BOUNDED (HAUSDORFF) CONVERGENCE: BASIC FACTS AND APPLICATIONS

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Abstract: We present a survey of some uses of a remarkable convergence on families of sets or functions. We evoke some of its applications and stress some calculus rules. The main novelty lies in the use of a notion of "firm" (or uniform) asymptotic cone to an unbounded subset of a normed space. This notion yields criteria for the study of boundedness properties.

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1. INTRODUCTION

It is the purpose of this paper to survey some properties of a convergence on sets and functions which has received a great deal of interest during the last two decades. We review some of its applications and show why this convergence is convenient. However, we leave apart the application to Hamilton–Jacobi equations which are dealt with in [63]. We also observe that when restricted to the space of continuous linear functions on a normed vector space X the convergence we consider reduces to convergence for the dual norm; this fact (and the abundance of terminologies) suggests to call this convergence "bounded convergence" or, in short, "b-convergence".

One of the reasons of the success of this convergence lies in its compatibility with the usual operations, provided some technical assumptions reminiscent to constraint qualification conditions in mathematical programming are satisfied. Such conditions already appeared in [44] in the finite dimensional case and in our very first investigations about this question which motivated our interest ([19], [53]); see also [4], [22], [25], [26], [38], [55], [56], [69], [72]. These assumptions involve openness or boundedness conditions. This fact justifies the focus we give to such questions.

The main novelty of the present paper is in the use of a concept of asymptotic cone introduced in [60] which bears some uniformity with respect to directions in a way reminiscent of the uniformity with respect to directions which is involved in the notion of Fréchet derivative (or semi-derivative [45], [50], also called B-derivative) or in the notion of Fréchet cone in the sense of [31], [33]. This concept replaces asymptotic compactness conditions which were used in [62].

As in [62], our methods are essentially geometric. Given an operation $*$ and some sort of variational convergence, in order to prove that $(f_n * g_n) \to f * g$ whenever the sequences of functions (f_n) and (g_n) are such that $(f_n) \to f$, $(g_n) \to g$, we reduce this question to several problems of set convergence: images, intersections, products. Each of these set-theoretical results yields a rule for convergence of functions. In particular, convergence of performance functions and of infimal convolutions are deduced from convergence of images (or sums) of sets. Such a study may have been conducted for other convergences, for instance the ones considered in [4], [9], [20], [24], [38], [42], [70], [72]. However, we believe bounded convergence is appropriate in such a respect and we do not look for completeness.

Other applications could benefit from our analysis. Regularization properties and well-posedness results are already considered in [26], [57]–[59], [61]; more attention could be given to nonconvex cases and to asymptotic methods.

The paper is organized as follows. The next section is devoted to preliminary material about convergences. The main novelties are contained in Section 4: conical enlargements, an expansion property and a notion of disjointness at infinity for non convex sets. Section 4 is also focused on the new notion of firm asymptotic cone to a subset of a normed vector space (n.v.s.). There this tool is applied to boundedness properties. These properties may play a role in obtaining a priori estimates for solving equations. They are crucial for ensuring that convergence properties of families of sets or functions are preserved under usual operations; a short account of this topic is given in section 5. Such properties are used in [63] to obtain stability and persistence properties of explicit solutions to first order Hamilton–Jacobi equations. Other applications to the convergence of functions are presented in [62] and in [73] where integral functionals and well-posedness questions are considered. In section 3 we evoke some other applications.

2. BOUNDED CONVERGENCE

Throughout this paper, unless otherwise stated, X and Y are real normed vector spaces (n.v.s.), U_X (resp. B_X) is the open (resp. closed) unit ball of X and S_X is the unit sphere in X. The closed (resp. open) ball with center x and radius r is denoted by $B(x,r)$ (resp. $U(x,r)$). For a subset A of X, *intA*, *cℓA* stand for the interior and the closure of A respectively. The product space $X \times Y$ is equipped with the max norm. In particular, one has $U_{X\times Y} = U_X \times U_Y$, $B_{X\times Y} = B_X \times B_Y$. The distance of $x \in X$ to a subset E of X is $d(x,E) := \inf\{d(x,w) : w \in E\}$, with $d(x,\varnothing) := \infty$. The *remoteness* of E is $d(0,E)$. We denote by $\mathbb{P}$ (resp. $\mathbb{R}_+$) the set of positive (resp. nonnegative) numbers.

Recall (see [3], [13], [24], [69]...) that a sequence (A_n) of subsets of X is said to *converge* to a subset A of X in the sense of *Painlevé–Kuratowski* if $\limsup_n A_n = A = \liminf_n A_n$, where $\limsup_n A_n$ is the set of limits of sequences (x_n) such that $x_k \in A_k$ for k in an infinite subset K of $\mathbb{N}$ and $\liminf_n A_n$ is the set of limits of sequences (x_n) such that $x_n \in A_n$ for each $n \in \mathbb{N}$. We write $(A_n) \to A$. Here we focus our attention to a somewhat stronger notion. It requires the definition of the *excess* of a subset A of X over another subset B of X which is given by

$$e(A,B) := \sup_{a\in A} d(a,B) \quad \text{if } A,B \neq \varnothing,$$

with $e(A,\varnothing) = \infty$ if $A \neq \varnothing$ and $e(\varnothing,B) = 0$ for any B. Then, for $p \in \mathbb{P}$, we set

$$e_p(A,B) := e(A \cap pU_X, B), \quad d_p(A,B) := \max(e_p(A,B), e_p(B,A)).$$

It is convenient to write symbolically $A \subset b-\liminf_n A_n$ if, for each $p \in \mathbb{P}$, $\left(e_p(A,A_n)\right)_n \to 0$ as $n \to \infty$ and $A \supset b$-$\limsup_n A_n$ if $\left(e_p(A_n,A)\right)_n \to 0$ for each $p \in \mathbb{P}$. We write $(A_n) \xrightarrow{b} A$ and we say that (A_n) *boundedly converges* (or b-converges) to A or that (A_n) converges to A for the *bounded (Hausdorff) topology* if $A \subset b-\lim\inf_n A_n$ and b-$\lim\sup_n A_n \subset A$. Let us note that $c\ell A \subset \lim\inf_n A_n$ whenever $A \subset b$-$\lim\inf_n A$ since then $A \subset \lim\inf_n A_n$ and since $\lim\inf_n A_n$ is closed. On the other hand, when $A \supset b$-$\lim\sup_n A_n$ then $c\ell A \supset \lim\sup_n A_n$. Thus, we get that $(A_n) \to c\ell A$ when $(A_n) \xrightarrow{b} A$. If X is finite dimensional, the reverse implication holds. The choice of the open unit ball of X in what precedes, rather than the closed unit ball, enables one to use the equalities

$$e_p(c\ell A, B) = e_p(A, B) = e_p(A, c\ell B) = e_p(c\ell A, c\ell B).$$

These equalities show that we could restrict our attention to the case the limit set is closed; then we get uniqueness of the set A such that $(A_n) \xrightarrow{b} A$ and we can write $A = b$-$\lim_n A_n$.

As for other variational convergences, one can pass from these convergences of sets to convergences of functions. Denoting by epi$f$ the epigraph of f, we set $e_p(f,g) := e_p(\text{epi} f, \text{epi}\, g)$. Accordingly, for a sequence (f_n) of functions from X to $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and a function f on X, we write $f \geq b-\lim\sup_n f_n$ if $\text{epi} f \subset b-\lim\inf_n (\text{epi} f_n)$ and $f \leq b-\lim\inf_n f_n$ if $\text{epi} f \supset b$-$\lim\sup(\text{epi} f_n)$. Of course, writing $(f_n) \xrightarrow{b} f$ when $(\text{epi} f_n) \xrightarrow{b} \text{epi} f$ means that $f \leq b$-$\lim\inf_n f_n$ and $f \geq b$-$\lim\sup_n f_n$; we say that (f_n) *b-converges* to f. This type of convergence which has been thoroughly studied in [4]-[6], [8]-[12], [18]-[26], [32], [38], [43], [54]–[58], [68]–[72]... is also called the *Attouch–Wets convergence, the bounded Hausdorff convergence* and the *epidistance convergence*; this last term is justified by the fact that b-convergence on the space $\mathcal{P}_c(X)$ of closed nonempty subsets of X arises from the distance d given by

$$d(A,B) := \sum_{p=1}^{\infty} 2^{-p} \min\{d_p(A,B), 1\}, \qquad A, B \in \mathcal{P}_c(X),$$

where $d_p(A,B) := \max(e_p(A,B), e_p(B,A))$ (see [5], [24]). This convergence has been studied (in Hilbert spaces) in analytical terms through the Moreau regularization in [8]. Pioneering contributions in this vein are due to Choquet, Moreau [46], Mosco [47]; the case of cones is considered in [28], [31], [33], [35].

