Chapter 13

Continuity of the Optimal Value Function in Quadratic Programming

In this chapter we will characterize the continuity property of the optimal value function in a general parametric QP problem. The lower semicontinuity and upper semicontinuity properties of the optimal value function are studied as well. Directional differentiability of the optimal value function in QP problems will be addressed in the next chapter.

13.1 Continuity of the Optimal Value Function

Consider the following general quadratic programming problem with linear constraints, which will be denoted by $QP(D, A, c, b)$,

$$\begin{cases} \text{Minimize } \ f(x, c, D) := \frac{1}{2}x^T Dx + c^T x \\ \text{subject to } \ x \in \Delta(A, b) := \{x \in R^n \ : \ Ax \geq b\} \end{cases} \tag{13.1}$$

depending on the parameter $\omega = (D, A, c, b) \in \Omega$, where

$$\Omega := R_S^{n \times n} \times R^{m \times n} \times R^n \times R^m.$$

The solution set of (13.1) will be denoted by $\mathrm{Sol}(D, A, c, b)$. The function $\varphi : \Omega \longrightarrow \overline{R}$ defined by

$$\varphi(\omega) = \inf\{f(x, c, D) \ : \ x \in \Delta(A, b)\}.$$

is the optimal value function of the parametric problem (13.1).

If $v^T Dv \geq 0$ (resp., $v^T Dv \leq 0$) for all $v \in R^n$ then $f(\cdot, c, D)$ is a convex (resp., concave) function and (13.1) is a convex (resp., concave) QP problem. If such conditions are not required then (13.1) is an *indefinite* QP problem (see Section 1.5).

In this section, complete characterizations of the continuity of the function φ at a given point are obtained. In Section 13.2, sufficient conditions for the upper and lower semicontinuity of φ at a given point will be established. For proving the results, we rely on some results due to Robinson (1975, 1977) on stability of the feasible region $\Delta(A, b)$ and the Frank-Wolfe Theorem.

Before obtaining the desired characterizations, we state some lemmas.

Lemma 13.1. *Let $A \in R^{m\times n}$, $b \in R^m$. The system $Ax \geq b$ is regular if and only if the multifunction $\Delta(\cdot) : R^{m\times n} \times R^m \longrightarrow 2^{R^n}$, defined by $\Delta(A', b') = \{x \in R^n : A'x \geq b'\}$, is lower semicontinuous at (A, b).*

Proof. Suppose that $Ax \geq b$ is a regular system and $x^0 \in R^n$ is such that $Ax^0 > b$. Obviously, $\Delta(A, b)$ is nonempty. Let V be an open subset in R^n satisfying $\Delta(A, b) \cap V \neq \emptyset$. Take $x \in \Delta(A, b) \cap V$. For every $t \in [0, 1]$, we set

$$x_t := (1 - t)x + tx^0.$$

Since $x_t \to x$ as $t \to 0$, there is $t_0 > 0$ such that $x_{t_0} \in V$. Since

$$Ax_{t_0} = (1 - t_0)Ax + t_0 Ax^0 > (1 - t_0)b + t_0 b = b,$$

there exists $\delta_{t_0} > 0$ such that

$$A'x_{t_0} > b'$$

for all $(A', b') \in R^{m\times n} \times R^m$ satisfying

$$\|(A', b') - (A, b)\| < \delta_{t_0}. \tag{13.2}$$

Thus $x_t \in \Delta(A', b')$ for every (A', b') fulfilling (13.2). Therefore $\Delta(\cdot)$ is lower semicontinuous at (A, b).

Conversely, if $\Delta(\cdot)$ is lower semicontinuous at (A, b) then there exists $\delta > 0$ such that $Ax \geq b'$ is solvable for every $b' \in R^m$ satisfying $b' > b$ and $\|b' - b\| < \delta$. This implies that $Ax > b$ is solvable. Thus $Ax \geq b$ is a regular system. □

Remark 13.1. If the inequality system $Ax \geq b$ is irregular then there exists a sequence $\{(A^k, b^k)\}$ in $R^{m\times n} \times R^m$ converging to (A, b) such that, for every k, the system $A^k x \geq b^k$ has no solutions. This fact follows from the results of Robinson (1977).

