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HOW CAN TEACHERS BUILD NOTIONS OF CONDITIONAL PROBABILITY AND INDEPENDENCE?

Chapter 9

"The probability changes unless you put it back in." – Middle school student reasoning about conditional probabilities in a without-replacement task. (Tarr & Jones, 1997, p. 54)

1. INTRODUCTION

Research offers an emerging description of students' thinking in conditional probability and independence. Each of these concepts is associated with precise mathematical definitions that convey their interrelatedness. With respect to conditional probability, Hogg and Tanis (1993) point out that in some random experiments there is interest only in those outcomes that are elements of a subset B of the sample space S. Under these circumstances, the *conditional probability of an event A given that event B has occurred*, P(A|B), is the probability of A considering as possible outcomes only those outcomes of the random experiment that are elements of B. That is, the probability of event A is evaluated under the conditions of a new sample space, one that has been conditioned by the occurrence of event B.

Hogg and Tanis also note that a special case of conditional probability occurs in a random experiment carried out in *without-replacement* situations. For example, consider an experiment where a gumball is selected and not replaced from a machine containing one red, one green, and one yellow gumball. The sample space immediately prior to the second draw will be a subset of the original sample space. The probability of "green," for example, will be conditional on the outcome of the first draw. If a green gumball is picked on the first draw, the probability of "green" given the event "green" on the first draw will be 0. On the other hand, if a red gumball is selected on the first draw, the probability of "green" given the event "red" on the first draw will be 0.5.

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In "without replacement" situations, such as those described above, conditional probabilities become particularly explicit because the reduction of the sample space can be visualized. In the research literature, it is often within the context of without-replacement situations (sampling *one object at a time*) that conditional probability problems occur (e.g., Falk, 1983; Falk, 1988; Borovcnik & Bentz, 1991) although social contexts have also been used (e.g., Tversky & Kahneman, 1982; Watson, this volume; Watson & Moritz, 2002) to assess students' understanding of conditional probability.

Based on these definitions it is generally accepted that students' "understanding of conditional probability" is demonstrated by their ability to recognize and adjust the probability of an event when it is changed by the occurrence of another event; that is, to "revise probability judgments as new information becomes available" (Borovcnik & Bentz, 1991, p. 90).

Some mathematical presentations (e.g., Borovcnik, Bentz & Kapadia, 1991; Hogg & Tanis, 1993) of independent events define A as an independent event of B if $P(A|B) = P(A)$, that is if the occurrence of event B does not change the probability of the occurrence of event A. It follows that independence represents a special case of conditional probability. In fact, Borovcnik and Bentz (1991) associate independence with the "unconditional probability" (p. 90) of event A; that is when occurrence of event B does not influence the probability of event A. Because of its relatedness to conditional probability, some researchers (e.g., Kelly & Zwiers, 1988; Ahlgren & Garfield, 1991) recommend introducing the concept of independence via the conditional probability definition because it is more intuitive for students (Shaughnessy, 1992). In contrast with conditional probability, tasks that have been used in research on students' thinking about independence (e.g., Cohen, 1957; Kahneman & Tversky, 1972; Shaughnessy, 1977) are largely associated with either observing a sequence of independent trials or with random experiments involving *with-replacement* situations. Within this context, an "understanding of independence" is demonstrated by students' ability to recognize and correctly explain when the occurrence of one event does not influence the probability of another event. The focus of this chapter is on research in conditional probability and independence that uses both with- and without-replacement tasks.

2. THE EMERGENCE OF CONDITONAL PROBABILITY AND INDEPENDENCE IN THE MIDDLE SCHOOL MATHEMATICS CURRICULUM

Recent world-wide curriculum reforms in school mathematics (e.g., Australian Curriculum Corporation, 1994; Department of Education and Science and the Welsh Office, 1991; National Council of Teachers of

Mathematics [NCTM], 1989, 2000) advocate broadening the scope of probability in the middle school mathematics curriculum and placing more emphasis on conceptual understanding. Such recommendations represent a departure from traditional curricula which placed only nominal emphasis on probability in school mathematics. Shaughnessy (1992) notes that although no comprehensive survey of how much probability is taught in schools has been undertaken, one could confidently say that, until recently, elementary and middle school students have had little or no opportunity to study probability concepts such as conditional probability and independence.

In the wake of new curriculum developments, conditional probability and independence have emerged in prominent curricular materials for middle school students including the *Connected Mathematics Project* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997), *Mathematics in Context* (National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1997-1998), *The National Numeracy Strategy* (Department for Education and Skills, 1997), *Chance and Data: Investigations* (Lovitt & Lowe, 1993) and NCTM-sponsored curriculum projects for middle school probability (Bright, Frierson, Tarr & Thomas, 2003; Zawojewski, 1991). Although some curricular documents (e.g., NCTM, 2000) restrict the formal study of conditional probability for students in Grades 9-12, Watson (1995) argues that "it would be a disservice to save conditional probability only for advanced students in the final years of high school" (p. 16). She advocates that conditional probability and independence be introduced in the middle school mathematics curriculum, and taught in an intuitive manner.

