Chapter 7

CUBIC AND OTHER EFFECTS

In this chapter we derive equations for cubic nonlinear effects. Some other effects not included in the general framework of Chapter 1 are also discussed.

1. CUBIC THEORY

1.1 Cubic Effects

By cubic theory we mean that effects of all terms up to the third power of the displacement and potential gradients or their products are included [6]. Cubic theory is an approximate theory for relatively weak nonlinearities, and can be obtained by expansions and truncations from the nonlinear theory in Chapter 1. From

$$X_M = \delta_{Mj} y_j - u_M , \tag{7.1-1}$$

by repeated use of the chain rule of differentiation, we obtain, to the second order in products of the derivative of u_M,

$$\begin{aligned} X_{M,i} &= \delta_{Mi} - u_{M,L} X_{L,i} = \delta_{Mi} - u_{M,L}(\delta_{Li} - u_{L,K} X_{K,i}) \\ &= \delta_{Mi} - u_{M,L}\delta_{Li} + u_{M,L} u_{L,K} X_{K,i} \\ &\cong \delta_{Mi} - u_{M,L}\delta_{Li} + u_{M,L} u_{L,K} \delta_{Ki} . \end{aligned} \tag{7.1-2}$$

From (1.1-16), retaining terms up to the second order in the derivative of u_M, we find

$$J \cong 1 + u_{K,K} + \frac{1}{2}(u_{K,K})^2 - \frac{1}{2} u_{K,L} u_{L,K} . \tag{7.1-3}$$

From (7.1-2) and (7.1-3)

$$\begin{aligned} JX_{L,i} \cong\ & \delta_{Li} - u_{L,R}\delta_{Ri} + \delta_{Li} u_{R,R} + u_{L,K} u_{K,R} \delta_{Ri} \\ & - u_{R,R} u_{L,K} \delta_{Ki} + \frac{1}{2}\delta_{Li} u_{K,K} u_{R,R} - \frac{1}{2}\delta_{Li} u_{K,R} u_{R,K} . \end{aligned} \tag{7.1-4}$$

From $(1.5\text{-}5)_2$, $(1.5\text{-}3)_2$, (1.5-11), and (7.1-4), retaining terms up to cubic in the small field variables, we obtain

$$F_{Lj} \cong \delta_{jM}\Big[\underset{2}{c}_{LMAB}\, u_{A,B} + e_{ALM}\phi_{,A} + \frac{1}{2}\underset{2}{c}_{LMAB}\, u_{K,A}u_{K,B}$$

$$+ \underset{2}{c}_{LKAB}\, u_{M,K}u_{A,B} + \frac{1}{2}\underset{3}{c}_{LMABCD}\, u_{A,B}u_{CD}$$

$$+ e_{ALK}u_{M,K}\phi_{,A} - \underset{1}{d}_{ABCLM}\, u_{B,C}\phi_{,A} - \frac{1}{2}b_{ABLM}\phi_{,A}\phi_{,B}$$

$$+ \frac{1}{2}\underset{2}{c}_{LRAB}\, u_{M,R}u_{K,A}u_{K,B} + \frac{1}{2}\underset{3}{c}_{LKABCD}\, u_{M,K}u_{A,B}u_{CD}$$

$$+ \frac{1}{2}\underset{3}{c}_{LMABcD}\, u_{A,B}u_{K,C}u_{K,D} + \frac{1}{6}\underset{4}{c}_{LMABCDEF}\, u_{A,B}u_{CD}u_{E,F}$$

$$- \underset{1}{d}_{ABCLK}\, u_{B,C}u_{M,K}\phi_{,A} - \frac{1}{2}\underset{1}{d}_{ABCLM}\, u_{K,B}u_{K,C}\phi_{,A}$$

$$- \frac{1}{2}\underset{2}{d}_{ABCDELM}\, u_{B,C}u_{D,E}\phi_{,A} - \frac{1}{2}b_{ABLK}u_{M,K}\phi_{,A}\phi_{,B}$$