Lemma 13.2 (cf. Robinson (1977), Lemma 3). *Let $A \in R^{m\times n}$. If the system $Ax \geq 0$ is regular then, for every $b \in R^m$, the system $Ax \geq b$ is regular.*

Proof. Assume that $Ax \geq 0$ is a regular and $\bar{x} \in R^n$ is such that $A\bar{x} > 0$. Setting $\bar{b} = A\bar{x}$, we have $\bar{b} > 0$. Let $b \in R^m$ be given arbitrarily. Then there exists $t > 0$ such that $t\bar{b} > b$. We have $A(t\bar{x}) = tA\bar{x} = t\bar{b}$. Therefore $A(t\bar{x}) > b$, hence the system $Ax \geq b$ is regular. □

The set

$$G := \{(D, A) \in R_S^{n\times n} \times R^{m\times n} : \mathrm{Sol}(D, A, 0, 0) = \{0\}\} \tag{13.3}$$

is open in $R_S^{n\times n} \times R^{m\times n}$. This fact can be proved similarly as Lemma 12.5. It is worthy to stress that Lemma 12.5 is applicable only to canonical QP problems while, in this chapter, the standard QP problems are considered.

Lemma 13.3. *If $\Delta(A, b)$ is nonempty and if* $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ *then, for every $c \in R^n$,* $\mathrm{Sol}(D, A, c, b)$ *is a nonempty compact set.*

Proof. Let $\Delta(A, b)$ be nonempty and $\mathrm{Sol}(D, A, 0, 0) = \{0\}$. Suppose that $\mathrm{Sol}(D, A, c, b) = \emptyset$ for some $c \in R^n$. By the Frank-Wolfe Theorem, there exists a sequence $\{x^k\}$ such that $Ax^k \geq b$ for every k and

$$f(x^k, c, D) = \frac{1}{2}(x^k)^T Dx^k + c^T x^k \to -\infty \quad \text{as} \quad k \to \infty.$$

It is clear that $\|x^k\| \to +\infty$ as $k \to \infty$. By taking a subsequence if necessary, we can assume that $\|x^k\|^{-1} x^k \to \bar{x} \in R^n$ and

$$f(x^k, c, D) = \frac{1}{2}(x^k)^T Dx^k + c^T x^k < 0 \quad \text{for every } k. \tag{13.4}$$

We have

$$A\frac{x^k}{\|x^k\|} \geq \frac{b}{\|x^k\|}.$$

Letting $k \to \infty$, we obtain $\bar{x} \in \Delta(A, 0)$. Dividing both sides of the inequality in (13.4) by $\|x^k\|^2$ and letting $k \to \infty$, we get $\bar{x}^T D\bar{x} \leq 0$. Since $\|\bar{x}\| = 1$, we have $\mathrm{Sol}(D, A, 0, 0) \neq \{0\}$. This contradicts the assumption $\mathrm{Sol}(D, A, 0, 0) = \{0\}$. Thus $\mathrm{Sol}(D, A, c, b)$ is nonempty for each $c \in R^n$.

Suppose, contrary to our claim, that $\mathrm{Sol}(D, A, c, b)$ is unbounded for some $c \in R^n$. Then there exists a sequence $\{x^k\} \subset \mathrm{Sol}(D, A, c, b)$ such that $\|x^k\| \to \infty$ as $k \to \infty$ and $\{\|x^k\|^{-1}x^k\}$ converges to a certain $\bar{x} \in R^n$. Taking any $x \in \Delta(A, b)$, we have

$$\frac{1}{2}(x^k)^T Dx^k + c^T x^k \leq \frac{1}{2}x^T Dx + c^T x, \tag{13.5}$$

$$Ax^k \geq b. \tag{13.6}$$

Dividing both sides of (13.5) by $\|x^k\|^2$, both sides of (13.6) by $\|x^k\|$, and letting $k \to \infty$, we obtain

$$\bar{x}^T D\bar{x} \leq 0, \quad A\bar{x} \geq 0.$$

Thus $\mathrm{Sol}(D, A, 0, 0) \neq \{0\}$, a contradiction. We have proved that, for every $c \in R^n$, the solution set $\mathrm{Sol}(D, A, c, b)$ is bounded. Fixing any $\bar{x} \in \mathrm{Sol}(D, A, c, b)$ one has

$$\mathrm{Sol}(D, A, c, b) = \{x \in \Delta(A, b) \,:\, f(x, c, D) = f(\bar{x}, c, D)\}.$$

Hence $\mathrm{Sol}(D, A, c, b)$ is a closed set and, therefore, $\mathrm{Sol}(D, A, c, b)$ is a compact set. □

We are now in a position to state our first theorem on the continuity of the optimal value function φ. This theorem gives a set of conditions which is necessary and sufficient for the continuity of φ at a point $\omega = (D, A, c, b)$ where φ has a finite value.

Theorem 13.1. *Let $(D, A, c, b) \in \Omega$. Assume that $\varphi(D, A, c, b) \neq \pm\infty$. Then, the optimal value function $\varphi(\cdot)$ is continuous at (D, A, c, b) if and only if the following two conditions are satisfied:*

(a) *the system $Ax \geq b$ is regular,*

(b) $\mathrm{Sol}(D, A, 0, 0) = \{0\}$.

