Chapter 11

Lower Semicontinuity of the KKT Point Set Mapping

Our aim in this chapter is to characterize the lower semicontinuity of the Karush-Kuhn-Tucker point set mapping in quadratic programming. Necessary and sufficient conditions for the lsc property of the KKT point set mapping in canonical QP problems are obtained in Section 11.1. The lsc property of the KKT point set mapping in standard QP problems under linear perturbations is studied in Section 11.2.

11.1 The Case of Canonical QP Problems

Consider the canonical QP problem of the form (10.1). The following statement gives a necessary condition for the lower semicontinuity of the multifunction (10.3).

Theorem 11.1. *Let $D \in R_S^{n\times n}$ and $A \in R^{m\times n}$ be given. If the multifunction $S(D, A, \cdot, \cdot)$ is lower semicontinuous at $(c, b) \in R^n \times R^m$, then the set $S(D, A, c, b)$ is finite.*

Proof. Setting

$$M = \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} c \\ -b \end{pmatrix},$$

and $s = n + m$, we consider the problem of finding a vector $z = \begin{pmatrix} x \\ \lambda \end{pmatrix} \in R^s$ satisfying

$$Mz + \bar{q} \geq 0, \quad z \geq 0, \quad z^T(Mz + \bar{q}) = 0. \tag{11.1}$$

For a nonempty subset $\alpha \subset \{1, 2, \dots, s\}$, $M_{\alpha\alpha}$ denotes the corresponding principal submatrix of M. If $p \in R^s$, then the column-vector with the components $(p_i)_{i\in\alpha}$ is denoted by p_α.

Let $z = (z_1, z_2, \dots, z_s)$ be a nonzero solution of (11.1), and let $J = \{j : z_j = 0\}$, $I = \{i : z_i > 0\}$. Since $z_J = 0$ and $(Mz+\bar{q})_I = 0$, we have $M_{II} z_I = -\bar{q}_I$. Therefore, if $\det M_{II} \neq 0$ then z is defined uniquely via $\bar{q}$ by the formulae

$$z_J = 0, \quad z_I = -M_{II}^{-1}(\bar{q}_I).$$

If $I \neq \emptyset$ and $\det M_{II} = 0$, then

$$\mathrm{Q}_I := \{q \in R^s : -q_I = M_{II} z_I \text{ for some } z \in R^s\}$$

is a proper subspace of R^s. By Baire's Lemma (Brezis (1987), p. 15), the union $\mathrm{Q} := \cup\{\mathrm{Q}_I : I \subset \{1, 2, \dots, s\},\ I \neq \emptyset,\ \det M_{II} = 0\}$ is nowhere dense. So there exists a sequence $q^k = \begin{pmatrix} c^k \\ -b^k \end{pmatrix}$ converging to $\bar{q} = \begin{pmatrix} c \\ -b \end{pmatrix}$ such that $q^k \notin \mathrm{Q}$ for all k.

Fix any $x \in S(D, A, c, b)$ and let $\varepsilon > 0$ be given arbitrarily. Since the multifunction $S(D, A, \cdot, \cdot)$ is lsc at (c, b), there exists $\delta_\varepsilon > 0$ such that

$$x \in S(D, A, c', b') + \varepsilon B_{R^n}$$

for all (c', b') satisfying $\max\{\|c' - c\|, \|b' - b\|\} < \delta_\varepsilon$. Consequently, for each k sufficiently large, there exists $x^k \in S(D, A, c^k, b^k)$ such that

$$\|x - x^k\| \leq \varepsilon. \tag{11.2}$$

Since $x^k \in S(D, A, c^k, b^k)$, there exists λ^k such that $z^k := \begin{pmatrix} x^k \\ \lambda^k \end{pmatrix}$ is a solution of the LCP problem

$$Mz + q^k \geq 0, \quad z \geq 0, \quad z^T(Mz + q^k) = 0.$$

We put $J_k = \{j : z_j^k = 0\}$, $I_k = \{i : z_i^k > 0\}$. If $I_k = \emptyset$ then $z^k = 0$. If $I_k \neq \emptyset$ then $\det M_{I_k I_k} \neq 0$, because $q^k \notin \mathrm{Q}$. Hence

$$z_{J_k}^k = 0, \quad z_{I_k}^k = -M_{I_k I_k}^{-1}\left(q_{I_k}^k\right). \tag{11.3}$$

Obviously, there exists a subset $I \subset \{1, 2, \dots, s\}$ and a subsequence $\{k_i\}$ of $\{k\}$ such that $I_{k_i} = I$ for all k_i. Let Z denote the set of all $z \in R^s$ such that there is a nonempty subset $I \subset \{1, \dots, s\}$ with the property that $\det M_{II} \neq 0$, $z_I = -M_{II}^{-1}(\bar{q}_I)$ and $z_J = 0$, where $J := \{1, \ldots, s\} \setminus I$. It is clear that Z is finite. From (11.3) it follows that the sequence $z_{I_{k_i}}^{(k_i)}$ converges to a point from the finite set $\widetilde{Z} := Z \cup \{0\}$. For every $z = \begin{pmatrix} \xi \\ \lambda \end{pmatrix}$ let $\mathrm{pr}_1(z) := \xi$. Since $\mathrm{pr}_1(z^{(k_i)}) = x^{(k_i)}$ and $\mathrm{pr}_1(\cdot)$ is a continuous function, the sequence $\{x^{(k_i)}\}$ has a limit $\bar{\xi}$ in the finite set $\widetilde{X} := \{\mathrm{pr}_1(z) : z \in \widetilde{Z}\}$. By (11.2), $x \in \widetilde{X} + \varepsilon B_{R^n}$. As this inclusion holds for every $\varepsilon > 0$, we have $x \in \widetilde{X}$. Thus $S(D, A, c, b) \subset \widetilde{X}$. We have shown that $S(D, A, c, b)$ is a finite set. □

The following examples show that the finiteness of $S(D, A, c, b)$ may not be sufficient for the multifunction $S(\cdot)$ to be lower semicontinuous at (D, A, c, b).

