Chapter 7

Upper-Lipschitz Continuity of the Solution Map in Affine Variational Inequalities

In this chapter we shall discuss two fundamental theorems due to Robinson (1979, 1981) on the upper-Lipschitz continuity of the solution map in affine variational inequality problems. The theorem on the upper-Lipschitz continuity of the solution map in linear complementarity problems due to Cottle et al. (1992) is also studied in this chapter. The Walkup-Wets Theorem (see Walkup and Wets (1969)), which we analyze in Section 7.1, is the basis for obtaining these results.

7.1 The Walkup-Wets Theorem

Let $\Delta \subset R^n$ be a nonempty subset. Let $\tau : R^n \to R^m$ be an affine operator; that is there exist a linear operator $A : R^n \to R^m$ and a vector $b \in R^m$ such that $\tau(x) = Ax + b$ for every $x \in R^n$. Define

$$\begin{aligned} \Delta(y) = \tau^{-1}(y) \cap \Delta \; &= \{x \in \Delta \,:\, \tau(x) = y\} \\ &= \{x \in \Delta \,:\, Ax + b = y\}. \end{aligned} \tag{7.1}$$

Definition 7.1. (See Walkup and Wets (1969), Definition 1) A subset $\Delta \subset R^n$ is said to have *property* $\mathcal{L}_j$ if for every affine operator $\tau : R^n \to R^m$, $m \in N$, with $\dim(\ker(\tau)) = j$, the inverse mapping

$y \to \Delta(y)$ is Lipschitz on its effective domain. This means that there exists a constant $\ell > 0$ such that

$$\Delta(y') \subset \Delta(y) + \ell\|y' - y\|\bar{B}_{R^n} \quad \text{whenever } \Delta(y) \neq \emptyset,\ \Delta(y') \neq \emptyset. \tag{7.2}$$

In the above definition, $\dim(\ker(\tau))$ denotes the dimension of the affine set

$$\ker(\tau) = \{x \in R^n : \tau(x) = 0\}.$$

The following theorem is a key tool for proving other results in this chapter.

Theorem 7.1 (The Walkup-Wets Theorem; see Walkup and Wets (1969), Theorem 1). *Let $\Delta \subset R^n$ be a nonempty closed convex set and let $j \in N$, $1 \le j \le n-1$. Then Δ is a polyhedral convex set if and only if it has property $\mathcal{L}_j$.*

In the sequel, we will use only one assertion of this theorem: *If Δ is a polyhedral convex set, then it has property $\mathcal{L}_j$.* A detailed proof of this assertion can be found in Mangasarian and Shiau (1987).

Corollary 7.1. *If $\Delta \subset R^n$ is a polyhedral convex set and if $\tau : R^n \to R^m$ is an affine operator, then there exists a constant $\ell > 0$ such that* (7.2)*, where $\Delta(y)$ is defined by* (7.1) *for all $y \in R^n$, holds.*

Proof. If $j := \dim(\ker(\tau))$ satisfies the condition $1 \le j \le n-1$, then the conclusion is immediate from Theorem 7.1. If $\dim(\ker(\tau)) = n$ then $\ker(\tau) = R^n$, and we have

$$\Delta(y) = \tau^{-1}(y) \cap \Delta = \begin{cases} \Delta & \text{if } y = 0, \\ \emptyset & \text{if } y \neq 0. \end{cases}$$

This shows that (7.2) is fulfilled with any $\ell > 0$. We now suppose that $\dim(\ker(\tau)) = 0$. Let $\tau(x) = Ax + b$, where $A : R^n \to R^m$ is a linear operator and $b \in R^m$. Since τ is an injective mapping, $Y := \tau(R^n)$ is an affine set in R^m with $\dim Y = n$, and that $n \le m$. Likewise, the set $Y_0 := A(R^n)$ is a linear subspace of R^m with $\dim Y_0 = n$. Let $\widetilde{A} : R^n \to Y_0$ be the linear operator defined by setting $\widetilde{A}x = Ax$ for every $x \in R^n$. It is easily shown that

$$\|\tau^{-1}(y') - \tau^{-1}(y)\| \le \|\widetilde{A}^{-1}\|\|y' - y\|$$

for every $y \in Y$ and $y' \in Y$. From this we deduce that (7.2) is satisfied with $\ell := \|\widetilde{A}^{-1}\|$. □

Remark 7.1. Under the assumptions of Corollary 7.1, for every $y \in R^m$, $\Delta(y)$ is a polyhedral convex set (possibly empty).

Remark 7.2. The conclusion of Theorem 7.1 is not true if one chooses $j = 0$. Namely, the arguments described in the final part of the proof of Corollary 7.1 show that any nonempty set $\Delta \subset R^n$ has property $\mathcal{L}_0$. Similarly, the conclusion of Theorem 7.1 is not valid if $j = n$.

Corollary 7.2. *For any nonempty polyhedral convex set $\Delta \subset R^n$ and any matrix $C \in R^{s \times n}$ there exists a constant $\ell > 0$ such that*

$$\Delta(C, d'') \subset \Delta(C, d') + \ell \|d'' - d'\| \bar{B}_{R^n} \tag{7.3}$$

whenever $\Delta(C, d')$ and $\Delta(C, d'')$ are nonempty; where

$$\Delta(C, d) := \{x \in \Delta \, : \, Cx = d\}$$

for every $d \in R^s$.

Proof. Set $\tau(x) = Cx$. Since

$$\Delta(C, y) = \tau^{-1}(y) \cap \Delta = \Delta(y)$$

where $\Delta(y)$ is defined by (7.1), applying Corollary 7.1 we can find $\ell > 0$ such that the Lipschitz continuity property stated in (7.3) is satisfied. $\square$

Corollary 7.3. *For any nonempty polyhedral convex set $\Delta \subset R^n$, any matrix $A \in R^{m \times n}$ and matrix $C \in R^{s \times n}$ there exists a constant $\ell > 0$ such that*

$$\Delta(A, C, b'', d'') \subset \Delta(A, C, b', d') + \ell(\|b'' - b'\| + \|d'' - d'\|) \bar{B}_{R^n} \tag{7.4}$$

whenever $\Delta(A, C, b', d')$ and $\Delta(A, C, b'', d'')$ are nonempty; where

$$\Delta(A, C, b, d) := \{x \in \Delta \, : \, Ax \geq b, \; Cx = d\}$$

for every $b \in R^m$ and $d \in R^s$.

Proof. Define

$$\widetilde{C} = \begin{pmatrix} A & -E \\ C & 0 \end{pmatrix} \in R^{(m+s) \times (n+m)},$$

where E denotes the unit matrix in $R^{m \times m}$ and 0 denotes the null in $R^{s \times m}$. Let

$$\widetilde{\Delta} = \{(x, w) \in R^n \times R^m \, : \, x \in \Delta, \; w \geq 0\}.$$

By Corollary 7.2, there exists $\ell > 0$ such that

$$\widetilde{\Delta}(\widetilde{C}, b'', d'') \subset \widetilde{\Delta}(\widetilde{C}, b', d') + \ell(\|b'' - b'\| + \|d'' - d'\|)\bar{B}_{R^{n+m}} \quad (7.5)$$

whenever $\widetilde{\Delta}(\widetilde{C}, b', d') \neq \emptyset$ and $\widetilde{\Delta}(\widetilde{C}, b'', d'') \neq \emptyset$, where

$$\widetilde{\Delta}(\widetilde{C}, b, d) := \left\{ (x, w) \in \widetilde{\Delta} \,:\, \widetilde{C}\begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \right\}.$$

Since

$$\begin{aligned} \Delta(A, C, b, d) &= \{x \in \Delta \,:\, \exists w \in R^m,\ w \geq 0,\ Ax - w = b,\ Cx = d\} \\ &= \mathrm{Pr}_{R^n}(\widetilde{\Delta}(\widetilde{C}, b, d)) \end{aligned}$$

where $\mathrm{Pr}_{R^n}(x, w) = x$ for every $(x, w) \in R^n \times R^m$, we see at once that (7.5) implies (7.4). □

7.2 Upper-Lipschitz Continuity with respect to Linear Variables

The notion of polyhedral multifunction was proposed by Robinson (see Robinson (1979, 1981). We now study several basic facts concerning polyhedral multifunctions.