The precise placement of conditional probability in the school mathematics curriculum cannot be determined by any single investigation but the results of several recent studies (Jones, Langrall, Thornton, & Mogill, 1999; Tarr, 1997; Tarr & Jones, 1997) lend credence to the notion that topics such as conditional probability and independence are indeed appropriate for the middle school mathematics curriculum and need not be deferred until students have developed robust skills in comparing fractions. In all of these recent studies, students used a variety of strategies to make correct conditional probability judgments following instruction, and many did so *without* the predominant use of fractions or numerical probabilities. Instead, they used their own invented strategies to make valid probability judgments.

3. CONDITIONAL PROBABILTIY AND INDEPENDENCE: CONCEPTIONS AND MISCONCEPTIONS

Research into middle school students' thinking in conditional probability has grown steadily (e.g., Fischbein & Gazit, 1984; Piaget & Inhelder, 1951/1975; Shaughnessy, 1992; Tarr & Jones, 1997; Tarr, 2002; Watson & Moritz, 2002; Yáñez, 2002) as has research on independence (e.g., Fischbein, Nello, & Marino, 1991; Green, 1983; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993). The findings of a number of these studies provide a helpful focus for this section (for a more detailed discussion of misconceptions in conditional probability and independence, see Jones & Thornton; Pratt; this volume).

Probabilistic Thinking in Conditional Probability

Fischbein and Gazit (1984) carried out a teaching experiment involving 285 students from Grades 5, 6 and 7. They found that, when students were asked to determine conditional probabilities in with- and without-replacement situations, the percentage of correct responses was generally lower for without-replacement situations, although approximately 24% of fifth-grade students correctly determined conditional probabilities in both with and without-replacement tasks. By way of contrast, the percentage of correct responses for sixth graders was 63% for with-replacement tasks and 43% for without-replacement tasks, and that for seventh graders was 89% for with-replacement tasks and 71% for without-replacement situations. Based on their analysis, Fischbein and Gazit identified two fundamental misconceptions in students' thinking in conditional probability:

1. Students did not realize that the sample space had changed in a without-replacement situation, and
2. Students found the probability of an event in a without-replacement situation by comparing the number of favorable outcomes for the event before and after the first trial rather than by making comparisons with the total number of outcomes (pp. 8-9).

In a related study, Tarr (2002) reported that students' conditional probability judgments were impaired by their misuse of the phrase “50-50 chance" in two distinct ways. In particular, when the sample space contained two elements, students often assumed each outcome had a "50-50 chance," even when the two events were not equally likely. Additionally, they applied the phrase to probability situations in which more than two outcomes in the sample space were equally likely to occur, and concluded that each event had a "50-50 chance." Both of these invalid uses of "50-50 chance" were problematic as students considered conditional probabilities in without-replacement situations. In particular, their persistent use of this phrase often impeded students' ability to recognize that the probabilities of all events changed in non-replacement situations.

The conclusions of Fischbein and Gazit (1984) and Tarr (2002) clearly suggest that a primary objective of instruction must be to develop the idea that *the sample space is changed in without-replacement situations.* Additionally, an instructional program must help students to *consider the composition of the sample space in relation to the total number of outcomes;* that is, although the number of elements of a target color may remain unchanged after sampling without-replacement, it is critical to consider how the entire sample space has been modified by the conditioning event.

Probabilistic Thinking about Independence

In relation to middle school students' thinking about independence, a key study was carried out by Fischbein et al. (1991) with 618 students in Grades 4-8. In this study, the researchers asked students to determine which event was more likely: obtaining three "heads" by tossing one coin three times or by tossing three coins simultaneously. Thirty-eight percent of fourth and fifth graders and 30% of junior high students, with no prior instruction in probability, responded that the probabilities were not equal. By a ratio of nearly 2:1, students at each grade level believed the probability of obtaining three heads, by tossing a single coin three times, was higher. Based on follow-up interviews, Fischbein et al. found that students harboured a pervasive belief that the outcomes of a coin toss can be controlled by the individual. The researchers concluded that such a belief is incompatible with the notion of independence, given that the probability of obtaining a head on each trial remains constant at 0.5.

Similar misconceptions were evident in the U.S. National Assessment of Educational Progress in Mathematics (Brown et. al., 1988) that asked students to state the most likely outcome on the next toss of a fair coin which had landed "TTTT" on four successive trials. Results indicated that only 47% of the seventh graders selected the correct alternative--heads and tails are equally likely. Slightly higher achievement was obtained in a study of 2,930 British students aged 11 to 16 years (Green, 1983). In this study a fair coin was flipped four times, each time landing heads up. When asked to name the most likely outcome of the fifth toss, 75% of all students, including 67% of 11-12 year olds, answered correctly that "heads is as likely as tails."