Proof. *Necessity:* First, suppose that $\varphi(\cdot)$ is continuous at $\omega := (D, A, c, b)$ and $\varphi(\omega) \neq \pm\infty$. If (a) is violated then, by Remark 13.1, there exists a sequence $\{(A^k, b^k)\}$ in $R^{m\times n} \times R^m$ converging to (A, b) such that, for every k, the system $A^k x \geq b^k$ has no solutions. Consider the sequence $\{(D, A^k, c, b^k)\}$. Since $\Delta(A^k, b^k) = \emptyset$,

$\varphi(D, A^k, c, b^k) = +\infty$ for every k. As $\varphi(\cdot)$ is continuous at ω and $\{(D, A^k, c, b^k)\}$ converges to ω, we have

$$\lim_{k\to\infty} \varphi(D, A^k, c, b^k) = \varphi(D, A, c, b) \neq \pm\infty.$$

We have arrived at a contradiction. Thus (a) is fulfilled.

Now we suppose that (b) fails to hold. Then there is a nonzero vector $\bar{x} \in R^n$ such that

$$A\bar{x} \geq 0, \quad \bar{x}^T D\bar{x} \leq 0. \tag{13.7}$$

Consider the sequence $\{(D^k, A, c, b)\}$, where $D^k := D - \frac{1}{k}E$, E is the unit matrix in $R^{n\times n}$. From the assumption $\varphi(\omega) \neq \pm\infty$ it follows that $\Delta(A, b)$ is nonempty. Then from (13.7) we can deduce that $\Delta(A, b)$ is unbounded. For every k, by (13.7) we have

$$\bar{x}^T D^k \bar{x} = \bar{x}^T (D - \frac{1}{k}E)\bar{x} < 0.$$

Hence, for any x belonging to $\Delta(A, b)$ and for any $t > 0$, we have $x + t\bar{x} \in \Delta(A, b)$ and

$$f(x + t\bar{x}, c, D^k) = \frac{1}{2}(x + t\bar{x})^T D^k (x + t\bar{x}) + c^T(x + t\bar{x}) \to -\infty$$

as $t \to \infty$. This implies that, for all k, $\mathrm{Sol}(D^k, A, c, b) = \emptyset$ and $\varphi(D^k, A, c, b) = -\infty$. We have arrived at a contradiction, because $\varphi(\cdot)$ is continuous at ω and $\varphi(\omega) \neq \pm\infty$. We have proved that (b) holds true.

Sufficiency: Suppose that (a), (b) are satisfied and

$$\{(D^k, A^k, c^k, b^k)\} \subset \Omega$$

is a sequence converging to ω. By Lemma 13.1, assumption (a) implies the existence of a positive integer k_0 such that $\Delta(A^k, b^k) \neq \emptyset$ for every $k \geq k_0$. From assumption (b) it follows that the set G defined by (13.3) is open. Hence there exists a positive integer $k_1 \geq k_0$ such that $\mathrm{Sol}(D^k, A^k, 0, 0) = \{0\}$ for every $k \geq k_1$. By Lemma 13.3, $\mathrm{Sol}(D^k, A^k, c^k, b^k) \neq \emptyset$ for every $k \geq k_1$. Therefore, for every $k \geq k_1$ there exists $x^k \in R^n$ satisfying

$$\varphi(D^k, A^k, c^k, b^k) = \frac{1}{2}(x^k)^T D x^k + (c^k)^T x^k, \tag{13.8}$$

$$A^k x^k \geq b^k. \tag{13.9}$$

Since $\varphi(\omega) \neq \pm\infty$, the Frank-Wolfe Theorem shows that

$$\mathrm{Sol}(D, A, c, b) \neq \emptyset.$$

Taking any $x^0 \in \mathrm{Sol}(D, A, c, b)$, we have

$$\varphi(D, A, c, b) = \frac{1}{2}(x^0)^T D x^0 + c^T x^0, \tag{13.10}$$

$$Ax^0 \geq b. \tag{13.11}$$

By Lemma 13.1, there exists a sequence $\{y^k\}$ in R^n converging to x^0 and

$$A^k y^k \geq b^k \quad \text{for every } k \geq k_1. \tag{13.12}$$

From (13.12) it follows that $y^k \in \Delta(A^k, b^k)$ for $k \geq k_1$. So

$$\varphi(D^k, A^k, c^k, b^k) \leq \frac{1}{2}(y^k)^T D^k y^k + (c^k)^T y^k. \tag{13.13}$$