A convenient way of expressing that a sequence (A_n) of subsets of X b-converges to A is: for any bounded sequence (a_n) of A one has $(d(a_n, A_n)) \to 0$ and for any bounded sequence (a_n) of X such that $a_n \in A_n$ for n large enough one has $(d(a_n, A)) \to 0$ (see [71]).

The following result shows how natural bounded convergence is; it also justifies the simplification of terminology we suggest.

Proposition 1. *Let* $f, f_n \in X^*$ $(n \in \mathbb{N})$. *Then*

$$\operatorname{epi} f \subset b\text{-}\liminf(\operatorname{epi} f) \Leftrightarrow \|f - f_n\| \to 0 \Leftrightarrow \operatorname{epi} f = b\text{-}\lim(\operatorname{epi} f_n).$$

Proof. Assume that $\operatorname{epi} f \subset b\text{-}\liminf(\operatorname{epi} f_n)$. Let $0 < \varepsilon < \rho < 1$. For every $n \in \mathbb{N}$ there exists $x_n \in U_X$ such that $\rho\|f_n\| \le \langle x_n, f_n \rangle$. Because the sequence $((x_n, \langle x_n, f \rangle))$ is bounded, it follows that $d((x_n, \langle x_n, f \rangle), \operatorname{epi} f_n) \to 0$. Hence there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \ge n_\varepsilon$ there exists $(u_n, t_n) \in \operatorname{epi} f_n$ with $\|x_n - u_n\| \le \varepsilon$ and $|\langle x_n, f \rangle - t_n| \le \varepsilon$. It follows that

$$\begin{aligned}\rho\|f_n\| &\le \langle x_n, f_n \rangle \le \langle x_n, f_n \rangle - \langle u_n, f_n \rangle + t_n - \langle x_n, f \rangle + \langle x_n, f \rangle \\ &\le \|f_n\| \cdot \|x_n - u_n\| + \varepsilon + \|f\| \le \varepsilon\|f_n\| + \varepsilon + \|f\|,\end{aligned}$$

and so $(\rho - \varepsilon)\|f_n\| \le \varepsilon + \|f\|$ for $n \ge n_\varepsilon$. Hence $(\rho - \varepsilon)\limsup\|f_n\| \le \varepsilon + \|f\|$. As ε and ρ are arbitrary such that $0 < \varepsilon < \rho < 1$, we obtain that $\limsup\|f_n\| \le \|f\|$. Now, let $(\rho_n) \uparrow 1$ and $(x_n) \subset U_X$ be such that $\rho_n\|f_n - f\| \le (f_n - f)(x_n)$ for every n. Once again, because the sequence $((x_n, \langle x_n, f \rangle))$ is bounded, we have that $d((x_n, \langle x_n, f \rangle), \operatorname{epi} f_n) \to 0$; there exists $((u_n, t_n)) \subset X$ such that $\langle u_n, f_n \rangle \le t_n$ for every n, $(\|x_n - u_n\|) \to 0$ and $(\langle x_n, f \rangle - t_n) \to 0$. But

$$\begin{aligned}\rho_n\|f_n - f\| &\le \langle x_n, f_n - f \rangle \le \langle x_n, f_n \rangle - \langle u_n, f_n \rangle + t_n - \langle x_n, f \rangle \\ &\le \|f_n\| \cdot \|x_n - u_n\| + t_n - \langle x_n, f \rangle.\end{aligned}$$

Since (f_n) is bounded, it follows that $(\|f_n - f\|) \to 0$. Assume now that $(\|f_n - f\|) \to 0$. Let $((x_n, t_n)) \subset \operatorname{epi} f$ be bounded; in particular, (x_n) is bounded. Let $s_n := \max\{t_n, \langle x_n, f_n \rangle\}$; of course, $(x_n, s_n) \in \operatorname{epi} f_n$. Then

$$\begin{aligned}d((x_n, t_n), \operatorname{epi} f_n) &\le \|(x_n, t_n) - (x_n, \Delta_n)\| = s_n - t_n = (\langle x_n, f_n \rangle - t_n)_+ \\ &\le (\langle x_n, f_n \rangle - \langle x_n, f \rangle)_+ + (\langle x_n, f \rangle - t_n)_+ \le \|f - f_n\| \cdot \|x_n\| \to 0.\end{aligned}$$

Hence $\operatorname{epi} f \subset b\text{-}\liminf(\operatorname{epi} f_n)$. Let now $((x_n, s_n))$ be bounded such that $(x_n, s_n) \in \operatorname{epi} f_n$ for every n; in particular (x_n) is bounded. Let $t_n := \max\{s_n, \langle x_n, f \rangle\}$; of course, $(x_n, t_n) \in \operatorname{epi} f$. Then

$$d((x_n,s_n),\operatorname{epi} f) \le \|(x_n,s_n)-(x_n,t_n)\| = t_n - s_n = (\langle x_n, f\rangle - s_n)_+$$
$$\le (\langle x_n, f\rangle - \langle x_n, f_n\rangle)_+ + (\langle x_n, f_n\rangle - s_n)_+ \le \|f - f_n\| \|x_n\| \to 0.$$

Hence $\operatorname{epi} f \supset b - \limsup(\operatorname{epi} f_n)$. □

The preceding result can be transposed to a somewhat more general (and in fact different) case. Here b-convergence of a sequence of operators means b-convergence of their graphs and $e_p(S,T) := e_p(\operatorname{gph} S, \operatorname{gph} T)$.

Proposition 2. *Let X,Y be normed vector spaces and $T,T_n : X \to Y$ $(n \in \mathbb{N})$ be continuous linear operators. Then*

$$\operatorname{gph} T \subset b - \liminf(\operatorname{gph} T_n) \Leftrightarrow \|T_n - T\| \to 0 \Leftrightarrow \operatorname{gph} T = b\text{-}\lim(\operatorname{gph} T_n).$$

Proof. As elsewhere in the paper, the product space $X \times Y$ is endowed with the box norm. Assume that $\operatorname{gph} T \subset b - \liminf(\operatorname{gph} T_n)$. Let $0 < \varepsilon < \rho < 1$. For every $n \in \mathbb{N}$ there exists $x_n \in U_X$ such that $\rho\|T_n\| \le \|T_n x_n\|$. Because the sequence $((x_n, Tx_n))$ is bounded, it follows that $d((x_n,Tx_n), \operatorname{gph} T_n) \to 0$. Hence there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \ge n_\varepsilon$ there exists $u_n \in X$ with $\|x_n - u_n\| \le \varepsilon$ and $\|Tx_n - T_n u_n\| \le \varepsilon$. It follows that

$$\rho\|T_n\| \le \|T_n x_n\| \le \|T_n x_n - T_n u_n\| + \|T_n u_n - Tx_n\| + \|Tx_n\|$$
$$\le \|T_n\| \|x_n - u_n\| + \|T_n u_n - Tx_n\| + \|T\| \le \varepsilon\|T_n\| + \varepsilon + \|T\|,$$

and so $(\rho - \varepsilon)\|T_n\| \le \varepsilon + \|T\|$ for $n \ge n_\varepsilon$. Hence $(\rho-\varepsilon)\limsup\|T_n\| \le \varepsilon + \|T\|$. Since ε and $1-\rho$ are arbitrarily close to 0, we obtain that $\limsup\|T_n\| \le \|T\|$. Now, let $(\rho_n) \uparrow 1$ and (x_n) in U_X be such that $\rho_n\|T_n - T\| \le \|(T_n - T)x_n\|$ for every n. Once again, because the sequence $((x_n, Tx_n))$ is bounded, we have that $d((x_n,Tx_n), \operatorname{gph} T_n) \to 0$; there exists $(u_n) \subset X$ such that $(\|x_n - u_n\|) \to 0$ and $(\|Tx_n - T_n u_n\|) \to 0$. But

$$\rho_n\|T_n - T\| \le \|(T_n - T)x_n\| \le \|T_n x_n - T_n u_n\| + \|T_n u_n - Tx_n\|$$
$$\le \|T_n\| \|x_n - u_n\| + \|T_n u_n - Tx_n\|.$$

Since (T_n) is bounded, it follows that $(\|T_n - T\|) \to 0$.

