ROLF BIEHLER

RECONSTRUCTION OF MEANING AS A DIDACTICAL TASK: THE CONCEPT OF FUNCTION AS AN EXAMPLE

The meaning of a mathematical concept differs in different contexts. We can find different practices related to the same mathematical concept, such as the concept of function. Physicists have a practice different from that of mathematicians. Qualitative functional thinking is necessary in many vocations, however, this is a fairly different practice from that of academic sciences with their explicit use of symbolic mathematical notation. Commonly, uses will differ even if people use the same definition of a concept. However, spheres of practice may also differ with regard to definitions of a mathematical concept. The notion of a functional relationship between magnitudes may still be much used in physics, whereas mathematicians tend to use a more general notion of correspondence between sets. If we speak of different meanings of "the same" concept, we can further analyse differences and commonalities. How can didactics of mathematics cope with these many meaning differences? The mathematics classroom should not be a closed and self-reproducing system developing its own concept meanings. Rather, the meanings that are to be constituted in the classroom should be related to practices and meanings outside school. But what are the important points of orientation?

All the various spheres of practice (academic mathematics is one of them) in which mathematics is used are, in principle, relevant sources of meaning for general education. What dimensions of meaning of a concept should curriculum designers ideally take into account? The meaning and the importance of the concept within the theoretical network of academic mathematics, its historical genesis and development, its uses for problem solving inside and outside mathematics, its prototypical interpretations, its roots in everyday thinking and language as well as different tools and representations for working with the concept are relevant. How these sources are exploited and given relative weight to is dependent on the social meaning attributed to mathematics education. The social meaning varies. For instance, the traditional German *Gymnasium* had to prepare students for university studies, and the traditional German *Volksschule* had to prepare students for various vocations (artisans, workers etc.). The meanings of mathematical concepts that were selected for the various student groups differed very much according to the various social functions of schools and according to assumptions concerning what these students were able and willing to learn under given societal and schooling conditions.

We know that teachers are important agents in the classroom constitution of mathematical meaning. The implementation of curricula by teachers is shaped by their beliefs, in particular, by what they consider to be important aspects of a concept's meaning. If teachers themselves share some of the rich meanings which are implicit in curriculum material and serve as a background for its design, it is more probable that the intended meanings of the concepts will be implemented in the classroom. If teachers are not explicitly trained or educated in this respect, they may tend to base their teaching on the meanings they have acquired elsewhere, namely the traditional meanings of school mathematics, or on the meanings of concepts they have acquired during their academic studies in mathematics—if they have had an academic mathematics education and still consider this orientation the most important source for their teaching. When students study academic mathematics, they are confronted with meanings of concepts that can be considered only as part of the overall meaning landscape. The many uses of the concepts in various disciplines and in societal practices (and also in history) can be considered as part of a very comprehensive meaning landscape of that concept. But usually, these uses and practices are not part of the consciousness of mathematics students, professors and educators.

As mathematics education, however, has to base its curricular decisions on a broader picture of mathematics than that of academic mathematics, we consider the *reconstruction of meaning,* the development of a *synthesising meaning landscape* of a mathematical concept to be an important task for the didactics of mathematics that could serve as a theoretical background for curriculum design and implementation. We will also speak of a *didactically reconstructed intended mathematics for schools.* In this paper, we will argue in favour of a more systematic approach to this problem, illustrating and exemplifying our own ideas with regard to the concept of function. In some points in educational history, we can well identify interesting attempts to construct intended mathematics for schools as a referent for constituting the *knowledge to be taught* in Chevallard's (1985) sense. We will start with discussing some attempts below that will also show that it is usually not just "academic mathematics" that functions as a referent for constituting knowledge to be taught.

1. MEANING OF FUNCTIONS IN THE CONTEXT OF DIDACTICALLY RECONSTRUCTED MATHEMATICS

1.1 Examples of didactically reconstructed mathematics

Reconstructions of meanings of functions were often embedded in more general attempts to reconstruct the meaning of mathematics in the context of reform attempts in mathematics education.

Well-known historical examples for such a reconstruction are Felix Klein's books on "Elementary mathematics from a higher standpoint" (Klein, 1925a; Klein, 1925b; Klein, 1933) where he described and synthesised a view and selected a content of mathematics for German *Gymnasium* teachers, who had already good knowledge in mathematics. A value system is implicit in his books. It is related to the reform efforts of restructuring school in the direction of giving more emphasis to geometrical aspects of meaning (intuition, *Anschauung*) and to applications. A reintroduction of geometrical and visual aspects was regarded as necessary for school mathematics and for users of mathematics at the same time. The arithmetisation and formalisation that had led to banning geometry from the foundations of mathematics was not considered to provide an acceptable basis. A particular expression of this reform was the emphasis on "functional thinking" as one of the major goals of school mathematics. Functional thinking was considered a fundamental idea that should integrate pure and applied aspects of mathematics and legitimise the introduction of calculus into the senior secondary curriculum. Calculus was considered the top level of functional thinking that should be taught in the senior grades of secondary schools but that had to be adequately prepared in junior grades prior to that. Klein' s books are a good prototype of reconstructed mathematics because they not only develop a "philosophy of mathematics", but rather a view of mathematical content from a certain "philosophical" perspective that is more or less explicit. Klein introduced some epistemological distinctions, namely the distinction between "precision mathematics" and "approximation mathematics" as a way to describe the difference between the ideal and exact world of mathematics and mathematics applied to reality and to *Anschauung*. Typically, Klein does not just present "school mathematics" but goes far beyond this level with regard to the content treated.