From (13.13) it follows that

$$\begin{aligned}
&\limsup_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \\
&\quad \leq \limsup_{k\to\infty} \left[\frac{1}{2}(y^k)^T D^k y^k + (c^k)^T y^k\right] \\
&\quad = \lim_{k\to\infty} \left[\frac{1}{2}(y^k)^T D^k y^k + (c^k)^T y^k\right].
\end{aligned}$$

Therefore, taking account of (13.10) and (13.11), we get

$$\limsup_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \leq \varphi(D, A, c, b). \tag{13.14}$$

We now claim that the sequence $\{x^k\}$, $k \geq k_1$, is bounded. Indeed, if it is unbounded then, by taking a subsequence if necessary, we can assume that $\|x^k\| \to \infty$ as $k \to \infty$ and $\|x^k\| \neq 0$ for all $k \geq k_1$. Then the sequence $\{\|x^k\|^{-1} x^k\}$, $k \geq k_1$, has a convergent subsequence. Without loss of generality we can assume that $\|x^k\|^{-1} x^k \to \hat{x}$, $\|\hat{x}\| = 1$. From (13.9) we have

$$A^k \frac{x^k}{\|x^k\|} \geq \frac{b^k}{\|x^k\|}.$$

Letting $k \to \infty$, we obtain

$$A\hat{x} \geq 0. \tag{13.15}$$

By (13.8) and (13.13),

$$\frac{1}{2}(x^k)^T D^k x^k + (c^k)^T x^k \le \frac{1}{2}(y^k)^T D^k y^k + (c^k)^T y^k. \qquad (13.16)$$

Dividing both sides of (13.16) by $\|x^k\|^2$ and taking limits as $k \to \infty$, we get

$$\hat{x}^T D \hat{x} \le 0. \qquad (13.17)$$

By (13.15) and (13.17), we have $\mathrm{Sol}(D, A, 0, 0) \ne \{0\}$. This contradicts (b). We have thus shown that the sequence $\{x^k\}$, $k \ge k_1$, is bounded; hence it has a convergent sequence. There is no loss of generality in assuming that $x^k \to \tilde{x} \in R^n$. By (13.8) and (13.9),

$$\lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = \frac{1}{2}\tilde{x}^T D \tilde{x} + c^T \tilde{x} = f(\tilde{x}, c, D), \qquad (13.18)$$

$$A\tilde{x} \ge b. \qquad (13.19)$$

From (13.19) it follows that $\tilde{x} \in \Delta(A, b)$. Hence

$$f(\tilde{x}, c, D) \ge \varphi(D, A, c, b).$$

Therefore, by (13.18),

$$\lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \ge \varphi(D, A, c, b). \qquad (13.20)$$

Combining (13.14) with (13.20) gives

$$\lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = \varphi(D, A, c, b).$$

This shows that φ is continuous at (D, A, c, b). The proof is complete. □

Example 13.1. Consider the problem $QP(D, A, c, b)$ where $m = 3,\ n = 2$,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \end{bmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It can be verified that $\varphi(D, A, c, b) = 0$, $\mathrm{Sol}(D, A, 0, 0) = \{0\}$, and the system $Ax \ge b$ is regular. By Theorem 13.1, φ is continuous at (D, A, c, b).

Example 13.2. Consider the problem $QP(D, A, c, b)$ where $m = n = 1$, $D = [1]$, $A = [0]$, $c = (1)$, $b = (0)$. It can be shown that

$\varphi(D, A, c, b) = 0$, and the system $Ax \geq b$ is irregular. By Theorem 13.1, φ is not continuous at (D, A, c, b).

Remark 13.2. If $\Delta(A, b)$ is nonempty then $\Delta(A, 0)$ is the recession cone of $\Delta(A, b)$. By definition, $\mathrm{Sol}(D, A, 0, 0)$ is the solution set of the problem $QP(D, A, 0, 0)$. So, verifying the assumption $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ is equivalent to solving one special QP problem.

Now we study the continuity of the optimal value function $\varphi(\cdot)$ at a point where its value is infinity. Let $\alpha \in \{+\infty, -\infty\}$ and $\varphi(D, A, c, b) = \alpha$. We say that $\varphi(\cdot)$ is continuous at (D, A, c, b) if, for every sequence $\{(D^k, A^k, c^k, b^k)\} \subset \Omega$ converging to (D, A, c, b),

$$\lim_{k \to \infty} \varphi(D^k, A^k, c^k, b^k) = \alpha.$$

The next theorem characterizes the continuity of φ at a point $\omega = (D, A, c, b)$ where φ has the value $-\infty$.