Now assume that $(\|T_n - T\|) \to 0$. Let $((x_n, Tx_n))$ be bounded (or equivalently, (x_n) be bounded). Then

$$d((x_n,Tx_n),\text{gph } T_n) \le \|(x_n,Tx_n)-(x_n,T_nx_n)\| = \|Tx_n - T_nx_n\| \le \|T-T_n\|\cdot\|x_n\| \to 0.$$

Hence $\text{gph } T \subset b\text{-}\liminf(\text{gph } T_n)$. Let now $((x_n,T_nx_n))$ be bounded (or equivalently, (x_n) be bounded). Then

$$d((x_n,T_nx_n),\text{gph } T) \le \|(x_n,T_nx_n)-(x_n,Tx_n)\| = \|T_nx_n - Tx_n\| \le \|T_n-T\|\cdot\|x_n\| \to 0.$$

Hence $\text{gph } T \supset b\text{-}\limsup(\text{gph } T_n)$. □

As noted in [62], b-convergence is a stringent condition. Therefore, it may be advisable to use compromises with weaker convergence notions, as done in [53], [4], [38]. For simplicity, we do not do that here.

3. APPLICATIONS

We devote the present section to some illustrations of the uses of bounded convergence; we just give a sample. We refer to [6], [7], [22], [24], [30], [49], [63], [73] for other applications.

3.1 Reinforced tangency

In [2] and its references, approximations of a subset E of a n.v.s. X around one of its points are considered. Outer firm approximations C of E at $e \in E$ are obtained in requiring that

$$C \supset b\text{-}\limsup_{t\to 0_+} \frac{1}{t}(E-e).$$

Clearly, such a set C, when closed, contains the tangent cone $T(E,e) = \limsup_{t\to 0_+} t^{-1}(E-e)$; but it enjoys better properties. In [17] (see also [16]), a notion of equicirca-tangent cone is introduced in order to prove open mapping theorems for multimappings. It involves a notion akin to

$$b-\liminf_{(t,e')\to(0_+,e),\, e'\in E} \frac{1}{t}(E-e').$$

Reinforced asymptotic approximation properties which bear some analogy with the preceding reinforced tangency will be considered later on.

Similar notions of approximations for functions can be defined and used.

3.2 Nonlinear conditioning and perturbations

It is not difficult to see that the functional $f \mapsto m_f := \inf f(X)$ from $\overline{\mathbb{R}}^X$ to $\overline{\mathbb{R}}$ is upper semicontinuous when $\overline{\mathbb{R}}^X$ is endowed with the topology associated with b-convergence. A more precise and quantitative result can be given. Given $f : X \to \overline{\mathbb{R}}$ such that $m_f := \inf f(X) \in \mathbb{R}$ and $S_f := \arg\min f \neq \emptyset$, a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ is said to be a *conditioner* for f if $\varphi(0) = 0$ and

$$\forall x \in X : d(x, S_f) \leq \varphi(f(x) - m_f).$$

f is said to be well-set if it has a conditioner which is a modulus (i.e. $\varphi(t) \to 0$ as $t \to 0$).

The following statement shows that one only gets a one-sided perturbation result for the set of minimizers. Other results are given in [11].

Theorem 3. ([57]) *Suppose S_f is nonempty and bounded. Suppose f is well-set, with an usc conditioner φ. Then there exists $r > 0$ and $\delta > 0$ such that for any function $g : X \to \mathbb{R} \cup \{\infty\}$ whose sublevel sets are connected satisfying $d_r(f, g) < \delta$ one has*

$$|m_g - m_f| \leq d_r(f, g), \quad e(S_g, S_f) \leq d_r(f, g) + \varphi(2d_r(f, g)).$$

3.3 Convergence of fixed points

Following Aubin, given $\lambda \in \mathbb{P}$, a complete metric space (X, d), and a nonempty subset U of X, one says that $F : X \rightrightarrows X$ is *pseudo-λ-Lipschitzian with respect to U* if

$$e(F(x) \cap U, F(x')) \leq \lambda d(x, x') \quad \forall x, x' \in U.$$

The following existence result is close to the Nadler fixed point theorem [48]. However, here we use the preceding weakening of the notion of Lipschitzian multimapping.

Proposition 4. ([21]) *Let $F : X \rightrightarrows X$ be a multimapping with closed values which is assumed to be pseudo-λ-Lispchitzian with respect to some ball $U(x_0, r)$ with $\lambda \in (0,1)$, $r > (1-\lambda)^{-1} d(x_0, F(x_0))$. Then the set Φ_F of fixed points of F is nonempty and*

$$d(x_0,\Phi_F) \leq (1-\lambda)^{-1} d(x_0, F(x_0)).$$

The following result gives a measure of the variation of the sets of fixed points of multimappings in terms of the variation of the graphs. Again it is a one-sided result. In [21] this result is applied to the variations of the sets of solutions to a differential inclusion.

Proposition 5. ([21]) *Let $F : X \rightrightarrows X$ be a multimapping with closed values which is pseudo-λ-Lispchitzian with respect to $U(x_0,r)$, and $\lambda \in (0,1)$. Then for any $s \in (0,r)$ and for any $G : X \rightrightarrows X$ with $e_s(G,F) < (1-\lambda)(1+\lambda)^{-1}(r-s)$, one has*

$$e_s(\Phi_G, \Phi_F) \leq (1-\lambda)^{-1}(1+\lambda) e_s(G,F).$$

3.4 Continuity of the Fenchel transform

In the sequel we denote by $\mathcal{F}(X)$ the set of proper lsc functions on X with values in $\mathbb{R} \cup \{+\infty\}$. The Fenchel–Legendre conjugate of $f \in \mathcal{F}(X)$ is

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(x^*) = \sup_{x \in X}(\langle x^*, x\rangle - f(x)),$$

where X^* is the topological dual of X. The continuity of the transform $f \mapsto f^*$ is important for a number of applications ([22], [27], [32], [63]...). It has been mostly studied under convexity assumptions.

Theorem 6. ([23], [54], [64]) *Let $f, f_n, g, g_n \in \mathcal{F}(X)$ $(n \in \mathbb{R})$, with f_n, g convex.*

(a) $f \leq b$-$\liminf_n f_n \Rightarrow f^* \geq b - \limsup_n f_n^*$ *if* $\sup_n d((0,0), \operatorname{epi} f_n) < \infty$.

(b) $g \geq b - \limsup_n g_n \Rightarrow g^* \leq b - \liminf_n g_n^*$.

(c) $(f_n) \xrightarrow{b} f \Rightarrow (f_n^*) \xrightarrow{b} f^*$.

However some conclusions can be drawn without convexity assumptions; note that the following statement can be converted into a continuity result in terms of uniform convergence on bounded subsets of the transforms.

