the process with each of the remaining colored shirts. This is a more efficient procedure than randomly matching shirts with pants (see Polaki, this volume). The development of the important concepts of relations, equivalence classes, mapping, and functions is also promoted through combinatorial activities. Furthermore, given the broad applicability of the combinatorial domain (e.g., chemistry, biology, physics), cross-disciplinary problems can be created within real-world contexts for students.

This chapter begins by exploring some elementary ideas of combinatorics and how they support children's development of beginning probability ideas and problem-solving skills. Consideration is then given to various types of combinatorial problems and the relevant difficulties they present children. A review of studies that have addressed children's combinatorial reasoning is presented in the second half of the chapter. The chapter concludes by looking at ways in which we might increase children's access to powerful ideas in combinatorics.

2. ELEMENTARY COMBINATORICS

The fundamental counting principle (DeGuire, 1991) is frequently cited in describing the combinatorial domain. This principle asserts that if one task can be performed in n ways and another task in m ways, then the number of ways of performing the two tasks is nm, with the principle extending to any number of tasks. The principle can also be viewed in terms of the Cartesian product of two given sets, A and B, which is the set formed by the combinations produced by pairing each member of A in turn with each member of B. The Cartesian product of two or more sets is also especially useful in constructing sample spaces. That is, if S_1 and S_2 denote sample spaces for two different probability experiments performed in succession, then a sample space for the combined experiment is the Cartesian product, S_1 x S_2 (Borovcnik, Bentz, & Kapadia, 1991; Polaki, this volume).

The process of linking items from discrete sets in a systematic manner to form all possible combinations has also been referred to as the "odometer strategy," so named because of its resemblance to a vehicle's odometer (English, 1988, 1990, 1991; Scardamalia, 1977). To illustrate this strategy, consideration is given to the selection of pairs of items, one from each of two discrete sets.

When combinations of two items are formed from two given sets with one item from each set, an item from one set is held constant while the items from the other set are varied systematically until all possible combinations with the constant item have been formed. A new constant item from the first set is then selected. The exhaustion of all constant items in the first set indicates the generation of all possible combinations.

There is, of course, more to the combinatorial domain than the aforementioned basic combinatorial operations. Batanero, Godino, and Navarro-Pelayo (1997; see also Batanero & Sanchez, this volume) identified a number of concepts and procedures important to the teaching and assessment of combinatorics. Most of these, listed below, are also linked to probability. For example, the sampling model is included in the combinatorial models, and the logical procedures of classification, systematic enumeration, inclusion/exclusion, and recurrence are fundamental in dealing with probabilistic situations. Likewise, the use of tree diagrams is a basic procedure in determining sample space.

Basic Combinatorial Concepts and Models

- Combinatorial operations: These comprise combinations, arrangements, permutations, and the associated concepts, notations, and formulae;
- Combinatorial models: These include the sampling model (population, sample, ordered/non-ordered sampling, replacement), the distribution model (correspondence and application), and the partition model (sets, subsets, union).

Combinatorial Procedures

- Logical procedures: Included here are classification, systematic enumeration, inclusion/exclusion principle, and recurrence;

- Graphical procedures: Common procedures here include tree diagrams and graphs;
- Numerical procedures: These include addition, multiplication, and division principles, combinatorial and factorial numbers, Pascal's triangle, and difference equations;
- Tabular procedures: The construction of tables and arrays are most common here;
- Algebraic procedures: These include the generation of functions. (Batanero, Godino, et al., 1997, p.240).