In a third study using this same item, Konold et al. (1993) found that only 70% of the undergraduates in a remedial mathematics course responded correctly. Moreover, in extensions to this item, Konold et al. asked students to state which of the following sequences was most likely and which was least likely to occur when a fair coin was tossed five times: (a) HHHTT, (b) THHTH, (c) THTTT, (d) HTHTH and (e) all four sequences are equally likely. In the most likely case, approximately 61% of the undergraduates responded correctly, but only about 35% responded correctly in the case of the least likely sequence. Clearly a substantial number of students who demonstrated some understanding of independence in the most likely case abandoned this thinking in the least likely case. Konold concluded that a conflict existed between the belief that a coin has an equal chance of coming up heads or tails and that roughly half heads and half tails are expected in a sample of coin flips. In the most likely case, Konold asserted that students justified all sequences were equally likely by use of an *outcome approach* in which they interpreted the problem as a request to predict what will happen; such students typically used statements such as "anything could happen" to justify the response "equally likely." Students who switched their response from "equally likely" changed their perspective from an outcome approach to a related heuristic known as *representativeness* – the belief that a sample or even a single outcome should reflect the parent population (Kahneman & Tversky, 1972). Such pervasive beliefs are powerful and demonstrate students' inability to deal consistently with the concept of independence. Thus, even when students seemingly exhibit an understanding of the concept of independence, the representativeness heuristic may still prevail.

Because students of all ages are prone to exhibit various misconceptions when observing a series of independent trials (Garfield & Ahlgren, 1988; Shaughnessy, 1992) or when considering with-replacement probability situations, an instructional program in independence must address these problematic features of probabilistic thinking. In particular, "representativeness" arguably represents the greatest impediment to developing an understanding of independence. Accordingly, instruction must provide experiences that will challenge the thinking of students who have adopted this powerful and pervasive heuristic. Moreover, instruction should develop the notion that the sample space is preserved in with-replacement situations as this represents a key to fostering an understanding of independence.

Frameworks for Describing Students' Reasoning

Although the investigations into students' thinking in conditional probability (e.g., Borovcnik & Bentz, 1991; Fischbein & Gazit, 1984) and independence (e.g., Fischbein et al., 1991; Green, 1983; Konold et al., 1993) depict various aspects of students' probabilistic thinking, none of these studies provided a coherent model of middle school students' thinking in conditional probability and independence. Several recent studies (Jones et al., 1996; Jones et al., 1997; Tarr & Jones, 1997; Jones et al., 1999) addressed this void by formulating and validating cognitive frameworks that capture the manifold-nature of students' probabilistic thinking. Consistent with cognitive research by neo-Piagetian theorists (e.g., Biggs & Collis, 1991), Tarr & Jones (1997) postulated that middle school students' thinking in conditional probability and independence could be described and predicted across four levels that represent a continuum from subjective thinking to numerical reasoning (see Figure 1). In particular, the four levels were in concert with the existence of substages or levels of thinking that recycle during maturational stages and reflect shifts in the structural complexity of students' thinking: Level 1 is associated with subjective thinking, Level 2 is seen as a transitional stage between subjective and naive quantitative thinking, Level 3 involves the use of informal quantitative thinking and Level 4 incorporates numerical reasoning. Students' probabilistic reasoning at each level is illustrated in relation to the conditional probability and independence tasks in Figure 2.

Level 1

Students exhibiting Level 1 thinking tend to rely on subjective judgments; they generally believe that they can control the outcome of an event, and they ignore relevant quantitative information in formulating probability judgments. These students' lack of quantitative referents leads them to form conditional probability judgments by constructing their own reality, by imposing their own system of regularity or by searching for patterns that do not exist. For example, when asked whether the chance of drawing a grape candy has changed (Conditional Probability Task, Figure 2), a student at this level may respond, "No, because grape is my favorite flavor and I really want a grape!" Notice that this judgment is made without regard to the changing number of grape candies in the jar. Additionally, these students often use their own recent experiences (availability heuristic, Tversky & Kahneman, 1983) when predicting the outcome of an event, and this leads them to believe that previous outcomes generally influence future outcomes. Thus, when predicting the outcome of the third flip (Independence Task,

Figure 2), a student exhibiting Level 1 might respond, “When I flip coins with my brother, I am good at getting three in a row so I think it will be red again.” Because of their tendency to rely on subjective judgments, to impose their own system of regularity, or to rely heavily on personal experiences, students at Level 1 do not focus on independence and conditional probability in any meaningful way.