Theorem 7. ([64]) *Let $f \in \mathcal{F}(X)$ be hypercoercive (i.e. $\lim_{\|x\| \to \infty} f(x)/\|x\| = \infty$) and bounded below. Then, for all $q, \varepsilon \in \mathbb{P}$ there exist $r, \delta \in \mathbb{P}$ such that*

$$e_r(f,g)<\delta \Rightarrow [\forall x^* \in qU_{X^*}\ \ g^*(x^*)>f^*(x^*)-\varepsilon] \Rightarrow [e_q(g^*,f^*)\leq \varepsilon].$$

In particular, if $f \leq b$-lim inf$_n$ f_n *then* $f^* \geq b$-lim sup$_n$ f_n^*.

4. BOUNDEDNESS PROPERTIES

We devote the present section to some concepts which will be used as key ingredients in some boundedness properties we need.

4.1 Apart subsets

Given a nonempty subset E of X and $\varepsilon \in \mathbb{P}$, the *conical* ε-*enlargement of* E is the set

$$C_\varepsilon(E) := \{x \in X : d(x,E) < \varepsilon \| x \|\} \cup \{0\}.$$

For $\alpha,\beta \in]0,1[$ and $\gamma := \alpha+\beta+\alpha\beta$ one has, whenever $0 \in E$,

$$C_\beta(C_\alpha(E)) \subset C_\gamma(E). \tag{1}$$

When $E \neq \{0\}$ is a cone, for $\alpha,\beta \in (0,1)$, one has the following inclusions:

$$\mathbb{R}_+(E \cap S_X + \alpha U_X) \subset C_{\alpha(1-\alpha)^{-1}}(E), \tag{2}$$

$$C_\beta(E) \subset \mathbb{R}_+(E \cap S_X + \beta(1-\beta)^{-1} U_X). \tag{3}$$

The notion of conical enlargement is thus especially useful when dealing with cones; for such subsets it is related to the notion of plastering due to Krasnoselski ([37]; see also [28], [31], [33], [35]). But it can be used for any subset.

The following definition recalls a notion introduced and used in [41], [60] which will be much used in the sequel.

Definition 8. *Two nonempty subsets* E, F *of* X *are said to be (asymptotically) apart if there exists* $\varepsilon \in \mathbb{P}$ *such that* $C_\varepsilon(E) \cap C_\varepsilon(F)$ *is bounded.*

Equivalently, the nonempty subsets E, F of X are apart if, and only if, there is no sequence (x_n) such that $(\|x_n\|) \to \infty$, $(\|x_n\|^{-1} d(x_n,E)) \to 0$,

$(\|x_n\|^{-1} d(x_n, F)) \to 0$. In the case E and F are cones, several other characterizations are given in [41] and [60]; we recall them for the reader's convenience. Their simple proofs are consequences of relations (1)–(3).

Lemma 9. *([41]) Given two cones P, Q in X, the following assertions are equivalent and hold if and only if P and Q are apart:*

a) *there exist $\alpha, \beta > 0$ such that $C_\alpha(P) \cap C_\beta(Q) = \{0\}$;*

b) *there exists $\gamma > 0$ such that $P \cap C_\gamma(Q) = \{0\}$;*

c) *there exists $\delta > 0$ such that $P \cap (Q \cap S_X + \delta U_X) = \varnothing$;*

d) *there exists $\varepsilon > 0$ such that $(P \cap S_X + \varepsilon U_X) \cap (Q \cap S_X + \varepsilon U_X) = \varnothing$;*

e) *there exists $\kappa > 0$ such that $\max(d(x,P), d(x,Q)) \geq \kappa \|x\|$ for each $x \in X$.*

These assertions are satisfied when P, Q are closed, $P \cap Q = \{0\}$ and one of the following conditions is satisfied:

i) *P (or Q) is locally compact (in particular if* $\operatorname{span} P$ *is finite dimensional);*

ii) *P (or Q) is weakly locally compact and P and Q are convex.*

When P and Q are convex, dual properties can be given in terms of polar cones.

4.2 Boundedness and expansion properties

The preceding notions can be used for studying boundedness questions. Let us recall that a multimapping $M : W \rightrightarrows X$ between two n.v.s. is said to be *bounding* if it transforms any bounded set into a bounded set (sometimes M is said to be bounded, but we prefer to avoid any confusion with the case the image of M is bounded). Let us say it is *quasi-bounding* if the remoteness of M is bounded over any bounded subset of its domain. It is easy to give examples showing that the latter condition is less exacting than the former one; in particular, the notion of bounding multimapping cannot be used when the values of M are unbounded, in particular when they are epigraphs. The following concepts have been used repeatedly but implicitly in [53], [62] and explicitly in [60]. In this last reference, by analogy with the case of proper maps, a quasi-expanding map was called *boundedly proper on E*. There is also a certain analogy between expansive maps and expanding maps as any expansive map is expanding (but the converse is not true).

Definition 10. *A map F from X to a normed vector space Y is said to be expanding (resp. quasi-expanding) on a subset E of X if the multimapping $M : y \rightrightarrows F^{-1}(y) \cap E$ is bounding (resp. quasi-bounding) from Y to X. It is said to be linearly expanding on E if there are $\alpha \in \mathbb{P}$, $\rho \in \mathbb{R}_+$ such that*

$$\|F(x)\| \geq \alpha \|x\| \text{ for all } x \in E \setminus \rho U_X.$$

It is said to be linearly quasi-expanding on E if there are $\alpha \in \mathbb{P}$, $\rho \in \mathbb{R}_+$ such that

$$F(E) \cap \alpha r U_Y \subset F(E \cap r U_X) \text{ for all } r > \rho.$$

Let us state easy characterizations of these properties.

Proposition 11. *The map $F : X \to Y$ is expanding on $E \subset X$ if, and only if,*

$$\forall r \in \mathbb{P}, \exists q \in \mathbb{P} : E \cap F^{-1}(rU_Y) \subset qU_X.$$

It is quasi-expanding on E if, and only if,

$$\forall r \in \mathbb{P}, \exists q \in \mathbb{P} : F(E) \cap rU_Y \subset F(E \cap qU_X). \tag{4}$$

Moreover, the mapping $F : X \to Y$ is expanding on a subset E of X if, and only if, any sequence (x_n) in E is bounded when $(F(x_n))$ is bounded. It is quasi-expanding on E if, and only if, for any bounded sequence (y_n) in $F(E)$ there exists a bounded sequence (x_n) in E such that $y_n = F(x_n)$ for each $n \in \mathbb{N}$.

We also have the following immediate implications.

Proposition 12.

(a) If F is expanding on E then it is quasi-expanding on E.
(b) If F is linearly expanding on E then it is expanding on E and linearly quasi-expanding on E.
(c) If F is linearly quasi-expanding on E then it is quasi-expanding on E.

For positive homogeneous maps, more can be said.

Proposition 13. *Suppose E is a cone and F is positively homogeneous. Then*

(a) *F is linearly expanding on E if, and only if, it is expanding if, and only if, there exists some $c \in \mathbb{P}$ such that $E \cap F^{-1}(U_Y) \subset cU_X$ if, and only if, there exists some $\alpha \in \mathbb{P}$ such that $\|F(x)\| \geq \alpha \|x\|$ for all $x \in E$. In such a case one has $F^{-1}(0) \cap E = \{0\}$.*

(b) *F is linearly quasi-expanding on E if, and only if, it is quasi-expanding if, and only if, there exists some $c \in \mathbb{P}$ such that $F(E) \cap U_Y \subset F(E \cap cU_X)$.*

Moreover, when $0 \in E$, F is linearly quasi-expanding if, and only if, F is open at 0 at a linear rate from E onto $F(E)$.

Example 1. *The preceding notions can be illustrated by the case $X = Y = \mathbb{R}$. In such a case, F is expanding if, and only if, F is coercive in the sense that $|F(x)| \to \infty$ when $|x| \to \infty$.*

Example 2. *Suppose $A : X \to Y$ is a linear isomorphism, $h : \mathbb{R}_+ \to \mathbb{R}$ is a function and $F(x) = h(\|x\|)A(x)$ for $x \in X$. If* $\liminf_{r \to \infty} h(r) > 0$, *then F is linearly expanding on X.*

The linear expansion property enjoys a useful stability property detected in [41] and [60].