Example 11.1. Consider the problem (P_ε) of minimizing the function

$$f_\varepsilon(x) = -\frac{1}{2}x_1^2 - x_2^2 + x_1 - \varepsilon x_2$$

on the set $\Delta = \{x \in R^2 : x \geq 0, -x_1 - x_2 \geq -2\}$. Note that Δ is a compact set with nonempty interior. Denote by $S(\varepsilon)$ the KKT point set of (P_ε). A direct computation using (10.2) gives $S(0) = \left\{(0,0), (1,0), (2,0), \left(\frac{5}{3}, \frac{1}{3}\right), (0,2)\right\}$, and

$$S(\varepsilon) = \left\{(2,0), \left(\frac{5+\varepsilon}{3}, \frac{1-\varepsilon}{3}\right), (0,2)\right\}$$

for $\varepsilon > 0$ small enough. For $U := \{x \in R^2 : \frac{1}{2} < x_1 < \frac{3}{2}, -1 < x_2 < 1\}$ we have $S(\varepsilon) \cap U = \emptyset$ for every $\varepsilon > 0$ small enough. Meanwhile, $S(0) \cap U = \{(1,0)\}$. Hence the multifunction $\varepsilon \mapsto S(\varepsilon)$ is not lsc at $\varepsilon = 0$.

Example 11.2. Consider the problem $(\widetilde{P}_\varepsilon)$ of minimizing the function

$$\widetilde{f}_\varepsilon(x) = \frac{1}{2}x_1^2 - x_2^2 - x_1 - \varepsilon x_2$$

on the set $\Delta = \{x \in R^2 : x \geq 0, -x_1 - x_2 \geq -2\}$. Denote by $\widetilde{S}(\varepsilon)$ the KKT point set of $(\widetilde{P}_\varepsilon)$. Using (10.2) we can show that $\widetilde{S}(0) = \{(1,0), (0,2)\}$, and $\widetilde{S}(\varepsilon) = \{(0,2)\}$ for every $\varepsilon > 0$. For $U := \{x \in R^2 : \frac{1}{2} < x_1 < \frac{3}{2}, -1 < x_2 < 1\}$ we have $\widetilde{S}(0) \cap U =$

$\{(1,0)\}$, but $\widetilde{S}(\varepsilon) \cap U = \emptyset$ for every $\varepsilon > 0$. Hence the multifunction $\varepsilon \mapsto \widetilde{S}(\varepsilon)$ is not lsc at $\varepsilon = 0$.

In the KKT point set $S(D, A, c, b)$ of (10.1) we distinguish three types of elements: (1) Local solutions of $QP(D, A, c, b)$; (2) Local solutions of $QP(-D, A, c, b)$ which are not local solutions of $QP(D, A, c, b)$; (3) Points of $S(D, A, c, b)$ which do not belong to the first two classes. Elements of the first type (of the second type, of the third type) are called, respectively, the *local minima*, the *local maxima*, and the *saddle points* of (10.1).

In Example 11.1, $(1, 0) \in S(0)$ is a local maximum of (P_0) which lies on the boundary of Δ. Similarly, in Example 11.2, $(1, 0) \in \widetilde{S}(0)$ is a saddle point of $\widetilde{P}_0$ which lies on the boundary of Δ. If such situations do not happen, then the set of the KKT points is lower semicontinuous at the given parameter.

Theorem 11.2. *Assume that the inequality system $Ax \geq b$, $x \geq 0$ is regular. If the set $S(D, A, c, b)$ is nonempty, finite, and in $S(D, A, c, b)$ there exist no local maxima and no saddle points of (10.1) which are on the boundary of $\Delta(A, b)$, then the multifunction $S(\cdot)$ is lower semicontinuous at (D, A, c, b).*

Proof. For proving the lower semicontinuity of $S(\cdot)$ at (D, A, c, b) it suffices to show that: For any $\bar{x} \in S(D, A, c, b)$ and for any neighborhood U of $\bar{x}$ there exists $\delta > 0$ such that $S(D', A', c', b') \cap U \neq \emptyset$ for every (D', A', c', b') satisfying

$$\max\{\|D' - D\|, \|A' - A\|, \|c' - c\|, \|b' - b\|\} < \delta.$$

First, suppose that $\bar{x}$ is a local minimum of (10.1). As $S(D, A, c, b)$ is a finite set, $\bar{x}$ is an isolated local minimum. Using Theorem 3.7 we can verify that, for any Lagrange multiplier $\bar{\lambda}$ of $\bar{x}$, the second-order sufficient condition in the sense of Robinson (1982) is satisfied at $(\bar{x}, \bar{\lambda})$. According to Theorem 3.1 from Robinson (1982), for each neighborhood U of $\bar{x}$ there exists $\delta > 0$ such that for every (D', A', c', b') satisfying