	LEVEL 1 (Subjective)	**LEVEL 2** (Transitional)	**LEVEL 3** (Informal Quantitative)	**LEVEL 4** (Numerical)
CONDITIONAL PROBABILITY	· Recognizes when "certain" and "impossible" events arise in replacement and non-replacement situations. · Generally uses subjective reasoning in considering the conditional probability of any event in a "with" or "without" replacement situation. · Ignores given numerical information in formulating predictions.	· Recognizes that the probabilities of *some* events change in a non-replacement situation, however recognition is incomplete and is usually confined to events that have previously occurred. · Inappropriate use of numbers in determining conditional probabilities. For example, when the sample space contains two outcomes, always assumes that the two outcomes are equally likely. · Representativeness acts as a confounding effect when making decisions about conditional probability. · May revert to subjective judgments.	· Recognizes that the probabilities of *all* events change in a non-replacement situation. · Keeps track of the complete composition of the sample space in judging the relatedness of two events in both replacement and non-replacement situations. · Can quantify, albeit imprecisely, changing probabilities in a non-replacement situation.	· Assigns numerical probabilities in replacement and non-replacement situations. · Uses numerical reasoning to compare the probabilities of events before and after each trial in replacement and non-replacement situations. · States the necessary conditions under which two events are related.
INDEPENDENCE	· Predisposition to consider that consecutive events are always related. · Pervasive belief that they can control the outcome of an event. · Uses subjective reasoning which precludes any meaningful focus on independence. · Exhibits unwarranted confidence in predicting successive outcomes.	· Shows some recognition as to whether consecutive events are related or unrelated. · Frequently uses a "representativeness" strategy, either a positive or negative recency orientation. · May also revert to subjective reasoning.	· Recognizes when the outcome of the first event does or does not influence the outcome of the second event. In replacement situations, sees the sample space as restored. · Can differentiate, albeit imprecisely, independent and dependent events in "with" and "without" replacement situations. · Some reversion to representativeness.	· Distinguishes dependent and independent events in replacement and non-replacement situations, using numerical probabilities to justify their reasoning. · Observes outcomes of successive trials but rejects a representativeness strategy. · Reluctance or refusal to predict outcomes when events are equally-likely.

Figure 1. A framework for assessing students' thinking in conditional probability and independence (Tarr & Jones, 1997)

Level 2

Students exhibiting Level 2 thinking are in transition between subjective and informal quantitative thinking. Although they sometimes make appropriate use of quantitative information in making conditional probability judgments, they are often distracted by irrelevant features. That is, students at this level tend to place too much faith in the distribution of previous outcomes when forming predictions; consequently, they are often prone to invoke a *representativeness* strategy (Shaughnessy, 1992), incorporating either a positive or negative recency orientation (see Jones & Thornton, this volume). For example, when predicting the outcome of the third flip (Independence Task, Figure 2), a student at this level may respond, "I think it will be white since it's really hard to get red three times in a row." In considering conditional probabilities, when they do utilize quantitative reasoning, their thinking is limited. Consequently, students at Level 2 are able to recognize that the probabilities of only *some events* change in non-replacement situations, and recognition is usually restricted to events that have previously occurred. Thus, Level 2 students might argue that the chance of drawing a grape candy has decreased because there are now fewer in the jar but they maintain that the chance of drawing each of the other flavored candies has remained unchanged "because there's still the same number of those in the jar."

CONDITIONAL PROBABILITY	A candy jar contains an assortment of flavors: 4 grape, 3 cherry, 2 apple, and 1 lemon candies. A grape candy is drawn and eaten. Has the chance of drawing another grape candy from the jar changed or is it the same chance as it was before? Has the chance of drawing a cherry candy changed? ...an apple candy? ...a lemon candy? Explain.
INDEPENDENCE TASK	A chip colored red on one side, white on the other is flipped repeatedly, landing with the red side facing up twice in a row. Which outcome is most likely for the third flip: red, white, or are these outcomes equally likely? Explain.

Figure 2. Sample tasks for conditional probability and independence

Level 3

By Level 3, students have gained an awareness of the role quantities play in forming conditional probability judgments. Although such students do not usually assign precise numerical probabilities, they often use relative frequencies, ratios, or some form of odds as an appropriate strategy in determining conditional probabilities in both with- and without-replacement situations. Students at Level 3 monitor the complete composition of the sample space and usually recognize that the conditional probabilities of *all events* change in nonreplacement situations. Thus, students at this level might argue that the probability of drawing a lemon candy (Conditional Probability Task, Figure 2) has increased since "lemon was 3 away from having the most but now it is only 2 away (from having the most)," and assert the probability of cherry has increased "because it is now tied with grape [for having the most] but it used to be behind." By keeping track of the sample space composition, students are able to recognize independent events in replacement situations but they sometimes revert to a representativeness strategy after observing a run on one outcome in a sequence of independent trials (Shaughnessy, 1992). Thus, in predicting the outcome of the third toss (Independence Task, Figure 2), students may explain, "It could be red or white because there is still one red and one white side on the chip."

Level 4

Students exhibiting Level 4 thinking can spontaneously assign numerical probabilities when interpreting probability situations. Because they are acutely aware of the role numbers play in forming probability judgments, they closely monitor the composition of the sample space and recognize its importance in determining whether two events are independent or dependent. Thus, students at this level are able to articulate the changing probabilities of drawing a grape candy (Conditional Probability Task, Figure 2) by saying, "It was 4 out of 10 before you drew the grape candy, but now it is only a 3 out of 9 chance for drawing grape *unless you put the grape candy back in the jar.*" Such a response reflects sophistication in probabilistic reasoning in that the students are able to state the conditions by which two events are dependent or independent. With respect to replacement situations, Level 4 students are less likely to succumb to use of a representativeness strategy even after observing a run of one outcome on a sequence of independent trials. Students at this level may use numerical probabilities to reject representativeness by stating, "It doesn't matter what happened before. It's always going to be a '50-50 chance' for red because there are two outcomes and they are equally likely" (Independence Task, Figure 2).