Lemma 14. *If $F : X \to Y$ is Lipschitzian and linearly expanding on a subset E of X, then there is a positive number δ such that F is linearly expanding on $C_\delta(E)$. Moreover, for any $\varepsilon > 0$ there exist $\delta, \sigma > 0$ such that $F(C_\delta(E) \setminus \sigma U_X) \subseteq C_\varepsilon(F(E))$.*

Let us quote some criteria from [41] and [60, Lemma 8].

Lemma 15. *Let P be a cone in X and let F be a continuous linear map from X to Y, with $N := \ker F$. Each of the following conditions is sufficient for F to be linearly expanding on P:*

a) *F is open onto its image and N and P are apart;*
b) *F is quasi-expanding on P and N and P are apart;*
c) *F is quasi-expanding on P, P is closed and N is finite dimensional with $N \cap P = \{0\}$;*
d) *P is closed, locally compact and $N \cap P = \{0\}$;*
e) *P is closed, P has a weakly compact base and $N \cap P = \{0\}$.*

Another connection between the two concepts introduced above is the following one (see [41, Lemma 2.2 c] for a quantitative proof in the case E and F are cones and [60] in the general case).

Lemma 16. *The subsets E and F of X are apart if and only if the map $L:(x,y)\mapsto x-y$ is linearly expanding on $E\times F$.*

We also need a notion which is a global variant of a property which has been widely used in nonsmooth analysis since its introduction in [34] and its use in [50], [52] in which the terminology has been coined.

Definition 17. *A mapping $F:X\to Y$ between two normed vector spaces is said to be metrically regular (resp. asymptotically metrically regular) on a subset E of X if there exists $\gamma>0$ such that $d(x,N)\le\gamma\|F(x)\|$ for $x\in E$ (resp. for $x\in E$ with $\|x\|$ large enough), where $N:=F^{-1}(0)$.*

When the closure of N contains 0 (in particular when N is nonempty and F is positively homogeneous), F is asymptotically metrically regular on E whenever F is linearly expanding on E. When X and Y are Banach spaces and F is linear, continuous and surjective F is metrically regular on X. Let us note the following simple facts which clarify some relationships between the preceding concepts.

Lemma 18. *Suppose $F:X\to Y$ is positively homogeneous. Let $N:=F^{-1}(0)$ and let C be a cone of X.*

(a) F is metrically regular on C if, and only if, $(d(x_n,N))\to 0$ whenever $(F(x_n))\to 0$ with $x_n\in C$ for each n.

(b) Suppose $C-N\subset C$ and $F(x-w)=F(x)$ for any $w\in N$, $x\in C$. If F is metrically regular on C then F is linearly quasi-expanding on C.

(c) Suppose $C-N\subset C$ and $x'-x\in N$ whenever x, $x'\in C$ and $F(x)=F(x')$. If F is quasi-expanding on C then F is metrically regular on C.

Proof.

(a) If F is not metrically regular on C there exists a sequence (x_n) in C such that $d(x_n,N)>n\|F(x_n)\|$ for each n. Since F is positively homogeneous and $F(x_n)\neq 0$, we may suppose that $n\|F(x_n)\|=1$ for each n. Then $(F(x_n))\to 0$ and $(d(x_n,N))$ does not converges to 0. The converse is obvious.

(b) Suppose there exists $\gamma > 0$ such that $d(x,N) \le \gamma \|F(x)\|$ for $x \in C$. Then for any $q \in \mathbb{P}$, $y \in F(C) \cap qU_Y$ and any $x \in F^{-1}(y) \cap C$ one can find $w \in N$ such that $\|w - x\| < \gamma q$, so that $y = F(x - w) \in F(\gamma q U_X \cap C)$ by the assumption $C - N \subset C$.

(c) Suppose F is quasi-expanding on C. Let $p \in \mathbb{P}$ be such that $F(C) \cap U_Y \subset F(C \cap pU_X)$. For each $x \in C$ and each $q > \|F(x)\|$ one has $F(q^{-1}x) = F(pu)$ for some $u \in C \cap U_X$, hence $q^{-1}d(x,N) = d(q^{-1}x, N) \le \|q^{-1}x - (q^{-1}x - pu)\| < p$ as $q^{-1}x - pu \in N$, and one gets $d(x,N) \le p\|F(x)\|$.

□

Part (a) of the preceding lemma can be used to show that if $F : X \to \mathbb{R}$ is positively homogenous and if $N_- := \{x \in X : F(x) \le 0\}$, then F satisfies $d(x, N_-) \le \gamma F(x)_+$ for some $\gamma > 0$ and each $x \in C$ if, and only if, $(d(x_n, N_-)) \to 0$ whenever $(F(x_n)_+) \to 0$ with $x_n \in C$ for each n, where $r_+ := \max(r, 0)$: it suffices to replace $F(\cdot)$ by $F(\cdot)_+$.

4.3 Firm asymptotic cones

Let E be a nonempty subset of the normed vector space X. We recall that the *asymptotic cone* (sometimes called the recession cone) of E is the cone $E_\infty := \limsup_{t \to +\infty} t^{-1}E$, consisting of all limits of sequences $(t_n^{-1}x_n)$, where $x_n \in E$ and $t_n \in \mathbb{P}$ with $(t_n) \to \infty$ (see [14], [15], [39]–[41], [69], [75] for the study of related properties).

The following definition, which is the central concept of [60], will be used here instead of the concept of asymptotic compactness used in [62] as a boundedness criteria. Recall that E is said to be *asymptotically compact* if for any sequence (x_n) of E such that $(\|x_n\|) \to \infty$ the sequence $(\|x_n\|^{-1}x_n)$ has a converging subsequence (see [29], [51], [76] for preliminary definitions).

Definition 19. *A cone C of X is a firm (outer) asymptotic cone of a subset E of X if for any $\varepsilon > 0$ there exists some $r > 0$ such that $E \setminus rU_X \subset C_\varepsilon(C)$.*

The following characterizations may be convenient.

Proposition 20. *For a subset E of X and a closed cone C in X, the following assertions are equivalent:*

a) *C is a firm asymptotic cone of E;*

b) $d(x,C)/\|x\| \to 0$ *as* $\|x\| \to \infty$ *with* $x \in E$;

c) *there exists a map* $h : E \to C$ *such that* $d(x,h(x))/\|x\| \to 0$ *as* $\|x\| \to \infty$ *with* $x \in E$.

Proof. The implications a)$\Rightarrow$b), c)$\Rightarrow$a) are direct consequences of the definitions. To prove that b)$\Rightarrow$c), given $c > 1$, for $x \in E$ we pick $h(x) \in C$ such that $\|h(x) - x\| \le cd(x,C)$ (considering separately the case $d(x,C) = 0$ and the case $d(x,C) > 0$). □

An interpretation of the preceding conditions in terms of bounded convergence can be given.

Proposition 21. *A cone* C *of* X *is a firm asymptotic cone of a subset* E *of* X *if, and only if,* b-$\limsup_{t\to\infty} t^{-1}E \subset C$.

Proof. Suppose C is a firm asymptotic cone of E. Given $p \in \mathbb{P}$ we have $e_p(t^{-1}E,C) \to 0$ as $t \to \infty$: otherwise, we could find $c > 0$, a sequence $(t_n) \to \infty$ and $x_n \in E$ such that $\|t_n^{-1}x_n\| < p$ and $d(t_n^{-1}x_n,C) > c$ and then we would have $\|x_n\| \ge ct_n \to \infty$, and $d(x_n,C) > ct_n > cp^{-1}\|x_n\|$, a contradiction.

