Chapter 10

Upper Semicontinuity of the KKT Point Set Mapping

We have studied QP problems in Chapters 1–4. Studying various stability aspects of QP programs is an interesting topic. Although the general stability theory in nonlinear mathematical programming is applicable to convex and nonconvex QP problems, the specific structure of the latter allows one to have more complete results.

In this chapter we obtain some conditions which ensure that a small perturbation in the data of a quadratic programming problem can yield only a small change in its Karush-Kuhn-Tucker point set. Convexity of the objective function and boundedness of the constraint set are not assumed. Obtaining *necessary* conditions for the upper semicontinuity of the KKT point set mapping will be our focus point. Sufficient conditions for the upper semicontinuity of the mapping will be developed on the framework of the obtained necessary conditions.

10.1 KKT Point Set of the Canonical QP Problems

Here we study QP problems of the canonical form:

$$\begin{cases} \text{Minimize} & f(x) := \frac{1}{2}x^T Dx + c^T x \\ \text{subject to} & x \in \Delta(A, b) := \{x \in R^n \, : \, Ax \geq b, \; x \geq 0\}, \end{cases} \tag{10.1}$$

where $D \in R_S^{n\times n}$, $A \in R^{m\times n}$, $c \in R^n$ and $b \in R^m$ are given data. In the sequel, sometime problem (10.1) will be referred to as $QP(D, A, c, b)$.

Recall that $\bar{x} \in R^n$ is a *Karush-Kuhn-Tucker point* of (10.1) if there exists a vector $\bar{\lambda} \in R^m$ such that

$$\begin{cases} D\bar{x} - A^T\bar{\lambda} + c \geq 0, \quad A\bar{x} - b \geq 0, \\ \bar{x} \geq 0, \quad \bar{\lambda} \geq 0, \\ \bar{x}^T(D\bar{x} - A^T\bar{\lambda} + c) + \bar{\lambda}^T(A\bar{x} - b) = 0. \end{cases} \tag{10.2}$$

The set of all the Karush-Kuhn-Tucker points of (10.1) is denoted by $S(D, A, c, b)$. In Chapter 3, we have seen that if $\bar{x}$ is a local solution of (10.1) then $\bar{x} \in S(D, A, c, b)$. This fact leads to the following standard way to solve (10.1): *Find first the set $S(D, A, c, b)$ then compare the values $f(x)$ among the points $x \in S(D, A, c, b)$.* Hence, one may wish to have some criteria for the (semi)continuity of the following multifunction

$$(D, A, c, b) \mapsto S(D, A, c, b). \tag{10.3}$$

In Section 10.2 we will obtain a necessary condition for the *upper semicontinuity* of the multifunction $S(\cdot, \cdot, c, b)$ at a given point $(D, A) \in R_S^{n\times n} \times R^{m\times n}$. In Section 10.3 we study a special class of QP problems for which the necessary condition obtained in this section is also a sufficient condition for the usc property of the multifunction in (10.3). This class contains some nonconvex QP problems. Sections 10.4 and 10.5 are devoted to sufficient conditions for the usc property of the multifunction in (10.3). In Section 10.5 we will investigate some questions concerning the usc property of the KKT point set mapping in a general QP problem.

Note that the upper Lipschitz property of the multifunction $S(D, A, \cdot, \cdot)$ with respect to the parameters (c, b) is a direct consequence of Theorem 7.3 in Chapter 7.

Since (10.2) can be rewritten as a linear complementarity problem, the study of continuity of the multifunction (10.3) is closely related to the study of continuity and stability of the solution map in linear complementarity theory (see Jansen and Tijs (1987), Cottle et al. (1992), Gowda (1992), Gowda and Pang (1992, 1994a)). However, when the data of (10.1) are perturbed, only some components of the matrix $M = M(D, A)$ (see formula (10.18) below) are perturbed. So, necessary conditions for (semi)continuity and stability of the Karush-Kuhn-Tucker point set cannot be derived from the corresponding results in linear complementarity theory (see, for example, Gowda and Pang (1992)) where all the components of M are perturbed.

10.2 A Necessary Condition for the usc Property of $S(\cdot)$

We now obtain a necessary condition for $S(\cdot,\cdot,c,b)$ to be upper semicontinuous at a given pair $(D,A) \in R_S^{n\times n} \times R^{m\times n}$.

Theorem 10.1. *Assume that the set $S(D,A,c,b)$ is bounded. If the multifunction $S(\cdot,\cdot,c,b)$ is upper semicontinuous at (D,A), then*

$$S(D,A,0,0) = \{0\}. \tag{10.4}$$

Proof. Arguing by contradiction, we assume that $S(D,A,c,b)$ is bounded, the multifunction $S(\cdot,\cdot,c,b)$ is usc at (D,A), but (10.4) is violated. The latter means that there is a nonzero vector $\hat{x} \in S(D,A,0,0)$. Hence there exists $\hat{\lambda} \in R^m$ such that

$$D\hat{x} - A^T\hat{\lambda} \geq 0, \quad A\hat{x} \geq 0, \tag{10.5}$$

$$\hat{x} \geq 0, \quad \hat{\lambda} \geq 0, \tag{10.6}$$

$$\hat{x}^T D\hat{x} = 0. \tag{10.7}$$

Setting

$$x_t = \frac{1}{t}\hat{x}, \quad \lambda_t = \frac{1}{t}\hat{\lambda}, \quad \text{for every } t \in (0,1), \tag{10.8}$$

we claim that there exist matrices $D_t \in R_S^{n\times n}$ and $A_t \in R^{m\times n}$ such that $D_t \to D$, $A_t \to A$ as $t \to 0$, and

$$D_t x_t - A_t^T\lambda_t + c \geq 0, \quad A_t x_t - b \geq 0, \tag{10.9}$$

$$x_t \geq 0, \quad \lambda_t \geq 0, \tag{10.10}$$

$$x_t^T(D_t x_t - A_t^T\lambda_t + c) + \lambda_t^T(A_t x_t - b) = 0. \tag{10.11}$$

Matrices D_t and A_t will be of the form

$$D_t = D + tD_0, \quad A_t = A + tA_0, \tag{10.12}$$

where matrices D_0 and A_0 are to be constructed. Since

$$\begin{aligned} D_t x_t - A_t^T \lambda_t + c &= \frac{1}{t}(D + tD_0)\hat{x} - \frac{1}{t}(A^T + tA_0^T)\hat{\lambda} + c \\ &= \frac{1}{t}(D\hat{x} - A^T\hat{\lambda}) + D_0\hat{x} - A_0^T\hat{\lambda} + c, \end{aligned}$$

and

$$\begin{aligned} A_t x_t - b &= \frac{1}{t}(A + tA_0)\hat{x} - b \\ &= \frac{1}{t}A\hat{x} + A_0\hat{x} - b, \end{aligned}$$

the following conditions, due to (10.5), imply (10.9):

$$D_0\hat{x} - A_0^T\hat{\lambda} + c \geq 0, \quad A_0\hat{x} - b \geq 0. \tag{10.13}$$

As $x_t = \frac{1}{t}\hat{x}$ and $\lambda_t = \frac{1}{t}\hat{\lambda}$, (10.6) implies (10.10). Taking account of (10.7), we have

$$\begin{aligned} & x_t^T(D_t x_t - A_t^T\lambda_t + c) + \lambda_t^T(A_t x_t - b) \\ &= \frac{1}{t}\hat{x}^T\left[\frac{1}{t}(D\hat{x} - A^T\hat{\lambda}) + D_0\hat{x} - A_0^T\hat{\lambda} + c\right] \\ &\qquad + \frac{1}{t}\hat{\lambda}^T\left(\frac{1}{t}A\hat{x} + A_0\hat{x} - b\right) \\ &= \frac{1}{t^2}\left(\hat{x}^T D\hat{x} - \hat{x}^T A^T\hat{\lambda} + \hat{\lambda}^T A\hat{x}\right) \\ &\quad + \frac{1}{t}\hat{x}^T\left(D_0\hat{x} - A_0^T\hat{\lambda} + c\right) + \frac{1}{t}\hat{\lambda}^T(A_0\hat{x} - b) \\ &= \frac{1}{t}\left[\hat{x}^T(D_0\hat{x} - A_0^T\hat{\lambda} + c) + \hat{\lambda}^T(A_0\hat{x} - b)\right]. \end{aligned}$$

So the following equality implies (10.11):

$$\hat{x}^T(D_0\hat{x} - A_0^T\hat{\lambda} + c) + \hat{\lambda}^T(A_0\hat{x} - b) = 0. \tag{10.14}$$

Let $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$, where $\hat{x}_i > 0$ for $i \in I \subset \{1, \ldots, n\}$, and $\hat{x}_i = 0$ for $i \notin I$. Since $\hat{x} \neq 0$, I must be nonempty. Fixing an $i_0 \in I$, we define A_0 as the $m \times n$-matrix whose i_0–th column is $\hat{x}_0^{-1}b$, and whose other columns consist solely of zeros. For this A_0 we have $A_0\hat{x} - b = 0$, hence the second inequality in (10.13) is satisfied, and condition (10.14) becomes the following one:

$$\hat{x}^T(D_0\hat{x} - A_0^T\hat{\lambda} + c) = 0.$$

We have to find a matrix $D_0 \in R_S^{n\times n}$ such that this condition and the first inequality in (10.13) are valid. For this purpose it is enough to find a symmetric matrix D_0 such that

$$D_0\hat{x} - w = 0, \tag{10.15}$$

where $w := A_0^T\hat{\lambda} - c \in R^n$.

If $w = (w_1, \ldots, w_n)$, then we put $D_0 = (d_{ij})$, $1 \le i,j \le n$, where

$$\begin{aligned} d_{ii} &:= \hat{x}_i^{-1}w_i \quad \text{for all } \ i \in I, \\ d_{i_0 j} = d_{j i_0} &:= \hat{x}_{i_0}^{-1}w_j \quad \text{for all } \ j \in \{1,2,\ldots,n\} \setminus I, \end{aligned}$$

and

$$d_{ij} := 0 \quad \text{for other pairs } \ (i,j), \ 1 \le i,j \le n.$$

A simple direct computation shows that this symmetric matrix D_0 satisfies (10.15).

We have thus constructed matrices A_0 and D_0 such that for x_t, λ_t, D_t and A_t defined by (10.8) and (10.12), the conditions (10.9)–(10.11) are satisfied. As a consequence, $x_t \in S(D_t, A_t, c, b)$. Since $S(D,A,c,b)$ is a bounded set, there exists a bounded open set Ω such that $S(D,A,c,b) \subset \Omega$. Since $D_t \to D$ and $A_t \to A$ as $t \to 0$, and the multifunction $S(\cdot,\cdot,c,b)$ is usc at (D,A), we have $x_t \in \Omega$ for all t sufficiently small. This is a contradiction, because $\|x_t\| = \dfrac{1}{t}\|\hat{x}\| \to \infty$ as $t \to 0$. The proof is complete. □

Observe also that, in general, (10.4) is not a sufficient condition for the upper semicontinuity of $S(\cdot)$ at (D,A,c,b).