$$\max\{\|D' - D\|, \|A' - A\|, \|c' - c\|, \|b' - b\|\} < \delta$$

there is a local minimum x' of the problem $QP(D', A', c', b')$ belonging to U. Since $x' \in S(D', A', c', b')$, we have $S(D', A', c', b') \cap U \neq \emptyset$, as desired. Now, suppose that $\bar{x}$ is a local maximum or a saddle point of (10.1). By our assumption, $\bar{x}$ belongs to the interior of $\Delta(A, b)$. Hence $\nabla f(\bar{x}) = D\bar{x} + c = 0$, or equivalently,

$$D\bar{x} = -c. \tag{11.4}$$

As $S(D, A, c, b)$ is finite, $\bar{x}$ is an isolated KKT point of (10.1). Then $\bar{x}$ must be the unique solution of the linear system (11.4). Therefore, the matrix D is nonsingular, and

$$\bar{x} = -D^{-1}c. \tag{11.5}$$

Since the system $Ax \geq b$, $x \geq 0$ is regular, using Lemma 3 from Robinson (1977) we can prove that there exist $\delta_0 > 0$ and an open neighborhood U_0 of $\bar{x}$ such that $U_0 \subset \Delta(A', b')$ for every (A', b') satisfying $\max\{\|A' - A\|, \|b' - b\|\} < \delta_0$. For any neighborhood U of $\bar{x}$, by (11.5) there exists $\delta \in (0, \delta_0)$ such that, for every (D', A', c', b') satisfying $\max\{\|D' - D\|, \|A' - A\|, \|c' - c\|, \|b' - b\|\} < \delta$, the matrix D' is nonsingular and $x' := -(D')^{-1}c'$ belongs to $U \cap U_0$. Since x' is an interior point $\Delta(A', b')$, this implies that $x' \in S(D', A', c', b')$. (It is easily seen that $\lambda' := 0$ is a Lagrange multiplier corresponding to x'.) We have thus shown that, for every (D', A', c', b') satisfying $\max\{\|D' - D\|, \|A' - A\|, \|c' - c\|, \|b' - b\|\} < \delta$, $S(D', A', c', b') \cap U \neq \emptyset$. The proof is complete. □

11.2 The Case of Standard QP Problems

In this section we consider the following QP problem

$$\begin{cases} \text{Minimize} & \frac{1}{2}x^T Dx + c^T x \\ \text{subject to} & x \in \Delta(A, b) \end{cases} \tag{11.6}$$

where $A \in R^{m \times n}$ and $D \in R_S^{n \times n}$ are given matrices, $b \in R^m$ and $c \in R^n$ are given vectors,

$$\Delta(A, b) = \{x \in R^n \, : \, Ax \geq b\}.$$

Recall that $x \in R^n$ is a Karush-Kuhn-Tucker point of (11.6) if there exists $\lambda \in R^m$ such that

$$\begin{cases} Dx - A^T\lambda + c = 0, \\ Ax \geq b, \quad \lambda \geq 0, \\ \lambda^T(Ax - b) = 0. \end{cases}$$

The KKT point set (resp., the local solution set, the solution set) of (11.6) are denoted by $S(D, A, c, b)$, (resp., $\text{loc}(D, A, c, b)$, $\text{Sol}(D, A, c, b)$).

We will study the lower semicontinuity of the multifunctions

$$(D', A', c', b') \mapsto S(D', A', c', b') \tag{11.7}$$

and

$$(c', b') \mapsto S(D, A, c', b'), \tag{11.8}$$

which will be denoted by $S(\cdot)$ and $S(D, A, \cdot, \cdot)$, respectively. It is obvious that if (11.7) is lsc at $(D, A, c, b) \in R^{n\times n}_S \times R^{m\times n} \times R^n \times R^m$ then (11.8) is lsc at $(c, b) \in R^n \times R^m$.

Necessary conditions for the lsc property of the multifunction (11.8) can be stated as follows.

Theorem 11.3. *Let $(D, A, c, b) \in R^{n\times n}_S \times R^{m\times n} \times R^n \times R^m$. If the multifunction $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b), then*

(a) *the set $S(D, A, c, b)$ is finite, nonempty, and*

(b) *the system $Ax \geq b$ is regular.*

Proof. (a) For each index set $I \subset \{1, \cdots, m\}$, we define a matrix $M_I \in R^{(n+|I|)\times(n+|I|)}$, where $|I|$ is the number of elements of I, by setting

$$M_I = \begin{bmatrix} D & -A_I^T \\ A_I & O \end{bmatrix}.$$

(If $I = \emptyset$ then we set $M_I = D$). Let

$$\mathrm{Q}_I = \Big\{(u, v) \in R^n \times R^m \ : \ \binom{u}{v_I} = M_I \binom{x}{\lambda_I} \text{ for some } (x, \lambda) \in R^n \times R^m\Big\},$$

and

$$\mathrm{Q} = \bigcup \{\mathrm{Q}_I \ : \ I \subset \{1, \cdots, m\}, \ \det M_I = 0\}.$$

If $\det M_I = 0$ then it is clear that Q_I is a proper linear subspace of $R^n \times R^m$. Since the number of the index sets $I \subset \{1, \ldots, m\}$ is finite, the set Q is nowhere dense in $R^n \times R^m$ according to the Baire Lemma (see Brezis (1987), p. 15). So there exists a sequence $\{(c^k, b^k)\}$ converging to the given point $(c, b) \in R^n \times R^m$ such that $(-c^k, b^k) \notin \mathrm{Q}$ for all k.