Definition 7.2. If $\Phi : R^n \to 2^{R^m}$ is a multifunction then its *graph* and *effective domain* are defined, respectively, by setting

$$\begin{aligned} \mathrm{graph}\Phi &= \{(x, y) \in R^n \times R^m \,:\, y \in \Phi(x)\}, \\ \mathrm{dom}\Phi &= \{x \in R^n \,:\, \Phi(x) \neq \emptyset\}. \end{aligned}$$

Definition 7.3. A set-valued mapping $\Phi : R^n \to 2^{R^m}$ is called a *polyhedral multifunction* if its graph can be represented as the union of finitely many polyhedral convex sets in $R^n \times R^m$.

The following statement shows that the *normal-cone operator* corresponding to a polyhedral convex set is a polyhedral multifunction.

Proposition 7.1. (See Robinson (1981)) *Suppose that $\Delta \subset R^n$ is a nonempty polyhedral convex set. Then the formula*

$$\Phi(x) = N_\Delta(x) \quad (x \in R^n)$$

defines a polyhedral multifunction $\Phi : R^n \to 2^{R^n}$.

Proof. Let $m \in N$, $A \in R^{n \times n}$ and $b \in R^m$ be such that $\Delta = \{x \in R^n : Ax \geq b\}$. Set $I = \{1, \ldots, m\}$. Let

$$F_\alpha = \{x \in R^n : A_\alpha x = b_\alpha, \ \ A_{I \setminus \alpha} x > b_{I \setminus \alpha}\}$$

be the pseudo-face of Δ corresponding to an index set $\alpha \subset I$. For every $x \in F_\alpha$ we have

$$T_\Delta(x) = \{v \in R^n : A_\alpha v \geq 0\}.$$

(See the proof of Theorem 4.2.) Since

$$N_\Delta(x) = \{\xi \in R^n : \langle \xi, v \rangle \leq 0 \ \ \forall v \in T_\Delta(x)\},$$

we have $\xi \in N_\Delta(x)$ if and only if the inequality $\langle \xi, v \rangle \leq 0$ is a consequence of the inequality system $A_\alpha v \geq 0$. Consequently, applying Farkas' Lemma (see Theorem 3.2) we deduce that $\xi \in N_\Delta(x)$ if and only if there exist $\lambda_1 \geq 0, \ldots, \ \lambda_m \geq 0$ such that

$$\xi = \sum_{i \in \alpha} \lambda_i (-A_i^T),$$

where A_i denotes the $i-$th row of matrix A. (Note that if $\alpha = \emptyset$ and $x \in F_\alpha$, then $x \in \text{int}\Delta$; hence $\xi = 0$ for every $\xi \in N_\Delta(x)$.) Define

$$\Omega_\alpha = \{(x, \xi) \in R^n \times R^n : x \in F_\alpha, \ \xi \in N_\Delta(x)\}.$$

Obviously, $\Omega_\alpha \subset \text{graph}\Phi$. Note that

$$\begin{aligned} \Omega_\alpha \ &= \{(x, \xi) \in R^n \times R^n : A_\alpha x = b_\alpha, \ A_{I \setminus \alpha} x > b_{I \setminus \alpha}, \\ &\qquad \xi = \textstyle\sum_{i \in \alpha} \lambda_i (-A_i^T) \ \text{ for some } \ \lambda_\alpha \in R_+^{|\alpha|}\}. \end{aligned}$$

is a convex set. Here $|\alpha|$ denotes the number of elements in α. It is easily seen that the topological closure $\overline{\Omega}_\alpha$ of Ω_α is given by the formula

$$\begin{aligned} \overline{\Omega}_\alpha \ &= \{(x, \xi) \in R^n \times R^n : A_\alpha x = b_\alpha, \ A_{I \setminus \alpha} x \geq b_{I \setminus \alpha}, \\ &\qquad \xi = \textstyle\sum_{i \in \alpha} \lambda_i (-A_i) \ \text{ for some } \ \lambda_\alpha \in R_+^{|\alpha|}\} \\ &= \text{Pr}_{R^n \times R^n} \{(x, \xi, \lambda_\alpha) \in R^n \times R^n \times R_+^{|\alpha|} : A_\alpha x = b_\alpha, \\ &\qquad A_{I \setminus \alpha} x \geq b_{I \setminus \alpha}, \ \textstyle\sum_{i \in \alpha} \lambda_i A_i^T + \xi = 0\}, \end{aligned}$$

where $\text{Pr}_{R^n \times R^n}(x, \xi, \lambda_\alpha) = (x, \xi)$. It is clear that the set in the last curly brackets is a polyhedral convex set. From this fact, the above formula for $\overline{\Omega}_\alpha$ and Theorem 19.1 in Rockafellar (1970) we deduce that $\overline{\Omega}_\alpha$ is a polyhedral convex set (see the proof of Theorem 4.3). Since $\Delta = \bigcup_{\alpha \subset I} F_\alpha$, we have

$$\text{graph}\Phi = \bigcup_{\alpha \subset I} \Omega_\alpha. \tag{7.6}$$

Observe that graphΦ is a closed set. Indeed, suppose that $\{(x^k, \xi^k)\}$ is a sequence satisfying $(x^k, \xi^k) \to (\bar{x}, \bar{\xi}) \in R^n \times R^n$, and $(x^k, \xi^k) \in$ graphΦ for every $k \in N$. On account of formula (1.12), we have

$$\langle \xi^k, y - x^k \rangle \leq 0 \quad \forall y \in \Delta, \ \forall k \in N.$$

Fixing any $y \in \Delta$ and taking limit as $k \to \infty$, from the last inequality we obtain $\langle \bar{\xi}, y - \bar{x} \rangle \leq 0$. Since this inequality holds for each $y \in \Delta$, we see that $\bar{\xi} \in N_\Delta(\bar{x})$. Hence $(\bar{x}, \bar{\xi}) \in$ graphΦ. We have thus proved that the set graphΦ is closed. On account of this fact, from (7.6) we deduce that

$$\text{graph}\Phi = \bigcup_{\alpha \subset I} \overline{\Omega}_\alpha.$$

This shows that graphΦ can be represented as the union of finitely many polyhedral convex sets. The proof is complete. □

The following statement shows that the solution map of a parametric affine variational inequality problem is a polyhedral multifunction (on the linear variables of the problem).

Proposition 7.2. *Suppose that $M \in R^{n \times n}$, $A \in R^{m \times n}$ and $C \in R^{s \times n}$ are given matrices. Then the formula*

$$\Phi(q, b, d) = \text{Sol}(\text{AVI}(M, q, \Delta(b, d))),$$

where $(q, b, d) \in R^n \times R^m \times R^s$, $\Delta(b, d) := \{x \in R^n : Ax \geq b, \ Cx = d\}$ and Sol(AVI$(M, q, \Delta(b, d))$) *denotes the solution set of problem* (6.1) *with $\Delta = \Delta(b, d)$, defines a polyhedral multifunction*

$$\Phi : R^n \times R^m \times R^s \to 2^{R^n}.$$

Proof. According to Corollary 5.2, $x \in \text{Sol}(\text{AVI}(M, q, \Delta(b, d)))$ if and only if there exist $\lambda = (\lambda_1, \ldots, \lambda_m) \in R^m$ and $\mu = (\mu_1, \ldots, \mu_s) \in R^s$ such that

$$\begin{cases} Mx - A^T\lambda - C^T\mu + q = 0, \\ Ax \geq b, \ Cx = d, \ \ \lambda \geq 0, \\ \lambda^T(Ax - b) = 0. \end{cases} \tag{7.7}$$