Key Elements Underlying Reasoning in Conditional Probability

Two substantive aspects emerged from our review of research into students' reasoning in conditional probability. The first relates to part-part and part-whole reasoning and the second relates to students' invented representations.

Part-Part versus Part-Whole Reasoning

Conditional probability judgments require the ability to make probability comparisons. There is conflicting evidence documenting middle school students' abilities to make correct probability comparisons. On the one hand, Piaget and Inhelder (1951/1975) concluded that children who lacked an understanding of part-whole relationships experienced difficulty comparing the likelihood of different events. On the other hand, Falk (1983) and Green (1983) identified numerous strategies that enabled students to make probability comparisons without an understanding of rational numbers and without assigning numerical probabilities. Using odds or other part-part comparisons, students in the Falk and Green studies were able to compare the likelihood of two events. In contrast to the assertions of Piaget and Inhelder (1951/1975), Falk (1983) and Green (1983) suggest that students do not need to reach the stage of formal operations in order to successfully make probability comparisons (see also Polaki, this volume).

In a pivotal study of 26 fifth-grade students, Tarr (1997) observed that prior to a 9-day instructional program students primarily used part-part comparisons rather than part-whole comparison when making conditional probability judgments. Although part-part comparisons enabled many Level 2 students (see Figure 1) to realize that the probabilities of some events change in without replacement situations, such strategies often limited students in recognizing that the probabilities of all events change in without-replacement situations. Because the total number of objects is critical to making part-whole comparisons, emphasis during instruction was placed on having students determine the total number of outcomes in the sample space. Analysis of video tapes taken during instruction revealed that students began to make part-whole comparisons after learning how to assign numerical probabilities in the initial lesson.

Invented Forms of Probability Representations

Given the lack of opportunity to learn probability (Shaughnessy, 1992), many students may not spontaneously assign conventional numerical probabilities in describing conditional probabilities. In the absence of a standard form for representing the probability of an event, students are likely to use alternative forms for stating and comparing probabilities. Some of these invented representations were associated with part-part comparisons while others were associated with part-whole comparisons and others were idiosyncratic representations. In particular, Tarr (1997) reported four invented probability notations, three of them exhibited prior to instruction while the other occurred during instruction. Students' invented forms of probability representations are illustrated in relation to the Conditional Probability Task in Figure 2.

In the first of these invented representations, students describe the probability of an event using the word "chance" as a unit of probability measure. Rather than identifying the total number of objects (candies) when assigning numerical probabilities to various events, these students consider each individual object as a unit of chance. Thus, in describing the conditional probabilities after a grape candy is drawn without replacement, students may explain, "Drawing a grape candy *has gone down one chance* because you've pulled a grape candy out. The lemon candy *has gone up one more chance* because you took the grape candy out." From a positive vantage point, this invented way of describing the probabilities of events may well take into account both the total number of ways the target event can occur as well as the total number of objects comprising the sample space. Despite such consideration of the total number of objects, the strategy nevertheless focuses on part-part comparisons, in particular the number of objects of the target event and its complement.

A second alternative form involves the use of relative frequencies, ratios or some form of odds to describe the probability of an event. Essentially, these students make part-part comparisons to determine whether the probability of an event has or has not changed. For example, students adopting this representation in the Conditional Probability Task (Figure 2) will argue that the probability of selecting a grape candy (on the second draw) has changed because "there were more grape candies than cherry candies and now there is the same number of each." Using this strategy, students keep track of the composition of the sample space after each trial and compare the number of favorable and unfavorable outcomes when making judgments about conditional probabilities. Moreover, they often monitor the relative ranks of events within the sample space and notice, for example, when the number of grape candies no longer exceeds the number of cherry candies.

A third alternative form of stating the probability of an event is essentially the conventional numerical representation. In this strategy, students compare the number of ways the target event could occur to the total number of possible outcomes but do so in a nonconventional manner. For example, they may describe the probability of selecting a lemon candy as a *"one of ten chance"* (i.e., one in ten chance) before a grape candy was drawn without replacement and a *"one of nine chance"* afterwards. This use of numerical probabilities was limited to contexts in which the sample space comprised only two events as was the case in the preceding example.

Interestingly, when more than two events comprise the sample space these students seemed unable to describe the probabilities of the events.