Another big historical event for the function concept in mathematics education was the new math reform where functions were reconstructed as examples of the general concept of mapping, or as a specific relation. New meanings were derived from this embedding, whereas traditional aspects of meaning as "relations between magnitudes" were devalued. We can interpret the dozens of books on "new math for teachers and parents" as attempts to constitute a type of didactically reconstructed mathematics, although the writers would have thought of it just as of an elementarised academic mathematics. The latter illusion is quite understandable. If we only look at their concept definitions and theorems, then their mathematics will often appear only as a subset of academic mathematics. But if we include looking at domains of application, at the surrounding conceptual structure of a concept, and at the tools and means of representation used with a concept, we begin to see the differences.

A recent example of what I would consider a type of didactically reconstructed mathematics is the intended school mathematics constructed for the NCTM Standards (National Council of Teachers of Mathematics, 1989). This book, however, is already very much concerned with intended school teaching and learning processes. Maybe we can consider the book edited by Steen (1990) a description of the related didactically reconstructed mathematics as such, and as somewhat more separated from teaching and learning methods. The new NCTM's didactically reconstructed mathematics and other contemporary ones often make connections to the new humanistic and descriptive views of mathematics that include the "social dimension", problem-solving and the investigative nature of mathematics. (Davis & Hersh, 1980; Ernest, 1994). It is not clear how far mathematicians will see this characterisation of mathematics as an extension of their own view, or whether we are faced with an example of an artificial mathematical culture whose relation to academic mathematics is still pretty opaque.

In the context of this reform movement, reconstructions of the meaning of the function concept have been performed (Romberg, Fennema & Carpenter, 1993). A didactical analysis concerning the meaning of the concept of function in various mathematical practices or cultures is however still lacking. This deficiency is pointed out by Williams (1993, p. 315) in relation to the above book: "What we have instead is a description of a unique ethereal culture that, it can be argued, does not currently exist. It is well described by the *Curriculum and Evaluation Standards for School Mathematics* of the National Council of Teachers of Mathematics" (see also Biehler, 1994).

Attempts at reconstructing meanings of the function concept with regard to school mathematics after new maths are prevalent in other countries, too. A prominent example of the search for meaning is Freudenthal's (1983) *Didactical Phenomenology of Mathematical Structures*. He states as goals of his program:

> *Phenomenology* of a mathematical concept, structure or idea means describing it in relation to the phenomena for which it has been created, and to which it has been extended in the learning process of mankind, and, as far as this description is concerned with the learning process of the young generation, it is *didactical phenomenology*, a way to show the teacher the places where the learners might step into the learning process of mankind. (p. ix)

From this general approach, he develops a didactical phenomenology of functions (pp. 491–578). These reconstructions are related to reform attempts in the Netherlands under the conception of *realistic mathematics education.*

1.2 Research for meaning reconstruction and the complementarity of the function concept

There have been several studies concerning the meaning of the function concept that are not so closely related to reform movements. Vollrath's paper (1989) provides an example of synthesising aspects of the meaning of functions that were particularly discussed in Germany. Sierpinska's (1992) study on the meaning of functions can also be considered as intending a re-construction. She states the objectives of such research:

> In our attempt to define the basic conditions for understanding functions we shall be guided by an exploration of the reference of the definition of this notion. We shall ask ourselves what is this reality this definition refers to, what objects are there to be identified, discriminated between, what kind of orders can be found that would bring about the enlargement of reality by way of insightful generalisations and syntheses. (p. 30).

The analysis of regularities in relationships between changing magnitudes constitutes an important source of functions, i.e. a central part of its meaning. This is substantiated by Sierpinska's (1992) contribution analysing the concept's historical development in mathematics, which was related to uses in physics and geometry who were major "partners" in development. One of her results is the suggestion that "students must become interested in variability and search for regularities before examples of well-behaved mathematical elementary functions and definitions are introduced." (p. 32). This attitude may constitute an epistemological obstacle for teachers who have been "brought up" in a "pure-mathematics-culture". Sfard (1992) emphasises the dual nature of mathematical concepts as process and object and develops the thesis of the primacy of the operational origin of mathematical concepts. Structural notions emerge by reification later. The computational process of starting from a number x and calculating a resulting value y is, according to Sfard, the major source of the function concept.