Fix any $\bar{x} \in S(D, A, c, b)$. Since $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b), one can find a subsequence $\{(c^{k_l}, b^{k_l})\}$ of $\{(c^k, b^k)\}$ and a sequence $\{x^{k_l}\}$ converging to $\bar{x}$ in R^n such that

$$x^{k_l} \in S(D, A, c^{k_l}, b^{k_l})$$

for all k_l. As $x^{k_l} \in S(D, A, c^{k_l}, b^{k_l})$, there exists $\lambda^{k_l} \in R^m$ such that

$$\begin{cases} Dx^{k_l} - A^T\lambda^{k_l} + c^{k_l} = 0, \\ Ax^{k_l} \geq b^{k_l}, \quad \lambda^{k_l} \geq 0, \\ (\lambda^{k_l})^T(Ax^{k_l} - b^{k_l}) = 0. \end{cases} \tag{11.9}$$

For every k_l, let $I_{k_l} := \{i \in \{1, \ldots, m\} : \lambda_i^{k_l} > 0\}$. (It may happen that $I_{k_l} = \emptyset$.) Since the number of the index sets $I \subset \{1, \ldots, m\}$ is finite, there must exist an index set $I \subset \{1, \cdots, m\}$ such that $I_{k_l} = I$ for infinitely many k_l. Without loss of generality we can assume that $I_{k_l} = I$ for all k_l. From (11.9) we deduce that

$$Dx^{k_l} - A_I^T\lambda_I^{k_l} + c^{k_l} = 0, \quad A_I x^{k_l} = b_I^{k_l}.$$

or, equivalently,

$$M_I \begin{pmatrix} x^{k_l} \\ \lambda_I^{k_l} \end{pmatrix} = \begin{pmatrix} -c^{k_l} \\ b_I^{k_l} \end{pmatrix}. \tag{11.10}$$

We claim that $\det M_I \neq 0$. Indeed, if $\det M_I = 0$ then, by (11.10) and by the definitions of Q_I and Q, we have

$$(-c^{k_l}, b^{k_l}) \in \mathrm{Q}_I \subset \mathrm{Q},$$

contrary to the fact that $(-c^k, b^k) \notin \mathrm{Q}$ for all k. We have proved that $\det M_I \neq 0$. By (11.10), we have

$$\begin{pmatrix} x^{k_l} \\ \lambda_I^{k_l} \end{pmatrix} = M_I^{-1} \begin{pmatrix} -c^{k_l} \\ b_I^{k_l} \end{pmatrix}.$$

Therefore

$$\lim_{l \to \infty} \begin{pmatrix} x^{k_l} \\ \lambda_I^{k_l} \end{pmatrix} = M_I^{-1} \begin{pmatrix} -c \\ b_I \end{pmatrix}. \tag{11.11}$$

If $I = \emptyset$ then formula (11.11) has the form

$$\lim_{l \to \infty} x^{k_l} = D^{-1}(-c). \tag{11.12}$$

From (11.11) it follows that the sequence $\{\lambda_I^{k_l}\}$ converges to some $\lambda_I \geq 0$ in $R^{|I|}$. Since the sequence $\{x^{k_l}\}$ converges to $\bar{x}$, from (11.11) and (11.12) it follows that

$$\begin{pmatrix} \bar{x} \\ \lambda_I \end{pmatrix} = M_I^{-1} \begin{pmatrix} -c \\ b_I \end{pmatrix}. \tag{11.13}$$

(Recall that $M_I = D$ if $I = \emptyset$). We set

$$Z = \left\{(x,\lambda) \in R^n \times R^m \;:\; \text{there exists } J \subset \{1,\cdots,m\} \text{ such that } \det M_J \neq 0 \text{ and } \binom{x}{\lambda_J} = M_J^{-1}\binom{-c}{b_J}\right\},$$

and

$$X = \{x \in R^n \;:\; \text{there exists } \lambda \in R^m \text{ such that } (x,\lambda) \in Z\}.$$

From the definitions of Z and X, we can deduce that X is a finite set (although Z may have infinitely many elements). We observe also that Z and X do not depend on the choice of $\bar{x}$. Actually, these sets depend only on the parameters (D, A, c, b). From (11.13) we have $\bar{x} \in X$. Since $\bar{x} \in S(D, A, c, b)$ can be chosen arbitrarily and since X is finite, we conclude that $S(D, A, c, b)$ is a finite set.

(b) If $Ax \geq b$ is irregular then there exists a sequence $\{b^k\}$ converging in R^n to b such that $\Delta(A, b^k)$ is empty for all k (Robinson (1977), Lemma 3). Clearly, $S(D, A, c, b^k) = \emptyset$ for all k. As $\{b^k\}$ converges to b, this shows that $S(D, A, \cdot, \cdot)$ cannot be lower semicontinuous at (c, b). The proof is complete. □

Examples 11.1 and 11.2 show that finiteness and nonemptiness of $S(D, A, c, b)$ together with the regularity of the system $Ax \geq b$, in general, does not imply that $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b).

Let $(D, A, c, b) \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Let $x \in S(D, A, c, b)$ and let $\lambda \in R^m$ be a Lagrange multiplier corresponding to x. We define $I = \{1, 2, \ldots, m\}$,

$$K = \{i \in I \;:\; A_i x = b_i,\; \lambda_i > 0\} \tag{11.14}$$

and

$$J = \{i \in I \;:\; A_i x = b_i,\; \lambda_i = 0\}. \tag{11.15}$$

It is clear that K and J are two disjoint sets (possibly empty).