Example 10.1. Consider the problem $QP(D,A,c,b)$ where

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = [0, \ -1], \quad b = (-1), \quad c = (0,0).$$

For each $t \in (0,1)$, let $A_t = [-t, \ -1]$. By direct computation using (10.2) we obtain

$$\begin{aligned} &S(D,A,0,0) = \{0\}, \quad S(D,A,c,b) = \{(0,0), \ (0,1)\}, \\ &S(D,A_t,c,b) = \left\{(0,0), \ (0,1), \ \left(\frac{1}{t},0\right), \ \left(\frac{t}{t^2+1}, \frac{1}{t^2+1}\right)\right\}. \end{aligned}$$

Thus, for any bounded open set $\Omega \subset R^2$ containing $S(D,A,c,b)$, the inclusion

$$S(D,A_t,c,b) \subset \Omega$$

fails to hold for $t > 0$ small enough. Since $A_t \to A$ as $t \to 0$, $S(\cdot)$ cannot be usc at (D, A, c, b).

In the next section we will study a special class of quadratic programs for which (10.4) is not only a necessary but also a sufficient condition for the upper semicontinuity of $S(\cdot)$ at a given point (D, A, c, b).

10.3 A Special Case

We now study those canonical QP problems for which the following condition (H) holds:

(H) *There exists* $\bar{x} \in R^n$ *such that* $A\bar{x} > 0, \quad \bar{x} \geq 0$.

Denote by $\mathcal{H}$ the set of all the matrices $A \in R^{m \times n}$ satisfying (H).

The next statement can be proved easily by applying Lemma 3 from Robinson (1977) and the Farkas Lemma (Theorem 3.2).

Lemma 10.1. *Each one of the following two conditions is equivalent to* (H):

(i) *There exists* $\delta > 0$ *such that, for every matrix* A' *satisfying* $\|A' - A\| < \delta$ *and for every* $b \in R^n$, *the system* $A'x \geq b$, $x \geq 0$ *is solvable.*

(ii) *For any* $\lambda \in R^n$, *if*

$$-A^T\lambda \geq 0, \quad \lambda \geq 0, \tag{10.16}$$

then $\lambda = 0$.

Obviously, (H) implies the existence of an $\hat{x} \in R^n$ satisfying $A\hat{x} > 0$, $\hat{x} > 0$. Thus $\Delta(A, 0)$ has nonempty interior. Now suppose that (H) is fulfilled and $b \in R^n$ is an arbitrarily chosen vector. Since $\Delta(A, b) + \Delta(A, 0) \subset \Delta(A, b)$ and, by Lemma 10.1, $\Delta(A, b)$ is nonempty, we conclude that $\Delta(A, b)$ is an unbounded set with nonempty interior. Besides, it is clear that there exists $\widetilde{x} \in R^n$ satisfying

$$A\widetilde{x} > b, \quad \widetilde{x} > 0.$$

The latter property is a specialization of the Slater constraint qualification (Mangasarian (1969), p. 78), and the Mangasarian-Fromovitz constraint qualification (called by Mangasarian the modified Arrow-Hurwicz-Uzawa constraint qualification) (Mangasarian

(1969), pp. 172-173). These well-known constraint qualifications play an important role in the stability analysis of nonlinear optimization problems.

In the sequence, the inequality system $Ax \geq b$, where $A \in R^{m\times n}$ and $b \in R^m$, is said to be *regular* if there exists $x^0 \in R^n$ such that $Ax^0 > b$.

As it has been noted in Section 5.4, a pair $(\bar{x}, \bar{\lambda}) \in R^n \times R^m$ satisfies (10.2) if and only if $\bar{z} := \begin{pmatrix} \bar{x} \\ \bar{\lambda} \end{pmatrix}$ is a solution to the following linear complementarity problem

$$Mz + q \geq 0, \quad z \geq 0, \quad z^T(Mz+q) = 0, \tag{10.17}$$

where

$$M = M(D,A) := \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix}, \quad q := \begin{pmatrix} c \\ -b \end{pmatrix}, \quad z = \begin{pmatrix} x \\ \lambda \end{pmatrix} \in R^{n+m}. \tag{10.18}$$

Denoting by $\mathrm{Sol}(M,q)$ the solution set of (10.17), we have

$$S(D,A,c,b) = \pi_1\left(\mathrm{Sol}(M,q)\right), \tag{10.19}$$

where $\pi_1 : R^{n+m} \to R^n$ is the linear operator defined by setting $\pi_1 \begin{pmatrix} x \\ \lambda \end{pmatrix} := x$ for every $\begin{pmatrix} x \\ \lambda \end{pmatrix} \in R^{m+n}$.

The notion of $\mathbf{R}_0$-matrix, which is originated to Garcia (1973), has proved to be useful in characterizing the upper semicontinuity property of the solution set of linear complementarity problems (see Jansen and Tijs (1987), Cottle et al. (1992), Gowda (1992), Gowda and Pang (1992), Oettli and Yen (1995, 1996a, 1996b)), and in studying other questions concerning these problems (see Cottle et al. (1992)). $\mathbf{R}_0$-matrices are called also pseudo-regular matrices (Gowda and Pang (1992), p. 78).

Definition 10.1. (See Cottle et al. (1992), Definition 3.8.7) A matrix $M \in R^{p\times p}$ is called an $\mathbf{R}_0$-matrix if the linear complementarity problem

$$Mz \geq 0, \quad z \geq 0, \quad z^TMz = 0, \qquad (z \in R^p),$$

has the unique solution $z = 0$.

Theorem 10.2. *Assume that $A \in \mathcal{H}$ and that $S(D,A,c,b)$ is bounded. If the multifunction $S(\cdot,\cdot,c,b)$ is upper semicontinuous at (D,A), then $M(D,A)$ is an $\mathbf{R}_0$-matrix.*

Proof. Since $S(D,A,c,b)$ is bounded and $S(\cdot,\cdot,c,b)$ is usc at (D,A), by Theorem 10.1, (10.4) holds. Let $\hat{z} = \begin{pmatrix} \hat{x} \\ \hat{\lambda} \end{pmatrix}$ be such that

$$M\hat{z} \geq 0, \quad \hat{z} \geq 0, \quad \hat{z}^T M\hat{z} = 0, \tag{10.20}$$

where $M = M(D,A)$. This means that the system (10.5)–(10.7) is satisfied. Hence, $\hat{x} \in S(D,A,0,0)$. Then $\hat{x} = 0$ by (10.4), and the system (10.5)–(10.7) implies

$$-A^T\hat{\lambda} \geq 0, \quad \hat{\lambda} \geq 0.$$

Since $A \in \mathcal{H}$, $\hat{\lambda} = 0$. Thus any $\hat{z}$ satisfying (10.20) must be zero. So $M(D,A)$ is an $\mathbf{R}_0$-matrix. □

Corollary 10.1. *Let $A \in \mathcal{H}$. If for every $(c,b) \in R^n \times R^m$ the multifunction $S(\cdot,\cdot,c,b)$ is upper semicontinuous at (D,A), then $M(D,A)$ is an $\mathbf{R}_0$-matrix.*

Proof. Consider problem (10.17), where $M = M(D,A)$ and q are defined via D, A, c, b by (10.18). Lemma 1 from Oettli and Yen (1995) shows that there exists $\bar{q} \in R^{n+m}$ such that $\mathrm{Sol}(M,\bar{q})$ is bounded. If $(\bar{c},\bar{b}) \in R^n \times R^m$ is the pair satisfying $\bar{q} = \begin{pmatrix} \bar{c} \\ -\bar{b} \end{pmatrix}$, then it follows from (10.19) that $S(D,A,\bar{c},\bar{b})$ is bounded. Since $S(\cdot,\cdot,\bar{c},\bar{b})$ is usc at (D,A), $M(D,A)$ is an $\mathbf{R}_0$-matrix by Theorem 10.2. □

The following statement gives a sufficient condition for the usc property of the multifunction $S(\cdot)$.

Theorem 10.3. *If $M(D,A)$ is an $\mathbf{R}_0$-matrix, then for any $(c,b) \in R^n \times R^m$ the set $S(D,A,c,b)$ is bounded, and the multifunction $S(\cdot)$ is upper semicontinuous at (D,A,c,b). If, in addition, $S(D,A,c,b)$ is nonempty, then there exist constants $\gamma > 0$ and $\delta > 0$ such that*

$$\begin{aligned} S(D',A',c',b') \subset\ & S(D,A,c,b) \\ & +\gamma(\|D'-D\| + \|A'-A\| + \|c'-c\| + \|b'-b\|)B_{R^n}, \end{aligned} \tag{10.21}$$

for all $(c',b') \in R^n \times R^m$, $D' \in R^{n\times n}$ and $A' \in R^{m\times n}$ satisfying $\|D'-D\| < \delta$, $\|A'-A\| < \delta$.

Proof. Since $M(D,A)$ is an $\mathbf{R}_0$-matrix, by Proposition 5.1 and Theorem 5.6 in Jansen and Tijs (1987) and the remarks before Theorem 2 of in Gowda (1992), $\mathrm{Sol}(M,q)$ is a bounded set, and the solution map $\mathrm{Sol}(\cdot)$ is usc at (M,q). It follows from (10.19) that

$S(D, A, c, b)$ is bounded. Let $\Omega \subset R^n$ be an arbitrary open set containing $S(D, A, b, c)$. By the upper semicontinuity of Sol$(\cdot)$ at (M, q), we have

$$\mathrm{Sol}(M', q') \subset \Omega \times R^m, \tag{10.22}$$

for all (M', q') in a neighborhood of (M, q). Using (10.19) and (10.22) we get $S(D', A', c', b') \subset \Omega$, for all (D', A', c', b') in a neighborhood of (D, A, c, b).

The upper Lipschitz property described in (10.21) follows from a result of Gowda (1992). Indeed, since $S(D, A, c, b)$ is nonempty, Sol(M, q) is nonempty. Since M is an $\mathbf{R}_0$-matrix, by Theorem 9 of Gowda (1992) there exist γ_0 and δ_0 such that

$$\mathrm{Sol}(M', q') \subset \mathrm{Sol}(M, q) + \gamma_0(\|M' - M\| + \|q' - q\|)B_{R^{n+m}} \tag{10.23}$$

for all $q' \in R^{n+m}$ and for all $M' \in R^{(n+m)\times(n+m)}$ satisfying $\|M' - M\| < \delta_0$. The inclusion (10.21) follows easily from (10.23) and (10.19). □

Combining Theorem 10.3 with Corollary 10.1 we get the following result.

Corollary 10.2. *If $A \in \mathcal{H}$, then for every $(c, b) \in R^n \times R^m$ the multifunction $S(\cdot, \cdot, c, b)$ is upper semicontinuous at (D, A) if and only if $M(D, A)$ is an $\mathbf{R}_0$-matrix.*

We now find necessary and sufficient conditions for $M(D, A)$ to be an $\mathbf{R}_0$-matrix. By definition, $M = M(D, A)$ is an $\mathbf{R}_0$-matrix if and only if the system

$$D\hat{x} - A^T\hat{\lambda} \geq 0, \quad A\hat{x} \geq 0, \tag{10.24}$$

$$\hat{x} \geq 0, \quad \hat{\lambda} \geq 0, \tag{10.25}$$

$$\hat{x}^T D\hat{x} = 0 \tag{10.26}$$

has the unique solution $(\hat{x}, \hat{\lambda}) = (0, 0)$.