These two positions really point to two sources of the notion of function admirably expressed by the mathematician Hermann Weyl (my translation, R.B.):

> Historically, the concept of function has a double root. Leading up to it are firstly the 'naturally given dependencies' ruling the material world which consist, on the one hand, in the fact that states and constitutions of real things are changeable in time, and on the other in the causal connection between cause and effect. A second root quite independent of the first lies in the arithmetico-algebraic operations. According to this, the analysis of old had in mind an expression which is formed from the independent variable by applying the four species and some less elementary transcedents a finite number of times, though these elementary operations have never been clearly and completely designated and historical growth has always pushed beyond to closely set boundaries without the agents of this development realising this every time. The point where these two sources which are at the outset quite foreign to one another begin to relate is the concept of the natural law. Its essence consists in the very fact that the natural law represents a naturally given dependency as a function constructed in a purely conceptual-arithmetical way. Galileo's laws of falling bodies are the first important examples. The modern growth of mathematics has led to the insight that the special algebraic principles of construction on which the analysis of old was based are much too narrow for a logico-natural and general development of analysis as well as when the role is considered which the function concept has to assume for the recognition of the laws governing what happens in the field of matter. General logical principles of construction must replace those algebraic ones. (Quoted in Weyl, 1917, p. 35-36)

I have taken the quote from the book of the IDM-Arbeitsgruppe Mathematiklehrerbildung (1981), who used it as an illustration for what they call the complementarity of the concept of function. The function concept is an excellent example of the complementarity of concepts in mathematics (Otte, 1984). In recent years, the didactical analysis of the concept of function has led to a revival of various characterisations of complementary aspects of functions. An early reference to the complementary aspects of the function concept as a mathematical object and as a thinking tool is Otte and Steinbring (1977). Other complementary characterisations are descriptive-relational vs. algorithmic-constructive (Richenhagen, 1990), geometrical-set theoretic-extensional vs. algebraic-analytical-intensional (Steiner, 1969), process and object respectively dynamical mapping vs. static relation (Sfard, 1992), co-variational versus correspondence aspects (Confrey & Smith, 1994), and, similarly, Vollrath (1989) who emphasises the distinction between horizontal (correspondence) and vertical (co-variation) aspects of functions. Concepts of complementarity have proved useful for empirical and constructive research on functions (Dubinsky & Harel, 1992; Romberg, Fennema & Carpenter, 1993).

2. RECONSTRUCTION OF MEANINGS OF THE CONCEPT OF FUNCTION

2.1 The context: Developing teachers' knowledge

I will now go into more detail concerning the function concept. As usual, ideas are shaped by the context they were developed in. The following ideas were developed in connection with pre-service courses for teachers on "the concept of function and functional thinking". The teacher-students already had a good mathematical background, but the intention was to enable them to reflect, enrich, and restructure the meaning they associated with the concept of function with regard to mathematics education. My selection of aspects was to emphasise a teaching of functions with technological support, applications outside mathematics and the general idea of functional relationship as contrasted to the limited view of functions normally taught in school. In particular, I will point to the meaning differences between functions in academic mathematics and what I consider as important for school teachers.

Empirical studies concerning teachers' knowledge of functions have to be based on an overall conception of knowledge on functions. For instance, Ruhama Even' s (1989, 1990, 1993) empirical study on teachers' knowledge of functions is based on an integrative analysis of what is considered as the meaning of the function concept in some part of the relevant didactical literature:

> As a result of this integration, six aspects seemed to be critical components of subject matter knowledge required to teach functions:
>
> - What is a function? (including image and definition of the concept of function, univalent property of functions, and arbitrariness of functions).
> - Different representations of functions.
> - Inverse function and composition of functions.
> - Knowledge about functions of the high school curriculum.
> - Different ways of approaching functions: point-wise, interval-wise, globally, and as entities.
> - Different kinds of knowledge and understanding of functions and mathematics. (Even, 1989, p. 212)

Many of Even's detailed results are interesting and point to a need to change teacher education in this area. However, her study represents a view of functions from a certain didactical discussion, whose strengths and weaknesses carry over to her study.

We have to broaden the perspective. I will describe a more extended meaning landscape for mathematics educators. It should also serve as a basis for further discussion on which aspects of the landscape are most important for teachers.

I will briefly discuss exponential functions as an example. Figure 1 contains a sketch of a semantic landscape for exponential functions. This is an example that has been considered in other recent projects, too (Confrey, 1991; Confrey & Smith, 1994). The exponential function is to provide a concrete example for important relations in the landscape. The network should give an impression of the conceptual complexity. I will not explain all individual elements and their importance in detail. That would be beyond the scope of this paper. Some comments must suffice. The picture contains theoretical mathematical aspects (difference, differential and functional equation, isomorphism between addition and multiplication, power series and number systems), relations to the dynamical systems and growth and decay processes, relations to other growth functions, to discrete models (geometric series), computational aspects (tables→slide rules→algorithms), relations to statistics and data analysis (curve fitting, log scales, data graphs), domains of application (radioactive decay and population explosion) and related general concepts relevant in applications (prediction, explanation, and model).
As compared to the normative view concerning teachers' knowledge on exponential functions on which Even (1989) based her study, the content of the above semantic network appears to be very ambitious as content for teachers to be learned. Compared with what secondary teachers have to learn in mathematics, far away from the elementary level, the landscape seems to be quite acceptable. Present teacher education does not yet provide sufficient preparation to enable teachers to develop such a complex system of meaning for themselves—in the first place. Even if all individual elements of the landscape were present in the teacher's mind, it is questionable whether he or she sees it as a whole, as a highly interrelated network of meaning as a background knowledge and meta-knowledge as a basis for teaching exponential functions in school.