We now obtain a sufficient condition for the lsc property of the multifunction $S(D, A, \cdot, \cdot)$ at a given point $(c, b) \in R^n \times R^m$.

Theorem 11.4. *Let $(D, A, c, b) \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Suppose that*

(i) *the set $S(D, A, c, b)$ is finite, nonempty,*

(ii) *the system $Ax \geq b$ is regular,* *and suppose that for every $x \in S(D, A, c, b)$ there exists a Lagrange multiplier λ corresponding to x such that at least one of the following conditions holds:*

(c1) $x \in \mathrm{loc}(D, A, c, b)$,

(c2) $J = K = \emptyset$,

(c3) $J = \emptyset$, $K \neq \emptyset$, *and the system* $\{A_i : i \in K\}$ *is linearly independent,*

(c4) $J \neq \emptyset$, $K = \emptyset$, *D is nonsingular and $A_J D^{-1} A_J^T$ is a positive definite matrix,*

where K and J are defined via (x, λ) by (11.14) *and* (11.15). *Then, the multifunction $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b).*

Proof. Since $S(D, A, c, b)$ is nonempty, in order to prove that $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b) we only need to show that, for any $x \in S(D, A, c, b)$ and for any open neighborhood V_x of x, there exists $\delta > 0$ such that

$$S(D, A, c', b') \cap V_x \neq \emptyset \tag{11.16}$$

for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$.

Let $x \in S(D, A, c, b)$ and let V_x be an open neighborhood of x. By our assumptions, there exists a Lagrange multiplier λ corresponding to x such that at least one of the four conditions (c1)-(c4) holds.

We first examine the case where (c1) holds, that is

$$x \in \mathrm{loc}(D, A, c, b).$$

Since $S(D, A, c, b)$ is finite by (i), $\mathrm{loc}(D, A, c, b)$ is finite. So x is an isolated local solution of (11.1). Using Theorem 3.7 we can verify that, for any Lagrange multiplier $\bar{\lambda}$ of $\bar{x}$, the second-order sufficient condition in the sense of Robinson (1982), Definition 2.1, is satisfied at $(\bar{x}, \bar{\lambda})$. By assumption (ii), we can apply Theorem 3.1 from Robinson (1982) to find an $\delta > 0$ such that

$$\mathrm{loc}(D, A, c', b') \cap V_x \neq \emptyset$$

for every $(c', b') \in R^n \times R^n$ with $\|(c', b') - (c, b)\| < \delta$. Since $\mathrm{loc}(D, A, c, b) \subset S(D, A, c', b')$, we conclude that (11.16) is valid for every (c', b') satisfying $\|(c', b') - (c, b)\| < \delta$.

Consider the case where (c2) holds, that is $A_i x > b_i$ for every $i \in I$. Since λ is a Lagrange multiplier corresponding to x, the system

$$Dx - A^T\lambda + c = 0, \quad Ax \geq b, \quad \lambda \geq 0, \quad \lambda^T(Ax - b) = 0$$

is satisfied. As $Ax > b$, from this we deduce that $\lambda = 0$. Hence the first equality in the above system implies that $Dx = -c$. Thus x is a solution of the linear system

$$Dz = -c \quad (z \in R^n). \tag{11.17}$$

Since $S(D, A, c, b)$ is finite, x is a locally unique KKT point of (11.6). Combining this with the fact that x is an interior point of $\Delta(A, b)$, we can assert that x is a unique solution of (11.17). Hence matrix D is nonsingular and we have

$$x = -D^{-1}c. \tag{11.18}$$

Since $Ax > b$, there exist $\delta_1 > 0$ and an open neighborhood $U_x \subset V_x$ of x such that $U_x \subset \Delta(A, b')$ for all $b' \in R^m$ satisfying $\|b' - b\| < \delta_1$. By (11.18), there exists $\delta_2 > 0$ such that if $\|c' - c\| < \delta_2$ and $x' = -D^{-1}c'$ then $x' \in U_x$. Set $\delta = \min\{\delta_1, \ \delta_2\}$. Let (c', b') be such that $\|(c', b') - (c, b)\| < \delta$. Since $x' := -D^{-1}c'$ belongs to the open set $U_x \subset \Delta(A, b')$, we deduce that

$$Dx' + c' = 0, \quad Ax' > b'.$$

From this it follows that $x' \in S(D, A, c', b')$. (Observe that $\lambda' = 0$ is a Lagrange multiplier corresponding to x'.) We have thus shown that (11.16) is valid for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$.