Proposition 10.1. *If $M = M(D, A)$ is an $\mathbf{R}_0$-matrix then $A \in \mathcal{H}$ and the following condition holds:*

$$\left[D\hat{x} \geq 0,\ A\hat{x} \geq 0,\ \hat{x} \geq 0,\ \hat{x}^T D\hat{x} = 0\right] \Longrightarrow \hat{x} = 0. \tag{10.27}$$

Proof. If $\hat{\lambda} \in R^m$ is such that $-A^T\hat{\lambda} \geq 0$, $\hat{\lambda} \geq 0$, then $(0, \hat{\lambda})$ is a solution of the system (10.24)–(10.26). If M is an $\mathbf{R}_0$-matrix then we must have $\hat{\lambda} = 0$. By Lemma 10.1, $A \in \mathcal{H}$. Furthermore, for any

$\hat{x} \in R^n$ satisfying $D\hat{x} \geq 0,\ A\hat{x} \geq 0,\ \hat{x} \geq 0$ and $\hat{x}^T D\hat{x} = 0$, it is clear that $(\hat{x}, 0)$ is a solution of (10.24)–(10.26). If M is an $\mathbf{R}_0$-matrix then $(\hat{x}, 0) = (0, 0)$. We have thus proved (10.27). □

The above proposition shows that the inclusion $A \in \mathcal{H}$ and the property (10.27) are necessary conditions for $M = M(D, A)$ to be an $\mathbf{R}_0$-matrix. Sufficient conditions for $M = M(D, A)$ to be an $\mathbf{R}_0$-matrix are given in the following proposition. Recall that a matrix is said to be *nonnegative* if each of its elements is a nonnegative real number.

Proposition 10.2. *Assume that $A \in \mathcal{H}$. The following properties hold:*

(i) *If A is a nonnegative matrix and D is an $\mathbf{R}_0$-matrix then $M(D, A)$ is an $\mathbf{R}_0$-matrix.*

(ii) *If D a positive definite or a negative definite matrix, then $M(D, A)$ is an $\mathbf{R}_0$-matrix.*

Proof. For proving (i), let D be an $\mathbf{R}_0$-matrix and let $(\hat{x}, \hat{\lambda})$ be a pair satisfying (10.24)–(10.26). Since A is a nonnegative matrix, the inequalities $D\hat{x} - A^T\hat{\lambda} \geq 0$ and $\hat{\lambda} \geq 0$ imply $D\hat{x} \geq A^T\hat{\lambda} \geq 0$. Hence (10.24)–(10.26) yield $D\hat{x} \geq 0,\ \hat{x} \geq 0,\ \hat{x}^T D\hat{x} = 0$. Since D is an $\mathbf{R}_0$-matrix, $\hat{x} = 0$. This fact and (10.24)–(10.26) imply $-A^T\hat{\lambda} \geq 0,\ \hat{\lambda} \geq 0$. Since $A \in \mathcal{H}$, $\hat{\lambda} = 0$ by Lemma 10.1. Thus $(\hat{x}, \hat{\lambda}) = (0, 0)$ is the unique solution of (10.24)–(10.26). Hence M is an $\mathbf{R}_0$-matrix. We omit the easy proof of (ii). □

Observe that in Proposition 10.2(i) the condition that A is a nonnegative matrix cannot be dropped.

Example 10.2. Let $n = 2,\ m = 1,\ D = \mathrm{diag}(1, -1),\ A = (1, -1)$. It is clear that D is an $\mathbf{R}_0$-matrix and the condition $A \in \mathcal{H}$ is satisfied with $\bar{x} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$. Meanwhile, M is not an $\mathbf{R}_0$-matrix. Indeed, one can verify that the pair $(\hat{x}, \hat{\lambda})$, where $\hat{x} = (1, 1)$ and $\hat{\lambda} = 1$, is a solution of the system (10.24)–(10.26).

Definition 10.2 (Murty (1972), p. 67). We say that $D = (d_{ij}) \in R^{n \times n}$ is a *nondegenerate matrix* if, for any nonempty subset $\alpha \subset \{1, \ldots, n\}$, the determinant of the principal submatrix $D_{\alpha\alpha}$ consisting of the elements d_{ij} $(i \in \alpha,\ j \in \alpha)$ of D is nonzero.

Every nondegenerate matrix is an $\mathbf{R}_0$-matrix (see Cottle et al. (1992), p. 180). It can be proved that the set of nondegenerate

$n \times n$-matrices is open and dense in $R^{n \times n}$. From the following simple observation it follows that symmetric nondegenerate $\mathbf{R}_0$-matrices form a dense subset in the set of all symmetric matrices.

Proposition 10.3. *For any matrix $D \in R^{n \times n}$ and for any $\epsilon > 0$ there exists a nonnegative diagonal matrix U^ϵ such that $D + U^\epsilon$ is a nondegenerate matrix, and $\|U^\epsilon\| \leq \epsilon$.*

Proof. The proposition is proved by induction on n. For $n = 1$, if $D = [d]$, $d \neq 0$, then we set $U^\epsilon = [0]$. If $D = [0]$ then we set $U^\epsilon = [\epsilon]$. Assume that the conclusion of the proposition is true for all indexes $n \leq k - 1$. Let $D = (d_{ij})$ be a $k \times k$-matrix which is *not* nondegenerate. Denote by D_{k-1} the left-top submatrix of the order $(k - 1) \times (k - 1)$ of D. By induction, there is a diagonal matrix $U^\epsilon_{k-1} = \mathrm{diag}(\alpha_1, \ldots, \alpha_{k-1})$ such that every principal minor of the matrix $D_{k-1} + U^\epsilon_{k-1}$ is nonzero, and $\|U^\epsilon_{k-1}\| \leq \epsilon$. The required matrix U^ϵ is sought in the form

$$U^\epsilon = \mathrm{diag}(\alpha_1, \ldots, \alpha_{k-1}, y),$$

where $y \in R$ is a parameter.

From the construction of U^ϵ it follows that all the determinants of the principal submatrices of $D + U^\epsilon$ which do not contain the element $d_{kk} + y$, are nonzero. Obviously, there are 2^{k-1} principal submatrices of $D + U^\epsilon$ containing the element $d_{kk} + y$. The determinant of each one of these submatrices has the form $\alpha_i y + \beta_i$, $1 \leq i \leq 2^{k-1}$, where α_i and β_i are certain real numbers. Moreover, α_i equals 1 or equals one of the principal minors of $D_{k-1} + U^\epsilon_{k-1}$. So $\alpha_i \neq 0$ for all i. Since the numbers $-\dfrac{\beta_i}{\alpha_i}$, $1 \leq i \leq 2^{k-1}$, cannot cover the segment $[0, \epsilon]$, there exists $\bar{y} \in [0, \epsilon]$ such that $\bar{y} \neq -\frac{\beta_i}{\alpha_i}$ for all i. From what has already been said, we conclude that for $U^\epsilon := \mathrm{diag}(\alpha_1, \ldots, \alpha_{k-1}, \bar{y})$ the matrix $D + U^\epsilon$ is nondegenerate. In addition, it is clear that $\|U^\epsilon\| \leq \epsilon$. The proof is complete. □

Remark 10.1. The property of being a nondegenerate matrix is not invariant under the operation of matrix conjugation. This means that even if D is nondegenerate and P is nonsingular, the matrix $P^{-1}DP$ still may have zero principal minors. Examples can be found even in $R^{2 \times 2}$. Consequently, a linear operator with a nondegenerate matrix in one basis may have a degenerate matrix in another basis.

It follows from Theorem 10.3 and Proposition 10.2 that the multifunction $S(\cdot)$ is usc at (D, A, c, b) if $A \in \mathcal{H}$, A is a nonnegative matrix and D is an $\mathbf{R}_0$-matrix. There are many nonconvex QP problems fulfilling these conditions. For example, in the quadratic programs whose objective functions are given by the formula

$$f(x) = c^T x + \sum_{i=1}^{s} \mu_i x_i^2 - \sum_{i=s+1}^{n} \mu_i x_i^2,$$

where $c \in R^n$, $1 \leq s < n$ and $\mu_i > 0$ for all i, D is an $\mathbf{R}_0$-matrix. Proposition 10.3 shows that the set of symmetric $\mathbf{R}_0$-matrices is dense in $R_S^{n \times n}$.

10.4 Sufficient Conditions for the usc Property of $S(\cdot)$

Consider problem (10.1) whose Karush-Kuhn-Tucker point set is denoted by $S(D, A, c, b)$. A necessary condition for the usc property of $S(\cdot)$ was obtained in Section 10.2. Sufficient conditions for having that property were given in Section 10.3 only for a special class of QP problems. Our aim in this section is to find sufficient conditions for the usc property of the multifunction $S(\cdot)$ which are applicable for larger classes of QP problems.

For a matrix $A \in R^{m \times n}$, the dual of the cone

$$\Lambda[A] := \{\lambda \in R^m \, : \, -A^T \lambda \geq 0, \ \lambda \geq 0\}$$

is denoted by $(\Lambda[A])^*$. By definition, $(\Lambda[A])^* = \{\xi \in R^m \, : \, \lambda^T \xi \leq 0 \ \ \forall \lambda \in \Lambda[A]\}$. The interior of $(\Lambda[A])^*$ is denoted by $\operatorname{int}(\Lambda[A])^*$. By Lemma 6.4,

$$\operatorname{int}(\Lambda[A])^* = \{\xi \in R^m \, : \, \lambda^T \xi < 0 \quad \forall \lambda \in \Lambda[A] \setminus \{0\}\}.$$

The proofs of Theorems 10.4–10.6 below are based on some observations concerning the structure of the Karush-Kuhn-Tucker system (10.2). It turns out that the desired stability property of the set $S(D, A, c, b)$ depends greatly on the behavior of the quadratic form $x^T D x$ on the recession cone of $\Delta(A, b)$ and also on the position of b with respect to the set $\operatorname{int}(\Lambda[A])^*$.

One can note that in Example 10.1 the solution set $\operatorname{Sol}(D, A, 0, 0)$ is empty. In the following theorem, such "abnormal" situation will be excluded.

Theorem 10.4. *If* $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ *and if* $b \in \mathrm{int}(\Lambda[A])^*$ *then, for any* $c \in R^n$, *the multifunction* $S(\cdot)$ *is upper semicontinuous at* (D, A, c, b).