As mathematical teacher education must capitalise on the teachers' ability to extend and reorganise their professional knowledge during their future life, we have to think of adequate measures to ensure that teachers not only improve their practical knowledge of teaching methods by way of experience, but also actively extend their mathematical meanings beyond those they have already learned during their studies within academic mathematics.

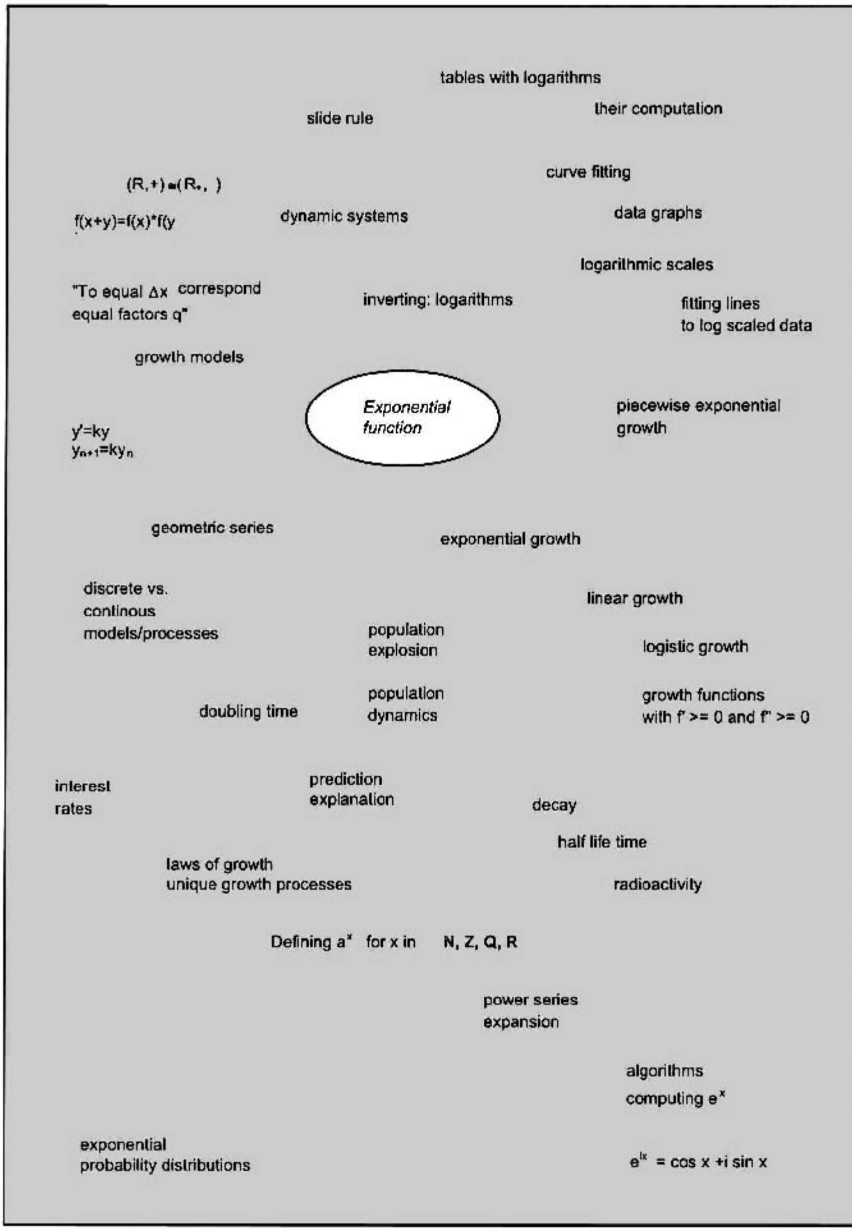

Figure 1: Semantic landscape for exponential functions

2.2 The meaning landscape in general

We need some distinctions concerning the conception of "meaning" that will enable us to structure our meaning landscape. I will use a variant of the epistemological triangle, which Steinbring (1994) uses to discuss the meaning of mathematical concepts. In Figure 2, I have drawn a variant that is most adequate for my current purpose. The epistemological triangle is based on the belief that the domains of application (a concept's uses inside and outside mathematics) are constitutive for what we may call meaning of a concept. Also the relation to other concepts, its role within a conceptual structure (a theory) and the tools and representations available for working with a concept are constitutive parts of the meaning. These dimensions constrain the problems for which the concept can be used. The epistemological triangle interpreted that way also implies a time dependence of meaning. Meaning may change by new applications, by new conceptual relations, or by new representations. I consider conceptualisations of knowledge like the *conceptual fields* (Vergnaud, 1990) and the *semantic fields* (Boero, 1992) as conceptualisations that are similar (see "Meanings of Meaning of Mathematics" in this volume for a more detailed account). Table 1 is an attempt to structure elements of a network according to the meaning components of the epistemological triangle: relations to other concepts (inside and outside mathematics), representations, and applications. I have used Sierpinska's (1992) study and Freudenthal's (1983, pp. 491–578) phenomenology as one of my sources for developing this "semantic landscape" for functions. I have added aspects that come from the practice of using functions in connection with statistics (in italics) and related to the new technologies (underlined), which are not covered by Freudenthal's and Sierpinska's analyses.