Combinatorial procedures are also applied to the *random experiment*, which is the starting point for the study of probability in the elementary grades. The two key aspects of a random experiment are the formulation of the experiment and the identification of all the possible outcomes or sample space (Batanero, Godino, et al., 1997; Hawkins et al., 1992; Langrall & Mooney, this volume; Nisbet, Jones, Langrall, & Thornton, 2000; Polaki, this volume). To describe simple experiments, we can simply list all the possible outcomes. However, to determine a sample space dealing with compound events (e.g., multiple trials of a random experiment) requires more complex enumeration processes than a sample space involving simple events (e.g., rolling a single die). The compound events require combinatorial reasoning. For example, to determine the relative frequencies of various sums (experimental probabilities) of tossing two or three dice, one would begin by systematically listing all of the outcome pairs (or triples). Once the sample space is (or was) produced, one could reason that the probabilities of the various sums are different because some sums would be generated by several combinations of numbers (see Section 5). For example, the sum of 7 occurs when a 5 and 2 are rolled, as well as a 6 and a 1, and a 4 and a 3 (as well as 2-5, 1-6, and 3-4, if ordered pairs are being listed). Likewise, a sum of 8 is produced by rolling a 6 and a 2, two 4s, and a 5 and a 3. In contrast, a sum of 11 can only be produced by rolling a 5 and a 6.

As Batanero, Godino, et al. (1997) note, many probability misconceptions are due to a lack of combinatorial reasoning where students incorrectly enumerate the sample space in a problem. This further highlights the importance of the inclusion of combinatorics in the elementary mathematics curriculum. The next section considers some of the combinatorial problem situations that can be used to develop children's combinatorial reasoning.

3. COMBINATORIAL PROBLEM SITUATIONS

There are several different types of problem situations in which combinatorial ideas are utilised (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Batanero & Sanchez, this volume; Dubois, 1984; Tarr & Lannin, this volume). It is not the intention here to review all of these classifications; rather, consideration is given to the more common problem types including those suitable for elementary school students. These problem types include the following:

- Problems that reflect the fundamental counting principle (DeGuire, 1991) and that make use of tree diagrams, systematic lists, and tables;
- Combinatorial configurations that involve (a) selections, (b) distributions, and (c) partitions (Batanero, Navarro-Pelayo, et al., 1997; Dubois, 1984).

Fundamental Counting Principle

- Application of the odometer strategy promotes more efficient problem solving in problems requiring the systematic testing of alternative solutions. For example:
- The sum of two numbers is 14. Their product is 45. What are the two numbers?
- Large towels cost $12 and regular towels cost $8. If I spent $76, what could I have bought?
- How many different three-digit numbers can be formed using only the digits, 3, 6, and 8?

Other problems include situations where items from two or more discrete sets are combined in all possible ways, such as the following:

- Sarah is making greetings cards. She has blue card, pink card, and yellow card, as well as gold and silver lettering. How many different types of cards can she make using a colored card and lettering?
- A sandwich bar sells brown rolls, white rolls, and multigrain rolls. It offers choices of chicken, beef, and seafood fillings, along with French and Italian dressings. How many different kinds of rolls can you buy, with each having one type of filling and one type of dressing?

Selections

Dubois' (1984) classification of combinatorial problems as "selections" emphasises the concept of sampling. In problems of this nature, a sample of y objects must be taken from a set of x (usually distinct) objects. Examples of the type below are important in developing early probability understandings. The problem below can easily be changed to allow each marble to be selected once only (i.e., selection without replacement).

- Sam has a bag containing four numbered marbles, with each marble showing one of these digits: 4, 6, 8, and 1. He asks his friend to select a marble from the bag and write down its number. He then tells his friend to put the marble back in the bag. His friend repeats this process until he has made a 3-digit number. How many different 3-digit numbers can his friend make?

Distributions

Problems that belong to this category involve the distribution of a set of n objects into m cells (Batanero, Navarro-Pelayo, et al., 1997, Kapur, 1970), as in the following example:

- Carla has three identical birthday invitations and has four different colored envelopes in which she can place them. She cannot place more than one invitation in the one envelope. How many ways can she place the three invitations in the envelopes?

In problems of the type above, conditions can be changed to generate other distribution situations, such as whether the items to be distributed are identical or not, whether the containers are identical or not, whether the items must be ordered, and so on.