Proof. Suppose the theorem were false. Then we could find an open set Ω containing $S(D, A, c, b)$, a sequence $\{(D^k, A^k, c^k, b^k)\}$ converging to (D, A, c, b) in $R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$, a sequence $\{x^k\}$ with the property that $x^k \in S(D^k, A^k, c^k, b^k)$ and $x^k \notin \Omega$ for every k. By the definition of KKT point, there exists a sequence $\{\lambda^k\} \subset R^m$ such that

$$D^k x^k - (A^k)^T\lambda^k + c^k \geq 0, \quad A^k x^k - b^k \geq 0, \tag{10.28}$$

$$x^k \geq 0, \quad \lambda^k \geq 0, \tag{10.29}$$

$$(x^k)^T(D^k x^k - (A^k)^T\lambda^k + c^k) + (\lambda^k)^T(A^k x^k - b^k) = 0. \tag{10.30}$$

We first consider the case where the sequence of norms $\{\|(x^k, \lambda^k)\|\}$ is bounded. As the sequences $\{\|x^k\|\}$ and $\{\|\lambda^k\|\}$ are also bounded, from $\{x^k\}$ and $\{\lambda^k\}$, respectively, one can extract converging subsequences $\{x^{k_i}\}$ and $\{\lambda^{k_i}\}$. Assume that $x^{k_i} \to x^0 \in R^n$ and $\lambda^{k_i} \to \lambda^0 \in R^m$ as $i \to \infty$. From (10.28)–(10.30) it follows that

$$\begin{aligned} &Dx^0 - A^T\lambda^0 + c \geq 0, \quad Ax^0 - b \geq 0, \\ &x^0 \geq 0, \quad \lambda^0 \geq 0, \\ &(x^0)^T(Dx^0 - A^T\lambda^0 + c) + (\lambda^0)^T(Ax^0 - b) = 0. \end{aligned}$$

Hence $x^0 \in S(D, A, c, b) \subset \Omega$. On the other hand, since $x^{k_i} \notin \Omega$ for all i and Ω is open, we have $x^0 \notin \Omega$, a contradiction.

We now turn to the case where the sequence $\{\|(x^k, \lambda^k)\|\}$ is unbounded. In this case, there exists a subsequence, which is denoted again by $\{\|(x^k, \lambda^k)\|\}$, such that $\|(x^k, \lambda^k)\| \to \infty$ and $\|(x^k, \lambda^k)\| \neq 0$ for every k. Let

$$z^k := \frac{(x^k, \lambda^k)}{\|(x^k, \lambda^k)\|} = \left(\frac{x^k}{\|(x^k, \lambda^k)\|}, \frac{\lambda^k}{\|(x^k \lambda^k)\|}\right). \tag{10.31}$$

Since $\|z^k\| = 1$, there is a subsequence of $\{z^k\}$, which is denoted again by $\{z^k\}$, such that $z^k \to \bar{z} \in R^n \times R^m$, $\|\bar{z}\| = 1$. Let $\bar{z} = (\bar{x}, \bar{\lambda})$. By (10.31),

$$\frac{x^k}{\|(x^k, \lambda^k)\|} \to \bar{x}, \quad \frac{\lambda^k}{\|(x^k, \lambda^k)\|} \to \bar{\lambda}.$$

Dividing both sides of (10.28) and (10.29) by $\|(x^k, \lambda^k)\|$, both sides of (10.30) by $\|(x^k, \lambda^k)\|^2$, and taking limits as $k \to \infty$, we obtain

$$D\bar{x} - A^T\bar{\lambda} \geq 0, \quad A\bar{x} \geq 0, \tag{10.32}$$

$$\bar{x} \geq 0, \quad \bar{\lambda} \geq 0, \tag{10.33}$$

$$\bar{x}^T(D\bar{x} - A^T\bar{\lambda}) + \bar{\lambda}^T A\bar{x} = 0. \tag{10.34}$$

By (10.32) and (10.33), $\bar{x} \in \Delta(A, 0) = \{x \in R^n : Ax \geq 0, x \geq 0\}$. Let us suppose for the moment that $\bar{x} \neq 0$. It is obvious that (10.34) can be rewritten as $\bar{x}^T D\bar{x} = 0$. If $x^T Dx \geq 0$ for all $x \in \Delta(A, 0)$ then $\bar{x} \in \mathrm{Sol}\,(D, A, 0, 0)$, contrary to the assumption $\mathrm{Sol}\,(D, A, 0, 0) = \{0\}$. If there exists $\hat{x} \in \Delta(A, 0)$ such that $\hat{x}^T D\hat{x} < 0$ then

$$\inf\{x^T Dx : x \in \Delta(A, 0)\} = -\infty,$$

because $\Delta(A, 0)$ is a cone. Thus $\mathrm{Sol}\,(D, A, 0, 0) = \emptyset$, contrary to the condition $\mathrm{Sol}\,(D, A, 0, 0) = \{0\}$. Therefore $\bar{x} = 0$.

As $\|(\bar{x}, \bar{\lambda})\| = 1$, from (10.32) and (10.33) it follows that $\bar{\lambda} \in \Lambda[A] \setminus \{0\}$. The assumption $b \in \mathrm{int}\,(\Lambda[A])^*$ implies

$$\bar{\lambda}^T b < 0. \tag{10.35}$$

Since $\|(x^k, \lambda^k)\| \to \infty$, $\dfrac{\lambda^k}{\|(x^k, \lambda^k)\|} \to \bar{\lambda}$ and $\|\bar{\lambda}\| = \|(\bar{x}, \bar{\lambda})\| = 1$, $\|\lambda^k\| \to \infty$. Using the obvious identity $(x^k)^T(A^k)^T\lambda^k = (\lambda^k)^T A^k x^k$ we can rewrite (10.30) as the following

$$(x^k)^T D^k x^k + (x^k)^T c^k = (\lambda^k)^T b^k. \tag{10.36}$$

If the sequence $\{x^k\}$ is bounded, then dividing both sides of (10.36) by $\|(x^k, \lambda^k)\|$ and letting $k \to \infty$ we obtain $\bar{\lambda}^T b = 0$, contrary to (10.35). So the sequence $\{x^k\}$ must be unbounded, and it has a subsequence, denoted again by $\{x^k\}$, such that $\|x^k\| \to \infty$, $\|x^k\| \neq 0$ for all k, and $\dfrac{x^k}{\|x^k\|} \to \hat{x}$ with $\|\hat{x}\| = 1$. For the sequence $\{(\lambda^k)^T b^k\}$ there are only two possibilities:

(α) There exists an integer i_0 such that

$$(\lambda^k)^T b^k \leq 0 \tag{10.37}$$

for all $k \geq i_0$, and

(β) For each i there exists an integer $k_i > i$ such that

$$(\lambda^{k_i})^T b^{k_i} > 0. \tag{10.38}$$

If case (α) arises, then (10.36) implies

$$(x^k)^T D^k x^k + (x^k)^T c^k \leq 0 \tag{10.39}$$

for all $k \geq i_0$. Dividing both sides of (10.39) by $\|x^k\|^2$ and letting $k \to \infty$ we get

$$\hat{x}^T D\hat{x} \leq 0. \tag{10.40}$$

By (10.28) and (10.29),

$$A^k x^k \geq b^k, \quad x^k \geq 0.$$

Dividing both sides of each of the last two inequalities by $\|x^k\|$ and letting $k \to \infty$ we obtain

$$A\hat{x} \geq 0, \quad \hat{x} \geq 0. \tag{10.41}$$

Since $0 \in \mathrm{Sol}\,(D, A, 0, 0)$, by (10.40) and (10.41) we have $\hat{x} \in \mathrm{Sol}\,(D, A, 0, 0)$, contrary to the condition $\mathrm{Sol}\,(D, A, 0, 0) = \{0\}$. Thus case (α) is impossible. If case (β) happens, then by dividing both sides of (10.38) by $\|(x^{k_i}, \lambda^{k_i})\|$ and letting $i \to \infty$ we obtain $\bar{\lambda}^T b \geq 0$, contrary to (10.35). The proof is complete, because neither (α) nor (β) can occur. □

Theorem 10.5. *If* $\mathrm{Sol}\,(-D, A, 0, 0) = \{0\}$ *and* $b \in -\mathrm{int}\,(\Lambda[A])^*$ *then, for any* $c \in R^n$, *the multifunction* $S(\cdot)$ *is upper semicontinuous at* (D, A, c, b).

Proof. Except for several small changes, this proof is very similar to the proof of Theorem 10.4. Suppose, contrary to our claim, that there is an open set $\Omega \subset R^n$ containing $S(D, A, c, b)$, a sequence $\{(D^k, A^k, c^k, b^k)\}$ converging to (D, A, c, b) in $R^{n\times n}_S \times R^{m\times n} \times R^n \times R^m$, a sequence $\{x^k\}$ with $x^k \in S(D^k, A^k, c^k, b^k)$ and $x^k \notin \Omega$ for every k. By the definition of KKT point, there is a sequence $\{\lambda^k\}$ satisfying (10.28)–(10.30). If the sequence of $\{\|(x^k, \lambda^k)\|\}$ is bounded then, arguing similarly as in the preceding proof, we will arrive at a contradiction. If the sequence $\{\|(x^k, \lambda^k)\|\}$ is unbounded then, without any loss of generality, we can assume that the sequence $\left\{\dfrac{(x^k, \lambda^k)}{\|(x^k, \lambda^k)\|}\right\}$ converges to a certain pair $(\bar{x}, \bar{\lambda})$ with $\|(\bar{x}, \bar{\lambda})\| = 1$. Dividing both sides of (10.28) and of (10.29) by $\|(x^k, \lambda^k)\|$, both sides of (10.30) by $\|(x^k, \lambda^k)\|^2$ and letting $k \to \infty$ we obtain (10.32)–(10.34). From (10.34) we have $\bar{x}^T(-D)\bar{x} = 0$. The assumption $\mathrm{Sol}\,(-D, A, 0, 0) = \{0\}$ forces $\bar{x} = 0$. Thus $\bar{\lambda} \in \Lambda[A] \setminus \{0\}$. Since $b \in -\mathrm{int}\,(\Lambda[A])^*$, we have

$$\bar{\lambda}^T b > 0. \tag{10.42}$$

Since $\|(x^k, \lambda^k)\| \to \infty$, $\dfrac{\lambda^k}{\|(x^k, \lambda^k)\|} \to \bar{\lambda}$, and $\|\bar{\lambda}\| = 1$, we must have $\|\lambda^k\| \to \infty$. Again, rewrite (10.30) in the form (10.36). If the sequence $\{x^k\}$ is bounded, we can divide both sides of (10.36) by $\|(x^k, \lambda^k)\|$ and let $k \to \infty$ to obtain $\bar{\lambda}^T b = 0$, which contradicts (10.42). Thus the sequence $\{x^k\}$ must be bounded, and it has a subsequence, denoted again by $\{x^k\}$, such that $\|x^k\| \to \infty$, $\|x^k\| \neq 0$ for all k, and $\dfrac{x^k}{\|x^k\|} \to \hat{x}$ with $\|\hat{x}\| = 1$.

If there exists an index i_0 such that (10.37) holds, then dividing both sides of (10.36) by $\|(x^k, \lambda^k)\|$ and taking limit as $k \to \infty$ we have $\bar{\lambda}^T b = 0$, contrary to (10.42).