Conversely suppose $e_p(t^{-1}E,C) \to 0$ as $t \to \infty$ for each $p \in \mathbb{P}$. Given $\varepsilon > 0$, let $t_\varepsilon > 0$ be such that $e_1(t^{-1}E,C) < \varepsilon$ for $t > t_\varepsilon$. Then, for $x \in E \setminus t_\varepsilon U_X$ and for $t > \|x\|$ we have $t^{-1}x \in U_X$ hence $d(t^{-1}x,C) < \varepsilon$ and $d(x,C) < \varepsilon t$. Since t is arbitrarily close to $\|x\|$, we get $d(x,C) \le \varepsilon\|x\|$ and $x \in C_\varepsilon(C)$. □

Of course, the preceding definition does not determine C uniquely: any cone D containing C is also a firm asymptotic cone. Thus, one is led to take as a firm asymptotic cone a cone which is as small as possible. The following result shows a limitation in this direction.

Proposition 22. *If* C *is a closed firm asymptotic cone of* E, *then* C *contains the asymptotic cone* E_∞ *of* E. *If* E *is asymptotically compact, then* E *is firmly semi-asymptotable in the sense that* E_∞ *is a firm asymptotic cone of* E.

Proof. Let $v \in E_\infty \setminus \{0\}$: there exists a sequence (e_n) in E and a sequence $(t_n) \to \infty$ in $\mathbb{P}$ such that $(t_n^{-1}e_n) \to v$. Then $(\|e_n\|) \to \infty$ and

$$d(v,C) = \lim_n d(t_n^{-1}e_n,C) = \lim_n t_n^{-1}d(e_n,C) = \|v\|\lim_n \|e_n\|^{-1}d(e_n,C) = 0,$$

so that $v \in C$. Suppose E is asymptotically compact and the asymptotic cone E_∞ of E is not a firm asymptotic cone of E. Then there exist $\varepsilon > 0$

and a sequence (e_n) of E such that $(\|e_n\|) \to \infty$ and $d(e_n, E_\infty) \geq \varepsilon \|e_n\|$. Since E is asymptotically compact, taking a subsequence if necessary, we may suppose that $(e_n/\|e_n\|)$ has a limit v. Then $v \in E_\infty$ and we get a contradiction: $d(\|e_n\|^{-1} e_n, v) \geq d(\|e_n\|^{-1} e_n, E_\infty) \geq \varepsilon$. □

Thus, in any finite dimensional space, the asymptotic cone is a firm asymptotic cone. On the other hand, in any infinite dimensional normed vector space X there exists a set E whose asymptotic cone is not a firm asymptotic cone.

Example 3. *Let E be the epigraph of a function $f : W \to \overline{\mathbb{R}}$ with nonempty domain in a n.v.s. W which is bounded below on bounded sets, and let $X = W \times \mathbb{R}$. If f is hypercoercive* (i.e., *$f(x)/\|x\| \to \infty$ as $\|x\| \to 0$*), *then E is firmly asymptotable and $E_\infty = \{0\} \times \mathbb{R}_+$.*

Example 4. *Let E be the epigraph of a function $f : W \to \overline{\mathbb{R}}$ which is bounded below on bounded sets, such that $\liminf_{\|w\| \to \infty} (f(w) - p(w))/\|w\| \geq 0$, where $p : W \to \mathbb{R}$ is a positively homogeneous function and let C be the epigraph of p in $X = W \times \mathbb{R}$. Then C is a firm asymptotic cone of E. In particular, if $c_\infty := \liminf_{\|w\| \to \infty} f(w)/\|w\| \in \mathbb{R}$, the set $C := \text{epi}\, c_\infty \|\cdot\|$ is a firm asymptotic cone of E. When $c_\infty \in \mathbb{P} \cup \{+\infty\}$, f is said to be super-coercive.*

Example 5. *Suppose there exist a bounded subset B of X and a closed cone C such that $E \subset B + C$. Then C is a firm asymptotic cone to E.*

Some calculus rules can be given (see [60, Prop. 13]).

A connection between the concept of firm asymptotic cone and the notion of apart subsets is as follows.

Proposition 23. ([60, Prop. 17]) *Let P and Q be firm asymptotic cones of subsets E and F of X respectively. If P and Q are apart, then E and F are apart.*

4.4 Applications to boundedness properties

Let us now show that the preceding concepts can be used for the study of boundedness properties. We only deal with mappings; boundedness properties of correspondences could be dealt with similarly. The first result we give is a simple consequence of Lemmas 12-14.

Lemma 24. *Let $F : X \to Y$ be a Lipschitzian, positively homogeneous map between two normed vector spaces and let E be a subset of X. Suppose K*

is a firm asymptotic cone of E *and that* F *is expanding on* K. *Then* F *is linearly expanding on* E.

The following proposition is closely related.

Proposition 25. *Under the following assumptions, a positively homogeneous mapping* $F : X \to Y$ *is expanding on a subset* E *of* X:

(a) E *has a firm asymptotic cone* K;

(b) K *and* $N := F^{-1}(0)$ *are apart;*

(c) F *is asymptotically metrically regular on* E.

In fact assumption (a) can be replaced with the following weaker condition:

(a') there exists $\alpha \in \mathbb{P}$ such that $E \cap C_\alpha(N)$ has a firm asymptotic cone K.

Proof. If the conclusion does not hold, one can find $r \in \mathbb{P}$, a sequence (x_n) of E such that $\|F(x_n)\| < r$ and $(\|x_n\|) \to \infty$. In view of (c), we have $(\|x_n\|^{-1} d(x_n, N)) \to 0$. Then, dropping a finite number of terms if necessary, we have $x_n \in E \cap C_\alpha(N)$ for each $n \in \mathbb{N}$. Then, by assumption (a'), we have $(\|x_n\|^{-1} d(x_n, K)) \to 0$. In view of the characterization given after Definition 8 of the property that K and N are apart, we get a contradiction. □

The preceding result can be specialized to the case E is a sub-level set $[f < q]$ of some function f on X. It can also be adapted to the case the function f has a *firm asymptotic approximation* φ on some subset S of X; by this we mean that $\liminf_{x \in S, \|x\| \to \infty} (f(x) - \varphi(x))/\|x\| \geq 0$.

Corollary 26. *Under the following assumptions, the map* F *is expanding on the sub-level set* $[f < q]$:

(a) *there exists* $\beta \in \mathbb{P}$ *such that* f *has a firm asymptotic approximation* φ *on* $[f < q] \cap C_\beta(N)$ *which is positively homogeneous;*

(b) *there exists* $\gamma \in \mathbb{P}$ *such that* $\varphi(x) \geq \gamma \|x\|$ *for each* $x \in C_\gamma(N)$;

(c) F *is asymptotically metrically regular on* $[f < q]$.

Proof. Let $\alpha \in (0, \min\{\beta, \gamma\})$ and let

$$K := \{x \in C_\beta(N) : \varphi(x) \leq \alpha \|x\|\}.$$

Since $K \cap C_\gamma(N) = \{0\}$ the sets K and N are apart. It remains to show that K is a firm asymptotic cone to $[f < q] \cap C_\alpha(N)$. If it is not the case, one can find $\delta \in \mathbb{P}$ and a sequence (x_n) in $[f < q] \cap C_\alpha(N)$ such that

$(\|x_n\|) \to \infty$ and $x_n \notin C_\delta(K)$ for each $n \in \mathbb{N}$. Then, by (a) and (b), there exists a sequence $(\varepsilon_n) \to 0_+$ such that

$$q > f(x_n) \geq \varphi(x_n) - \varepsilon_n \|x_n\| \geq (\gamma - \varepsilon_n)\|x_n\|,$$

a contradiction. □

The preceding proposition can be applied to the case of the sum $S : X^2 \to X$ given by $S(x, y) = x + y$. We note that S is metrically regular: for any $(x, y) \in X^2$ we have

$$d((x,y),N) \leq d((x,y),\frac{1}{2}(x-y,y-x)) = \frac{1}{2}\|x+y\| = \frac{1}{2}\|S(x,y)\|.$$

Since when P (resp. Q) is a firm asymptotic cone of A (resp. B), the cone $P \times Q$ is a firm asymptotic cone of $A \times B$, and since $P \times Q$ is apart from $N := \ker S$ when P and $-Q$ are apart, as easily seen, we get the following result. Another (simple, direct) proof is provided in [60, Prop. 20]; still another proof can be derived from Lemmas 14 and 16.