Partitions

Partition problems entail breaking a set of n objects into m subsets, which, as Batanero, Navarro-Pelayo, et al. (1997) indicate, is in bijective (or one-to-one) correspondence with distribution problems. A partition problem is of the form:

- James has 6 spare flag stickers displaying Australia, USA, France, Italy, the Netherlands, and New Zealand. He decides to share these flags between his two friends, Samantha and Penny. In how many ways could he share the flags?

4. PROBLEM DIFFICULTY

The research of Batanero, Godino, et al. (1997) showed that the three types of combinatorial configurations (i.e., selections, distributions, partitions) are not of equal difficulty, even after instruction on combinatorics. Drawing upon Fischbein and Gazit's (1988) earlier research, Batanero, Navarro-Pelayo, et al. (1997) explored other task factors that influence problem difficulty with 14-15 year-olds. These factors included the type of combinatorial operation, namely, permutations (where the order of item placement matters), combinations, and arrangements (with and without repetition). The other factors were the nature of the elements to be combined (digits, letters, people, and objects) and the values given to the parameters n and m.

Before instruction, there was little difference in the difficulty level of the three types of combinatorial configurations. Interestingly, the main type of difficulty before instruction was the students' inability to systematically list items. After the students had received instruction in combinatorics (as part of their regular school curriculum), there was a reduction in the difficulty level of selection problems, arrangements problems, and permutations with repetition, but not so for the partition and distribution problems. Individual student interviews indicated that many students also failed to see two combinatorial problems with a different combinatorial configuration as equivalent, even when both problems employed the same combinatorial operation. This highlights the need to address the underlying mathematical features of combinatorial problems so that students can recognize related problem structures. This point is revisited in a later section.

5. CHILDREN'S COMBINATORIAL REASONING

Research on children's combinatorial reasoning has not been prolific, despite its role in the development of early probability ideas. The work of Piaget (e.g., Inhelder & Piaget, 1958) is probably the most cited, where the establishment of a combinatorial system plays a central role in his theory of cognitive development. This system is considered evident in one's ability to "link a set of base associations or correspondences with each other in all possible ways so as to draw from them the relationships of implication, disjunction, exclusion etc." (Inhelder & Piaget, 1958, p.107). Piaget and his associates conducted a number of studies charting the development of combinatoric operations in the formation of propositional logic. These studies included the "coloring liquids" experiment (Inhelder & Piaget, 1958) and the "colored counters" task (Piaget & Inhelder, 1951/1975).

In the coloring liquids experiment, children were presented with four containers of different chemical substances that they were to mix in all possible ways. The colorless, odorless liquids were perceptually identical (beaker 1 contained diluted sulphuric acid, beaker 2 contained water, beaker 3 was oxygenated water, and beaker 4, thiosulphate). A bottle with a dropper that contained potassium iodide was also included. Given that oxygenated water oxidises potassium iodide in an acid medium, combining the two liquids will produce a yellow color (this was demonstrated to the child). The child was given the four containers and the dropper bottle, and asked to reproduce the yellow color.

The colored counters task was part of an investigation into children's development of the idea of chance. In another task, children were presented with sets of counters: each set was a different color, and the children were asked to create as many different pairs of counters as possible. The children's performance in both experiments suggested that preoperational children generate combinations only in an empirical manner by randomly associating two elements at a time (i.e., there is a lack of systematic method). Not until the concrete-operational stage is there evidence of some systematic method in generating combinations, albeit a rather limited system involving only a one-to-many multiplicative correspondence between one item and all others.

The experiments indicated that 9- to 11-year-old children can generate two-by-two and three-by-three combinations, but without a systematic procedure. As the formal-operational period is entered, a change is evident in both combinatorial methods and reasoning. Children now have a systematic method for generating m x n combinations and are able to reason propositionally in forming their combinations. For example, when considering a possible combination, they are able to entertain hypothetical statements such as,

> If this liquid in beaker 4 is water, when you mix it with liquids in beakers 1 and 3, it wouldn't completely prevent the yellow solution from forming.