Following instruction, most students' use of alternative forms for stating numerical probabilities was largely replaced with more conventional ways of describing the probability of an event: using ratios or odds, or formal numerical probabilities. Nevertheless, other students either adopted or continued to use invented forms of representing probabilities. Remarkably, one invented strategy was exhibited only in postinstructional assessments; these students combined the use of percents and ratios to create a "hybrid" form of numerical probability. For example, after assigning the probability of drawing each individual piece of candy in the Conditional Probability Task (Figure 2), these students might describe the conditional probability of drawing an apple candy as “20% out of 90%” since 10% of the entire sample space was, in essence, removed by the occurrence of the conditioning event. It should be noted that although the strategy is not mathematically correct, this invented form of communicating the conditional probability of an event seems to have meaning to the student. Given that no such strategy was demonstrated during the instructional program, the students' strategy for stating probabilities is evidence that students continue to invent their own representations even when standard forms are negotiated during instruction.

4. THE IMPACT OF INSTRUCTION ON STUDENTS' PROBABILISTIC REASONING

In this section we focus specifically on research findings and implications related to the teaching of conditional probability. Recent teaching experiments (Castro, 1998; Fischbein & Gazit, 1984; Jones et al., 1999; Kiczek & Maher, 2001; Tarr, 1997) have documented growth in students' understanding of conditional probability and independence. In addition to providing insights into the development of students' probabilistic thinking, this research has identified learning environments, teaching strategies, learning tasks, and assessment activities that have the potential to contribute to theory and practice in the teaching and learning of probability.

Impact of Instruction on Students' Probabilistic Reasoning

Shaughnessy (1992) noted that research has “not been particularly concerned with the influence of instruction on the misconceptions of stochastics” (p. 483) and this statement remains largely true, especially as it relates to the impact of instruction on student understanding of conditional probability and independence.

As discussed earlier in this chapter, Fischbein and Gazit (1984) were the first to specifically address the impact of instruction in conditional probability. Due to poor student performance following implementation of the instructional program, they cautioned against introducing these concepts prior to sixth grade. However, two important limitations to this study have caused researchers to question this conclusion: (a) there was no preassessment of student performance that allowed for examining growth in student understanding, and (b) there is little evidence to confirm the extent to which the instructional intervention was implemented as intended (Shaughnessy, 1992).

As evidence that instruction can impact student understanding at earlier grade levels, Jones et al. (1999) conducted an instructional intervention that significantly increased third grade students' understanding of probability concepts. Using a small-group teaching experiment format for 16 instructional sessions, Jones et al. documented growth in student learning of the concepts of sample space, theoretical probability of an event, and probability comparisons. However, student understanding of conditional probability appeared to lag behind that of the other constructs, with only 1 of 37 students able to use informal quantitative or numerical reasoning in conditional probability. These results suggest that an understanding of sample space and theoretical probability of an event are requisite to developing an understanding of conditional probability.

Further evidence of the impact of instruction on student learning can be found in Tarr (1997). This teaching experiment focused exclusively on fifth grade students' understanding of conditional probability and independence concepts. Utilizing an instructional design that was informed by a research-based framework of students' probabilistic reasoning (Tarr & Jones, 1997), the study reported statistically significant growth in student learning in conditional probability and independence. Specifically, in conditional probability 19 of 26 students were coded at Level 1 or 2 before instruction, whereas following instruction 22 of 26 students exhibited thinking at Level 3 or Level 4; sustained learning was evidenced in retention assessments seven weeks following instruction. Similar growth patterns were found with regard to student thinking in independence. Moreover, statistically significant differences on measures of conditional probability and independence were also found.

Learning Environments and Instructional Strategies

A key characteristic of the aforementioned teaching experiments (Fischbein & Gazit, 1984; Jones et al., 1999; Tarr, 1997) is the value placed on students' probabilistic reasoning during instruction. Each instructional program was designed so that students predicted the outcome of a particular experiment, collected data for a number of trials, and re-examined their predictions based on their empirical evidence and renewed understanding of the situation. Such environments lead to deeper understanding of key ideas by encouraging discussion and reflection among students regarding possible misconceptions. Similarly, other researchers (e.g., Kiczek & Maher, 2001; Koirala, 2003; Stohl & Tarr, 2002) document the importance of providing students with opportunities to collaborate on probability problems to enable them to overcome initial misconceptions and negotiate shared meanings.

Using a related model of instruction, Castro (1998) compared the impact of two different instructional orientations: (a) an environment that focused on eliciting student thinking and encouraging reflection on probabilistic ideas (referred to as, "conceptual change"), and (b) "traditional instruction" that centered on a clear, linear presentation of mathematical ideas without considering student conceptions and misconceptions. Castro found that misconceptions in conditional probability and independence were more resilient among those receiving "traditional instruction" than in classes that focused on "conceptual change." For example, students experiencing "traditional instruction" were more likely to retain *representativeness* strategies than students experiencing instruction that confronted misconceptions.

Learning Tasks and Assessment Activities

The above results highlight the importance of intertwining assessment and instruction. In particular, the findings of Castro (1998), Jones et al. (1999), and Tarr (1997) indicate that instructional tasks should elicit particular student conceptions and misconceptions, enabling students to reflect on the validity of their probabilistic intuitions, and providing teachers access to student thinking. Engaging students in carefully designed tasks allows the teacher to formally and informally assess student thinking and inform instructional decision-making.