Assume that for each i, there exists an integer $k_i > i$ such that (10.38) holds. From (10.36) and (10.38) it follows that

$$(x^{k_i})^T D_{k_i} x^{k_i} + (x^{k_i})^T c^{k_i} \geq 0 \tag{10.43}$$

for all i. Dividing both sides of (10.43) by $\|x^{k_i}\|^2$ and taking limit as $i \to \infty$ we get $\hat{x}^T D \hat{x} \geq 0$ or, equivalently,

$$\hat{x}^T (-D) \hat{x} \leq 0. \tag{10.44}$$

By (10.28) and (10.29), $A_{k_i} x^{k_i} \geq b^{k_i}$, $\quad x^{k_i} \geq 0$. Dividing both sides of each of the last two inequalities by $\|x^{k_i}\|$ and taking limits we obtain (10.41). Properties (10.41), (10.44), and the inclusion $0 \in \mathrm{Sol}(-D, A, 0, 0)$ yield $\hat{x} \in \mathrm{Sol}(-D, A, 0, 0)$, contrary to the condition $\mathrm{Sol}(-D, A, 0, 0) = \{0\}$. Thus, in all possible cases we have arrived at a contradiction. The proof is complete. □

Our third sufficient condition for the stability of the Karush-Kuhn-Tucker point set can be formulated as follows.

Theorem 10.6. *If $S(D, A, 0, 0) = \{0\}$ and $\Lambda[A] = \{0\}$ then, for any $(c, b) \in R^n \times R^m$, the multifunction $S(\cdot)$ is upper semicontinuous at (D, A, c, b).*

Proof. Repeat the arguments in the proof of Theorem 10.4 until reaching the system (10.32)–(10.34). Since $S(D, A, 0, 0) = \{0\}$, we have $\bar{x} = 0$, hence (10.32)–(10.34) imply $-A^T \bar{\lambda} \geq 0$, $\bar{\lambda} \geq 0$. By $\|\bar{\lambda}\| = \|(\bar{x}, \bar{\lambda})\| = 1$, one has $\bar{\lambda} \in \Lambda[A] \setminus \{0\}$, contrary to the assumption that $\Lambda[A] = \{0\}$. □

10.5 Corollaries and Examples

We now consider some corollaries of the results established in the preceding section and give several illustrative examples.

Corollary 10.3. *If $\Lambda[A] = \{0\}$ and if the matrix D is a positive definite (or negative definite) then, for any pair $(c, b) \in R^n \times R^m$, the multifunction $S(\cdot)$ is upper semicontinuous at (D, A, c, b).*

Proof. If D is positive definite, then $S(D, A, 0, 0) = \mathrm{Sol}(D, A, 0, 0) = \{0\}$. So our assertion follows from Theorem 10.6.

If D is negative definite, then $S(D, A, 0, 0) = \mathrm{Sol}\,(-D, A, 0, 0) = \{0\}$, and again the assertion follows from Theorem 10.6. □

We proceed to show that the condition $b \in \mathrm{int}\,(\Lambda[A])^*$ in Theorem 10.4 is equivalent to the regularity of the following system of linear inequalities

$$Ax \geq b, \quad x \geq 0. \tag{10.45}$$

Lemma 10.2. *System* (10.45) *is regular if and only if* $b \in \mathrm{int}\,(\Lambda[A])^*$.

Proof. Assume (10.45) is regular, i.e. there exists x^0 such that $Ax^0 > b$, $x^0 > 0$. Let $q := Ax^0 - b > 0$ and let $\bar{\lambda}$ be any vector from $\Lambda[A] \setminus \{0\}$, that is $A^T\bar{\lambda} \leq 0, \bar{\lambda} \geq 0$, and $\bar{\lambda} \neq 0$. Then

$$\bar{\lambda}^T b = \bar{\lambda}^T(Ax^0 - q) = (x^0)^T A^T \bar{\lambda} - \bar{\lambda}^T q \leq -\lambda^T q < 0.$$

Hence $b \in \mathrm{int}\,(\Lambda[A])^*$.

Conversely, assume that $b \in \mathrm{int}\,(\Lambda[A])^*$. Suppose for a moment that (10.45) is irregular. Since the system $Ax > b$, $x \geq 0$ has no solutions, for any sequence $b^k \to b$ with $b^k > b$ for all k, the systems

$$Ax \geq b^k, \quad x \geq 0$$

have no solutions. By Theorem 2.7.9 from Cottle et al. (1992), which is a corollary of the Farkas Lemma, there exists $\lambda^k \in R^m$ such that

$$-A^T\lambda^k \geq 0, \quad \lambda^k \geq 0, \quad (\lambda^k)^T b^k > 0. \tag{10.46}$$

Since $\lambda^k \neq 0$, without loss of generality, we can assume that $\|\lambda^k\| = 1$ for every k, and $\lambda^k \to \bar{\lambda}$ with $\|\bar{\lambda}\| = 1$. Taking limits in (10.46) as $k \to \infty$ we get

$$-A^T\bar{\lambda} \geq 0, \quad \bar{\lambda} \geq 0, \quad \bar{\lambda}^T b \geq 0.$$

Hence $\bar{\lambda} \in \Lambda[A] \setminus \{0\}$, and the inequality $\bar{\lambda}^T b \geq 0$ contradicts the assumption $b \in \mathrm{int}\,(\Lambda[A])^*$. We have thus proved that (10.45) is regular. □

Corollary 10.4. *If* (10.45) *is regular and if* $\Delta(A, b)$ *is bounded, then the multifunction* $S(\cdot)$ *is upper semicontinuous at* (D, A, c, b).

Proof. Since $\Delta(A, b)$ is nonempty, bounded, and $\Delta(A, b)+\Delta(A, 0) \subset \Delta(A, b)$, we have $\Delta(A, 0) = \{0\}$. Since (10.45) is regular, by Lemma 10.2 we have $b \in \operatorname{int}(\Lambda[A])^*$. Applying Theorem 10.4 we get the desired result. □

We have the following sufficient condition for stability of the KKT point set in QP problems with bounded constraint sets.

Corollary 10.5. *If* $\Delta(A, 0) = \{0\}$ *and* $\lambda^T b \neq 0$ *for all* $\lambda \in \Lambda[A] \setminus \{0\}$ *then, for any* $c \in R^n$, *the multifunction* $S(\cdot)$ *is upper semicontinuous at* (D, A, c, b).

Proof. Obviously, the condition $\Delta(A, 0) = \{0\}$ implies

$$S(D, A, 0, 0) = \operatorname{Sol}(D, A, 0, 0) = \operatorname{Sol}(-D, A, 0, 0) = \Delta(A, 0) = \{0\}. \tag{10.47}$$

Since $\Lambda[A]$ is a convex cone, the assumption $\lambda^T b \neq 0$ for all $\lambda \in \Lambda[A] \setminus \{0\}$ implies that one of the following two cases must occur:

(i) $\lambda^T b < 0$ for all $\lambda \in \Lambda[A] \setminus \{0\}$,

(ii) $\lambda^T b > 0$ for all $\lambda \in \Lambda[A] \setminus \{0\}$.

In the first case, the desired conclusion follows from (10.47) and Theorem 10.4. In the second case, the conclusion follows from (10.47) and Theorem 10.5. □

The following two examples show that the obtained sufficient conditions for stability can be applied to nonconvex QP problems.

Example 10.3. Consider problem (10.1) where $n = 2$, $m = 1$,

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$A = \left[-\frac{1}{2}, -1\right], \; b = (-1), \; c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have $\Delta(A, 0) = \{0\}$, $\operatorname{Sol}(D, A, 0, 0) = \{0\}$ and $b \in \operatorname{int}(\Lambda[A])^*$. By Theorem 10.4, $S(\cdot)$ is usc at (D, A, c, b).

Example 10.4. Consider problem (10.1) where $n = 2$, $m = 1$,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; A = [-1, 0], \; b = (-1), \; c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

An easy computation shows that

$$S(D, A, 0, 0) = \{0\}, \; \operatorname{Sol}(-D, A, 0, 0) = \{0\}, \; \text{and } b \in -\operatorname{int}(\Lambda[A])^*.$$

The multifunction $S(\cdot)$ is usc at (D, A, c, b) by Theorem 10.5.

The next two examples show that when the condition $b \in \text{int}\,(\Lambda[A])^*$ is violated the conclusion of Theorem 10.4 may hold or may not hold, as well.

Example 10.5. Let $D = [1]$, $A = [0]$, $b = (1)$, $c = (0)$, $A_t = [-t]$, where $t \in (0, 1)$. It is easily seen that

$$S(D, A, 0, 0) = \{0\}, \quad \text{Sol}\,(D, A, 0, 0) = \{0\}, \quad S(D, A, c, b) = \emptyset,$$
$$S(D, A_t, c, b) = \left\{\frac{1}{t}\right\}, \quad \Lambda[A] = R_+, \quad b \notin \text{int}\,(\Lambda[A])^*.$$

We have $S(D, A, c, b) \subset \Omega$, where $\Omega = \emptyset$. Since $A_t \to A$ and the inclusion $S(D, A_t, c, b) \subset \Omega$ cannot hold for sufficiently small $t > 0$, $S(\cdot)$ cannot be usc at (D, A, c, b).

Example 10.6. Let $D = [-1]$, $A = [-1]$, $b = (1)$, $c = (0)$. It is easy to verify that

$$S(D, A, 0, 0) = \{0\}, \quad \text{Sol}\,(D, A, 0, 0) = \{0\}, \quad S(D, A, c, b) = \emptyset,$$
$$\Lambda[A] = R_+, \quad b \notin \text{int}\,(\Lambda[A])^*.$$

The map $S(\cdot)$ is usc at (D, A, c, b). Indeed, since $S(-D, A, 0, 0) = \{0\}$ and $b \in -\text{int}\,(\Lambda[A])^*$, Theorem 10.5 can be applied.

The following two examples show that if $b \notin -\text{int}\,(\Lambda[A])^*$ then the conclusion of Theorem 10.5 may hold or may not hold, as well.

Example 10.7. Let D, A, c, b be defined as in Example 10.5. In this case we have

$$S(D, A, 0, 0) = \{0\}, \quad \text{Sol}\,(-D, A, 0, 0) = \{0\},$$
$$\Lambda[A] = R_+, \quad b \notin -\text{int}\,(\Lambda[A])^*.$$

As it has been shown in Example 10.5, the map $S(\cdot)$ is not usc at (D, A, c, b).

Example 10.8. Let $D = [1]$, $A = [-1]$, $b = (-1)$, $c = (0)$. It is a simple matter to verify that

$$S(D, A, 0, 0) = \{0\}, \quad \text{Sol}\,(-D, A, 0, 0) = \{0\},$$
$$\Lambda[A] = R_+, \quad b \notin -\text{int}\,(\Lambda[A])^*.$$

The fact that $S(\cdot)$ is usc at (D, A, c, b) follows from Theorem 10.4, because $\text{Sol}\,(D, A, 0, 0) = \{0\}$ and $b \in \text{int}\,(\Lambda[A])^*$.

10.6 USC Property of $S(\cdot)$: The General Case

In this section we obtain necessary and sufficient conditions for the stability of the Karush-Kuhn-Tucker point set in a general QP problem.