This is a consideration of possibility rather than reality since the event involving the formation of the yellow solution is not seen in reality (Flavell, 1963).

Children's performance in these Piagetian tasks suggests that the combinatorial system does not emerge until well into the stage of formal operations. However, as with several of Piaget's experiments, the equipment and instructions given were scientific and abstract (Carey, 1985) for children; a feature which is likely to have masked their abilities in the combinatorial domain. More recent research, which has employed child-appropriate materials and meaningful task contexts, has indicated that young children are able to link items from discrete sets in a systematic manner to form all possible combinations of items (e.g., English, 1991; 1993).

In one such study (English, 1991), 50 children aged between 4.5 years and 9.8 years were individually administered a series of 7 novel tasks that involved the dressing of cardboard toy bears (placed on stands) in all possible different outfits. In these tasks an outfit comprised a colored top and a colored pair of pants (or same-colored tops and skirts with different numbers of buttons, for two of the tasks). For each task, the child was provided with more items than were needed to form all possible combinations. The goal of the problem tasks was to dress the bears such that each bear had a different outfit (in terms of colors for the first five tasks, and in terms of total number of buttons for the remaining tasks). No assistance was given to the child by the researcher.

The tasks increased in complexity from the initial two tasks (i.e., Tasks 2 and 3; Task 1 was a familiarisation task) through to the final task (Task 7). Task 2 (2 sets of tops x 3 sets of pants) and Task 3 (3 sets of tops x 2 sets of pants) were of comparable difficulty, each involving 6 combinations. Task 4 was made more complex by increasing the number of possible combinations from 6 to 9 (3 sets of tops x 3 sets of pants). Each of the remaining tasks required 6 combinations but there were additional features that increased their complexity. Task 5 included a constraint on goal attainment, namely, the instruction to give the third bear in the line of bears a blue top (while still forming all possible combinations). Tasks 6 and 7 engaged the children in working with number combinations instead of color. Task 7 had a hidden constraint, namely, two combinations derived from different items had the same total number of buttons (i.e., one-button top/three-button skirt; and two-button top/two-button skirt). One of these combinations thus had to be discarded. This is in contrast to the earlier probability example where each of the combinations derived from tossing 2 (or 3) dice must be taken into account in determining the probabilities of the various sums.

The results of the study revealed a series of solution paths used by the children in solving the set of problem tasks. These paths ranged from random item selection through to a systematic pattern in item choice, reflecting increasing sophistication in solution procedure. The most efficient procedure, namely, the odometer strategy, involves repeating the selection of an item until all possible combinations with that item have been formed (e.g., red top/blue pants; red top/yellow pants; red top/red pants). Upon exhaustion of the item (e.g., red top), a new "constant" item is chosen and the systematic matching process repeated. The selection of items to combine with each new constant item displays a systematic cyclic pattern (e.g., blue pants, yellow pants, red pants; blue pants, yellow pants, red pants). Children who develop this strategy know when they have solved the task and conclude that no further combinations can be formed (e.g., "I know I can't make any more outfits because there are three different tops and I've used each pair of pants three times").

While the 4- to 6-year-old children did not display significant learning across the set of tasks, the 7- to 9-year-olds demonstrated considerable improvement in their solution strategies, with all but 4 of the 26 children adopting more efficient, systematic procedures (without adult intervention). Another interesting finding is that the most efficient solution procedure (i.e., the odometer strategy) emerged as a means of verifying task solution prior to its being used to generate a solution (English, 1991). That is, on task completion or apparent completion, children would use an odometer procedure to determine whether further combinations could be made from the remaining items. For example, children might select a red top from the remaining red tops, place it in front of them and then systematically match it with a pair of pants from each of the remaining sets of pants. This procedure would be repeated with each of the remaining sets of tops. Each trial combination would normally be held in the children's hands or placed in front of them. All but one of the children who adopted this procedure during the course of task execution initially used the procedure for checking purposes. In all, 29 of the 50 children used the odometer strategy to form trial combinations with their unused items. It thus appears that a significant component in children's adoption of this strategy is their ability to use it in a verifying or checking capacity (see Polaki, this volume, for a related discussion on children's checking procedures).