Theorem 13.2. *Let $(D, A, c, b) \in \Omega$ and $\varphi(D, A, c, b) = -\infty$. Then, the optimal value function φ is continuous at (D, A, c, b) if and only if the system $Ax \geq b$ is regular.*

Proof. Suppose that $\varphi(D, A, c, d) = -\infty$ and φ is continuous at (D, A, c, b) but the system $Ax \geq b$ is irregular. By Remark 13.1, there exists a sequence $\{(A^k, b^k)\}$ in $R^{m \times n} \times R^m$ converging to (A, b) such that, for every k, the system $A^k x \geq b^k$ has no solutions. Since $\Delta(A^k, b^k) = \emptyset$, $\varphi(D, A^k, c, b^k) = +\infty$ for every k. Therefore, $\lim_{k \to \infty} \varphi(D, A^k, c, b^k) = +\infty$. On the other hand, since φ is continuous at (D, A, c, b) and since

$$(D, A^k, c, b^k) \longrightarrow (D, A, c, b) \quad \text{as} \;\; k \to \infty,$$

we obtain

$$+\infty = \lim_{k \to \infty} \varphi(D, A^k, c, b^k) = \varphi(D, A, c, b) = -\infty,$$

a contradiction. Thus $Ax \geq b$ must be a regular system.

Conversely, suppose that $\varphi(D, A, c, d) = -\infty$ and the system $Ax \geq b$ is regular. Let $\{(D^k, A^k, c^k, b^k)\} \subset \Omega$ be a sequence converging to (D, A, c, b). By the assumption, $\varphi(D, A, c, b) = -\infty$, hence there is a sequence $\{x^i\}$ in R^n such that $Ax^i \geq b$ and

$$f(x^i, c, D) \longrightarrow -\infty \quad \text{as} \quad i \to \infty. \tag{13.21}$$

By Lemma 13.1, for every i, there exists a sequence $\{y^{i_k}\}$ in R^n with the property that

$$A^k y^{i_k} \geq b^k, \tag{13.22}$$

$$\lim_{k\to\infty} y^{i_k} = x^i. \tag{13.23}$$

By (13.22),

$$\varphi(D^k, A^k, c^k, b^k) \leq \frac{1}{2}(y^{i_k})^T D^k y^{i_k} + (c^k)^T y^{i_k}. \tag{13.24}$$

From (13.23) and (13.24) it follows that

$$\limsup_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \leq \frac{1}{2}(x^i)^T D x^i + c^T x^i. \tag{13.25}$$

Combining (13.25) with (13.21), we obtain

$$\limsup_{k\to\infty} \varphi(D^k, A^k, c, b^k) = -\infty.$$

This implies that

$$\lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = -\infty = \varphi(D, A, c, b).$$

Thus φ is continuous at (D, A, c, b). The proof is complete. $\square$

The following theorem characterizes the continuity of φ at a point $\omega = (D, A, c, b)$ where φ has the value $+\infty$.

Theorem 13.3. *Let $(D, A, c, b) \in \Omega$ and $\varphi(D, A, c, b) = +\infty$. Then, the optimal value function φ is continuous at (D, A, c, b) if and only if* $\mathrm{Sol}(D, A, 0, 0) = \{0\}$.

Proof. Suppose that $\varphi(D, A, c, b) = +\infty$ and that φ is continuous at (D, A, c, b) but $\mathrm{Sol}(D, A, 0, 0) \neq \{0\}$. Then there exists a nonzero vector $\bar{x} \in R^n$ such that

$$A\bar{x} \geq 0, \quad \bar{x}^T D \bar{x} \leq 0.$$

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$. We define a matrix $M \in R^{m\times n}$ by setting $M = [m_{ij}]$, where

$$m_{ij} = \bar{x}_j \quad \text{for } 1 \leq i \leq m,\ 1 \leq j \leq n.$$

Let

$$D^k = D - \frac{1}{k}E, \quad A^k = A + \frac{1}{k}M,$$

where E is the unit matrix in $R^{n\times n}$. Consider the sequence

$$\{(D^k, A^k, c, b)\}.$$

A simple computation shows that

$$A^k\bar{x} > 0 \quad \text{for every } k.$$

By Lemma 13.2, for every k the system $A^k x \geq b$ is regular. Let z be a solution of the system $A^k x \geq b$. Since $A^k\bar{x} > 0$ and

$$\bar{x}^T D^k \bar{x} = \bar{x}^T D \bar{x} - \frac{\bar{x}^T\bar{x}}{k} < 0$$

for every k, we have

$$f(z + t\bar{x}, c, D^k) = \frac{1}{2}(z + t\bar{x})^T D^k (z + t\bar{x}) + c^T(z + t\bar{x}) \to -\infty$$

as $t \to \infty$. Since $z + t\bar{x} \in \Delta(A^k, b)$ for every k and for every $t > 0$, $\mathrm{Sol}(D^k, A^k, c, b) = \emptyset$. We have arrived at a contradiction, because φ is continuous at (D, A, c, b) and

$$-\infty = \lim_{k\to\infty} \varphi(D^k, A^k, c, b) = \varphi(D, A, c, b) = +\infty.$$