Proposition 27. *Let A and B be two nonempty subsets of X and let P (resp. Q) be a firm asymptotic cone of A (resp. B). If P and $-Q$ are apart then the mapping $S : (x, y) \mapsto x + y$ is expanding on $A \times B$.*

5. CONTINUITY OF SOME OPERATIONS

We are in a position to give some persistence and stability results for usual operations on sets and functions.

5.1 Continuity of some operations with sets

The most obvious results concern products and unions for which a direct easy analysis leads to the following statement.

Proposition 28. ([62, Lemma 21 (e)]) *Suppose $(A_n) \xrightarrow{b} A$, $(B_n) \xrightarrow{b} B$. Then $(A_n \times B_n) \xrightarrow{b} A \times B$. If A, B, A_n, B_n are subsets of the same space then $(A_n \cup B_n) \xrightarrow{b} A \cup B$.*

For intersections, a convexity argument and a qualification condition ([50], [53], [66], [67], [75]) have to be used.

Proposition 29. *([62, Prop. 27 (e)]) Suppose* $(A_n)\xrightarrow{b}A$, $(B_n)\xrightarrow{b}B$ *where* A, A_n, B, B_n *are closed convex subsets of a Banach space* X *and* $X=\mathbb{R}_+(A-B)$. *Then* $(A_n\cap B_n)\xrightarrow{b}A\cap B$.

In fact a quantitative result can be given. Assuming that

$$sU_X \subset A\cap rU_X - B\cap rU_X \tag{5}$$

for some $r,s\in\mathbb{P}$ (what occurs when $X=\mathbb{R}_+(A-B)$), we will show that for each $p\in\mathbb{P}$ and any $A',B'\subset X$ we have

$$e_p(A'\cap B', A\cap B)\le \frac{p+r+s}{s+\max(e_p(A',A),e_p(B',B))}(e_p(A',A)+e_p(B',B)), \tag{6}$$

and that if $p\ge r$ and if $e_p(A,A')+e_p(B,B')<s$, we have

$$d_p(A'\cap B', A\cap B)\le s^{-1}(p+r+s)(d_p(A',A)+d_p(B',B)) \tag{7}$$

Proof. Let $x'\in A'\cap B'\cap pU_X$ and let $t>e_p(A',A)+e_p(B',B)$. We can find $y\in A$, $z\in B$ such that $\|y-x'\|+\|z-x'\|<t$. Relation (5) ensures that there exists $(a,b)\in(A\cap rU_X)\times(B\cap rU_X)$ such that

$$st^{-1}(z-y)=a-b.$$

Then $x:=(s+t)^{-1}(sy+ta)=(s+t)^{-1}(sz+tb)$ belongs to $A\cap B$ and since $\|a-x'\|<p+r$

$$\|x-x'\|\le(s+t)^{-1}s\|y-x'\|+(s+t)^{-1}t\|a-x'\|<(s+t)^{-1}t(s+p+r).$$

Since t is arbitrarily close to $e_p(A',A)+e_p(B',B)$, we obtain (6).

Now let us suppose A' and B' are such that $e_p(A,A')+e_p(B,B')<t<s$. Then, for $s'\in(t,s)$ we have

$$s'B_X\subset A\cap rU_X - B\cap rU_X \subset A'\cap(r+t)U_X - B'\cap(r+t)U_X + tU_X.$$

Using the Rådström's cancellation rule, we get

$$(s'-t)B_X\subset \mathrm{cl}\left(A'\cap(r+t)U_X - B'\cap(r+t)U_X\right).$$

Then the openness result of [67, Lemma 1.0] ensures that

$$(s'-t)U_X \subset A' \cap (r+t)U_X - B' \cap (r+t)U_X.$$

Thus the first part of the proof can be applied with A, B interchanged with A', B' and r, s replaced by $r+t$ and $s'-t$ respectively:

$$e_p(A \cap B, A' \cap B') \leq \frac{p+r+s'}{s'-t+e_p(A,A')+e_p(B,B')}(e_p(A,A')+e_p(B,B')).$$

Since s' and t can be chosen arbitrarily close to s and $e_p(A,A')+e_p(B,B')$ respectively, we get (7). □

Using the diagonal mapping, and the product rule, the preceding statement can be considered as a special case of a result about inverse images under a continuous linear map (see [62, Lemma 24]). On the other hand, the inverse image by a linear continuous map $L: X \to Y$ of a subset D of Y is obtained as the projection on X of the intersection $L \cap (X \times D)$, where L is identified with its graph and $p_X|_L$ is an isomorphism from L onto X. In fact, a direct analysis yields a quantitative result which reveals a kind of Lipschitzian behavior.

Proposition 30. ([19, Cor. 2.4], [62, Lemma 24]) *Let D be a closed convex subset of Y. Assume*

$$sU_Y \subset L(rU_X) - D$$

for some $r, s > 0$; this condition is satisfied when X, Y are complete and $Y = \mathbb{R}_+(L(X)-D)$. Then, for $t \in (0,s)$, $p > 0$, $q > \max(p\|L\|, r\|L\| + s)$ and D', D'' closed convex subsets of Y with $d_q(D,D') < t$, $d_q(D,D'') < t$ one has

$$d_p(L^{-1}(D'), L^{-1}(D'')) \leq \frac{p+r}{s-t} d_q(D', D'').$$

In particular, for a sequence (D_n) of closed convex subsets of Y one has

$$(D_n) \xrightarrow{b} D \Rightarrow (L^{-1}(D_n)) \xrightarrow{b} L^{-1}(D).$$

The study of b-convergence of images can be eased by the use of the criteria for the expansion property we displayed above. Here we introduce a slight refinement of condition (4), and of conditions (14) and (15) of [62]. We say that a map $F: X \to Y$ is *approximately quasi-expanding on a sequence* (E_n) of subsets of X if for each $\varepsilon > 0$ and each $q \in \mathbb{P}$ there exist

$p \in \mathbb{P}$ and $k \in \mathbb{N}$ such that $F(D_m) \cap qU_Y \subset F(D_m \cap pU_X) + \varepsilon U_Y$ for each $m \geq k$, where $D_m := \bigcup_{n \geq m} E_n$. This condition is satisfied if F is *expanding on* (E_n) in the following sense: for each $q \in \mathbb{P}$ there exist $p \in \mathbb{P}$ and $k \in \mathbb{N}$ such that $D_k \cap F^{-1}(qU_Y) \subset pU_X$. We also need an extension of the notion of firm asymptotic cone: we say that K is a *firm asymptotic cone to a sequence* (E_n) *of subsets of* X if for each $\varepsilon > 0$ there exist $r \in \mathbb{P}$ and $k \in \mathbb{N}$ such that $E_n \setminus rU_X \subset C_\varepsilon(K)$ for $n \geq k$. When the sequence (E_n) is constant, we recover the definition above.

Proposition 31. *Let* E, E_n $(n \in \mathbb{N})$ *be subsets of* X *and let* $F : X \to Y$ *be Lipschitzian on bounded sets.*

(a) *Suppose that* b*-*$\limsup_n E_n \subset E$. *Then* b*-*$\limsup_n F(E_n) \subset F(E)$ *provided that the map* F *is approximately quasi-expanding on* (E_n).

(b) *Suppose that* $E \subset b$*-*$\liminf_n E_n$. *Then* $F(E) \subset b$*-*$\liminf_n F(E_n)$ *provided* F *is quasi-expanding on* E.

(c) $F(E) \subset b$*-*$\liminf_n F(E_n)$ *provided that* $E \subset b$*-*$\liminf_n E_n$, F *is positively homogeneous, asymptotically metrically regular on* E *and* E *has a firm asymptotic cone* K *which is apart from* $N := F^{-1}(0)$.