Let $I = \{1, \dots, m\}$. For each index set $\alpha \subset I$, we define

$$\begin{aligned} Q_\alpha \ = \mathrm{Pr}_1\Big(\{(x,q,b,d,\lambda,\mu) \ : \ & Mx - A^T\lambda - C^T\mu + q = 0, \\ & A_\alpha x = b_\alpha,\ A_{I\setminus\alpha}x \geq b_{I\setminus\alpha}, \\ & Cx = d,\ \lambda_\alpha \geq 0,\ \lambda_{I\setminus\alpha} = 0\}\Big), \end{aligned} \tag{7.8}$$

where

$$\mathrm{Pr}_1(x,q,b,d,\lambda,\mu) = (x,q,b,d)$$

for all $(x,q,b,d,\lambda,\mu) \in R^n \times R^n \times R^m \times R^s \times R^m \times R^s$. Hence Q_α is a polyhedral convex set. Note that

$$\mathrm{graph}\Phi = \bigcup_{\alpha\subset I} Q_\alpha. \tag{7.9}$$

Indeed, for each $(x,q,b,d) \in \mathrm{graph}\Phi$ we have

$$x \in \mathrm{Sol}(\mathrm{AVI}(M,q,\Delta(b,d))).$$

So there exist $\lambda = (\lambda_1, \dots, \lambda_m) \in R^m$ and $\mu = (\mu_1, \dots, \mu_s) \in R^s$ satisfying (7.7). Let $\alpha = \{i \in I \ : \ A_i x = b_i\}$. For every $i \in I \setminus \alpha$, we have $A_i x > b_i$. Then from the equality $\lambda_i(A_i x - b_i) = 0$ we deduce that $\lambda_i = 0$ for every $i \in I \setminus \alpha$. On account of this remark, we see that (x,q,b,d,λ,μ) satisfies all the conditions described in the curly braces in formula (7.8). This implies that $(x,q,b,d) \in Q_\alpha$. We thus get

$$\mathrm{graph}\Phi \subset \bigcup_{\alpha\subset I} Q_\alpha.$$

Since the reverse inclusion is obvious, we obtain formula (7.9), which shows that graphΦ can be represented as the union of finitely many polyhedral convex sets. □

Theorem 7.2. (See Robinson (1981), Proposition 1) *If $\Phi : R^n \to 2^{R^m}$ is a polyhedral multifunction, then there exists a constant $\ell > 0$ such that for every $\bar{x} \in R^n$ there is a neighborhood $U_{\bar{x}}$ of $\bar{x}$ satisfying*

$$\Phi(x) \subset \Phi(\bar{x}) + \ell\|x - \bar{x}\|\bar{B}_{R^m} \quad \forall x \in U_{\bar{x}}. \tag{7.10}$$

Definition 7.4. (See Robinson (1981)) Suppose that $\Phi : R^n \to 2^{R^m}$ is a multifunction and $\bar{x} \in R^n$ is a given point. If there exist $\ell > 0$ and a neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that property (7.10) is valid, then Φ is said to be *locally upper-Lipschitz* at $\bar{x}$ with the Lipschitz constant ℓ.

The locally upper-Lipschitz property is weaker than the locally Lipschitz property which is described as follows.

Definition 7.5. A multifunction $\Phi : R^n \to 2^{R^m}$ is said to be *locally Lipschitz* at $\bar{x} \in R^n$ if there exist a constant $\ell > 0$ and a neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that

$$\Phi(x) \subset \Phi(u) + \ell\|x - u\|\bar{B}_{R^m} \quad \forall x \in U_{\bar{x}}, \forall u \in U_{\bar{x}}.$$

If there exists a constant $\ell > 0$ such that

$$\Phi(x) \subset \Phi(u) + \ell\|x - u\|\bar{B}_{R^m}$$

for all x and u from a subset $\Omega \subset R^n$, then Φ is said to be *Lipschitz* on Ω.

From Theorem 7.2 it follows that if Φ is a polyhedral multifunction then it is locally upper-Lipschitz at any point in R^n with the same Lipschitz constant. Note that the *diameter* $\text{diam}U_{\bar{x}} := \sup\{\|y - x\| : x \in U_{\bar{x}},\ y \in U_{\bar{x}}\}$ of neighborhood $U_{\bar{x}}$ depends on $\bar{x}$ and it can change greatly from one point to another.

Proof of Theorem 7.2.

Since Φ is a polyhedral multifunction, there exist nonempty polyhedral convex sets $Q_j \subset R^n \times R^m$ $(j = 1, \ldots, k)$ such that

$$\text{graph}\Phi = \bigcup_{j \in J} Q_j, \tag{7.11}$$

where $J = \{1, \ldots, k\}$. For each $j \in J$ we consider the multifunction $\Phi_j : R^n \to 2^{R^m}$ defined by setting

$$\Phi_j(x) = \{y \in R^m : (x, y) \in Q_j\}. \tag{7.12}$$

Obviously, $\text{graph}\Phi_j = Q_j$. From (7.11) and (7.12) we deduce that

$$\text{graph}\Phi = \bigcup_{j \in J} \text{graph}\Phi_j, \quad \Phi(x) = \bigcup_{j \in J} \Phi_j(x).$$

CLAIM 1. *For each $j \in J$ there exists a constant $\ell_j > 0$ such that*

$$\Phi_j(x) \subset \Phi_j(u) + \ell_j\|x - u\|\bar{B}_{R^m} \tag{7.13}$$

whenever $\Phi_j(x) \neq \emptyset$ *and* $\Phi_j(u) \neq \emptyset$. *(This means that* Φ_j *is Lipschitz on its effective domain.)*

For proving the claim, consider the linear operator $\tau : R^n \times R^m \to R^n$ defined by setting $\tau(x, y) = x$ for every $(x, y) \in R^n \times R^m$. Let

$$Q_j(x) = \{z \in Q_j \, : \, \tau(z) = x\}. \tag{7.14}$$

By Corollary 7.1, there exists $\ell_j > 0$ such that

$$Q_j(x) \subset Q_j(u) + \ell_j \|x - u\| \bar{B}_{R^{n+m}} \tag{7.15}$$

whenever $Q_j(x) \neq \emptyset$ and $Q_j(u) \neq \emptyset$. From (7.12) and (7.14) it follows that

$$Q_j(x) = \{x\} \times \Phi_j(x) \quad \forall x \in R^n. \tag{7.16}$$

In particular, $Q_j(x) \neq \emptyset$ if and only if $\Phi_j(x) \neq \emptyset$. Given any $x \in R^n$, $u \in R^n$ and $y \in \Phi_j(x)$, from (7.15) and (7.16) we see that there exist $v \in \Phi(u)$ such that

$$\|(x, y) - (u, v)\| \leq \ell_j \|x - u\|.$$

Since $\|(x, y) - (u, v)\| = (\|x - u\|^2 + \|y - v\|^2)^{1/2}$, the last inequality implies that $\|y - v\| \leq \ell_j \|x - u\|$. From what has already been proved, it may be concluded that (7.13) holds whenever $\Phi_j(x) \neq \emptyset$ and $\Phi_j(u) \neq \emptyset$.

We set $\ell = \max\{l_j \, : \, j \in J\}$. The proof will be completed if we can establish the following fact.

CLAIM 2. *For each* $\bar{x} \in R^n$ *there exists a neighborhood* $U_{\bar{x}}$ *of* $\bar{x}$ *such that* (7.10) *holds.*

Let $\bar{x} \in R^n$ be given arbitrarily. Define

$$J_0 = \{j \in J \, : \, \bar{x} \in \mathrm{dom}\Phi_j\}, \quad J_1 = J \setminus J_0.$$

Since $\mathrm{dom}\Phi_j = \tau(Q_j)$, where τ is the linear operator defined above, we see that $\mathrm{dom}\Phi_j$ is a polyhedral convex set. This implies that the set $\bigcup_{j \in J_1} \mathrm{dom}\Phi_j$ is closed. (Note that if $J_1 = \emptyset$ then this set is empty.) As $\bar{x} \notin \bigcup_{j \in J_1} \mathrm{dom}\Phi_j$, there must exist $\varepsilon > 0$ such that the neighborhood $U_{\bar{x}} := B(\bar{x}, \varepsilon)$ of $\bar{x}$ does not intersect the set

$$\bigcup_{j \in J_1} \mathrm{dom}\Phi_j.$$

Let $x \in U_{\bar{x}}$. If $x \notin \bigcup_{j\in J_0} \mathrm{dom}\Phi_j$, then

$$\Phi(x) = \left(\bigcup_{j\in J_0} \Phi_j(x)\right) \bigcup \left(\bigcup_{j\in J_1} \Phi_j(x)\right) = \emptyset.$$