The discussion of all the elements of the table, the relation between elements and what teacher-students do know or should know about that and why cannot be done within the scope of this paper. In the following sections, I will only comment on some aspects.

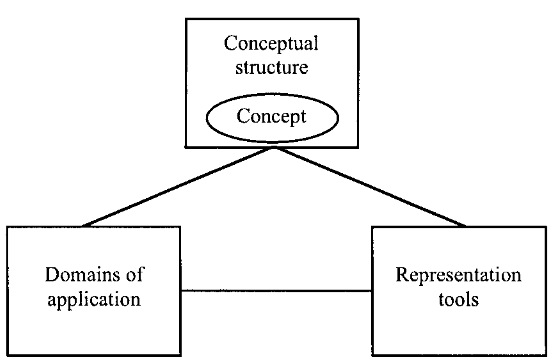

Figure 2: A variant of the epistemological triangle

2.3 Relations within mathematics

A didactically reconstructed meaning landscape has to overcome a compartmentalisation that is typical for the experience of a mathematics teacher-student. Algebra, calculus, differential equations and statistics are different courses in the university life of a mathematics student. Moreover, an average stochastics course in Germany would probably not cover regression and correlation. It is even more likely that students are not aware of the fact that the notions of correlation and conditional expectation can be regarded as generalisations of the concept of function, as a tool for analysing relations between magnitudes or "variables", as a statistician would say. Another generalisation of the operational aspect is that functions can be defined by algorithms or computer programs—extending the repertoire of algebra and of analytical expressions.

Generally, differential equations belong to a different cognitive compartment than functions, and students are not aware of the intimate historical relations existing between the emergence of the function concept and differential equations. The idea of an "unknown" function that is characterised by equations was pretty important for the constitution of functions as mathematical objects of study. Moreover, there are relations relevant to school mathematics that are no longer paid attention to in academic mathematics, in which a certain mathematical practice is already assumed. The different uses and meanings of variables are a good example: their use as unknowns in the context of solving equations, their use in describing rules for functions, and their use as symbols that signify variable magnitudes.

2.4 Representations

Computers provide plentiful new representations for functions that can be valuable for meaning development and for extending the range of applications. A reflection about the scope of different representations is something that has to be stimulated in teacher education courses. Often however, teachers have not yet become part of a practical mathematics culture where computer use for problem solving (similar to the practice of engineers) is common. This is why reflection and new experiences are necessary. I will discuss some aspects in more detail.

2.4.1 Language of functions and graphs

An important didactical idea for developing the function concept beyond algebraically defined functions consists in asking students to qualitatively sketch curves that describe features of processes (see Höfler, 1910, for a such an approach in history; and Swan, 1982, for a modern conception). Höfler was part of Klein's reform movement that intended to put relatively more emphasis again on the geometrical aspect of the complementary duality of functions. If students are asked by teachers to sketch a curve of the dependence of the water level upon time when various bottles are uniformly filled, then teachers should also know how this problem could be solved with advanced mathematical means: The sectional area as a function of time or height has to be integrated to get the volume as a function of time, and so forth. If spherical bottles are being used, it should be clear how to determine the volume of parts of a sphere. Integration in this context is a special case of solving a differential equation and teachers should be aware of the relation of these elementary integration tasks to differential or difference equations.

Related concepts in mathematics	**Related concepts in science and applications**
(mathematical) relation univalence of relation asymmetry of variables variables (unknowns) in algebra equation proportionality algorithm differential equation functional equation sequence mapping, operator *correlation* *conditional distribution* *regression*	law causal relation dependence interdependence, interaction [← curves] [← motion/change in time], change in general variable magnitudes relations between magnitudes equation between magnitudes data tables *data graphs* *time series* *strength of a relationship* machine constructed relation
Representations	**Applications**
symbolic: algebraic equation analytical expression implicit definition, "properties" algorithm computer program manipulable object in software graphs: standard Cartesian graph computer based Cartesian graphs (manipulable scales and zooming) various other graphs tables: standard tables interactive spreadsheet tables multiple linked representations	prediction description interpolation extrapolation *data reduction* determining (*estimating)*parameters interpreting parameters modelling range of validity *univalence as an idealisation* *deviation from model* *goodness of fit* dynamical systems

Table 1: Elements of the semantic landscape of the function concept

The idea of a function as an arbitrary free hand curve underlies this didactical approach. It was a central idea in history besides considering functions given by expressions. The arbitrariness of the free hand curve is more limited than the arbitrariness of the so-called Dirichlet definition of an arbitrary correspondence. Klein (1933) intended to mathematise the intuitive notion of "free hand curve" from a mathematical point of view. Klein obviously felt that this notion may provide a more adequate background for functions in school mathematics than the more general definition of Dirichlet. This is an interesting aspect of Klein's didactically reconstructed mathematics, which, by the way, did not really survive in history.

Students may ask their teacher whether it is possible to find a "formula" for every free hand curve. Teachers should know something about the problem of finding analytical expressions for arbitrary (continuous) curves, i.e., that the concept of algebraic formula had to be extended in history in the direction of "analytical expressions", which included infinite series, integrals and other things. In the sense of the earlier quotation from Hermann Weyl, teachers may begin to appreciate that the modern mathematical language, that algorithmic and programming language representations of functions have again extended the repertoire of constructive building blocks usable to reproduce the various relations that can be found in the real world.