Given matrices $A \in R^{m\times n}$, $F \in R^{s\times n}$, $D \in R_S^{n\times n}$, and vectors $c \in R^n$, $b \in R^m$, $d \in R^s$, we consider the following general indefinite QP problem $QP(D, A, c, b, F, d)$:

$$\begin{cases} \text{Minimize } f(x) := \frac{1}{2}x^T Dx + c^T x \\ \text{subject to } x \in R^n, \ Ax \geq b, \ Fx \geq d \end{cases} \tag{10.48}$$

In what follows, *the pair* (F, d) *is not subject to change.* So the set $\Delta(F, d) := \{x \in R^n : Fx \geq d\}$ is fixed. Define $\Delta(A, b) = \{x \in R^n : Ax \geq b\}$ and recall (see Definition 3.1 and Corollary 3.2) that $\bar{x} \in \Delta(A, b) \cap \Delta(F, d)$ is said to be a Karush-Kuhn-Tucker point of $QP(D, A, c, b, F, d)$ if there exists a pair $(\bar{u}, \bar{v}) \in R^m \times R^s$ such that

$$\begin{aligned} &D\bar{x} - A^T\bar{u} - F^T\bar{v} + c = 0, \\ &A\bar{x} \geq b, \quad \bar{u} \geq 0, \\ &F\bar{x} \geq d, \quad \bar{v} \geq 0, \\ &\bar{u}^T(A\bar{x} - b) + \bar{v}^T(F\bar{x} - d) = 0. \end{aligned}$$

The KKT point set and the solution set of (10.48) are denoted, respectively, by $S(D, A, c, b, F, d)$ and $\mathrm{Sol}(D, A, c, b, F, d)$.

If $s = n$, $d = 0$, and F is the unit matrix in $R^{n\times n}$, then problem (10.48) has the following canonical form (10.1). In agreement with the notation of the preceding sections, we write $S(D, A, c, b)$ instead of $S(D, A, c, b, F, d)$, and $\mathrm{Sol}(D, A, c, b)$ instead of $\mathrm{Sol}(D, A, c, b, F, d)$ if (10.48) has the canonical form. The upper semicontinuity of the multifunction

$$p' \mapsto S(p'), \ p' = (D', A', c', b') \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m, \tag{10.49}$$

has been studied in Sections 10.3–10.5. This property can be interpreted as the *stability* of the KKT point set $S(D, A, c, b)$ with respect to the change in the problem parameters. In this section we are interested in finding out how the results proved in Sections 10.3–10.5 can be extended to the case of problem (10.48). We will obtain some necessary and sufficient conditions for the upper semicontinuity of the multifunction

$$p' \mapsto S(p', F, d),\ p' = (D', A', c', b') \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m. \tag{10.50}$$

As for the canonical problem, the obtained results can be interpreted as the necessary and sufficient *conditions for the stability* of the Karush-Kuhn-Tucker point set $S(D, A, c, b, F, d)$ with respect to the change in the problem parameters.

Our proofs are based on several observations concerning the system of equalities and inequalities defining the KKT point set. It is worthy to stress that the proofs in the preceding sections cannot be applied to the case of problem (10.48). This is because, unlike the case of the canonical problem (10.1), $\Delta(F, d)$ may fail to be a cone with nonempty interior and the vertex 0. So we have to use some new arguments. Fortunately, the proof schemes in the preceding sections will be useful also for the case of problem (10.48).

Theorem 10.7 below deals with the case where $\Delta(F, d)$ is a polyhedral cone with a vertex x^0, where $x^0 \in R^n$ is an arbitrarily given vector. Theorem 10.8 works for the case where $\Delta(F, d)$ is an arbitrary polyhedral set, but the conclusion is weaker than that of Theorem 10.7.

For any $M \in R^{r\times n}$ and $q \in R^r$, the set $\{x \in R^n : Mx \geq q\}$ is denoted by $\Delta(M, q)$. For $F \in R^{s\times n}$ and $A \in R^{m\times n}$, we abbreviate the set

$$\{(u, v) \in R^m \times R^s : A^T u + F^T v = 0,\ u \geq 0,\ v \geq 0\}$$

to $\Lambda[A, F]$. Note that

$$\begin{aligned}&\operatorname{int}(\Lambda[A, F])^* \\ &= \{(\xi, \eta) : \xi^T u + \eta^T v < 0\ \ \forall (u, v) \in \Lambda[A, F] \setminus \{(0, 0)\}\}.\end{aligned}$$

The next two remarks clarify some points in the assumption and conclusion of Theorem 10.7 below.

Remark 10.2. If there is a point $x^0 \in R^n$ such that $F(x^0) = d$ then $\Delta(F, d) = x^0 + \Delta(F, 0)$. Conversely, for any $x^0 \in R^n$ and any polyhedral cone K, there exists a positive integer s and a matrix $F \in R^{s\times n}$ such that $x^0 + K = \Delta(F, d)$, where $d := F(x^0)$.

Remark 10.3. If $\Delta(F, d)$ and $\Delta(A, b)$ are nonempty, then $\Delta(F, 0)$ and $\Delta(A, 0)$, respectively, are the recession cones of $\Delta(F, d)$ and

$\Delta(A, b)$. By definition, $S(D, A, 0, 0, F, 0)$ is the Karush-Kuhn-Tucker point set of the following QP problem

$$\text{Minimize} \quad x^T D x \quad \text{subject to} \quad x \in R^n, \; Ax \geq 0, \; Fx \geq 0,$$

whose constraint set is the intersection $\Delta(A, 0) \cap \Delta(F, 0)$.

Theorem 10.7. *Assume that the set $S(p, F, d)$, where $p = (D, A, c, b)$, is bounded and there exists $x^0 \in R^n$ such that $F(x^0) = d$. If the multifunction* (10.50) *is upper semicontinuous at p then*

$$S(D, A, 0, 0, F, 0) = \{0\}. \tag{10.51}$$

Proof. Suppose, contrary to our claim, that there is a nonzero vector $\bar{x} \in S(D, A, 0, 0, F, 0)$. By definition, there exists a pair $(\bar{u}, \bar{v}) \in R^m \times R^s$ such that

$$D\bar{x} - A^T\bar{u} - F^T\bar{v} = 0, \tag{10.52}$$

$$A\bar{x} \geq 0, \quad \bar{u} \geq 0, \tag{10.53}$$

$$F\bar{x} \geq 0, \quad \bar{v} \geq 0, \tag{10.54}$$

$$\bar{u}^T A\bar{x} + \bar{v}^T F\bar{x} = 0. \tag{10.55}$$

For every $t \in (0, 1)$, we set

$$x_t = x^0 + \frac{1}{t}\bar{x}, \quad u_t = \frac{1}{t}\bar{u}, \quad v_t = \frac{1}{t}\bar{v}, \tag{10.56}$$

where x^0 is given by our assumptions. We claim that there exist matrices $D_t \in R_S^{n\times n}$, $A_t \in R^{m\times n}$ and vectors $c_t \in R^n$, $b_t \in R^m$ such that

$$\max\{\|D_t - D\|, \|A_t - A\|, \|c_t - c\|, \|b_t - b\|\} \to 0 \quad \text{as } t \to 0,$$

and

$$D_t x_t - A_t^T u_t - F^T v_t + c_t = 0, \tag{10.57}$$

$$A_t x_t \geq b_t, \quad u_t \geq 0, \tag{10.58}$$

$$F x_t \geq d, \quad v_t \geq 0, \tag{10.59}$$

$$u_t^T(A_t x_t - b_t) + v_t^T(F x_t - d) = 0. \tag{10.60}$$

The matrices D_t, A_t and the vectors c_t, b_t will have the following representations

$$D_t = D + tD_0, \quad A_t = A + tA_0 \tag{10.61}$$

$$c_t = c + tc_0, \quad b_t = b + tb_0, \tag{10.62}$$

where the matrices D_0, A_0 and the vectors c_0, b_0 are to be constructed. First we observe that, due to (10.54) and (10.56), (10.59) holds automatically. Clearly,

$$\begin{aligned} A_t x_t - b_t &= (A + tA_0)\left(x^0 + \frac{\bar{x}}{t}\right) - (b + tb_0) \\ &= t(A_0 x^0 - b_0) + \frac{1}{t} A\bar{x} + A_0\bar{x} + Ax^0 - b \end{aligned}$$

and

$$\begin{aligned} & u_t^T(A_t x_t - b_t) + v_t^T(F x_t - d) \\ &\quad = \frac{\bar{u}^T}{t}\left[t(A_0 x^0 - b_0) + \frac{1}{t} A\bar{x} + A_0\bar{x} + Ax^0 - b\right] \\ &\qquad + \frac{\bar{v}^T}{t}\left[F\left(x^0 + \frac{\bar{x}}{t}\right) - d\right] \\ &\quad = \bar{u}^T(A_0 x^0 - b_0) + \frac{1}{t^2}(\bar{u}^T A\bar{x} + \bar{v}^T F\bar{x}) + \frac{\bar{u}^T}{t}(A_0\bar{x} + Ax^0 - b). \end{aligned}$$

Therefore, by (10.53) and (10.55), if we have

$$A_0\bar{x} + Ax^0 - b = 0 \tag{10.63}$$

and

$$A_0 x^0 - b_0 = 0, \tag{10.64}$$

then (10.58) and (10.60) will be fulfilled. By (10.52),

$$\begin{aligned} D_t x_t \quad & -A_t^T u_t - F^T v_t + c_t \\ &= (D + tD_0)\left(x^0 + \frac{\bar{x}}{t}\right) - (A + tA_0)^T \frac{\bar{u}}{t} - F^T \frac{\bar{v}}{t} + c + tc_0 \\ &= \frac{1}{t}(D\bar{x} - A^T\bar{u} - F^T\bar{v}) + t(D_0 x^0 + c_0) + Dx^0 \\ &\qquad + D_0\bar{x} - A_0^T\bar{u} + c, \\ &= t(D_0 x^0 + c_0) + Dx^0 + D_0\bar{x} - A_0^T\bar{u} + c. \end{aligned}$$

Hence, if we have

$$Dx^0 + D_0\bar{x} - A_0^T\bar{u} + c = 0 \tag{10.65}$$

and

$$D_0 x^0 + c_0 = 0, \tag{10.66}$$

then (10.57) will be fulfilled.

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$, where $\bar{x}^i \neq 0$ for $i \in I$ and $\bar{x}_i = 0$ for $i \notin I$, $I \subset \{1, \ldots, n\}$. Since $\bar{x} \neq 0$, I is nonempty. Fixing an index

$i_0 \in I$, we define A_0 as the $m \times n$-matrix in which the $i_0 - th$ column is $\bar{x}_{i_0}^{-1}(b - Ax^0)$, and the other columns consist solely of zeros. Let $b_0 = A_0 x^0$. One can verify immediately that (10.63) and (10.64) are satisfied; hence conditions (10.58) and (10.60) are fulfilled. From what has been said it follows that our claim will be proved if we can construct a matrix $D_0 \in R_S^{n \times n}$ and a vector c_0 satisfying (10.65) and (10.66). Let $D_0 = (d_{ij})$, where d_{ij} $(1 \leq i, j \leq n)$ are defined by the following formulae:

$$\begin{aligned} d_{ii} &= \bar{x}_i^{-1} \left(A_0^T \bar{u} - Dx^0 - c \right)_i \quad \forall i \in I, \\ d_{i_0 j} &= d_{j i_0} = \bar{x}_{i_0}^{-1} \left(A_0^T \bar{u} - Dx^0 - c \right)_j \quad \forall j \in \{1, \ldots, n\} \setminus I, \end{aligned}$$

and $d_{ij} = 0$ for other pairs (i, j), $1 \leq i, j \leq n$. Here $(A_0^T \bar{u} - Dx^0 - c)_k$ denotes the k-th component of the vector $A_0^T \bar{u} - Dx^0 - c$. Since D_0 is a symmetric matrix, $D_0 \in R_S^{n \times n}$. If we define $c_0 = -D_0 x^0$ then (10.66) is satisfied. A direct computation shows that (10.65) is also satisfied.