$$+ \frac{1}{2}\underset{1}{a}_{ABCDLM}\, u_{c,D}\phi_{,A}\phi_{,B} + \frac{1}{6}\underset{3}{d}_{ABCLM}\, \phi_{,A}\phi_{,B}\phi_{,C} \Big], \tag{7.1-5}$$

and

$$\mathcal{P}_L \cong e_{LBC}u_{B,C} - \underset{2}{\chi}_{AL}\, \phi_{,A} + \frac{1}{2}e_{LBC}u_{K,B}u_{K,C}$$

$$- \frac{1}{2}\underset{1}{d}_{LBCDE}\, u_{B,C}u_{D,E} - b_{ALCD}u_{C,D}\phi_{,A}$$

$$+ \frac{1}{2}\underset{3}{\chi}_{ABL}\, \phi_{,A}\phi_{,B} - \frac{1}{2}\underset{1}{d}_{LBCDE}\, u_{B,C}u_{K,D}u_{K,E}$$

$$- \frac{1}{6}\underset{2}{d}_{LBCDEFG}\, u_{B,C}u_{D,E}u_{F,G} - \frac{1}{2}b_{ALCD}u_{K,C}u_{K,D}\phi_{,A}$$

$$+ \frac{1}{2}\underset{1}{a}_{ALCDEF}\, u_{C,D}u_{E,F}\phi_{,A} + \frac{1}{2}\underset{3}{d}_{ABLDE}\, u_{D,E}\phi_{,A}\phi_{,B} - \frac{1}{6}\underset{4}{\chi}_{ABCL}\, \phi_{,A}\phi_{,B}\phi_{,C}. \tag{7.1-6}$$

From $(1.5\text{-}5)_3$,(1.5-10), and (7.1-4):

$$M_{Lj} \cong \varepsilon_0 \delta_{jM} \Big[\phi_{,L}\phi_{,M} - \frac{1}{2}\phi_{,K}\phi_{,K}\delta_{LM} - \phi_{,K}\phi_{,M}u_{K,L}$$
$$- \phi_{,K}\phi_{,M}u_{L,K} + \phi_{,L}\phi_{,M}u_{K,K} - \phi_{,L}\phi_{,K}u_{K,M} \quad (7.1\text{-}7)$$
$$+ \phi_{,K}\phi_{,R}u_{R,K}\delta_{LM} + \frac{1}{2}\phi_{,K}\phi_{,K}u_{L,M} - \frac{1}{2}\phi_{,R}\phi_{,R}u_{K,K}\delta_{LM} \Big],$$

and

$$\varepsilon_0 J C^{-1}_{KL} \mathcal{E}_K \cong \varepsilon_0 \big[-\phi_{,L} + \phi_{,K}u_{L,K} - \phi_{,L}u_{K,K} + \phi_{,K}u_{K,L}$$
$$- \phi_{,M}u_{L,K}u_{K,M} + \phi_{,K}u_{M,M}u_{L,K} - \frac{1}{2}\phi_{,L}u_{K,K}u_{M,M} \quad (7.1\text{-}8)$$
$$+ \frac{1}{2}\phi_{,L}u_{K,M}u_{M,K} - \phi_{,M}u_{L,K}u_{M,K} + \phi_{,M}u_{M,L}u_{K,K} - \phi_{,M}u_{M,K}u_{K,L} \big].$$

Note that the fourth-order material constants are needed for a complete description of the cubic effects.