(d) *If* $E = b$*-*$\lim_n E_n$ *and if* F *is positively homogeneous, metrically regular on* $\bigcup_n E_n$ *and* E, *if* E *and* (E_n) *have a firm asymptotic cone* K *which is apart from* $N := F^{-1}(0)$, *then* $F(E) = b$*-*$\lim_n F(E_n)$.

Proof. (a) (Compare with [62, Prop. 8 (d)] when F is linear.) Let $q \in \mathbb{P}$ and $\varepsilon > 0$ be given. Since F is approximately quasi-expanding on (E_n), setting $D_m := \underset{h \geq m}{\cup} E_m$, we can find $p \in \mathbb{P}$ and some $k \in \mathbb{N}$ such that $F(D_m) \cap qU_Y \subset F(D_m \cap pU_X) + \frac{1}{2}\varepsilon U_Y$ for each $m \geq k$. Let κ be the Lipschitz rate of F on $(p+1)U_X$ and let $\delta := \min(\varepsilon/2\kappa, 1)$. Let $m \geq k$ be such that $E_n \cap pU_X \subset E + \delta U_X$ for $n \geq m$. Then, for $n \geq m$ we have

$$F(E_n) \cap qU_Y \subset F(D_m) \cap qU_Y \subset F(D_m \cap pU_X) + \frac{1}{2}\varepsilon U_Y$$
$$\subset F(E + \delta U_X) + \frac{1}{2}\varepsilon U_Y \subset F(E) + \varepsilon U_Y.$$

The proof of part (b) is simpler and is omitted (see [62, Prop. 1 (b)]). Part (c) is a consequence of part (b) and of Proposition 25.

(d) It is a consequence of parts (a) and (c) and of an adaptation of Proposition 25 which consists in proving that under the assumptions of (d) the map F is expanding on (E_n), and of course, on E. If the conclusion does not hold, one can find $q \in \mathbb{P}$, a sequence (x_p) of X such that $x_p \in E_{n(p)} \setminus pU_X$, $\|F(x_p)\| < q$ for each $p \in \mathbb{N}$, with $n(p) \to \infty$ as $p \to \infty$. Let $\alpha \in (0,1)$ be such that $C_\alpha(K) \cap C_\alpha(N) = \{0\}$. Since F is metrically regular on $\bigcup_n E_n$, we have $(\|x_p\|^{-1} d(x_p, N)) \to 0$, hence $x_p \in C_\alpha(N)$ for $p \in \mathbb{N}$ large enough. On the other hand, since K is a firm asymptotic cone to (E_n), we have $x_p \in C_\alpha(K)$ for $p \in \mathbb{N}$ large enough. This is a contradiction. □

The convergence of sums of sets is a special case of the preceding statement.

Corollary 32. *Let A, A_n, B, B_n $(n \in \mathbb{N})$ be subsets of X such that $(A_n) \xrightarrow{b} A$, $(B_n) \xrightarrow{b} B$. Suppose P (resp. Q) is a firm asymptotic cone to A and (A_n) (resp. B and (B_n)) and P and $-Q$ are apart. Then $(A_n + B_n) \xrightarrow{b} A + B$.*

5.2 Continuity of some operations on functions

The preceding results can be adapted to epigraphs of functions in order to get results about usual operations. The most immediate application concerns composition.

Proposition 33. *Let W, Z be two Banach spaces, let $A : W \to Z$ be a continuous linear map and let g be a closed proper convex function on Z such that $Z = \mathbb{R}_+ \operatorname{dom} g + A(W)$. If (g_n) is a sequence of closed proper convex functions on Z which b-converges to g, then $(g_n \circ A)$ b-converges to $g \circ A$.*

Proof. Let $X := W \times \mathbb{R}$, $Y := Z \times \mathbb{R}$, let D (resp. D_n) be the epigraph of g (resp. g_n) and let $L : X \to Y$ be given by $L(x,r) := (A(x), r)$. Then the epigraph of $g \circ A$ (resp. $g_n \circ A$) is $L^{-1}(D)$ (resp. $L^{-1}(D_n)$). Since the qualification condition of the statement easily implies that $Y = \mathbb{R}_+ D + L(X)$ there exist $r, s > 0$ such that $sU_Y \subset L(rU_X) - D$. Thus the conclusion follows from Proposition 30. □

The case of marginal functions can be deduced from the case of images of sets; in particular the convergence of the infimal convolution of two functions can be derived from the convergence of the sequence of the sum of two sets.

As a sample of what can be obtained with functions, let us give a result for infimal convolutions and recall the following result for sums (see [18], [19], [26], [53], [62], [74], [77]...); in the finite dimensional case such a result has been obtained by McLinden–Bergstrom [44].

Proposition 34. *Suppose that f, f_n, g, g_n are closed proper convex functions on the Banach space X satisfying*

$$X = \mathbb{R}_+(\operatorname{dom} f - \operatorname{dom} g).$$

Then, if $(f_n) \xrightarrow{b} f$, $(g_n) \xrightarrow{b} g$ *one has* $(f_n + g_n) \xrightarrow{b} f + g$.

For the infimal convolution of two functions given by $(f \square g)(x) := \inf_{w \in X} (f(w) + g(x - w))$ we devise a direct proof inspired by Corollary 32.

Proposition 35. *Let f, f_n, g, g_n be functions on the normed vector space X. Suppose that f, g are bounded below on bounded subsets and have asymptotic firm approximations p, q respectively which are positively homogeneous and for which there exist $\alpha, \beta \in \mathbb{P}$ such that*

$$p(u) + q(v) \geq \alpha \min(\|u\|, \|v\|) - \beta \|u + v\| \qquad \forall u, v \in X. \tag{8}$$

If $f \geq b$-$\limsup_n f_n$ and $g \geq b$-$\limsup_n g_n$, then $f \square g \geq b$-$\limsup_n f_n \square g_n$.

Let us note that relation (8) is satisfied whenever there exist $\gamma, \lambda \in \mathbb{P}$ such that q is λ-Lipschitzian and

$$p(u) + q(-u) \geq \gamma \|u\| \qquad \forall u \in X.$$

In fact, in such a case, for any $u, v \in X$ we have

$$p(u) + q(v) \geq p(u) + q(-u) - \lambda \|u + v\| \geq \gamma \|u\| - \lambda \|u + v\|.$$

Proof. Let F, F_n, G, G_n be the strict epigraphs of f, f_n, g, g_n respectively, so that $F + G$ (resp. $F_n + G_n$) is the strict epigraph of $f \square g$ (resp. $f_n \square g_n$). The assertion amounts to show that $F + G \subset b$-$\liminf_n (F_n + G_n)$. In view of Proposition 31 it suffices to show that $S : (x, r, y, s) \mapsto (x + y, r + s)$ is expanding on $F \times G$. Suppose, on the contrary, that there exist $s \in \mathbb{P}$ and a sequence $((x_n, r_n, y_n, s_n))$ in $F \times G$ such that $(\|(x_n, r_n, y_n, s_n)\|) \to \infty$ and $(\|(x_n + y_n, r_n + s_n)\|)$ is bounded. Since f and g are bounded below on bounded subsets, the sequence $(\|(x_n, y_n)\|)$ cannot have a bounded subsequence. Thus $(\|x_n\|) \to \infty$ and $(\|y_n\|) \to \infty$. Then there exist $(\varepsilon_n) \to 0_+$ such that

$$r_n + s_n \geq f(x_n) + g(y_n) \geq p(x_n) + q(y_n) - \varepsilon_n \|x_n\| - \varepsilon_n \|y_n\|$$
$$\geq \alpha \min(\|x_n\|, \|y_n\|) - \beta \|x_n + y_n\| - \varepsilon_n \|x_n\| - \varepsilon_n \|y_n\|.$$

Because $\|x_n\| / \|y_n\| \to 1$ we obtain the contradiction $(r_n + s_n) \to \infty$. □

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