So the inclusion (7.10) is valid. If $x \in \bigcup_{j\in J_0} \mathrm{dom}\Phi_j$, then we have

$$\Phi(x) = \bigcup_{j\in J_0} \Phi_j(x) = \bigcup_{j\in J_0'} \Phi_j(x),$$

where $J_0' = \{j \in J_0 : x \in \mathrm{dom}\Phi_j\}$. For each $j \in J_0'$, according to Claim 1, we have

$$\Phi_j(x) \subset \Phi_j(\bar{x}) + \ell_j \|x - \bar{x}\| \bar{B}_{R^m} \subset \Phi(\bar{x}) + \ell \|x - \bar{x}\| \bar{B}_{R^m}.$$

Therefore

$$\Phi(x) = \bigcup_{j\in J_0'} \Phi_j(x) \subset \Phi(\bar{x}) + \ell \|x - \bar{x}\| \bar{B}_{R^m}.$$

Claim 2 has been proved. □

Remark 7.3. From the proof of Theorem 7.2 it is easily seen that Φ is Lipschitz on the set $\bigcap_{j\in J} \mathrm{dom}\Phi_j$ with the Lipschitz constant ℓ.

Combining Theorem 7.2 with Proposition 7.2 we obtain the next result on upper-Lipschitz continuity of the solution map in a general AVI problem where the linear variables are subject to perturbation.

Theorem 7.3. *Suppose that $M \in R^{n\times n}$, $A \in R^{m\times n}$ and $C \in R^{s\times n}$ are given matrices. Then there exists a constant $\ell > 0$ such that the multifunction $\Phi : R^n \times R^m \times R^s \to 2^{R^n}$ defined by the formula*

$$\Phi(q, b, d) = \mathrm{Sol}(\mathrm{AVI}(M, q, \Delta(b, d))),$$

where $(q, b, d) \in R^n \times R^m \times R^s$ and $\Delta(b, d) := \{x \in R^n : Ax \geq b,\ Cx = d\}$, is locally upper-Lipschitz at any point $(\bar{q}, \bar{b}, \bar{d}) \in R^n \times R^m \times R^s$ with the Lipschitz constant ℓ.

Applying Theorem 7.3 to the case where the constraint set $\Delta(b, d)$ of the problem AVI$(M, q, \Delta(b, d))$ is fixed (i.e., the pair (b, d) is not subject to perturbations), we have the following result.

Corollary 7.4. *Suppose that $M \in R^{n\times n}$ is a given matrix and $\Delta \subset R^n$ is a nonempty polyhedral convex set. Then there exists a constant $\ell > 0$ such that the multifunction $\Phi : R^n \to 2^{R^n}$ defined by the formula*

$$\Phi(q) = \mathrm{Sol}(\mathrm{AVI}(M, q, \Delta)),$$

where $q \in R^n$, is locally upper-Lipschitz at any point $\bar{q} \in R^n$ with the Lipschitz constant ℓ.

7.3 Upper-Lipschitz Continuity with respect to all Variables

Our aim in this section is to study some results on locally upper-Lipschitz continuity of the multifunction $\Phi : R^{n\times n} \times R^n \to 2^{R^n}$ defined by the formula

$$\Phi(M, q) = \mathrm{Sol}(\mathrm{AVI}(M, q, \Delta)),$$

where $\mathrm{Sol}(\mathrm{AVI}(M, q, \Delta))$ denotes the solution set of the problem (6.1). First we consider the case where Δ is a polyhedral convex cone. Then we consider the case where Δ is an arbitrary nonempty polyhedral convex set.

The following theorem specializes to Theorem 7.5.1 in Cottle et al. (1992) about the solution map in parametric linear complementarity problems if $\Delta = R^n_+$.

Theorem 7.4. *Suppose that $\Delta \subset R^n$ is a polyhedral convex cone. Suppose that $M \in R^{n\times n}$ is a given matrix and $q \in R^n$ is a given vector. If M is copositive on Δ and*

$$q \in \mathrm{int}\left([\mathrm{Sol}(\mathrm{AVI}(M, 0, \Delta))]^+\right), \tag{7.17}$$

then there exist constants $\varepsilon > 0$, $\delta > 0$ and $\ell > 0$ such that if $(\widetilde{M}, \widetilde{q}) \in R^{n\times n} \times R^n$, $\widetilde{M}$ is copositive on Δ, and if

$$\max\{\|\widetilde{M} - M\|,\ \|\widetilde{q} - q\|\} < \varepsilon, \tag{7.18}$$

then the set $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta))$ is nonempty,

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \subset \delta \bar{B}_{R^n}, \tag{7.19}$$

and

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \subset \mathrm{Sol}(\mathrm{AVI}(M, q, \Delta)) + \ell(\|\widetilde{M} - M\| + \|\widetilde{q} - q\|)\bar{B}_{R^n}. \tag{7.20}$$

Proof. Suppose that M is copositive on Δ and (7.17) is satisfied. Since Δ is a polyhedral convex cone, we see that for every $(\widetilde{M}, \widetilde{q}) \in R^{n\times n} \times R^n$ the problem $\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)$ is an GLCP. In particular, $\mathrm{AVI}(M, 0, \Delta)$ is a GLCP problem and we have

$$\mathrm{Sol}(\mathrm{AVI}(M, 0, \Delta)) = \{v \in \Delta : Mv \in \Delta^+,\ \langle Mv, v\rangle = 0\}.$$

Since Sol(AVI($M,0,\Delta$)) is a closed cone, Lemma 6.4 shows that (7.17) is equivalent to the following condition

$$q^T v > 0 \quad \forall v \in \mathrm{Sol}(\mathrm{AVI}(M,0,\Delta)) \setminus \{0\}. \tag{7.21}$$

CLAIM 1. *There exists* $\varepsilon > 0$ *such that if* $(\widetilde{M},\widetilde{q}) \in R^{n\times n} \times R^n$, $\widetilde{M}$ *is copositive on* Δ, *and if* (7.18) *holds, then the set* Sol(AVI($\widetilde{M},\widetilde{q},\Delta$)) *is nonempty.*

Suppose Claim 1 were false. Then we could find a sequence $\{(M^k,q^k)\}$ in $R^{n\times n} \times R^n$ such that M^k is copositive on Δ for every $k \in N$, $(M^k,q^k) \to (M,q)$ as $k \to \infty$, and Sol(AVI(M^k,q^k,Δ)) $= \emptyset$ for every $k \in N$. According to Theorem 6.5, we must have

$$q^k \notin \mathrm{int}\left([\mathrm{Sol}(\mathrm{AVI}(M^k,0,\Delta))]^+\right) \quad \forall k \in N.$$

Applying Lemma 6.4 we can assert that for each $k \in N$ there exists $v^k \in$ Sol(AVI($M^k,0,\Delta$))$\setminus\{0\}$ such that $(q^k)^T v^k \le 0$. Then we have

$$v^k \in \Delta, \quad M^k v^k \in \Delta^+, \quad \langle M^k v^k, v^k\rangle = 0, \tag{7.22}$$

for every $k \in N$. Without loss of generality we can assume that

$$\frac{v^k}{\|v^k\|} \to \bar{v} \in R^n, \quad \|\bar{v}\| = 1.$$

From (7.22) it follows that

$$\frac{v^k}{\|v^k\|} \in \Delta, \quad M^k \frac{v^k}{\|v^k\|} \in \Delta^+, \quad \left\langle M^k \frac{v^k}{\|v^k\|}, \frac{v^k}{\|v^k\|} \right\rangle = 0.$$

Taking limits as $k \to \infty$ we obtain

$$\bar{v} \in \Delta, \quad M\bar{v} \in \Delta^+, \quad \langle M\bar{v},\bar{v}\rangle = 0.$$

This shows that $\bar{v} \in$ Sol(AVI($M,0,\Delta$)). Since $(q^k)^T v^k \le 0$, we see that $(q^k)^T \dfrac{v^k}{\|v^k\|} \le 0$ for every $k \in N$. Letting $k \to \infty$ yields $q^T\bar{v} \le 0$. Since $\bar{v} \in$ Sol(AVI($M,0,\Delta$)) $\setminus \{0\}$, the last inequality contradicts (7.21). We have thus justified Claim 1.