1.2 Quadratic Effects

If we keep terms up to the second order of the gradients only, we obtain the quadratic or second-order theory below:

$$F_{Lj} \cong \delta_{jM} \Big[\underset{2}{c}_{LMAB} u_{A,B} + e_{ALM}\phi_{,A} + \frac{1}{2}\underset{2}{c}_{LMAB} u_{K,A}u_{K,B}$$
$$+ \underset{2}{c}_{LKAB} u_{M,K}u_{A,B} + \frac{1}{2}\underset{3}{c}_{LMABCD} u_{A,B}u_{CD} \quad (7.1\text{-}9)$$
$$+ e_{ALK}u_{M,K}\phi_{,A} - \underset{1}{d}_{ABCLM} u_{B,C}\phi_{,A} - \frac{1}{2}b_{ABLM}\phi_{,A}\phi_{,B} \Big],$$

$$\mathcal{P}_L \cong e_{LBC}u_{B,C} - \underset{2}{\chi}_{AL} \phi_{,A} + \frac{1}{2}e_{LBC}u_{K,B}u_{K,C}$$
$$- \frac{1}{2}\underset{1}{d}_{LBCDE} u_{B,C}u_{D,E} - b_{ALCD}u_{C,D}\phi_{,A} + \frac{1}{2}\underset{3}{\chi}_{ABL} \phi_{,A}\phi_{,B}, \quad (7.1\text{-}10)$$

$$M_{Lj} \cong \varepsilon_0 \delta_{jM} [\phi_{,L}\phi_{,M} - \frac{1}{2}\phi_{,K}\phi_{,K}\delta_{LM}$$
$$- \phi_{,K}\phi_{,M}u_{K,L} - \phi_{,K}\phi_{,M}u_{L,K}], \quad (7.1\text{-}11)$$

$$\varepsilon_0 J C^{-1}_{KL} \mathcal{E}_K \cong \varepsilon_0 \big[-\phi_{,L} + \phi_{,K}u_{L,K} - \phi_{,L}u_{K,K} + \phi_{,K}u_{K,L} \big], \quad (7.1\text{-}12)$$

where the third-order material constants are needed.

2. NONLOCAL EFFECTS

2.1 Nonlocal Theory

Nonlocality comes from the consideration of long-range interactions. In one-dimensional lattice dynamics it has been shown that nonlocal theory includes, besides the interactions between neighboring atoms, interactions among non-neighboring atoms as well [41]. Nonlocality in constitutive relations is needed in modeling certain phenomena. Consider an electroelastic body V. Within the linear theory of piezoelectricity the nonlocal constitutive relations are given by [42]

$$\begin{aligned} T_{ij}(\mathbf{x}) &= \int_V [c_{ijkl}(\mathbf{x},\mathbf{x}')S_{kl}(\mathbf{x}') - e_{kij}(\mathbf{x},\mathbf{x}')E_k(\mathbf{x}')]dV(\mathbf{x}'), \\ D_i(\mathbf{x}) &= \int_V [e_{ikl}(\mathbf{x},\mathbf{x}')S_{kl}(\mathbf{x}') + \varepsilon_{ik}(\mathbf{x},\mathbf{x}')E_k(\mathbf{x}')]dV(\mathbf{x}'). \end{aligned} \tag{7.2-1}$$

As a special case, when the nonlocal material moduli are Dirac delta functions, Equations (7.2-1) reduce to the classical constitutive relations in (2.1-11). Substitution of (7.2-1) into the equation of motion and the charge equation results in integral-differential equations which are usually difficult to solve.

2.2 Thin Film Capacitance

In the following we give an example of what is probably the simplest nonlocal problem [43]. Consider an unbounded dielectric plate as shown in Figure 7.2-1. The plate is electroded and a voltage is applied. We want to obtain its capacitance from the nonlocal theory.

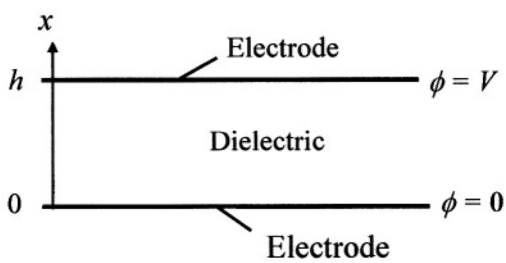

Figure 7.2-1. A thin dielectric plate.