The representation of a curve by a formula is also a relevant question when the shape of all kinds of things is to be mathematically expressed: the field of computer graphics provides myriads of applications for this basic idea. In summary, teachers have to be enabled to add and integrate meaning to functions from their knowledge of several separate courses of academic mathematics they may have attended.

2.4.2 Different representations of functions

Working with different representations and relating them to each other is regarded as a basic element of a meaningful teaching and learning of functions. A "classical" aspect is the geometrical meaning of the coefficients of standard functions such as parabolas. For instance, Even (1989, pp. 127) assesses teacher-students' knowledge in this domain. However, interpreting the "subject matter meaning" of coefficients would be a further, often neglected step. Also, more complicated functions are also relevant. For instance, the following equation for logistic growth has to be interpreted according to the various coefficients (*K*, for instance, is the level of saturation).

$$f(x) = \frac{K}{1 + e^{-b(x-a)}}$$

This family of functions can be parameterised quite differently, and teacher-students have only limited experience in choosing an algebraic representation so as to make it better interpretable. Also, how a function as a whole "depends" on its parameters is an important dimension of meaning that is needed in various domains of application.

2.4.3 Functions and equations

The following equations express typical relations between quantities in geometry.

$$A = a \cdot b$$

$$C = 2\pi r$$

Equations in other domains of application are similar. An example is the basic equation in electricity between intensity of current, resistance and voltage:

$$V = R \cdot I$$

An important element of practice consists in interpreting these equations as relations between quantities without any unidirectionality of the function concept in the first place. Their interpretation as functions, however, is also very important but it is not unique. Each equation can be interpreted in various ways. The electricity formula can be interpreted, among other things, as

$(I,R) \mapsto V$, where $V = I \cdot R$, as a function of two variables
$I \mapsto V$, where $R = const.$,
$V \mapsto I$, where $R = const.$,
$R \mapsto I$, where $V = const.$, and $I = \frac{V}{R}$
$R \mapsto V$, where $I = const.$, and $V = I \cdot R$

Each interpretation may correspond to a different situation or problem in reality. Similar interpretations can be done with the geometrical formulas. There are several studies showing that such a flexible functional interpretation of formulas is an important prop required to understand the scientific use and meaning of formulas (for instance, Kriesi, 1981). This qualification is also relevant in pure mathematics where it may pay to see a formula from a new functional perspective. Re-evaluating and re-discovering this practice for school mathematics was also an achievement of didactical research (see Harten et al., 1986).

2.4.4 New tools for working with functions

Software broadens the range of operations that can effectively be performed with functions. Geometrical aspects of meaning conquer more importance. Some of the necessary shifts and problems in teachers' knowledge in these new conditions have been studied by Zbiek (1992).

Teachers have to be also aware of the following problems. Using software for dealing with mathematical notions and theories leads to the problem of a "computational transposition" (Balacheff, 1993): there are shifts of meaning due to transforming knowledge to another representational system. Software, for instance, usually does not handle functions as Platonic objects. They could be represented as finite list of numbers or pixels that approximate the exact values. A generating algorithm could lie behind it, or not. In addition, every software tool has its own set of admissible operations with functions that also determine the "meaning" of functions in this context. The computational transposition can be the source of "meaning conflicts" when students are working with the software. Winkelmann

(1988) provides an instructive overview about the many different implementations of functions in various pieces of software.

2.5 Central historical domains of application

Motion and curves formed essentially important domains of application in history. In the above Table 1, the place of curves and motion could be within mathematics or within applications, just as kinematics and geometry have the same status as applied mathematics from the point of view of formal symbolic mathematics. This leads to the general question what kind of historical knowledge on the development of meaning of a concept including epistemological obstacles is helpful and necessary for the didactics of mathematics and for teachers. Various approaches to this problem can be found elsewhere (Jahnke, Knoche, & Otte, 1996); a particular use of a historical context for meaning development is made by Bartolini Bussi (this volume). Historical domains of application may contribute to making the state of current academic mathematics more understandable than contemporary concept applications do, which already depend on that level of development. Teachers' knowledge on historical domains of application may have a specific cultural value as such and contribute to guaranteeing a cultural continuity in meaning transmission.

In the literature known to the author, there seems to be a certain bias in the historiography of the function concept, namely concentrating on the "pre-history" that led to the modern Dirichlet or Peano (set theoretic) definition of functions. From the standpoint of applied mathematics, other definitions and meanings were still co-existent. Also, the relation to related concepts such as correlative relation as contrasted to functional relation seems to be usually neglected in the historiography of the function concept.

2.5.1 Functions and curves

Curves were one of the key contexts in which the concept of function emerged. The univalence requirement and other factors like the relative marginal role of curves in new math as compared to other instances of the general concept of "mapping", led to a situation where curves and functions became quite separate things in mathematics education. Cartesian function graphs are, now, just one representation of the concept of function, whereas the idea that the concept of function is used to study curves, which are genuine geometrical objects with an existence of their own, independent of the concept of function, was nearly forgotten or at least devalued in mathematics education. The forgotten meanings and relations had to be reconstructed in didactical research (see Weth, 1993). Computers contribute to the possibility of using kinematic curves presented by animated computer graphics as a new meaningful context for learning the function concept (Stowasser et al., 1994). Computer use has also extended the relevance of "curves" in various directions. For instance, CAD (computer aided design) uses computer based mathematical representations of all kinds of curves and surfaces. In addition, fractal curves have added quite a new visual world to ours.