We have thus constructed matrices D_0, A_0 and vectors c_0, b_0 such that for x_t, u_t, v_t, D_t, A_t, c_t, b_t defined by (10.56), (10.61) and (10.62), conditions (10.57)–(10.60) are satisfied. Consequently, $x_t \in S(D_t, A_t, c_t, b_t, F, d)$. Since $S(p, F, d)$ is bounded, there is a bounded open set $\Omega \subset R^n$ such that $S(p, F, d) \subset \Omega$. Since

$$\max\{\|D_t - D\|, \|A_t - A\|, \|c_t - c\|, \|b_t - b\|\} \to 0$$

as $t \to 0$ and the multifunction $p' \mapsto S(p', F, d)$ is usc at $p = (D, A, c, b)$, $x_t \in \Omega$ for all sufficiently small t. This is impossible, because $\|x_t\| = \|x^0 + \dfrac{\bar{x}}{t}\| \to \infty$ as $t \to 0$. The proof is complete. □

Remark 10.4. If $d = 0$ then $\Delta(F, d)$ is a cone with the vertex 0. In order to verify the assumptions of Theorem 10.7, one can choose $x^0 = 0$. In particular, this is the case of the canonical problem (10.1). Applying Theorem 10.7 we obtain the following necessary condition for the upper semicontinuity of the multifunction (10.49): *If $S(p)$, $p = (D, A, c, b)$ is bounded and if the multifunction $p' \mapsto S(p')$, $p' = (D', A', c', b')$, is usc at p, then $S(D, A, 0, 0) = \{0\}$.* Thus Theorem 10.8 above extends Theorem 10.1 to the case where $\Delta(F, d)$ can be any polyhedral convex cone in R^n, merely the standard cone R_+^n.

In the sequel, $S(D, A)$ denotes the set of all $x \in R^n$ such that there exists $u = u(x) \in R^m$ satisfying the following system:

$$Dx - A^T u = 0, \quad Ax \geq 0, \quad u \geq 0, \quad u^T A x = 0.$$

Remark 10.5. From the definition it follows that $S(D, A) = S(D, A, 0, 0, F, 0)$, where $s = n$ and $F = 0 \in R^{n \times n}$.

Theorem 10.8. *Assume that $\Delta(F, d)$ is nonempty and $S(p, F, d)$, where $p = (D, A, c, b)$, is bounded. If the multifunction* (10.50) *is upper semicontinuous at p then*

$$S(D, A) \cap \Delta(F, 0) = \{0\}. \tag{10.67}$$

Remark 10.6. Observe that (10.51) implies (10.67). Indeed, suppose that (10.51) holds. The fact that $0 \in S(D, A) \cap \Delta(F, 0)$ is obvious. So, if (10.67) does not hold then there exists $\bar{x} \in S(D, A) \cap \Delta(F, 0)$, $\bar{x} \neq 0$. Taking $\bar{u} = u(\bar{x})$, $\bar{v} = 0 \in R^s$, we see at once that the system (10.52)–(10.55) is satisfied. This means that $\bar{x} \in S(D, A, 0, 0, F, 0) \setminus \{0\}$, contrary to (10.51). Note that, in general, (10.67) does not imply (10.51).

Remark 10.7. If there exists x^0 such that $Fx^0 = d$ then, of course, $x^0 \in \Delta(F, d) = \{x \in R^n : Fx \geq d\}$. In particular, $\Delta(F, d) \neq \emptyset$. Thus Theorem 10.8 can be applied to a larger class of problems than Theorem 10.7. However, Remark 10.6 shows that the conclusion of Theorem 10.8 is weaker than that of Theorem 10.7. One question still unanswered is whether the assumptions of Theorem 10.8 always imply (10.51).

Proof of Theorem 10.8.

Assume that $\Delta(F, d)$ is nonempty, $S(D, A, c, b, F, d)$ is bounded and the multifunction $S(\cdot, F, d)$ is usc at p but (10.67) is violated. Then, there is a nonzero vector $\bar{x} \in S(D, A) \cap \Delta(F, 0)$. Hence there exists $\bar{u} \in R^m$ such that

$$D\bar{x} - A^T\bar{u} = 0, \tag{10.68}$$

$$A\bar{x} \geq 0, \quad \bar{u} \geq 0, \tag{10.69}$$

$$\bar{u}^T A\bar{x} = 0, \tag{10.70}$$

$$F\bar{x} \geq 0. \tag{10.71}$$

Let x^0 be an arbitrary point of $\Delta(F, d)$. Setting

$$x_t = x^0 + \frac{1}{t}\bar{x}, \quad u_t = \frac{1}{t}\bar{u}$$

for every $t \in (0, 1)$, we claim that there exist matrices

$$D_t \in R_S^{n \times n}, \quad A_t \in R^{m \times n}$$

and vectors $c_t \in R^n$, $b_t \in R^m$ such that

$$\max\{\|D_t - D\|, \|A_t - A\|, \|c_t - c\|, \|b_t - b\|\} \to 0$$

as $t \to 0$, and

$$\begin{aligned} &D_t x_t - A_t^T u_t - F^T 0 + c_t = 0, \\ &A_t x_t \geq b_t, \quad u_t \geq 0, \\ &F x_t \geq d, \\ &u_t^T (A_t x_t - b_t) + 0^T (F x_t - d) = 0. \end{aligned}$$

The matrices D_t, A_t and vectors c_t, b_t are defined by (10.61) and (10.62), where D_0, A_0, c_0, b_0 are constructed as in the proof of Theorem 10.7. Using (10.68)–(10.71) and arguing similarly as in the preceding proof we shall arrive at a contradiction. □

The following theorem gives three sufficient conditions for the upper semicontinuity of the multifunction (10.50). These conditions express some requirements on the behavior of the quadratic form $x^T D x$ on the cone $\Delta(A, 0) \cap \Delta(F, 0)$ and the position of the vector (b, d) relatively to the set $\text{int}(\Lambda[A, F])^*$.

Theorem 10.9. *Suppose that one of the following three pairs of conditions*

$$\text{Sol}(D, A, 0, 0, F, 0) = \{0\}, \quad (b, d) \in \text{int}\,(\Lambda[A, F])^*, \tag{10.72}$$

$$\text{Sol}(-D, A, 0, 0, F, 0) = \{0\}, \quad (b, d) \in -\text{int}\,(\Lambda[A, F])^*, \tag{10.73}$$

and

$$S(D, A, 0, 0, F, 0) = \{0\}, \quad \text{int}\,(\Lambda[A, F])^* = R^m \times R^s, \tag{10.74}$$

is satisfied. Then, for any $c \in R^n$ (and also for any $b \in R^m$ if (10.74) takes place), the multifunction $p' \mapsto S(p', F, d)$, where $p' = (D', A', c', b')$, is upper semicontinuous at $p = (D, A, c, b)$.

Proof. On the contrary, suppose that one of the three pairs of conditions (10.72)–(10.74) is satisfied but, for some $c \in R^n$ (and also for some $b \in R^m$ if (10.74) takes place), the multifunction $p' \mapsto S(p', F, d)$ is not usc at $p = (D, A, c, b)$. Then there exist an open subset $\Omega \subset R^n$ containing $S(p, F, d)$, a sequence $p^k = (D^k, A^k, c^k, b^k)$ converging to p in $R_S^{n \times n} \times R^{m \times n} \times R^n \times R^m$, and a sequence $\{x^k\}$ such that, for each k, $x^k \in S(p^k, F, d)$ and $x^k \notin \Omega$. By the definition of KKT point, for each k there exists a pair $(u^k, v^k) \in R^m \times R^s$ such that

$$D^k x^k - (A^k)^T u^k - F^T v^k + c^k = 0, \tag{10.75}$$

$$A^k x^k \geq b^k, \quad u^k \geq 0, \tag{10.76}$$

$$Fx^k \geq d, \quad v^k \geq 0, \tag{10.77}$$

$$(u^k)^T(A^k x^k - b^k) + (v^k)^T(Fx^k - d) = 0. \tag{10.78}$$

If the sequence $\{(x^k, u^k, v^k)\}$ is bounded, then the sequences $\{x^k\}$, $\{u^k\}$ and $\{v^k\}$ are also bounded. Therefore, without loss of generality, we can assume that the sequences $\{x^k\}$, $\{u^k\}$ and $\{v^k\}$ converge, respectively, to some points $x^0 \in R^n$, $u^0 \in R^m$ and $v^0 \in R^s$, as $k \to \infty$. Letting $k \to \infty$, from (10.75)–(10.78) we get

$$\begin{aligned}
&Dx^0 - A^T u - F^T v + c = 0,\\
&Ax^0 \geq b, \quad u^0 \geq 0,\\
&Fx^0 \geq d, \quad v^0 \geq 0,\\
&(u^0)^T(Ax^0 - b) + (v^0)^T(Fx^0 - d) = 0.
\end{aligned}$$

Hence $x^0 \in S(p, F, d) \subset \Omega$. On the other hand, since $x^k \notin \Omega$ for each k, we must have $x^0 \notin \Omega$, a contradiction. We have thus shown that the sequence $\{(x^k, u^k, v^k)\}$ must be unbounded. By considering a subsequence, if necessary, we can assume that $\|(x^k, u^k, v^k)\| \to \infty$ and, in addition, $\|(x^k, u^k, v^k)\| \neq 0$ for all k. Since the sequence of vectors

$$\frac{(x^k, u^k, v^k)}{\|(x^k, u^k, v^k)\|} = \left(\frac{x^k}{\|(x^k, u^k, v^k)\|}, \frac{u^k}{\|(x^k, u^k, v^k)\|}, \frac{v^k}{\|(x^k, u^k, v^k)\|} \right)$$

is bounded, it has a convergent subsequence. Without loss of generality, we can assume that

$$\frac{(x^k, u^k, v^k)}{\|(x^k, u^k, v^k)\|} \to (\bar{x}, \bar{u}, \bar{v}) \in R^n \times R^m \times R^s, \quad \|(\bar{x}, \bar{u}, \bar{v})\| = 1. \tag{10.79}$$

Dividing both sides of (10.75), (10.76) and (10.77) by $\|(x^k, u^k, v^k)\|$, both sides of (10.78) by $\|(x^k, u^k, v^k)\|^2$, and letting $k \to \infty$, by (10.79) we obtain

$$D\bar{x} - A^T\bar{u} - F^T\bar{v} = 0, \tag{10.80}$$

$$A\bar{x} \geq 0, \ \bar{u} \geq 0, \tag{10.81}$$

$$F\bar{x} \geq 0, \ \bar{v} \geq 0, \tag{10.82}$$

$$\bar{u}^T A\bar{x} + \bar{v}^T F\bar{x} = 0. \tag{10.83}$$

We first consider the case where (10.72) is fulfilled. It is evident that (10.80)–(10.83) imply

$$\bar{x}^T D\bar{x} = 0, \quad A\bar{x} \geq 0, \quad F\bar{x} \geq 0. \tag{10.84}$$