The problem is one-dimensional. The boundary-value problem is

$$\frac{dD}{dx} = 0, \quad 0 < x < h,$$
$$D = \varepsilon_0 E + P, \quad 0 < x < h,$$
$$P = \varepsilon_0 \chi \int_0^h K(x', x) E(x') dx', \quad 0 < x < h, \tag{7.2-2}$$
$$E = -\frac{d\phi}{dx}, \quad 0 < x < h,$$
$$\phi(0) = 0, \quad \phi(h) = V.$$

When the kernel function $K(x', x)$ has the following special form

$$K(x', x) = \delta(x' - x), \tag{7.2-3}$$

Equation (7.2-2) reduces to the usual classical form. χ is the dimensionless relative electric susceptibility which differs from the one in (1.5-11) by a factor of ε_0. The dielectric material of the capacitor is assumed to be homogeneous and isotropic. Hence $K(x', x)$ must be invariant under translation and inversion. We have

$$K(x', x) = K(x' - x) = K(x - x'). \tag{7.2-4}$$

$K(x', x)$ should have a localized behavior, large near $x' = x$ and decaying away from there. We chose the following kernel function

$$K(x' - x) = \frac{1}{2\alpha} e^{-\frac{|x' - x|}{\alpha}}, \quad \alpha > 0, \tag{7.2-5}$$

where α is a microscopic parameter with the dimension of a length. It is a characteristic length of microscopic interactions. It is easy to verify that $K(x', x)$ has the following properties:

$$\lim_{\substack{\alpha \to 0^+ \\ x \neq x'}} K = 0, \quad \int_{-\infty}^{+\infty} K dx = 1. \tag{7.2-6}$$

Hence

$$\lim_{\alpha \to 0^+} K = \delta(x' - x), \tag{7.2-7}$$

which shows that $K(x', x)$ does include the local form as a limit case. We also note that the above $K(x', x)$ is the fundamental solution of the following differential operator

$$-\alpha^2 \frac{\partial^2 K}{\partial x^2} + K = \delta(x' - x). \tag{7.2-8}$$

Integrating $(7.2\text{-}2)_1$ once, with $(7.2\text{-}2)_{2,3}$ we obtain

$$D = \varepsilon_0 E(x) + \varepsilon_0 \chi \int_0^h K(x',x)E(x')dx' = -\sigma_e, \tag{7.2-9}$$

where σ_e is an integration constant which physically represents the surface free charge density on the electrode at $x = h$. Equation (7.2-9) can be written as

$$E(x) = -\chi \int_0^h K(x',x)E(x')dx' - \frac{\sigma}{\varepsilon_0}, \tag{7.2-10}$$

which is a Fredholm integral equation of the second kind for the electric field E. Instead of solving (7.2-10) directly, we proceed as follows. With (7.2-8), we differentiate (7.2-10) with respect to x twice and obtain

$$\begin{aligned}
\frac{d^2E(x)}{dx^2} &= -\chi \int_0^h \frac{\partial^2 K(x',x)}{\partial x^2} E(x')dx' \\
&= -\chi \int_0^h \frac{1}{\alpha^2}[K(x',x) - \delta(x'-x)]E(x')dx' \\
&= \frac{1}{\alpha^2}[-\chi \int_0^h K(x',x)E(x')dx' - \frac{\sigma_e}{\varepsilon_0}] \\
&\quad + \frac{1}{\alpha^2}\chi \int_0^h \delta(x'-x)E(x')dx' + \frac{1}{\alpha^2}\frac{\sigma_e}{\varepsilon_0} \\
&= \frac{1}{\alpha^2}[E(x) + \chi E(x) + \frac{\sigma_e}{\varepsilon_0}].
\end{aligned} \tag{7.2-11}$$

Hence a solution E of the integral equation (7.2-10) also satisfies the following differential equation

$$\alpha^2 \frac{d^2E}{dx^2} - (1+\chi)E = \frac{\sigma_e}{\varepsilon_0}. \tag{7.2-12}$$