CLAIM 2. *There exist* $\varepsilon > 0$ *and* $\delta > 0$ *such that if* $(\widetilde{M},\widetilde{q}) \in R^{n\times n} \times R^n$, $\widetilde{M}$ *is copositive on* Δ, *and if* (7.18) *holds, then inclusion* (7.19) *is satisfied.*

To obtain a contradiction, suppose that there exist a sequence $\{(M^k, q^k)\}$ in $R^{n\times n} \times R^n$ and a sequence $\{x^k\}$ in R^n such that M^k is copositive on Δ for every $k \in N$, $x^k \in \mathrm{Sol}(\mathrm{AVI}(M^k, q^k, \Delta))$ for every $k \in N$, $(M^k, q^k) \to (M, q)$ as $k \to \infty$, and $\|x^k\| \to +\infty$ as $k \to \infty$. Since $x^k \in \mathrm{Sol}(\mathrm{AVI}(M^k, q^k, \Delta))$, we see that

$$x^k \in \Delta, \quad M^k x^k + q^k \in \Delta^+, \quad \langle M^k x^k + q^k, x^k \rangle = 0, \tag{7.23}$$

for every $k \in N$. There is no loss of generality in assuming that

$$\frac{x^k}{\|x^k\|} \to \bar{v} \in R^n, \quad \|\bar{v}\| = 1.$$

From (7.23) it follows that

$$\bar{v} \in \Delta, \quad M\bar{v} \in \Delta^+, \quad \langle M\bar{v}, \bar{v} \rangle = 0.$$

From this we conclude that $\bar{v} \in \mathrm{Sol}(\mathrm{AVI}(M, 0, \Delta))$. Since

$$\langle M^k x^k + q^k, x^k \rangle = 0$$

and since $0^+\Delta = \Delta$ and M^k is copositive on Δ, we have

$$-(q^k)^T x^k = -\langle q^k, x^k \rangle = \langle M^k x^k, x^k \rangle \geq 0.$$

Then

$$q^T \bar{v} = \lim_{k\to\infty} \left((q^k)^T \frac{x^k}{\|x^k\|} \right) \leq 0.$$

This contradicts (7.21). Claim 2 has been proved.

Now we are in a position to show that there exist $\varepsilon > 0$, $\delta > 0$ and $\ell > 0$ such that if $(\widetilde{M}, \widetilde{q}) \in R^{n\times n} \times R^n$, $\widetilde{M}$ is copositive on Δ, and if (7.18) holds, then $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \neq \emptyset$ and (7.19), (7.20) are satisfied.

Combining Claim 1 with Claim 2 we see that there exist $\varepsilon > 0$ and $\delta > 0$ such that if $(\widetilde{M}, \widetilde{q}) \in R^{n\times n} \times R^n$, $\widetilde{M}$ is copositive on Δ, and if (7.18) holds, then $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \neq \emptyset$ and (7.19) is satisfied. According to Corollary 7.4, for the given matrix M and vector q, there exist a constant $\ell_M > 0$ and a neighborhood U_q of q such that

$$\mathrm{Sol}(\mathrm{AVI}(M, q', \Delta)) \subset \mathrm{Sol}(\mathrm{AVI}(M, q, \Delta)) + \ell_M \|q' - q\| \bar{B}_{R^n} \tag{7.24}$$

for every $q' \in U_q$. Let $(\widetilde{M}, \widetilde{q}) \in R^{n\times n} \times R^n$ be such that $\widetilde{M}$ is copositive on Δ and (7.18) holds. Select any $\widetilde{x} \in \mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta))$. Setting

$$\bar{q} = \widetilde{q} + (\widetilde{M} - M)\widetilde{x} \tag{7.25}$$

we will show that

$$\widetilde{x} \in \mathrm{Sol}(\mathrm{AVI}(M, \bar{q}, \Delta)). \tag{7.26}$$

Since

$$\langle \widetilde{M}\widetilde{x} + \widetilde{q}, x - \widetilde{x} \rangle \geq 0 \quad \forall x \in \Delta,$$

using (7.25) we deduce that

$$\begin{aligned} 0 \leq \langle \widetilde{M}\widetilde{x} + \widetilde{q}, x - \widetilde{x} \rangle &= \langle \widetilde{M}\widetilde{x} + \bar{q} - \widetilde{M}\widetilde{x} + M\widetilde{x}, x - \widetilde{x} \rangle \\ &= \langle M\widetilde{x} + \bar{q}, x - \widetilde{x} \rangle \end{aligned}$$

for every $x \in \Delta$. This shows that (7.26) is valid. From (7.18), (7.19) and (7.25) it follows that

$$\|\bar{q} - q\| \leq \|\widetilde{q} - q\| + \|\widetilde{M} - M\|\|\widetilde{x}\| \leq \varepsilon(1 + \delta).$$

Consequently, choosing a smaller $\varepsilon > 0$ if necessary, we can assert that $\bar{q} \in U_q$ *whenever* $(\widetilde{M}, \widetilde{q}) \in R^{n\times n} \times R^n$, $\widetilde{M}$ *is copositive on* Δ, (7.18) *holds.* Hence from (7.24) and (7.26) we deduce that there exists $x \in \mathrm{Sol}(\mathrm{AVI}(M, q, \Delta))$ such that

$$\begin{aligned} \|\widetilde{x} - x\| &\leq \ell_M \|\bar{q} - q\| \\ &\leq \ell_M(\|\widetilde{q} - q\| + \|\widetilde{M} - M\|\|\widetilde{x}\|) \\ &\leq \ell_M(\|\widetilde{q} - q\| + \delta\|\widetilde{M} - M\|) \\ &\leq \ell(\|\widetilde{q} - q\| + \|\widetilde{M} - M\|), \end{aligned}$$

where $\ell = \max\{\ell_M, \delta\ell_M\}$. We have thus obtained (7.20). The proof is complete. □

Our next goal is to establish the following interesting result on AVI problems with positive semidefinite matrices.

Theorem 7.5. (See Robinson (1979), Theorem 2) *Let* $M \in R^{n\times n}$ *be a positive semidefinite matrix,* Δ *a nonempty polyhedral convex set in* R^n, *and* $q \in R^n$. *Then the following two properties are equivalent:*

(i) *The solution set* $\mathrm{Sol}(\mathrm{AVI}(M, q, \Delta))$ *is nonempty and bounded;*

(ii) *There exists* $\varepsilon > 0$ *such that for each* $\widetilde{M} \in R^{n\times n}$ *and each* $\widetilde{q} \in R^n$ *with*

$$\max\{\|\widetilde{M} - M\|,\ \|\widetilde{q} - q\|\} < \varepsilon, \tag{7.27}$$

the set $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta))$ *is nonempty.*

For proving the above theorem we shall need the following three auxiliary lemmas in which *it is assumed that* $M \in R^{n\times n}$ *is a positive semidefinite matrix,* $\Delta \subset R^n$ *is a nonempty polyhedral convex set, and* $q \in R^n$. We set $M\Delta = \{Mx : x \in \Delta\}$.

Lemma 7.1. (See, for instance, Best and Chakravarti (1992)) *For any* $\bar{v} \in R^n$, *if* $\bar{v}^T M \bar{v} = 0$ *then* $(M + M^T)\bar{v} = 0$.