We now suppose that (c3) holds. First, we establish that the matrix $M_K \in R^{(n+|K|)\times(n+|K|)}$ defined by setting

$$M_K = \begin{bmatrix} D & -A_K^T \\ A_K & 0 \end{bmatrix},$$

where $|K|$ denotes the number of elements in K, is nonsingular. To obtain a contradiction, suppose that M_K is singular. Then there exists a nonzero vector $(v, w) \in R^n \times R^{|K|}$ such that

$$M_K \begin{pmatrix} v \\ w \end{pmatrix} = \begin{bmatrix} D & -A_K^T \\ A_K & 0 \end{bmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 0.$$

This implies that

$$Dv - A_K^T w = 0, \quad A_K v = 0. \tag{11.19}$$

Since the system $\{A_i : i \in K\}$ is linearly independent by (c3), from (11.19) it follows that $v \neq 0$. As $A_{I\setminus K}x > b_{I\setminus K}$ and $\lambda_K > 0$, there exists $\delta_3 > 0$ such that $A_{I\setminus K}(x + tv) \geq b_{I\setminus K}$ and $\lambda_K + tw \geq 0$ for every $t \in [0, \delta_3]$. By (11.19), we have

$$\begin{cases} D(x+tv) - A_K^T(\lambda_K + tw) + c = 0, \\ A_K(x+tv) = b_K, \quad \lambda_K + tw \geq 0, \\ A_{I\setminus K}(x+tv) \geq b_{I\setminus K}, \quad \lambda_{I\setminus K} = 0 \end{cases} \tag{11.20}$$

for every $t \in [0, \delta_3]$. From (11.20) we deduce that $x+tv \in S(D, A, c, b)$ for all $t \in [0, \delta_3]$. This contradicts the assumption that $S(D, A, c, b)$ is finite. We have thus proved that M_K is nonsingular. From the definition of K it follows that

$$\begin{cases} Dx - A_K^T\lambda_K + c = 0, \\ A_Kx = b_K, \quad \lambda_K > 0, \\ A_{I\setminus K}x > b_{I\setminus K}, \quad \lambda_{I\setminus K} = 0. \end{cases}$$

The last system can be rewritten equivalently as follows

$$M_K\begin{pmatrix} x \\ \lambda_K \end{pmatrix} = \begin{pmatrix} -c \\ b_K \end{pmatrix}, \quad \lambda_K > 0, \quad \lambda_{I\setminus K} = 0, \quad A_{I\setminus K}x > b_{I\setminus K}. \tag{11.21}$$

As M_K is nonsingular, (11.21) yields

$$\begin{pmatrix} x \\ \lambda_K \end{pmatrix} = M_K^{-1}\begin{pmatrix} -c \\ b_K \end{pmatrix}, \quad \lambda_K > 0, \quad \lambda_{I\setminus K} = 0, \quad A_{I\setminus K}x > b_{I\setminus K}.$$

So there exists $\delta > 0$ such that if $(c', b') \in R^n \times R^m$ is such that $\|(c', b') - (c, b)\| < \delta$, then the formula

$$\begin{pmatrix} x' \\ \lambda'_K \end{pmatrix} = M_K^{-1}\begin{pmatrix} c' \\ b'_K \end{pmatrix}$$

defines a vector $(x', \lambda'_K) \in R^n \times R^{|K|}$ satisfying the conditions

$$x' \in V_x, \quad \lambda'_K > 0, \quad A_{I\setminus K}x' > b'_{I\setminus K}.$$

We see at once that vector x' defined in this way belongs to the set

$$S(D, A, c', b') \cap V_x$$

and $\lambda' := (\lambda'_K, \lambda'_{I\setminus K})$, where $\lambda'_{I\setminus K} = 0$, is a Lagrange multiplier corresponding to x'. We have shown that (11.16) is valid for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$.

Finally, suppose that (c4) holds. In this case, we have

$$Dx + c = 0, \quad A_J x = b_J, \quad \lambda_J = 0, \quad A_{I\setminus J} x > b_{I\setminus J}, \quad \lambda_{I\setminus J} = 0. \tag{11.22}$$

To prove that there exists $\delta > 0$ such that (11.16) is valid for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$, we consider the following system of equations and inequalities of variables $(z, \mu) \in R^n \times R^m$:

$$\begin{cases} Dz - A_J^T \mu_J + c' = 0, \quad A_J z \geq b'_J, \quad \mu_J \geq 0, \\ A_{I\setminus J} z \geq b'_{I\setminus J}, \quad \mu_{I\setminus J} = 0, \quad \mu_J^T (A_J z - b'_J) = 0. \end{cases} \tag{11.23}$$

Since D is nonsingular, (11.23) is equivalent to the system

$$\begin{cases} z = D^{-1}(-c' + A_J^T \mu_J), \quad A_J z \geq b'_J, \quad \mu_J \geq 0, \\ A_{I\setminus J} z \geq b'_{I\setminus J}, \quad \mu_{I\setminus J} = 0, \quad \mu_J^T (A_J z - b'_J) = 0. \end{cases} \tag{11.24}$$

By (11.22), $A_{I\setminus J} x > b_{I\setminus J}$. Hence there exist $\delta_4 > 0$ and an open neighborhood $U_x \subset V_x$ of x such that $A_{I\setminus J} z \geq b'_{I\setminus J}$ for any $z \in U_x$ and $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta_4$. Consequently, for every (c', b') satisfying $\|(c', b') - (c, b)\| < \delta_4$, the verification of (11.16) is reduced to the problem of finding $z \in U_x$ and $\mu_J \in R^{|J|}$ such that (11.24) holds. Here $|J|$ denotes the number of elements in J. We substitute z from the first equation of (11.24) into the first inequality and the last equation of that system to get

$$\begin{cases} A_J D^{-1} A_J^T \mu_J \geq b'_J + A_J D^{-1} c', \quad \mu_J \geq 0, \\ \mu_J^T (A_J D^{-1} A_J^T \mu_J - b'_J - A_J D^{-1} c') = 0. \end{cases} \tag{11.25}$$