Additionally, the frameworks used by Jones et al. (1999) and Tarr (1997) suggest that teachers' knowledge of the levels of students' probabilistic reasoning guided their questioning and their selection of instructional tasks. For example, when Tarr observed students reasoning that some probabilities remained the same in nonreplacement situations, he designed a task where one Milky Way bar was drawn without replacement from a bag of 3 Milky Way, 2 Butterfinger, and 1 Snickers candy bars. Then he focused student discussion on the probability of drawing a Snickers bar, given that the Milky Way bar had been removed from the bag. This drew student attention to the change in the total number of objects in the bag and the subsequent changes in probability. For Jones et al. and Tarr, the use of formative assessments of student understanding precipitated the construction of specific tasks designed to encourage deeper understanding of particular probabilistic ideas.

5. IMPLICATIONS FOR TEACHING AND LEARNING: FOSTERING UNDERSTANDING

Recent teaching experiments in probability provide several implications for the teaching and learning of conditional probability and independence. In particular, the previously discussed research-based knowledge of students' probabilistic reasoning can inform the planning, implementation, and evaluation of instructional programs.

The Design of Problem Tasks

Instructional plans in conditional probability and independence should include assessment tasks and key questions that elicit students' thinking and serve as a foundation for subsequent whole-class discussions. Such tasks should be set in contexts that are familiar to middle school students and promote small-group and whole-class discussions. For example, sampling candy bars without-replacement from a bag of Halloween candy or selecting names of students to be among the first to go to lunch represent appropriate contexts for the study of conditional probability. Likewise, by sampling with replacement in these same contexts, students can explore the concept of independence. Additionally, problem contexts requiring students to analyze whether a game is fair or unfair are viable avenues for eliciting students' reasoning and engaging them in mathematical discourse. For example, students can investigate whether any sequence of three flips of a colored chip (Independence Task, Figure 2) is more likely than another. Points could be assigned to each outcome (e.g., 1 point for obtaining Red-White-Red, 1 point for obtaining Red-Red-Red) with the winner being first to score 5 points. Data from many games can be subsequently analyzed and used as the focus of a whole-class discussion.

Such instructional tasks should encourage students to examine novel situations and provoke cognitive conflict among students. For example, consider a common probabilistic misconception among adults (Shaughnessy, 1992; Tversky & Kahneman,1982; Utts, 2003; Watson & Moritz, 2002), namely that $P(A|B) = P(B|A)$. Students could be encouraged to examine the difference in real-world situations similar to the one provided by Utts: the probability of testing positive for steroid use given that you actually used steroids, and the probability that you used steroids given that you test positive. We would expect that students would respond to this task in a manner similar to how most adults do, viewing these probabilities as the same. However, examining a related situation regarding the results when a single die is rolled could encourage students to revisit their initial reasoning. Teachers might ask students to examine the difference in the probabilities of two situations: (a) the die displays a six, given the result is even, and (b) the result is even, given the die displays a six. In the former case (a), the conditional probability is 1/3; in the latter case (b) the conditional probability is 1. Discussion of the difference between these two results can encourage students to re-evaluate their initial conception that $P(A|B) = P(B|A)$.

As stated earlier, the rich descriptions of students' thinking conveyed in the framework (see Figure 1) can aid teachers when designing tasks that could elicit student misconceptions and create cognitive conflict. Such tasks should be designed to focus on particular conceptions and misconceptions with regard to conditional probability and independence.

Relating Notions of Sample Space and Probability of an Event to Conditional Probability and Independence

From a teaching and learning perspective, it is apparent that the key to understanding conditional probability lies in making connections between sample space and the probability of an event. By fostering students' understanding of "sample space" and "probability of an event" it is possible to develop a predisposition to monitor the composition of the sample space, to make probability comparisons, and to determine that the probability of all events change in non-replacement situations. This assertion is consistent with Jones, Langrall, Thornton, & Mogill (1996) who concluded that third-grade children's ability to connect "sample space" and "probability of an event," and their willingness to use numbers in describing probability situations were key factors in facilitating the learning of conditional probability.

Understanding the role of the sample space in making conditional probability judgments is a distinguishing characteristic of students' thinking at Levels 3 and 4 of the framework (Figure 1). In particular, consideration of the total number of objects better enables students to recognize that the probability of all events changes in nonreplacement situations. Accordingly, teachers might find it helpful to encourage students to make multiple comparisons – including comparisons to the whole – when teaching conditional probability. For example, in relation to the Conditional Probability Task (Figure 2), teachers can focus student attention on the changing number of elements comprising the sample space by asking the following question *before and after* a grape candy is selected: "How many total candies are there?" Such a question focuses students' attention on the total number of objects as a basis for assigning numerical probabilities. By doing so, students become aware that although the number of lemon candies stays the same, the total number of candies decreases from 10 to 9 given the conditioning event.