Proof. Consider the unconstrained quadratic program

$$\min\left\{ f(x) := \frac{1}{2}x^T(M + M^T)x \; : \; x \in R^n \right\}.$$

From our assumptions it follows that

$$\begin{aligned}\frac{1}{2}x^T(M + M^T)x = x^T Mx \;\; &\geq 0 \\ &= \bar{v}^T M \bar{v} \\ &= \frac{1}{2}\bar{v}^T(M + M^T)\bar{v}\end{aligned}$$

for every $x \in R^n$. Hence $\bar{v}$ is a global solution of the above problem. By Theorem 3.1 we have

$$0 = \nabla f(\bar{v}) = (M + M^T)\bar{v},$$

which completes the proof. □

Lemma 7.2. *The inclusion*

$$q \in \mathrm{int}((0^+\Delta)^+ - M\Delta) \tag{7.28}$$

holds if and only if

$$\forall v \in 0^+\Delta \setminus \{0\} \;\; \exists x \in \Delta \;\; \text{such that} \;\; \langle Mx + q, v\rangle > 0. \tag{7.29}$$

Proof. *Necessity:* Suppose that (7.28) holds. Then there exists $\varepsilon > 0$ such that

$$B(q, \varepsilon) \subset (0^+\Delta)^+ - M\Delta. \tag{7.30}$$

To obtain a contradiction, suppose that there exists $\bar{v} \in 0^+\Delta \setminus \{0\}$ such that

$$\langle Mx + q, \bar{v} \rangle \leq 0 \quad \forall x \in \Delta.$$

By (7.30), for every $q' \in B(q, \varepsilon)$ there exist $w \in (0^+\Delta)^+$ and $x \in \Delta$ such that $q' = w - Mx$. So we have

$$\langle q' - q, \bar{v} \rangle \geq \langle w, \bar{v} \rangle \geq 0 \quad \forall q' \in B(q, \varepsilon).$$

This clearly forces $\bar{v} = 0$, which is impossible.

Sufficiency: On the contrary, suppose that (7.29) is valid, but (7.28) is false. Then there exists a sequence $\{q^k\} \subset R^n$ such that $q^k \notin (0^+\Delta)^+ - M\Delta$ for all $k \in N$, and $q^k \to q$. From this we deduce that

$$(M\Delta + q^k) \cap (0^+\Delta)^+ = \emptyset \quad \forall k \in N.$$

Since $M\Delta + q^k$ and $(0^+\Delta)^+$ are two disjoint polyhedral convex sets, by Theorem 11.3 from Rockafellar (1970) there exists a hyperplane separating these sets properly. Since $(0^+\Delta)^+$ is a cone, by Theorem 11.7 from Rockafellar (1970) there exists a hyperplane which separates the above two sets properly and passes through the origin. So there exists $v^k \in R^n$ with $\|v^k\| = 1$ such that

$$\langle v^k, Mx + q^k \rangle \leq 0 \leq \langle v^k, w \rangle \quad \forall x \in \Delta,\ \forall w \in (0^+\Delta)^+. \tag{7.31}$$

(Actually, the above-mentioned hyperplane is defined by the formula $H = \{z \in R^n : \langle v^k, z \rangle = 0\}$). Without loss of generality we can assume that $v^k \to \bar{v} \in R^n$, $\|\bar{v}\| = 1$. From (7.31) it follows that

$$\langle \bar{v}, Mx + q \rangle \leq 0 \quad \forall x \in \Delta \tag{7.32}$$

and

$$\langle \bar{v}, w \rangle \geq 0 \quad \forall w \in (0^+\Delta)^+. \tag{7.33}$$

By Theorem 14.1 from Rockafellar (1970), from (7.33) it follows that $\bar{v} \in 0^+\Delta$. Combining this with (7.32) we see that (7.29) is false, which is impossible. □

Lemma 7.3. (See Gowda-Pang (1994a), Theorem 7) *The solution set* Sol(AVI(M, q, Δ)) *is nonempty and bounded if and only if* (7.28) *holds.*

Proof. *Necessity:* To obtain a contradiction, suppose that the set Sol(AVI(M, q, Δ)) is nonempty and bounded, but (7.28) does not hold. Then, by Lemma 7.2 there exists $\bar{v} \in 0^+\Delta \setminus \{0\}$ such that

(7.32) holds. Select a point $x^0 \in \mathrm{Sol}(\mathrm{AVI}(M,q,\Delta))$. For each $t > 0$, we set $x_t = x^0 + t\bar{v}$. Since $\bar{v} \in 0^+\Delta$, we have $x_t \in \Delta$ for every $t > 0$. Substituting x_t for x in (7.32) we get

$$\langle \bar{v}, Mx^0 + q\rangle + t\langle \bar{v}, M\bar{v}\rangle \le 0 \quad \forall t > 0.$$

This implies that $\langle \bar{v}, M\bar{v}\rangle \le 0$. Besides, since M is positive semidefinite, we have $\langle \bar{v}, M\bar{v}\rangle \ge 0$. So

$$\langle \bar{v}, M\bar{v}\rangle = 0. \tag{7.34}$$

By Lemma 7.1, from (7.34) we obtain

$$(M + M^T)\bar{v} = 0. \tag{7.35}$$

Fix any $x \in \Delta$. On account of (7.32), (7.34), (7.35) and the fact that $x^0 \in \mathrm{Sol}(\mathrm{AVI}(M,q,\Delta))$, we have

$$\begin{aligned}
\langle Mx_t + q, x - x_t\rangle &= \langle Mx^0 + q + tM\bar{v}, x - x^0 - t\bar{v}\rangle \\
&= \langle Mx^0 + q, x - x^0\rangle + t\langle M\bar{v}, x - x^0\rangle \\
&\quad -t\langle Mx^0 + q, \bar{v}\rangle - t^2 \underbrace{\langle M\bar{v}, \bar{v}\rangle}_{=0} \\
&= \langle Mx^0 + q, x - x^0\rangle - t\underbrace{\langle \bar{v}, Mx + q\rangle}_{\le 0} \\
&\quad -t\langle \underbrace{(M + M^T)\bar{v}}_{=0}, x^0\rangle \\
&\ge \langle Mx^0 + q, x - x^0\rangle \\
&\ge 0.
\end{aligned}$$

Since this holds for every $x \in \Delta$, $x_t \in \mathrm{Sol}(\mathrm{AVI}(M,q,\Delta))$. As the last inclusion is valid for each $t > 0$, we conclude that $\mathrm{Sol}(\mathrm{AVI}(M,q,\Delta))$ is unbounded, a contradiction.

Sufficiency: Suppose that (7.28) holds. We have to show that the set $\mathrm{Sol}(\mathrm{AVI}(M,q,\Delta))$ is nonempty and bounded. By (7.28),

$$q \in (0^+\Delta)^+ - M\Delta.$$

Hence there exist $w \in (0^+\Delta)^+$ and $\bar{x} \in \Delta$ such that $q = w - M\bar{x}$. Since $M\bar{x} + q = w \in (0^+\Delta)^+$, for every $v \in 0^+\Delta$ it holds

$$\langle M\bar{x} + q, v\rangle = \langle w, v\rangle \ge 0.$$

Since M is a positive semidefinite matrix, we see that both conditions (i) and (ii) in Theorem 6.1 are satisfied. Hence the set

Sol(AVI(M, q, Δ)) is nonempty. To show that Sol(AVI(M, q, Δ)) is bounded we suppose, contrary to our claim, that there exists a sequence $\{x^k\}$ in Sol(AVI(M, q, Δ)) such that $\|x^k\| \to +\infty$. There is no loss of generality in assuming that $x^k \neq 0$ for each $k \in N$, and

$$\frac{x^k}{\|x^k\|} \to \bar{v} \in R^n, \quad \|\bar{v}\| = 1.$$

Let $m \in N$, $A \in R^{m \times n}$ and $b \in R^m$ be such that $\Delta = \{x \in R^n : Ax \geq b\}$. Since $Ax^k \geq b$ for every $k \in N$, dividing the inequality by $\|x^k\|$ and letting $k \to \infty$ we obtain $A\bar{v} \geq 0$. This shows that $\bar{v} \in 0^+\Delta$. We have

$$\langle Mx^k + q, x - x^k \rangle \geq 0 \quad \forall x \in \Delta \ \forall k \in N.$$

Hence

$$\langle Mx^k + q, x \rangle \geq \langle Mx^k, x^k \rangle + \langle q, x^k \rangle \quad \forall x \in \Delta \ \forall k \in N. \tag{7.36}$$

Dividing the last inequality by $\|x^k\|^2$ and letting $k \to \infty$ we get $0 \geq \langle M\bar{v}, \bar{v} \rangle$. Since M is positive semidefinite, from this we see that $\langle M\bar{v}, \bar{v} \rangle = 0$. Thus, by Lemma 7.1 we have

$$M\bar{v} = -M^T\bar{v}. \tag{7.37}$$

Fix a point $x \in \Delta$. Since $\langle Mx^k, x^k \rangle \geq 0$ for every $k \in N$, (7.36) implies that

$$\langle Mx^k + q, x \rangle \geq \langle q, x^k \rangle \quad \forall k \in N.$$

Dividing the last inequality by $\|x^k\|$ and letting $k \to \infty$ we obtain

$$\langle M\bar{v}, x \rangle \geq \langle q, \bar{v} \rangle.$$

Combining this with (7.37) we can assert that

$$\langle Mx + q, \bar{v} \rangle \leq 0 \quad \forall x \in \Delta.$$

Since $\bar{v} \in (0^+\Delta) \setminus \{0\}$, from the last fact and Lemma 7.2 it follows that (7.28) does not hold. We have thus arrived at a contradiction. The proof is complete. □

Proof of Theorem 7.5.