The research on young children's development of a basic combinatorial system (English, 1991; 1993) did not include Piaget's propositional reasoning, that is, subjects' ability to consider "the relationships of implication, disjunction, exclusion etc." (Inhelder & Piaget, 1958, p.107). Nevertheless, the findings do indicate that, with the use of hands-on materials within a meaningful context, young children are able to produce independently a systematic procedure for forming $m \times n$ combinations prior to the stage of formal operations postulated by Piaget and Inhelder. The findings support the inclusion of the combinatorial domain as a topic of investigation in the elementary school.

In another study, English (1999) investigated 32 ten-year-old children's structural understanding of combinatorial problems when presented in various task situations. The children were examined in terms of their ability to: (a) identify the structural (or relational) properties of elementary combinatorial problems (2-dimensional [AxB] and 3-dimensional [AxBxC]), and (b) represent and solve the problems. The children's ability to reason analogically was also explored with respect to: (a) determining the structural similarities and differences between problems, (b) solving new related problems, and (c) posing their own problems.

The findings highlighted, among others, the issue of solution accuracy as opposed to structural understanding. The majority of the children could solve the problems in a variety of ways and could represent the problems symbolically. However, they had difficulties in explaining fully the two-dimensional structure of the *A*x*B* problems and could rarely identify the cross-multiplication feature of these problems. Although a good proportion of the children recorded multiplication statements, there was nevertheless a sizeable number who favoured repeated addition, irrespective of the type of graphic representation they employed (e.g., drawings, systematic listing, tree diagrams). Children's symbolic representations for the 3-dimensional problems also suggested they lacked a complete understanding of the problems' structure. A multistep multiplication statement was rarely recorded for these problems; the majority of those who chose multiplication recorded one-step statements only (e.g., 3x2).

In a similar vein, the children's graphic representations of the problems at times gave mixed messages about their structural understanding. For example, some children displayed knowledge of the odometer strategy in their graphic representations yet recorded addition statements only. Other children did not demonstrate knowledge of this strategy, yet recorded multiplication statements for each problem.

The foregoing research studies indicate that, when given meaningful problem situations, children are able to independently develop powerful combinatorial ideas. Clearly, mathematics curricula in the elementary and middle schools need to include novel problem experiences that encourage children to explore combinatorial ideas and processes, without direct teacher instruction.

6. CHILDREN'S COMBINATORIAL REASONING AND PROBABILITY REASONING

The importance of children's combinatorial reasoning in analyzing sample space has been evident in several studies (e.g., Benson & Jones, 1999; Johnson, Jones, Thornton, Langrall, & Rous 1998; Nisbet et al., 2000; Zimmermann & Jones, 2002). The work of Jones and his colleagues has revealed children's difficulties in recognizing and constructing valid sample spaces and simulations for two-dimensional or compound-event problems (i.e., problems that involve performing two random experiments or performing one random experiment twice)

In the study by Jones, Langrall, Thornton, & Mogill (1999), 37 third-grade children participated in an instructional program in probability. About

43% of the program focused on sample space, 45% on probability of an event and probability comparisons, and the remainder on conditional probability. Fifteen of the 37 children exhibited what the authors referred to as the "sample-space misconception" immediately prior to instruction. These children could not list all the outcomes in a simple (one-stage) experiment and did not recognize the possibility that *all* outcomes could occur; rather, they made subjective statements as to why only particular outcomes would happen (e.g., "Red, because it's my favourite color."). For five of the 37 students, this misconception persisted even after the instructional program had ended.