2.5.2 Functions and the study of motion

The historical emergence of the function concept is intimately related to the study of motion (kinetically and dynamically). Therefore, concepts of calculus and of differential equations were closely related to the new concept of function (Youschkevitch, 1976). These meaningful relations were also in the foreground, when Felix Klein favoured the reform of school mathematics under the banner of functional thinking. The concept of function was seen from the perspective of its meaning in calculus and uses of calculus in the sciences. Interestingly, these relations have been newly evaluated and re-defined in the didactical value systems recently (Kaput, 1993; Kaput, 1994). A newly conceptualised integration of the function concept, the study of motion, and preparatory calculus is being developed under the heading of "the mathematics of change". Time-dependent functions are now considered to be a very important prototype for developing an important element of the meaning of functions and also of the concept of a variable (Freudenthal, 1983; Weigand, 1988). Interestingly, there was a historical controversy about this question whether it makes sense to develop calculus without motion: "in point of intellectual conviction and certainty, the fluxional calculus is decidedly superior [to the French and German versions]; to think of calculus 'without motion' was akin to thinking of 'war without bloodshed, gardening without spades'" (From O. Gregory's 11th edition of C. Hutton's *Course of Mathematics* of 1837; quoted in Howson, 1982, p. 251).

Laws of motion are different from descriptions of motion as time dependent functions. The idea that local causes (forces) "act" at a point to influence the next "step" in a particle's movement is a basic idea underlying differential equations and dynamic systems in general. It is intimately related to the co-variational aspect of functions.

The historical expulsion of "time" from mathematics is challenged by the above suggestions. The current division of labor between disciplines that has brought forth new isles of meaning may not be the relevant separation for mathematics at teacher education and school level. Even if a reunification may be illusionary, teachers should be aware of interfaces, borderlines, and (historical) relationships as a background of their teaching in school.

2.6 Functions as models

If functions are used in a modelling context, all the concepts I have listed under the heading of "applications" and "related concepts in science" become relevant. Science teachers may be better acquainted with these concepts, especially if they have learned scientific research as a process together with some epistemological reflections. For mathematics teachers, however, *proof* as a condition for truth and established knowledge is most important, and the validity of other types of knowledge is difficult for them to judge. I will discuss some aspects of this problem in more detail.

2.6.1 Curve fitting

Curve fitting can be discussed in a purely mathematical context focusing on methods of fitting. Function plotter software has provided new possibilities for doing curve fitting easily. Family of curves described by a parameter set (for instance, family of parabolas) can be used as a repertoire to select from. Geometrical transformations acquire a new relevance in this context, because changing a parameter value into another can be interpreted as geometrical transformation of curves. Systems of equations for unknown parameters are another aspect. In older books for applied mathematics, the distinction whether a function should pass through all points or only "near" the points is basic under the unifying topic of fitting curves to data. Today, courses in mathematics do not necessarily cover these relations and meanings. From the perspective of applied mathematics and the sciences, concepts such as *interpolation* and *extrapolation* and the notion of the *quality of fit* and *range of validity* are important.

Relations to statistical methods (regression, methods of least squares) are also relevant. If a function fits the data well, on what basis can we extrapolate and how far? Teachers should know something about the scientific critique of curve fitting when it is practised without models from which the family of functions can be derived. Nevertheless, such fitted curves can yield excellent predictions (without understanding) in many cases. Even hand-fitted curves may be acceptable for certain purposes, there is no need for complicated fitting methods in every case. They may unjustifiably suggest the application of scientific methods. In sum, many of the above concepts and values do not belong to academic mathematics, but rather to practical mathematics, but they are nevertheless highly relevant for mathematics teachers. What is the domain of validity of extrapolation and interpolation? Do continuous functions describe the "nature" of the relation, or not? Should genuine discrete models be used instead? These are some of the components of the teachers' system of meaning for functions.

Teachers should also know something of the problems of using certain classes of functions for fitting curves: what are the limitations of polynomials? For many, including some software designers, the next "easy" choice beyond linear functions would be quadratic functions. However, polynomials are often not adequate and Splines are preferable. What is the basic idea of Splines? What about Bezier curves that are the underlying curves in many drawing programs?

In the context of curve fitting, geometrical aspects and geometrical classification of functions are acquiring a new meaning. It can be the case that functions having different algebraic representations are nevertheless very near to each other in small intervals and vice versa. Algebraic "near" is different from geometrical "near".