We first prove the implication (i) ⇒ (ii). To obtain a contradiction, suppose that Sol(AVI(M, q, Δ)) is nonempty and bounded, while there exists a sequence $(M^k, q^k) \in R^{n\times n} \times R^n$ such that $(M^k, q^k) \to (M, q)$ and

$$\mathrm{Sol}(\mathrm{AVI}(M^k, q^k, \Delta)) = \emptyset \quad \forall k \in N. \tag{7.38}$$

Since Δ is nonempty, for $j \in N$ large enough, the set

$$\Delta_j := \Delta \cap \{x \in R^n : \|x\| \le j\}$$

is nonempty. Without restriction of generality we can assume that $\Delta_j \ne \emptyset$ for every $j \in N$. By the Hartman-Stampacchia Theorem (Theorem 5.1) we can find a point, denoted by $x^{k,j}$, in the solution set $\mathrm{Sol}(\mathrm{AVI}(M^k, q^k, \Delta_j))$. We have

$$\langle M^k x^{k,j} + q^k, x - x^{k,j}\rangle \ge 0 \quad \forall x \in \Delta_j. \tag{7.39}$$

Note that

$$\|x^{k,j}\| = j \quad \forall j \in N. \tag{7.40}$$

Indeed, if $\|x^{k,j}\| < j$ then there exists $\mu > 0$ such that $\bar{B}(x^{k,j}, \mu) \subset \bar{B}(0, j)$. Hence from (7.39) it follows that

$$\langle M^k x^{k,j} + q^k, x - x^{k,j}\rangle \ge 0 \quad \forall x \in \Delta \cap \bar{B}(x^{k,j}, \mu).$$

By Proposition 5.3, this implies that $x^{k,j} \in \mathrm{Sol}(\mathrm{AVI}(M^k, q^k, \Delta))$, which is impossible because (7.38) holds. Fixing an index $j \in N$ we consider the sequence $\{x^{k,j}\}_{k\in N}$. From (7.40) we deduce that this sequence has a convergent subsequence. There is no loss of generality in assuming that

$$\lim_{k\to\infty} x^{k,j} = x^j, \quad x^j \in R^n, \quad \|x^j\| = j. \tag{7.41}$$

Letting $k \to \infty$ we deduce from (7.39) that

$$\langle Mx^j + q, x - x^j\rangle \ge 0 \quad \forall x \in \Delta_j. \tag{7.42}$$

On account of (7.41), without loss of generality we can assume that

$$\frac{x^j}{\|x^j\|} \to \bar{v} \in R^n, \quad \|\bar{v}\| = 1.$$

Let us fix a point $x \in \Delta$. It is clear that there exists an index $j_x \in N$ such that $x \in \Delta_j$ for every $j \ge j_x$. From (7.42) we deduce that

$$\langle Mx^j + q, x - x^j\rangle \ge 0 \quad \forall j \ge j_x.$$

Hence

$$\langle Mx^j + q, x\rangle \geq \langle Mx^j, x^j\rangle + \langle q, x^j\rangle \quad \forall j \geq j_x. \tag{7.43}$$

As in the last part of the proof of Lemma 7.3, we can show that $\bar{v} \in (0^+\Delta) \setminus \{0\}$ and deduce from (7.43) the following inequality

$$\langle Mx + q, \bar{v}\rangle \leq 0.$$

Since the latter holds for every $x \in \Delta$, applying Lemma 7.2 we see that the inclusion (7.28) cannot hold. According to Lemma 7.3, the last fact implies that the set Sol(AVI(M, q, Δ)) cannot be nonempty and bounded. This contradicts our assumption.

We now prove the implication (ii) $\Rightarrow$ (i). Suppose that there exists $\varepsilon > 0$ such that if matrix $\widetilde{M} \in R^{n\times n}$ and vector $\widetilde{q} \in R^n$ satisfy condition (7.27) then the set Sol(AVI$(\widetilde{M}, \widetilde{q}, \Delta)$) is nonempty. Consequently, for any $\widetilde{q} \in R^n$ satisfying $\|\widetilde{q} - q\| < \varepsilon$, the set Sol(AVI$(M, \widetilde{q}, \Delta)$) is nonempty. Let $\widetilde{x} \in$ Sol(AVI$(M, \widetilde{q}, \Delta)$). For any $v \in 0^+\Delta$ we have

$$(M\widetilde{x} + \widetilde{q})^T v = \langle M\widetilde{x} + \widetilde{q}, (\widetilde{x} + v) - \widetilde{x}\rangle \geq 0.$$

Hence $M\bar{x} + \widetilde{q} \in (0^+\Delta)^+$. So we have $\widetilde{q} \in (0^+\Delta)^+ - M\Delta$. Since this inclusion is valid for each $\widetilde{q}$ satisfying $\|\widetilde{q} - q\| < \varepsilon$, we conclude that

$$q \in \operatorname{int}((0^+\Delta)^+ - M\Delta).$$

By Lemma 7.3, the set Sol(AVI(M, q, Δ)) is nonempty and bounded. The proof is complete. □

Let us consider three illustrative examples.

Example 7.1. Setting $\Delta = [0, +\infty) \subset R^1$, $M = (-1)$, and $q = 0$, we have Sol(AVI(M, q, Δ)) $= \{0\}$. Note that matrix M is not positive semidefinite. Taking $\widetilde{M} = M$ and $\widetilde{q} = -\theta$, where $\theta > 0$, we check at once that Sol(AVI$(\widetilde{M}, \widetilde{q}, \Delta)$) $= \emptyset$. So, for this AVI problem, property (i) in Theorem 7.5 holds, but property (ii) does not hold. This example shows that, in Theorem 7.5, one cannot omit the assumption that M is a positive semidefinite matrix.

Example 7.2. Setting $\Delta = (-\infty, +\infty) = R^1$, $M = (0)$, and $q = 0$, we have Sol(AVI(M, q, Δ)) $= \Delta$. So property (i) in Theorem 7.5 does not hold for this example. Taking $\widetilde{M} = (0)$ and $\widetilde{q} = \theta$, where $\theta > 0$, we have Sol(AVI$(\widetilde{M}, \widetilde{q}, \Delta)$) $= \emptyset$. This shows that, for the

AVI problem under consideration, property (ii) in Theorem 7.5 fails to hold.

Example 7.3. Setting $\Delta = [1, +\infty) \subset R^1$, $M = (0)$, and $q = 0$, we have $\mathrm{Sol}(\mathrm{AVI}(M, q, \Delta)) = \Delta$. Taking $\widetilde{M} = M$ and $\widetilde{q} = \theta$, where $\theta > 0$, we see that $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) = \{1\}$. But taking $\widetilde{M} = (-\theta)$ and $\widetilde{q} = 0$, where $\theta > 0$, we see that $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) = \emptyset$. So, for this problem, both the properties (i) and (ii) in Theorem 7.5 do not hold.

In connection with Theorem 7.5, it is natural to raise the following open question.

QUESTION: Is it true that property (i) in Theorem 7.5 implies that there exists $\varepsilon > 0$ such that if matrix $\widetilde{M} \in R^{n\times n}$ and vector $\widetilde{q} \in R^n$ satisfy condition (7.27) then the set $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta))$ is bounded (may be empty)?