Although facility in assigning numerical probabilities may help students make conditional probability judgments, it is not sufficient for developing students' understanding of independence. The key to fostering students' understanding of independence lies in making probability comparisons after each independent trial. In essence, students who examine probabilities "compared to before," or "compared to last time," are often subsequently able to realize when the probability of an event has or has not changed. As an example, in the Conditional Probability Task (Figure 2), teachers might ask, "What is the probability of selecting a lemon candy on the first draw? What is the probability of selecting a lemon candy on the second draw? How do the probabilities compare?" By way of contrast, on the Independence Task (Figure 2) teachers might ask, "What is the probability of obtaining 'red' on the first flip? What is the probability of obtaining 'red' on the second flip? How do the probabilities compare?" Additionally, teachers might be encouraged to have students compare the composition of the sample space before and after each trial in order to recognize that the sample space remains unchanged in replacement situations and that the probability of all events is likewise unchanged. Thus, on the Conditional Probability Task (Figure 2) teachers might ask, "How would the probabilities of each event change if the grape candy were *replaced* after being selected?"

Focusing on the Two Concepts Simultaneously

Given the relatedness of conditional probability and independence, several researchers (e.g., Ahlgren & Garfield, 1991) recommend introducing independence as a special case of conditional probability because it is more intuitive for students (Shaughnessy, 1992). This assertion is consistent with recent findings (Tarr, 1997) in which student understanding of both concepts was fostered as a result of discussion that focused on the distinctions between these two concepts. By learning how to describe the probability of an event, students were poised to make subsequent comparisons in deciding whether the probability of all events had or had not changed. More importantly, by heightening their attention on the composition of the sample space after each trial, students were able to recognize that the sample space is restored in replacement situations and is changed when sampling without replacement. Thus, it is recommended that teachers promote discussion of the two concepts within one instructional segment, with particular focus on the sample space in each sampling context.

Using Simulations to Build and Enhance Understanding of Conditional Probability and Independence

There is a growing body of evidence (e.g., delMas & Bart, 1989; Pratt, 2000; Pratt, this volume; Stohl & Tarr, 2002; Yáñez, 2002) to support the role of simulations as a means of fostering understanding of some probability concepts. Surprisingly, there is little evidence that simulations foster students' thinking in conditional probability. Yáñez (2002) reported that university engineering students struggled with modelling random experiments involving conditional events; they lacked confidence in the simulation method and experienced difficulty interpreting graphs of the relative frequencies for estimating conditional probabilities.

By way of contrast, it appears that students' thinking in independence can be developed by promoting links between data and chance. Specifically, results of several studies (e.g., delMas & Bart, 1989; Pratt, 2000; Pratt, this volume; Tarr, 1997) suggest that simulations may be fruitful in challenging students' use of a representativeness strategy which is in conflict with the concept of independence. Data from individual simulations can be pooled and the combined results can serve to challenge students' use of the representativeness heuristic. In a surprising result, Tarr (1997) found that repeated exposure to random experiments during assessment interviews may have produced learning among control group students, particularly when the results of individual trials did not turn out as they predicted. More precisely, after repeated flips of a colored chip did not yield a "representative" sequence, several students from the control group seemed to have learned that small samples do not necessarily reflect the parent population. Some of them may also have learned that events can occur against the odds and that a colored chip does not have a "memory." This finding is similar to delMas & Bart (1989) who reported that students exposed to computer-generated simulation data became less likely to believe they could predict outcomes of random experiments and less likely to adopt a representativeness strategy.

Some caution on the use of simulations is warranted. Middle school students are often eager to predict the outcome of a single trial and small samples of simulation data may, in fact, serve to validate flawed probabilistic reasoning. Thus, teachers should instead help students to focus on predictions over the long term rather than on predictions of individual outcomes. For example, instead of having students predict which flavor of candy will be drawn in any single trial (Conditional Probability Task, Figure 2), it is more useful for them to predict which flavor will be drawn most often when the experiment is carried out repeatedly. Larger pools of data are more likely to reflect theoretical probabilities, and trends in data can be used to challenge faulty predictions. Additionally, Shaughnessy, Canada, and Ciancetta, (2003) advocate carrying out repeated trials of an experiment to develop student intuition for a "range of outcomes," and how the probability of an outcome is situated within the distribution of outcomes for an experiment. This important focus on distribution of outcomes is also shared by Watson and Kelly (2003). For further suggestions on the use of technology to foster students' probabilistic reasoning, see Pratt, this volume.

6. CONCLUSION

The emergence of conditional probability and independence in the middle school mathematics curriculum presents new challenges for students and teachers. Recent research supports the notion that these topics are both important and appropriate for middle school students. Furthermore, instruction that is informed by researched-based knowledge of students' thinking in conditional probability and independence can foster a coordinated understanding of both key concepts.

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SECTION IV

TEACHING AND LEARNING PROBABILITY IN THE HIGH SCHOOL