Let $S := A_J D^{-1} A_J^T$ and $q' := -b'_J - A_J D^{-1} c'$. We can rewrite (11.25) as follows

$$S\mu_J + q' \geq 0, \quad \mu_J \geq 0, \quad (\mu_J)^T (S\mu_J + q') = 0. \tag{11.26}$$

Problem of finding $\mu_J \in R^{|J|}$ satisfying (11.26) is the linear complementarity problem defined by the matrix $S \in R^{|J|\times|J|}$ and the vector $q' \in R^{|J|}$. By assumption (c4), S is a positive definite matrix, that is $y^T S y > 0$ for every $y \in R^{|J|} \setminus \{0\}$. Then S is a *P-matrix*. The latter means that every principal minor of S is positive (see Cottle et al. (1992), Definition 3.3.1). According to Cottle et al. (1992), Theorem 3.3.7, for each $q' \in R^{|J|}$, problem (11.26) has a unique solution $\mu_J \in R^{|J|}$. Since D is nonsingular, from (11.22) it follows that

$$A_J D^{-1}(-c) - b_J = 0.$$

Setting $q = -b_J - A_J D^{-1}c$ we have $q = 0$. Substituting $q' = q = 0$ into (11.26) we find the unique solution $\bar{\mu}_J = 0 = \lambda_J$. By Theorem 7.2.1 from Cottle et al. (1992), there exist $\ell > 0$ and $\varepsilon > 0$ such that for every $q' \in R^{|J|}$ satisfying $\|q' - q\| < \varepsilon$ we have

$$\|\mu_J - \lambda_J\| \leq \ell \|q' - q\|.$$

Therefore

$$\|\mu_J\| = \|\mu_J - \lambda_J\| \leq \ell \|b'_J - b_J + A_J D^{-1}(c' - c)\|.$$

From this we conclude that there exists $\delta \in (0, \delta_4]$ such that if (c', b') satisfies the condition $\|(c', b') - (c, b)\| < \delta$, then the vector

$$z = D^{-1}(-c' + A_J^T \mu_J),$$

where μ_J is the unique solution of (11.26), belongs to U_x. From the definition of μ_J and z we see that system (11.24), where $\mu_{I \setminus J} := 0$, is satisfied. Then $z \in S(D, A, c', b')$. We have thus shown that, for any (c', b') satisfying $\|(c', b') - (c, b)\| < \delta$, property (11.16) is valid. The proof is complete. $\square$

To verify condition (c1), we can use Theorem 3.5.

We now consider three examples to see how the conditions (c1)–(c4) can be verified for concrete QP problems.

Example 11.3. (See Robinson (1980), p. 56) Let

$$f(x) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - x_1 \quad \text{for all } \ x = (x_1, x_2) \in R^2. \tag{11.27}$$

Consider the QP problem

$$\min\{f(x) : x = (x_1, x_2) \in R^2, x_1 - 2x_2 \geq 0, x_1 + 2x_2 \geq 0\}. \tag{11.28}$$

For this problem, we have

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}, \quad c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$S(D, A, c, b) = \left\{ (1, 0), \left(\frac{4}{3}, \frac{2}{3} \right), \left(\frac{4}{3}, -\frac{2}{3} \right) \right\},$$

$$\mathrm{loc}(D, A, c, b) = \left\{ \left(43, \frac{2}{3}\right), \left(\frac{4}{3}, -\frac{2}{3}\right) \right\}.$$

For any feasible vector $x = (x_1, x_2)$ of (11.28), we have $x_1 \geq 2|x_2|$. Therefore

$$f(x) + \frac{2}{3} = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - x_1 + \frac{2}{3} \geq \frac{3}{8}x_1^2 - x_1 + \frac{2}{3} \geq 0. \qquad (11.29)$$

For $\bar{x} := \left(\frac{4}{3}, \frac{2}{3}\right)$ and $\hat{x} := \left(\frac{4}{3}, -\frac{2}{3}\right)$, we have $f(\bar{x}) = f(\hat{x}) = -\frac{2}{3}$. Hence from (11.29) it follows that $\bar{x}$ and $\hat{x}$ are the solutions of (11.28). Actually,

$$\mathrm{Sol}(D, A, c, b) = \mathrm{loc}(D, A, c, b) = \{\bar{x},\ \hat{x}\}.$$

Setting $\widetilde{x} = (1, 0)$ we have $\tilde{x} \in S(D, A, c, b) \setminus \mathrm{loc}(D, A, c, b)$. Note that $\widetilde{\lambda} := (0, 0)$ is a Lagrange multiplier corresponding to $\tilde{x}$. We check at once that conditions (i) and (ii) in Theorem 11.4 are satisfied and, for each KKT point $x \in S(D, A, c, b)$, either (c1) or (c2) is satisfied. Theorem 11.4 shows that the multifunction $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b).

Example 11.4. Let $f(\cdot)$ be defined by (11.27). Consider the QP problem

$$\min\{f(x)\ :\ x = (x_1, x_2) \in R^2, x_1 - 2x_2 \geq 0, x_1 + 2x_2 \geq 0, x_1 \geq 1\}.$$

For this problem, we have

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $\bar{x}$, $\hat{x}$, $\widetilde{x}$ be the same as in the preceding example. Note that $\widetilde{\lambda} := (0, 0, 0)$ is a Lagrange multiplier corresponding to $\widetilde{x}$. We have

$$S(D, A, c, b) = \{\widetilde{x},\ \bar{x},\ \hat{x}\}, \quad \mathrm{Sol}(D, A, c, b) = \mathrm{loc}(D, A, c, b) = \{\bar{x},\ \hat{x}\}.$$

Clearly, for $x = \bar{x}$ and $x = \hat{x}$, assumption (c1) is satisfied. It is easily seen that, for the pair $(\widetilde{x}, \widetilde{\lambda})$, we have $K = \emptyset$, $J = \{3\}$. Since $A_J = (1\ 0)$ and $D^{-1} = D$, we get $A_J D^{-1} A_J^T = 1$. Thus (c4) is satisfied. By Theorem 11.4, $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b).