Overcoming this misconception was one of the key patterns in producing growth in probabilistic thinking. Case studies showed how children progressed from being unable to list all the outcomes of a sample space to spontaneously listing outcomes for simple and subsequently, compound experiments. For example, when asked to list the outcomes in a game where two chips (red on one side and white on the other) were tossed, one student wrote RR, RW, WR, and WW. The student explained that she "started with red and kept matching" (Jones et al. 1999, p. 507). Her strategy illustrated the "odometer" strategy cited previously in this chapter.

The use of written communication in developing children's probabilistic thinking was investigated in a study by Johnson, Jones, Thornton, Langrall, and Rous (1998). A class of fifth-grade students participated in a 5-week "Probability Writing Program," which engaged the children in discussing and solving a series of probability problems. Included in these problems were sample-space tasks, such as the Locker Problem:

– Ann can't remember her locker combination. She remembers that the first number is a 1 or a 2; the second number is a 3 or a 4; and the third number is a 5 or a 6. If she guesses, what is the chance that she will open her locker on the first try? (p. 222)

Following class discussions on their solutions, the children completed journal entries for the problem and the class teacher subsequently responded in writing to the children's entries. Pre- and Post-Probability Assessments were undertaken, together with an Initial Writing Assessment and a Final Writing Assessment. The writing assessments invited the children to respond to a probability question and provide detailed justification for their thinking. These pre- and post-program assessments showed that the students made significant gains in probability thinking as well as in their writing abilities.

In particular, the strong growth shown by several of the students seemed traceable to "a series of critical interactions" that took place between the teacher and the students through the medium of their journals (Johnson et al., p. 214). For example, the ongoing written exchanges between the teacher and a child in working the Locker Problem led the student to progress from randomly listing the possible combinations (135, 146, 235, 136, 245, 236, 145, 246) through to using the odometer strategy in solving the second component of the problem:

- Finally, Ann remembers that the first number of her locker combination is 2. She also knows that the second number is a 3 or a 4; and the third number is a 5 or a 6. What is Ann's chance now of opening her locker on the first try? (p. 222)

The student listed the possibilities (2-3-5, 2-3-6, 2-4-5, 2-4-6) and explained that "There are 4 possible combinations. There used to be 8 but we found out what the first number is so that eliminated 4 numbers. The possibility has changed because it used to be 1 right number out of 8 but now it's 1 out of 4 because she knows what the first number is." (Johnson et al., p. 215).

The studies addressed in this section indicate that children have difficulty with basic probability ideas because they are not able or not willing to construct combinatorial type outcomes. Because these children do not exhaust the sample space or, alternatively, duplicate possibilities, they fail to determine the probabilities of particular outcomes in two- and three-stage compound experiments.

7. INCREASING CHILDREN'S ACCESS TO POWERFUL COMBINATORIAL AND PROBABILITY IDEAS

The *Principles and Standards* (NCTM, 2000) highlights the importance of providing children with opportunities to engage in the mathematical processes of representation, reasoning, abstraction, generalization, and forming connections. Combinatorial problems can help children construct meaningful representations, reason mathematically, and abstract and generalize mathematical concepts (Sriraman & English, 2004). Furthermore, as research has shown, combinatorial problems lend themselves to a variety of solution approaches, enabling children with minimal content knowledge to work towards a solution.

In a "cautionary note," however, Gardiner (1991) indicated that the educational value of basic discrete mathematics lies in the fact that it forces students to *think* about important elementary processes such as systematic counting. Yet, he warns that this feature can be easily undermined by teachers who believe they should "help" students to solve problems by reducing the solutions to a number of "manageable and predictable steps, or rules, *requiring an absolute minimum of thinking*" (p. 12). This leads us to the first of several recommendations for increasing children's access to powerful combinatorial ideas.