2.6.2 Univalence of functions and modelling

The meaning of univalence as a characterising property of functions is often discussed in relation to distinguishing functions from more general relations. In history, there have been several reasons for giving up the possibility of multivalent symbols such as $\sqrt{\ }$. Also, the curve of a circle is no longer considered as a function because it does not satisfy the "vertical line test". However, curves like the circle can be modelled as functions on a higher level (as mappings from [0, 1] into the plane). Many of the tests with students and teachers Even (1989) refers to are related to the univalence in the above context of meaning. But a further important context is the use of functions in modelling: Here, univalence is an idealisation or model assumption, and there are many cases, where there are several varying values that are associated with one value of the independent variable. Concepts and techniques from statistics are required for modelling in these situations. A deterministic function model (for every x there is exactly one y) has to be epistemologically distinguished from a mixed statistical-functional model where for every x several y are possible and where we can assume a probability distribution for the possible y's. This distribution in general is dependent of the variable x. However, teacher-students have usually not had enough experience in adequate domains of application to appreciate this. The same applies to many didacticians who have done research on the concept of function. In this context, relating functional dependence and correlational dependence adds to the meaning of functions. The *strength* of a relationship is a new perspective in addition to the *form* of a relationship that is expressed by usual functions (Biehler, 1995).

2.7 Various prototypical interpretation

A classification and identification of prototypical ways of interpreting functions (prototypical domains of application) which summarise essential aspects of the meaning (s) of functions would be helpful for meaning development. We can consider Vollrath's (1989) analysis in this perspective. I will add some aspects that are important in the context of modelling and statistics.

Epistemological distinctions should include that functions can be used to express:

- natural laws,
- causal relations,
- constructed relations,
- descriptive relations,
- data reductions.

These distinctions are quite important to avoid misinterpretations. The relation between the quantity and price of a certain article is a constructed relation: it is imposed by fiat (Davis & Hersh, 1980, pp. 70). Using a parabola to describe the path of a cannon ball has the character of a physical (natural) law. Contrary to this use, a parabola used in curve fitting may just provide a data summary of the curvature in a limited interval. Using functions for describing time dependent processes are different from using functions for expressing causal relations: time is not a "cause" for a certain movement. Also, scientists have partly abandoned the concept of causal relation in favour of mere "functional relation" between two quantities (Sierpinska, 1992). This may be due to philosophical reasons but also to simple pragmatical ones: If we have a 1–1 correspondence, we can invert the cause-effect functional relation to infer the "causes" from the effects.

In many statistical applications, functions are used to describe structure in a set of data that cannot be interpreted as a natural law: "Cartesian curve-fitting uses data to determine (comprehend) the structure (curves=laws) governing the universe. Statistical orientation uses curves (regularities) to determine (comprehend) the structure of concrete sets of data—data about phenomena that are important to understand in their own right." (Wainer & Thissen, 1981, p. 195). For instance, the graph in Figure 3 shows the synchronic relation between fuel prices and fuel consumption per inhabitant and per year in various countries of the world.

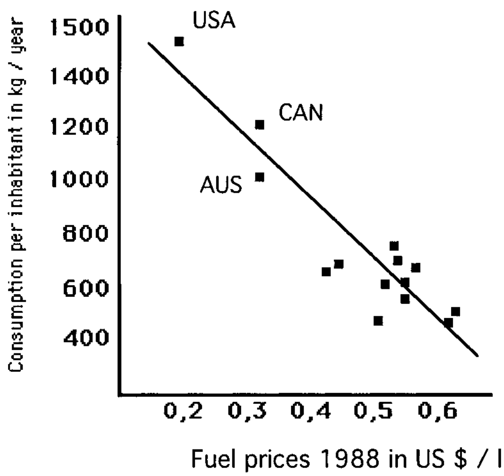

Figure 3: Fuel consumption and fuel prices in various countries
(Data from Weizsäcker, 1992)

If we interpret functions in a causal or natural law sense here, i.e., in the sense of "when we change x, then this results in the following change of y" this will be misleading: we have no direct evidence how the change of fuel price in one country would effect its fuel consumption. We would need diachronic data for that purpose. The above graph can only indicate some evidence. A second remark concerning the above figure: If we exclude North America and Australia from the graph, the rest of the data are only weakly correlated. Statistics requires a very flexible practice of fitting functions to data: excluding points from an analysis or fitting curves only to subsets can be successful tactics. These uses usually are not part of teachers' views of the meaning of functions. Functions are often still taught as if probability and statistics had never been invented.

We will finish our analyses of the meanings of the function concept vis-à-vis teacher education with these remarks. Although a lot has still to be done in doing further research respectively in synthesising research findings under our perspective of meaning reconstruction we hope that we were able to point to important further directions and extensions of current work.

3. SUMMARY AND CONCLUSION

The paper has started with arguing in favour of the thesis that we can re-interpret research and development work in mathematics education as "meaning construction" or "meaning reconstruction". The need is related to the differences between school and academic mathematics, and the situation that school mathematics cannot and should not take over the meaning of concepts in the context of academic mathematics. In the second part, we have looked at the concept of function as an example. A mathematics teacher in-service education course that stimulates enrichment and reorganisation of the meaning teacher-students associate with the notion of function, has provided a concrete context. Relevant but often neglected elements of a meaning landscape of functions have been sketched. The results may help to broaden the background on which we design studies on teachers' knowledge and beliefs about functions.

In addition to this, the paper argued for a systematic approach to the re-construction of the meanings of concepts as an important didactical task. A related research program should aim at knowledge that is less context-bound than knowledge on mathematical meanings that was developed in and for the context of designing concrete curricula and teacher education programs in some concrete reform movement in a very limited period in history.

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