Conversely, assume that $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ and

$$\{(D^k, A^k, c^k, b^k)\} \subset \Omega$$

is a sequence converging to (D, A, c, b). We shall show that

$$\liminf_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = +\infty.$$

Suppose that $\liminf\limits_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) < +\infty$. Without loss of generality we can assume that

$$\liminf_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = \lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) < +\infty.$$

Then, there exist a positive integer k_1 and a constant $\gamma \geq 0$ such that

$$\varphi(D^k, A^k, c^k, b^k) \leq \gamma$$

for every $k \geq k_1$. As $\mathrm{Sol}(D, A, 0, 0) = \{0\}$, we can assume that there is an positive integer k_2 such that $\mathrm{Sol}(D^k, A^k, 0, 0) = \{0\}$ for every $k \geq k_2$. By Lemma 13.3 we can assume that

$$\mathrm{Sol}(D^k, A^k, c^k, b^k) \neq \emptyset$$

for every $k \geq k_2$. Hence there exists a sequence $\{x^k\}$ in R^n such that, for every $k \geq k_2$, we have

$$\varphi(D^k, A^k, c^k, b^k) = \frac{1}{2}(x^k)^T D^k x^k + (c^k)^T x^k \leq \gamma, \tag{13.26}$$

$$A^k x^k \geq b^k. \tag{13.27}$$

We now prove that $\{x^k\}$ is a bounded sequence. Suppose, contrary to our claim, that the sequence $\{x^k\}$ is unbounded. Without loss of generality we can assume that $\|x^k\| \neq 0$ for every k and that $\|x^k\| \to \infty$ as $k \to \infty$. Then the sequence $\{\|x^k\|^{-1}x^k\}$ has a convergent subsequence. We can assume that the sequence itself converges to a point $x^0 \in R^n$ with $\|x^0\| = 1$. By (13.27) we have

$$A^k \frac{x^k}{\|x^k\|} \geq \frac{b^k}{\|x^k\|},$$

hence

$$Ax^0 \geq 0. \tag{13.28}$$

By dividing both sides of the inequality in (13.26) by $\|x^k\|^2$ and taking the limits as $k \to \infty$, we get

$$(x^0)^T D x^0 \leq 0. \tag{13.29}$$

From (13.28) and (13.29) we deduce that $\text{Sol}(D, A, 0, 0) \neq \{0\}$. This contradicts our assumption. Thus the sequence $\{x^k\}$ is bounded, and it has a convergent subsequence. Without loss of generality we can assume that $\{x^k\}$ converges to $\bar{x} \in R^n$. Letting $k \to \infty$, from (13.27) we obtain

$$A\bar{x} \geq b.$$

This means that $\Delta(A, b) \neq \emptyset$. We have arrived at a contradiction because $\varphi(D, A, c, b) = +\infty$. The proof is complete. □

From Theorems 13.1–13.3 it follows that conditions (a), (b) in Theorem 13.1 are sufficient for the function $\varphi(\cdot)$ to be continuous at the given parameter value (D, A, c, b).

13.2 Semicontinuity of the Optimal Value Function

As it has been shown in the preceding section, continuity of the optimal value function holds under a special set of conditions. In some situations, only the upper semicontinuity or the lower semicontinuity of that function is required. So we wish to have simple sufficient conditions for the upper semicontinuity and the lower semicontinuity of φ at a given point. Such conditions are given in this section.

A sufficient condition for the upper semicontinuity of the function $\varphi(\cdot)$ at a given parameter value is given in the following theorem.

Theorem 13.4. *Let $(D, A, c, b) \in \Omega$. If the system $Ax \geq b$ is regular then $\varphi(\cdot)$ is upper semicontinuous at (D, A, c, b).*

Proof. As $Ax \geq b$ is regular, we have $\Delta(A, b) \neq \emptyset$. Hence

$$\varphi(D, A, c, b) < +\infty.$$

Let $\{(D^k, A^k, c^k, b^k)\} \subset \Omega$ be a sequence converging to (D, A, c, b). Since $\varphi(D, A, c, b) < +\infty$, there is a sequence $\{x^i\}$ in R^n such that $Ax^i \geq b$ and

$$f\left(x^i, c, D\right) = \frac{1}{2}\left(x^i\right)^T Dx^i + c^T x^i \longrightarrow \varphi(D, A, c, b) \quad \text{as} \quad i \to \infty.$$