Example 11.5. Let $f(x)$ be as in (11.27). Consider the QP problem

$$\min\{f(x) : x = (x_1, x_2) \in R^2, x_1 - 2x_2 \geq 0, x_1 + 2x_2 \geq 0, x_1 \geq 2\}.$$

For this problem, we have

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix},$$

$$S(D, A, c, b) = \{(2, 0), (2, 1), (2, -1)\},$$
$$\mathrm{Sol}(D, A, c, b) = \mathrm{loc}(D, A, c, b) = \{(2, 1), (2, -1)\}.$$

Let $\bar{x} = (2, -1)$, $\hat{x} = (2, 1)$, $\widetilde{x} = (2, 0)$. Note that $\widetilde{\lambda} := (0, 0, 1)$ is a Lagrange multiplier corresponding to $\widetilde{x}$. For $x = \bar{x}$ and $x = \hat{x}$, we see at once that (c1) is satisfied. For the pair $(\widetilde{x}, \widetilde{\lambda})$, we have $K = \{3\}$, $J = \emptyset$. Since

$$\{A_i : i \in K\} = \{A_3\} = \{(1\ 0)\},$$

assumption (c3) is satisfied. According to Theorem 11.4, $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b).

The idea of the proof of Theorem 11.4 is adapted from Robinson (1980), Theorem 4.1, and the proof of Theorem 11.2. In Robinson (1980), some results involving Schur complements were obtained.

Let $(D, A, c, b) \in R_S^{n \times n} \times R^{m \times n} \times R^n \times R^m$. Let $x \in S(D, A, c, b)$ and let $\lambda \in R^m$ be a Lagrange multiplier corresponding to x. We define K and J by (11.14) and (11.15), respectively. Consider the case where both the sets K and J are nonempty. If the matrix

$$M_K = \begin{bmatrix} D & -A_K^T \\ A_K & 0 \end{bmatrix} \in R^{(n+|K|) \times (n+|K|)}$$

is nonsingular, then we denote by S_J the *Schur complement* (see Cottle et al. (1992), p. 75) of M_K in the following matrix

$$\begin{bmatrix} D & -A_K^T & -A_J^T \\ A_K & 0 & 0 \\ A_J & 0 & 0 \end{bmatrix} \in R^{(n+|K|+|J|) \times (n+|K|+|J|)}.$$

This means that

$$S_J = [A_J\ 0] M_K^{-1} [A_J\ 0]^T.$$

Note that S_J is a symmetric matrix (see Robinson (1980), p. 56). Consider the following condition:

(c5) *$J \neq \emptyset$, $K \neq \emptyset$, the system $\{A_i : i \in K\}$ is linearly independent, $v^T D v \neq 0$ for every nonzero vector v satisfying $A_K v = 0$, and S_J is positive definite.*

Modifying some arguments of the proof of Theorem 11.4 we can show that if $J \neq \emptyset$, $K \neq \emptyset$, the system $\{A_i : i \in K\}$ is linearly independent, and $v^T D v \neq 0$ for every nonzero vector v satisfying $A_K v = 0$, then M_K is nonsingular.

It can be proved that the assertion of Theorem 11.4 remains valid if instead of (c1)–(c4) we use (c1)–(c3) and (c5). The method of dealing with (c5) is similar to that of dealing with (c4) in the proof of Theorem 11.4. Up to now we have not found any example of QP problems of the form (11.1) for which there exists a pair (x, λ), $x \in S(D, A, c, b)$ and λ is a Langrange multiplier corresponding to x, such that (c1)–(c4) are not satisfied, but (c5) is satisfied. Thus the usefulness of (c5) in characterizing the lsc property of the multifunction $S(D, A, \cdot, \cdot)$ is to be investigated furthermore. This is the reason why we omit (c5) in the formulation of Theorem 11.4.

We observe that the sufficient condition in Theorem 11.2 for the lsc property of the following multifunction

$$(D', A', c', b') \to S(D', A', c', b'), \tag{11.30}$$

where $(D', A', c', b') \in R_S^{n \times n} \times R^{m \times n} \times R^n \times R^m$, can be reformulated equivalently as follows.

Theorem 11.5. *Let $(D, A, c, b) \in R_S^{n \times n} \times R^{m \times n} \times R^n \times R^m$. Suppose that*

(i) *the set $S(D, A, c, b)$ is finite, nonempty,*

(ii) *the system $Ax \geq b$ is regular,*

and suppose that for every $x \in S(D, A, c, b)$ at least one of the following conditions holds:

(c1) $x \in \mathrm{loc}(D, A, c, b)$,

(c2) $Ax > b$.

Then, multifunction (11.30) is lower semicontinuous at (D, A, c, b).

It is easy to check that (c2) in the above theorem is equivalent to (c2) in Theorem 11.4.

11.3 Commentaries

The material of this chapter is taken from Tam and Yen (1999) and Lee et al. (2002b, 2002c).