The next example shows that property (i) in Theorem 7.5 does not imply that the solution sets $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta))$, where $(\widetilde{M}, \widetilde{q})$ is taken from a neighborhood of (M, q), are uniformly bounded.

Example 7.4. (See Robinson (1979), pp. 139–140) Let $\Delta = [0, +\infty) \subset R^1$, $M = (0)$, and $q = 1$. It is clear that

$$\mathrm{Sol}(\mathrm{AVI}(M, q, \Delta)) = \{0\}.$$

Taking $\widetilde{M} = (-\mu)$ and $\widetilde{q} = 1$, where $\mu > 0$, we have

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) = \left\{0, \frac{1}{\mu}\right\}.$$

From this we conclude that there exist no $\varepsilon > 0$ and $\delta > 0$ such that if matrix $\widetilde{M} \in R^{1\times 1}$ and vector $\widetilde{q} \in R^1$ satisfy condition (7.27) then $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \subset \delta \bar{B}_{R^1}$.

The following theorem is one of the main results on solution stability of AVI problems. One can observe that this theorem and Theorem 7.4 are independent results.

Theorem 7.6. (See Robinson (1979), Theorem 2) *Suppose that $\Delta \subset R^n$ is a nonempty polyhedral convex set. Suppose that $M \in R^{n\times n}$ is a given matrix and $q \in R^n$ is a given vector. If M is a positive semidefinite matrix and if the solution set* $\mathrm{Sol}(\mathrm{AVI}(M, q, \Delta))$ *is nonempty and bounded, then there exist constants $\varepsilon > 0$, $\delta > 0$ and $\ell > 0$ such that if $(\widetilde{M}, \widetilde{q}) \in R^{n\times n} \times R^n$, $\widetilde{M}$ is positive semidefinite, and if*

$$\max\{\|\widetilde{M} - M\|,\ \|\widetilde{q} - q\|\} < \varepsilon, \tag{7.44}$$

then the set $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta))$ *is nonempty,*

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \subset \delta \bar{B}_{R^n}, \tag{7.45}$$

and

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \subset \mathrm{Sol}(\mathrm{AVI}(M, q, \Delta)) + \ell(\|\widetilde{M} - M\| + \|\widetilde{q} - q\|)\bar{B}_{R^n}. \tag{7.46}$$

Proof. Since M is positive semidefinite and $\mathrm{Sol}(\mathrm{AVI}(M, q, \Delta))$ is nonempty and bounded, by Lemmas 7.2 and 7.3 we have

$$\forall v \in 0^+\Delta \setminus \{0\} \ \exists x \in \Delta \ \text{ such that } \ \langle Mx + q, v\rangle > 0. \tag{7.47}$$

Moreover, according to Theorem 7.5, there exists $\varepsilon_0 > 0$ such that for each matrix $\widetilde{M} \in R^{n\times n}$ and each $\widetilde{q} \in R^n$ satisfying

$$\max\{\|\widetilde{M} - M\|, \ \|\widetilde{q} - q\|\} < \varepsilon_0,$$

the set $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta))$ is nonempty. We claim that there exist constants $\varepsilon > 0$ and $\delta > 0$ such that (7.45) holds for every $(\widetilde{M}, \widetilde{q}) \in R^{n\times n} \times R^n$ satisfying condition (7.44) and the requirement that $\widetilde{M}$ is a positive semidefinite matrix. Indeed, if the claim were false we would find a sequence $\{(M^k, q^k)\}$ in $R^{n\times n} \times R^n$ and a sequence $\{x^k\}$ in R^n such that M^k is positive semidefinite for every $k \in$, $(M^k, q^k) \to (M, q)$, $x^k \in \mathrm{Sol}(\mathrm{AVI}(M^k, q^k, \Delta))$ for every $k \in N$, and $\|x^k\| \to +\infty$ as $k \to \infty$. For each $x \in \Delta$, we have

$$\langle M^k x^k + q^k, x - x^k\rangle \geq 0 \quad \forall k \in N. \tag{7.48}$$

Without loss of generality we can assume that $x^k \neq 0$ for every $k \in N$, and

$$\frac{x^k}{\|x^k\|} \to \bar{v} \in R^n, \quad \|\bar{v}\| = 1.$$

It is easily seen that $\bar{v} \in (0^+\Delta)^+$. From (7.48) it follows that

$$\langle M^k x^k + q^k, x\rangle \geq \langle M^k x^k, x^k\rangle + \langle q^k, x^k\rangle \quad \forall k \in N. \tag{7.49}$$

Dividing the last inequality by $\|x^k\|^2$ and letting $k \to \infty$ we get $0 \geq \langle M\bar{v}, \bar{v}\rangle$. Since M is positive semidefinite, from this we see that $\langle M\bar{v}, \bar{v}\rangle = 0$. By Lemma 7.1 we have

$$M\bar{v} = -M^T\bar{v}. \tag{7.50}$$

Fix a point $x \in \Delta$. Since M^k is positive semidefinite, we have $\langle M^k x^k, x^k \rangle \geq 0$ for every $k \in N$. Hence (7.49) implies that

$$\langle M^k x^k + q^k, x \rangle \geq \langle q^k, x^k \rangle \quad \forall k \in N.$$

Dividing the last inequality by $\|x^k\|$ and letting $k \to \infty$ we obtain

$$\langle M\bar{v}, x \rangle \geq \langle q, \bar{v} \rangle.$$

Combining this with (7.50) we get

$$\langle Mx + q, \bar{v} \rangle \leq 0 \quad \forall x \in \Delta.$$

Since $\bar{v} \in (0^+\Delta)^+ \setminus \{0\}$, the last fact contradicts (7.47). Our claim has been proved. We can now proceed analogously to the proof of Claim 3 in the proof of Theorem 7.4 to find the required constants $\varepsilon > 0$, $\delta > 0$ and $\ell > 0$. □

7.4 Commentaries

As it has been noted in Robinson (1981), p. 206, the class of polyhedral multifunctions is closed under finite addition, scalar multiplication, and finite composition. This means that if $\Phi : R^n \to 2^{R^m}$, $\Psi : R^m \to 2^{R^s}$, $\Phi_j : R^n \to 2^{R^m}$ $(j = 1, \ldots, m)$ are some given polyhedral multifuntions and $\lambda \in R$ is a given scalar, then the formulae

$$(\lambda\Phi)(x) = \lambda\Phi(x) \quad (\forall x \in R^n),$$

$$(\Phi_1 + \ldots + \Phi_k)(x) = \Phi_1(x) + \ldots + \Phi_k(x) \quad (\forall x \in R^n),$$

and

$$(\Psi \circ \Phi)(x) = \Psi(\Phi(x)) \quad (\forall x \in R^n),$$

create new polyhedral multifunctions which are denoted by $\lambda\Phi$, $\Phi_1 + \ldots + \Phi_k$ and $\Psi \circ \Phi$, respectively.

The proof of Theorem 7.4 is similar in spirit to the proof of Theorem 7.5.1 in Cottle et al. (1992).

The 'elementary' proof of the results of Robinson (see Theorems 7.5 and 7.6) on the solution stability of AVI problems with positive semidefinite matrices given in this chapter is new. We hope that it can expose furthermore the beauty of these results. The original proof of Robinson is based on a general solution stability theorem for variational inequalities in Banach spaces (see Robinson (1979), Theorem 1).

Results presented in this chapter deal only with upper-Lipschitz continuity properties of the solution map of parametric AVI problems. For multifunctions, the lower semicontinuity, the upper semicontinuity, the openness, the Aubin property, the metric regularity, and the single-valuedness are other interesting properties which have many applications (see Aubin and Frankowska (1990), Mordukhovich (1993), Rockafellar and Wets (1998), and references therein). It is of interest to characterize these properties of the solution map in parametric AVI problems (in particular, of the solution map in parametric LCP problems). Some results in this direction have been obtained (see, for instance, Jansen and Tijs (1987), Gowda (1992), Donchev and Rockafellar (1996), Oettli and Yen (1995), Gowda and Sznajder (1996)). We will study the lower semicontinuity and the upper semicontinuity the solution map of parametric AVI problems in Chapter 18.