By Lemma 13.1 and by the regularity of the system $Ax \geq b$, for each i one can find a sequence $\{y^{i_k}\}$ in R^n such that $A^k y^{i_k} \geq b^k$ and

$$\lim_{k \to \infty} y^{i_k} = x^i.$$

Since $y^{i_k} \in \Delta(A^k, b^k)$,

$$\varphi(D^k, A^k, c^k, b^k) \leq f(y^i, c^k, D^k).$$

This implies that

$$\limsup_{k \to \infty} \varphi(D^k, A^k, c^k, b^k) \leq f(x^i, c, D).$$

Taking limits in the last inequality as $i \to \infty$, we obtain

$$\limsup_{k \to \infty} \varphi(D^k, A^k, c^k, b^k) \leq \varphi(D, A, c, b).$$

We have proved that $\varphi(\cdot)$ is upper semicontinuous at (D, A, c, b).
□

The next example shows that the regularity condition in Theorem 13.4 does not guarantee the lower semicontinuity of φ at (D, A, c, b).

Example 13.3. Consider the problem $QP(D, A, c, b)$ where $m = n = 1$, $D = [0]$, $A = [1]$, $c = (0), b = (0)$. It is clear that $Ax \geq 0$ is regular, $\text{Sol}(D, A, c, b) = \Delta(A, b) = \{x : x \geq 0\}$, and $\varphi(D, A, c, b) = 0$. Consider the sequence $\{(D^k, A, c, b)\}$, where $D^k = D - \left[\frac{1}{k}\right]$. We have $\varphi(D^k, A, c, b) = -\infty$ for every k, so

$$\liminf_{k\to\infty} \varphi(D^k, A, c, b) < \varphi(D, A, c, b).$$

Thus φ is not lower semicontinuous at (D, A, c, b).

The following example is designed to show that the regularity condition in Theorem 13.4 is sufficient but not necessary for the upper semicontinuity of φ at (D, A, c, b).

Example 13.4. Choose a matrix $A \in R^{m\times n}$ and a vector $b \in R^m$ such that $\Delta(A, b) = \emptyset$ (then the system $Ax \geq b$ is irregular). Fix an arbitrary matrix $D \in R_S^{n\times n}$ and an arbitrary vector $c \in R^n$. Since $\varphi(D, A, c, b) = +\infty$, for any sequence $\{(D^k, A^k, c^k, b^k)\}$ converging to (D, A, c, b), we have

$$\limsup_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \leq \varphi(D, A, c, b).$$

Thus φ is upper semicontinuous at (D, A, c, b).

A sufficient condition for the lower semicontinuity of the function $\varphi(\cdot)$ is given in the following theorem.

Theorem 13.5. *Let* $(D, A, c, b) \in \Omega$. *If* $\text{Sol}(D, A, 0, 0) = \{0\}$ *then* $\varphi(\cdot)$ *is lower semicontinuous at* (D, A, c, b).

Proof. Assume that $\text{Sol}(D, A, 0, 0) = \{0\}$. Let

$$\{(D^k, A^k, c^k, b^k)\} \subset \Omega$$

be a sequence converging to (D, A, c, b). We claim that

$$\liminf_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \geq \varphi(D, A, c, b).$$

Indeed, suppose that

$$\liminf_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) < \varphi(D, A, c, b).$$

Without loss of generality we can assume that

$$\liminf_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = \lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k).$$

Then there exist an index k_1 and a real number γ such that $\gamma < \varphi(D, A, c, b)$ and

$$\varphi(D^k, A^k, c^k, b^k) \le \gamma \quad \text{for every} \;\; k \ge k_1.$$

Since $\varphi(D^k, A^k, c^k, b^k) < +\infty$, we must have $\Delta(A^k, b^k) \neq \emptyset$ for every $k \ge k_1$. Since $\mathrm{Sol}(D, A, 0, 0) = \{0\}$, there exists an integer $k_2 \ge k_1$ such that

$$\mathrm{Sol}(D^k, A^k, 0, 0) = \{0\}$$

for every $k \ge k_2$. As $\Delta(A^k, b^k) \neq \emptyset$, by Lemma 13.3 we have $\mathrm{Sol}(D^k, A^k, c^k, b^k) \neq \emptyset$ for every $k \ge k_2$. Hence there exists a sequence $\{x^k\}$ such that we have $A^k x^k \ge b^k$ for every $k \ge k_2$, and

$$\frac{1}{2}(x^k)^T D^k x^k + (c^k)^T x^k = \varphi(D^k, A^k, c^k, b^k) \le \gamma.$$