If $\bar{x} \neq 0$ then, taking account of the fact that the constraint set $\Delta(A,0) \cap \Delta(F,0)$ of $QP(D,A,0,0,F,0)$ is a cone, one can deduce from (10.84) that either $\mathrm{Sol}(D,A,0,0,F,0) = \emptyset$ or

$$\bar{x} \in \mathrm{Sol}(D,A,0,0,F,0).$$

This contradicts the first condition in (10.72). Thus $\bar{x} = 0$. Then it follows from (10.80)–(10.83) that $(\bar{u},\bar{v}) \in \Lambda[A,F] \setminus \{(0,0)\}$. Since $(b,d) \in \mathrm{int}\,(\Lambda[A,F])^*$ by (10.72),

$$\bar{u}^T b + \bar{v}^T d < 0. \tag{10.85}$$

Consider the sequence $\{(u^k)^T b^k + (v^k)^T d\}$. By (10.75) and (10.78),

$$(x^k)^T D^k x^k + (c^k)^T x^k = (u^k)^T b^k + (v^k)^T d. \tag{10.86}$$

If for each positive integer i there exists an integer k_i such that $k_i > i$ and

$$(u^{k_i})^T b^{k_i} + (v^{k_i})^T d > 0 \tag{10.87}$$

then, dividing both sides of (10.87) by $\|(x^{k_i}, u^{k_i}, v^{k_i})\|$ and letting $i \to \infty$, we have

$$\bar{u}^T b + \bar{v}^T d \geq 0,$$

contrary to (10.85). Consequently, there must exist a positive integer i_0 such that

$$(u^k)^T b^k + (v^k)^T d \leq 0 \quad \text{for every } k \geq i_0. \tag{10.88}$$

If the sequence $\{x^k\}$ is bounded then, dividing both sides of (10.86) by $\|(x^k, u^k, v^k)\|$ and letting $k \to \infty$, we get $\bar{u}^T b + v^T d = 0$, contrary to (10.85). Thus $\{x^k\}$ is unbounded. We can assume that $\|x^k\| \to \infty$ and $\|x^k\| \neq 0$ for each k. Then $\left\{ \frac{x^k}{\{x^k\|} \right\}$ is bounded. We can assume that

$$\frac{x^k}{\|x^k\|} \to \hat{x} \quad \text{with } \|\hat{x}\| = 1.$$

Combining (10.86) with (10.88) gives

$$(x^T)^k D^k x^k + (c^k)^T x^k \leq 0 \quad \text{for every } k \geq i_0. \tag{10.89}$$

Dividing both sides of (10.89) by $\|x^k\|^2$ and letting $k \to \infty$, we obtain

$$\hat{x}^T D \hat{x} \leq 0. \tag{10.90}$$

By (10.76) and (10.77),

$$A^k x^k \geq b^k, \quad F x^k \geq d.$$

Dividing both sides of each of the last inequalities by $\|x^k\|$ and letting $k \to \infty$, we have

$$A\hat{x} \geq 0, \quad F\hat{x} \geq 0. \tag{10.91}$$

Combining (10.90) with (10.91), we can assert that

$$\mathrm{Sol}(D, A, 0, 0, F, 0) \neq \{0\},$$

contrary to the first condition in (10.72). Thus we have proved the theorem for the case where (10.72) is fulfilled.

Now we turn to the case where condition (10.73) is fulfilled. We deduce (10.84) from (10.80)–(10.83). If $\bar{x} \neq 0$ then from (10.84) we get $\mathrm{Sol}(-D, A, 0, 0, F, 0) \neq \{0\}$, which contradicts the first condition in (10.73). Thus $\bar{x} = 0$. From (10.80)–(10.83) it follows that $(\bar{u}, \bar{v}) \in \Lambda[A, F] \setminus \{(0, 0)\}$. By the second condition in (10.73),

$$\bar{u}^T b + v^T d > 0. \tag{10.92}$$

Consider the sequence $\{(u^k)^T b^k + (v^k)^T d\}$. We have (10.86). If there exists a positive integer i_0 such that (10.88) is valid then, dividing both sides of (10.88) by $\|(x^k, u^k, v^k)\|$ and letting $k \to \infty$, we obtain $\bar{u}^T b + \bar{v}^T d \leq 0$, contrary to (10.92). Therefore, for each positive integer i, one can find an integer $k_i > i$ such that (10.87) holds. If the sequence $\{x^k\}$ is bounded then, dividing both sides of (10.86) by $\|(x^k, u^k, v^k)\|$ and letting $k \to \infty$, we have $\bar{u}^T b + \bar{v}^T d = 0$, contrary to (10.92). Thus the sequence $\{x^k\}$ is unbounded. We can assume that $\|x^k\| \to \infty$ and $\|x^k\| \neq 0$ for all k. Since the sequence $\left\{\frac{x^k}{\|x^k\|}\right\}$ is well defined and bounded, without loss of generality, we can assume that

$$\frac{x^k}{\|x^k\|} \to \hat{x} \quad \text{with } \|\hat{x}\| = 1.$$

Combining (10.86) with (10.87) gives

$$(x^{k_i})^T D^{k_i} x^{k_i} + (c^{k_i})^T x^{k_i} > 0 \quad \text{for all } i. \tag{10.93}$$

Dividing both sides of (10.93) by $\|x^{k_i}\|^2$ and letting $i \to \infty$, we obtain $\hat{x}^T D\hat{x} \geq 0$ or, equivalently,

$$\hat{x}^T(-D)\hat{x} \leq 0. \tag{10.94}$$

By (10.76) and (10.77),

$$A^{k_i}x^{k_i} \geq b^{k_i}, \quad Fx^{k_i} \geq d. \tag{10.95}$$

Dividing both sides of each of the inequalities in (10.95) by $\|x^{k_i}\|$ and letting $i \to \infty$, we have

$$A\hat{x} \geq 0, \quad F\hat{x} \geq 0. \tag{10.96}$$

Combining (10.94) with (10.96) yields $\mathrm{Sol}(-D, A, 0, 0, F, 0) \neq \{0\}$, contrary to the first condition in (10.73). This proves the theorem in the case where (10.73) is fulfilled.

Now let us consider the last case where (10.74) is assumed. From (10.80)–(10.83) we have $\bar{x} \in S(D, A, 0, 0, F, 0)$. By the first condition in (10.73), $\bar{x} = 0$. Then it follows from (10.80)–(10.83) that

$$A^T\bar{u} + F^T\bar{v} = 0, \quad \bar{u} \geq 0, \quad \bar{v} \geq 0, \quad \|(0, \bar{u}, \bar{v})\| = 1.$$

Therefore, $(\bar{u}, \bar{v}) \in \Lambda[A, F] \setminus \{(0,0)\}$. Since $\bar{u}^T\bar{u} + \bar{v}^T\bar{v} > 0$, then $(\bar{u}, \bar{v}) \notin \mathrm{int}(\Lambda[A, F])^*$. This contradicts the second condition in (10.74).

We have thus proved that if one of the pairs of conditions (10.72)–(10.74) is fulfilled, then the conclusion of the theorem must hold true. □

We now proceed to show how the sufficient conditions (10.72) and (10.73) look like in the case of the canonical problem (10.1). As in Section 10.4, for any $A \in R^{n \times n}$, $\Lambda[A] = \{\lambda \in R^m : -A^T\lambda \geq 0,\ \lambda \geq 0\}$. We have

$$\mathrm{int}(\Lambda[A])^* = \{\xi \in R^m : \lambda^T\xi < 0 \quad \forall \lambda \in \Lambda[A] \setminus \{0\}\}.$$

Lemma 10.3. *Suppose that, in problem* (10.48), $s = n,\ d = 0$, *and* F *is the unit matrix in* $R^{n \times n}$. *Then the following statements hold:*

(a_1) *If* $b \in \mathrm{int}\,(\Lambda[A])^*$ *then* $(b, 0) \in \mathrm{int}\,(\Lambda[A, F])^*$;

(a_2) *If* $\mathrm{Sol}\,(D, A, 0, 0) = \{0\}$ *then* $\mathrm{Sol}\,(D, A, 0, 0, F, 0) = \{0\}$;

(a_3) *If* $b \in -\mathrm{int}\,(\Lambda[A])^*$ *then* $(b, 0) \in -\mathrm{int}\,(\Lambda[A, F])^*$;

(a_4) *If* $\mathrm{Sol}\,(-D, A, 0, 0) = \{0\}$ *then* $\mathrm{Sol}\,(-D, A, 0, 0, F, 0) = \{0\}$.

Proof. If $b \in \mathrm{int}(\Lambda[A])^*$ then

$$\lambda^T b < 0 \quad \text{for all } \lambda \in \Lambda[A] \setminus \{0\}. \tag{10.97}$$

For any $(u, v) \in \Lambda[A, F] \setminus \{0\}$ we have

$$A^T u + F^T v = 0, \quad u \geq 0, \quad v \geq 0.$$

This yields

$$-A^T u = v \geq 0, \quad u \geq 0, \quad u \neq 0,$$

hence $u \in \Lambda[A] \setminus \{0\}$. By (10.97), $b^T u + 0^T v = b^T u = u^T b < 0$. This shows that $(b, 0) \in \mathrm{int}(\Lambda[A, F])^*$. Statement (a_1) has been proved. It is clear that (a_3) follows from (a_1).

For proving (a_2) and (a_4) it suffices to note that, under our assumptions,

$$\mathrm{Sol}(D, A, 0, 0) = \mathrm{Sol}(D, A, 0, 0, F, 0)$$

and

$$\mathrm{Sol}(-D, A, 0, 0) = \mathrm{Sol}(-D, A, 0, 0, F, 0).$$

□

We check at once that Theorems 10.4 and 10.5 follow from Theorem 10.10 and Lemma 10.3.

10.7 Commentaries

The material of this chapter is taken from Tam and Yen (1999, 2000), Tam (2001a).

Several authors have made efforts in studying stability properties of the QP problems. Daniel (1973) established some basic facts about the solution stability of a QP problem whose objective function is a positive definite quadratic form. Guddat (1976) studied continuity properties of the solution set of a convex QP problem. Robinson (1979) obtained a fundamental result (see Theorem 7.6 in Chapter 7) on the stable behavior of the solution set of a monotone affine generalized equation (an affine variational inequality in the terminology of Gowda and Pang (1994), which yields a fact on the Lipschitz continuity of the solution set of a convex QP problem. Best and Chakravarti (1990) obtained some results on the continuity and differentiability of the optimal value function in a perturbed convex QP problem. By using the linear complementarity theory, Cottle, Pang and Stone (1992), studied in detail the stability of convex QP problems. Best and Ding (1995) proved a result on the continuity of the optimal value function in a convex QP problem. Auslender and Coutat (1996) established some results on stability and differential stability of generalized linear-quadratic programs, which include convex QP problems as a special case. Several attempts have been made to study the stability of nonconvex

QP problems (see, for instance, Klatte (1985), Tam (1999), Tam (2001a, 2001b, 2002)).

The proof of Theorem 10.1 is based on a construction developed by Oettli and Yen (1995, 1996a) for linear complementarity problems, homogeneous equilibrium problems, and quasi-complementarity problems.