Foster Independent Thinking

Children should be given opportunities to explore combinatorial problem situations without direct instruction. The rich and meaningful contexts in which these problems can be couched means that children have sufficient resources to tackle the problems unassisted. However, appropriate teacher questioning as children work on the problems can promote children's combinatorial understanding. For example, asking children to explain and justify their solutions can lead them to reject some of their original ideas, or to modify, refine, or consolidate their original arguments (Maher & Martino, 1996).

Encourage Flexibility in Approaches and Representations

Being able to work flexibly with different representational forms is an increasingly important skill in today's world. Indeed, *representational fluency* has been shown to be at the heart of an understanding of many of the key constructs in elementary mathematics and science (Cobb, Yackel, & McClain, 2000; Lesh & Heger, 2001). Fluency with representational systems is essential to mathematical learning at all levels.

As previously noted, combinatorial situations lend themselves to a variety of solution approaches and representations. When presented with novel combinatorial problems, children will naturally display a number of different solution approaches, as the research cited here has shown. At the same time, children will adopt various representations in solving these problems including the use of drawings, tables, systematic and unsystematic listings, and concrete models. It is important that children be given the freedom to use different representations and approaches, and that they be encouraged to describe and explain their actions. In doing so, children can identify the similarities and differences between their own representational forms and those of other children.

Focus on Problem Structures

One of the major goals of mathematics education is that children see the connections and relationships between mathematical ideas and apply this understanding to the solution of new problems (Fuson, 1992; Hiebert, 1992; NCTM, 2000). If children are to make the appropriate links to new learnings, they need to construct understandings that comprise the structural relations between ideas, not the superficial surface details (English, 1997). That is, children need to identify the important structural properties of a problem situation: these being determined primarily by how the quantities in the problem are related to each other, rather than by what the quantities themselves are (Novick, 1988). A common finding in many of the studies on combinatorics is that students have difficulty in identifying related problem structures. As a consequence, students' ability to transfer their learning to new combinatorial situations is limited. It is thus imperative that children's combinatorial experiences include problems that vary contextually but are essentially isomorphic in their mathematical structure (Sriraman & English, 2004). Furthermore, as indicated in the studies of English (1991, 1993), the inclusion of additional features, such as a constraint on goal attainment, can help children become more robust and more flexible in their application of combinatorial knowledge.

Encourage Sharing of Solutions

It is recommended that children share their solutions to combinatorial problems with their peers. Children should describe and explain how they arrived at their solutions and why they consider their solutions to be effective ones. This practice of sharing means that the solutions children generate must hold up under the scrutiny of others. When children don't have to produce something sharable, they can frequently "settle for second best" (English & Lesh, 2003). In addition, when children share their solutions, they provide us with insights into their combinatorial understanding and also provide important opportunities for their peers to give constructive feedback.

Provide Problem-Posing Opportunities

The ability to pose problems (in addition to solving them) is becoming increasingly important in today's society (Brown & Walter, 1993; English, 1998; English, 2003). In the study cited earlier (English, 1999), the 32 ten-year-old children were invited to pose their own problem using two of the given problems as a base. The children had considerable difficulty here: 41% were either unable to create a sensible problem, or pose a different problem type, such as subtraction (when the original was addition). Twenty-eight percent of the children could construct an appropriate problem statement, but were unable to pose an appropriate question, rendering the problem insolvable. This can be seen in the following example:

> John had a green toothbrush, a blue toothbrush, and a purple toothbrush, and he had blue toothpaste, red toothpaste, and white toothpaste. How many times can he use them?