The sequence $\{x^k\}$ must be bounded. Indeed, if $\{x^k\}$ is unbounded then, without loss of generality, we can assume that $\|x^k\| \neq 0$ for every k and $\|x^k\| \to \infty$ as $k \to \infty$. Then the sequence $\{\|x^k\|^{-1} x^k\}$ has a convergent subsequence. We can assume that this sequence itself converges to a vector $v \in R^n$ with $\|v\| = 1$. Since

$$A^k \frac{x^k}{\|x^k\|} \ge \frac{b^k}{\|x^k\|} \quad \text{for every } k \ge k_2,$$

we have $Av \ge 0$. On the other hand, since for each $k \ge k_2$ it holds

$$\frac{1}{2}\frac{(x^k)^T}{\|x^k\|} D^k \frac{x^k}{\|x^k\|} + (c^k)^T \frac{x^k}{\|x^k\|} \le \frac{\gamma}{\|x^k\|^2},$$

we deduce that

$$v^T D v \le 0.$$

Combining all the above we get $v \in \mathrm{Sol}(D, A, 0, 0) \setminus \{0\}$, a contradiction. We have thus proved that the sequence $\{x^k\}$ is bounded. Without loss of generality we can assume that $x^k \to \bar{x} \in R^n$. Since $A^k x^k \ge b^k$ for every k, we get $A\bar{x} \ge b$. Since

$$\frac{1}{2}(x^k)^T D^k x^k + (c^k)^T x^k \le \gamma,$$

we have

$$f(\bar{x}, c, D) = \frac{1}{2}\bar{x}^T D \bar{x} + c^T \bar{x} \le \gamma.$$

As $\gamma < \varphi(D, A, c, b)$, we see that $f(\bar{x}, c, D) < \varphi(D, A, c, b)$. This is an absurd because $\bar{x} \in \Delta(A, b)$. We have thus proved that $\varphi(\cdot)$ is lower semicontinuous at (D, A, c, b). □

The next example shows that the condition $\text{Sol}(D, A, 0, 0) = \{0\}$ in Theorem 13.5 does not guarantee the upper semicontinuity of φ at (D, A, c, b).

Example 13.5. Consider the problem $QP(D, A, c, b)$ where $m = n = 1$, $D = [1]$, $A = [0]$, $c = (0), b = (0)$. It is clear that $\text{Sol}(D, A, 0, 0) = \{0\}$. Consider the sequence $\{(D, A, c, b^k)\}$, where $b^k = (\frac{1}{k})$. We have $\varphi(D, A, c, b) = 0$ and $\varphi(D, A, c, b^k) = +\infty$ for all k (because $\Delta(A, b^k) = \emptyset$ for all k). Therefore

$$\limsup_{k \to \infty} \varphi(D, A, c, b^k) = +\infty > 0 = \varphi(D, A, c, b).$$

Thus φ is not upper semicontinuous at (D, A, c, b).

The condition $\text{Sol}(D, A, 0, 0) = \{0\}$ in Theorem 13.5 is sufficient but not necessary for the lower semicontinuity of φ at (D, A, c, b).

Example 13.6. Consider the problem $QP(D, A, c, b)$ where $m = n = 1$, $D = [-1]$, $A = [1]$, $c = (1), b = (0)$. It is clear that $\text{Sol}(D, A, 0, 0) = \emptyset$. Since $\varphi(D, A, c, b) = -\infty$, for any sequence $\{(D^k, A^k, c^k, b^k)\}$ converging to (D, A, c, b), we have

$$\liminf_{k \to \infty} \varphi(D^k, A^k, c^k, b^k) \geq \varphi(D, A, c, b).$$

Thus φ is lower semicontinuous at (D, A, c, b).

13.3 Commentaries

The results presented in this chapter are due to Tam (2002).

Lemma 13.1 is a well-known fact (see, for example, Robinson (1975), Theorem 1, and Bank et al. (1982), Theorem 3.1.5).

In Best and Chakravarti (1990) and Best and Ding (1995) the authors have considered convex quadratic programming problems and obtained some results on the continuity and differentiability of the optimal value function of the problem as a function of a parameter specifying the magnitude of the perturbation. In Auslender and Coutat (1996), similar questions for the case of linear-quadratic programming problems were investigated. Continuity and Lipschitzian properties of the function $\varphi(D, A, \cdot, \cdot)$ (the matrices D and A are fixed) were studied in Bank et al. (1982), Bank and Hansel (1984), Klatte (1985), Rockafellar and Wets (1998).

We have considered indefinite QP problems and obtained several results on the continuity, the upper and lower semicontinuity of the optimal value function φ at a given point ω. In comparison with the preceding results of Best and Chakravarti (1990), Best and Ding (1995), the advantage here is that the quadratic objective function is allowed to be indefinite.

The obtained results can be used for analyzing algorithms for solving the indefinite QP problems.