Only 32% of the children could create a solvable problem, many of which were set in one of the contexts of the given problems. This is of concern, given that problem posing — like its companion, problem solving — is a fundamental part of learning and doing mathematics. Problem posing is involved in creating new problems from old ones, as well as in reformulating given problems. Also like problem solving, problem posing is a natural part of our everyday lives (English, 2003). The benefits of incorporating problem-posing experiences within the mathematics curriculum are numerous. Winograd (1991) noted that students' original problem creations can serve as a viable and readily accessible source of content for students' mathematical learning. Students appear more motivated to pose and solve problems in which they have a vested interest; student-generated problems are more likely to connect mathematics to the students' interests, which is often not the case with standard textbook problems (Silver, 1994). At the same time, problem-posing experiences can lessen students' mathematics anxiety and lead to a more positive disposition towards the discipline (Brown & Walter, 1993; Healy, 1993; Silver, 1994).

By including problem posing in children's experiences with combinatorics, we can increase their access to the combinatorial concepts and procedures identified earlier in this chapter, and enhance their understanding of combinatorial problem structures. Furthermore, when children create their own combinatorial problems, they need to consider the problem design, that is, the components that will make up the problem, such as the known and unknown information, the goal to be attained, and any imposed constraints or conditions on achieving the goal (Moses, Bjork, & Goldenberg 1993). This understanding of problem design enables children to differentiate mathematical problems from nonmathematical problems, good problems from poor, and solvable from nonsolvable problems. In addition, understanding problem design enables children to provide more effective feedback on their peers' problem creations (English, 2003).

Provide Novel Probability Problems

Novel probability problems that utilise combinatorial ideas provide rich opportunities for children to predict, experiment, and analyse probabilistic situations (Jones, Langrall, Thornton, & Mogill, 1999). Two examples of such problems appear below:

a) Sarah is making greeting cards. She has a blue card, a pink card, and a yellow card, as well as gold and silver lettering. She makes as many different cards as possible and picks one at random to send to you. Is it more likely that you will get a "blue card" or a card with "gold lettering"? Explain your response.
b) Sam and his brother Ryan have a favourite game. Sam has 2 red smarties and one green smartie in his left hand and 1 red smartie and two green smarties in his right hand. He lets Ryan choose one smartie from each hand without looking. Is Ryan more likely to choose two smarties of the same color or two smarties of different colors? Justify your response.

Notice in the second example above that some combinations can occur in more than one way. Hence, children have to take this into consideration in determining which situation (two smarties of the same color or two smarties of different colors) is the more likely.

8. CONCLUDING POINTS

Combinatorics comprises a rich structure of significant mathematical principles that underlie several other areas of study including probability, computation, and counting. The domain also serves other disciplines such as biology, chemistry, and physics. As such, combinatorics has an important role to play in the elementary school mathematics curriculum and should go hand-in-hand with children's experiences in probability.

As the research cited here has indicated, even young children are able to work effectively with combinatorial situations when these are couched within meaningful contexts. Indeed, the real-world applications of combinatorics enable problems to be created that are appealing and meaningful, while at the same time, challenging to young children. Such problems lend themselves to a variety of approaches and representational forms. Furthermore, combinatorial problems facilitate the development of enumeration processes, as well as conjectures, generalisations, and systematic thinking.

It is thus imperative that we create learning environments that will facilitate children's development of powerful combinatorial ideas. A number of suggestions for fostering this learning have been presented. In particular, a focus on combinatorial problem structures needs attention. This is especially important across all problem types to enable children to develop conceptual understanding, transfer their learning to related situations, and create new problems for sharing with others.

Also in need of greater attention is how the use of computer technology can promote students' combinatorial reasoning. Stohl and Tarr (2002) have shown how the social aspects of learning, together with students' interactions with microworld tools, can challenge students' misconceptions in probability. In exploring simulations with combinatorial problems, Abramovich and Pieper (1997) have shown how a spreadsheet can enable students to focus on the patterns that emerge by decreasing the emphasis on pen and paper calculations. Indeed, as Stohl and Tarr have argued, there is a need to investigate the effects of long-term, sustained interaction with "dynamic, multi-representational software" on children's understanding of basic combinatorial ideas (p. 335).

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SECTION III

TEACHING AND LEARNING PROBABILITY IN THE MIDDLE SCHOOL