The general solution to (7.2-12) can be obtained easily. It has two exponential terms from the corresponding homogeneous equation, and a constant term which is the particular solution. The general solution contains two new integration constants. These two integration constants result from the differentiation in obtaining the differential equation (7.2-12) from the original integral equation (7.2-10). Hence the solution to (7.2-12) may not satisfy (7.2-10). Therefore we substitute the general solution to (7.2-12) back into (7.2-10), which determines the two new integration constants. Then, with the boundary conditions $(7.2\text{-}2)_{5,6}$, we can determine σ_e and another integration constant resulting from integrating E for ϕ, and thus obtain the nonlocal electric potential distribution ϕ

$$\phi = [\frac{x}{h} + \frac{\chi}{kh} \frac{\sinh k(x - \frac{h}{2}) + \sinh \frac{kh}{2}}{\cosh \frac{kh}{2} + k\alpha \sinh \frac{kh}{2}}]$$

$$\times (1 + \frac{2\chi}{kh} \frac{\tanh \frac{kh}{2}}{1 + k\alpha \tanh \frac{kh}{2}})^{-1} V, \qquad (7.2\text{-}13)$$

$$\phi_0 = \frac{x}{h} V,$$

where ϕ_0 is the classical local solution, and

$$k = \frac{\sqrt{1+\chi}}{\alpha}. \qquad (7.2\text{-}14)$$

The nonlocal electric field distribution E and the local solution E_0 are

$$E = [1 + \chi \frac{\cosh k(x - \frac{h}{2})}{\cosh \frac{kh}{2} + k\alpha \sinh \frac{kh}{2}}](1 + \frac{2\chi}{kh} \frac{\tanh \frac{kh}{2}}{1 + k\alpha \tanh \frac{kh}{2}})^{-1} E_0, \qquad (7.2\text{-}15)$$

$$E_0 = -\frac{V}{h}.$$

Denoting the capacitance per unit electrode area from the local theory by C_0 and the one from the nonlocal theory by C, we have

$$C = (1 + \frac{2\chi}{kh} \frac{\tanh \frac{kh}{2}}{1 + k\alpha \tanh \frac{kh}{2}})^{-1} C_0, \quad C_0 = \frac{\varepsilon_0 (1+\chi)}{h}. \qquad (7.2\text{-}16)$$

With the expression of k in (7.2-14), we write $(7.16\text{-}1)_1$ in the following form:

$$\frac{C}{C_0} = (1 + \frac{\chi}{\sqrt{1+\chi}\frac{h}{2\alpha}} \frac{\tanh(\sqrt{1+\chi}\frac{h}{2\alpha})}{1 + \sqrt{1+\chi}\tanh(\sqrt{1+\chi}\frac{h}{2\alpha})})^{-1}. \qquad (7.2\text{-}17)$$

It is seen that the thin film capacitance from the nonlocal theory differs from the result of the local theory. The nonlocal solution depends on the ratio $h/2\alpha$ of the film thickness to the microscopic characteristic length. From (7.2-17) we immediately have

$$C/C_0 < 1, \tag{7.2-18}$$

which shows that the nonlocal result is smaller than the local result. From (7.2-17) we also have the following limit behavior

$$\lim_{\frac{h}{\alpha}\to\infty} \frac{C}{C_0} = 1, \tag{7.2-19}$$

which shows that when the film thickness is large compared to the microscopic characteristic length, the nonlocal solution approaches the local solution. We also have the limit

$$\lim_{\frac{h}{\alpha}\to 0} \frac{C}{C_0} = \frac{1}{1+\chi} < 1, \tag{7.2-20}$$

which shows that the nonlocal and local solutions differ more for materials with larger χ. We plot C/C_0 from (7.2-17) as a function of $h/2\alpha$ for values of $\chi = 1$, 10, and 100 in Figure 7.2-2. It is seen that for a film with a moderate value of $\chi = 100$, when the thickness $h/2\alpha \approx 10$, there is a deviation of about 10% from the local theory which has a fixed value of 1. The figure shows that $C/C_0 < 1$ and the deviation from 1 becomes larger as h becomes smaller and disappears when h is large:

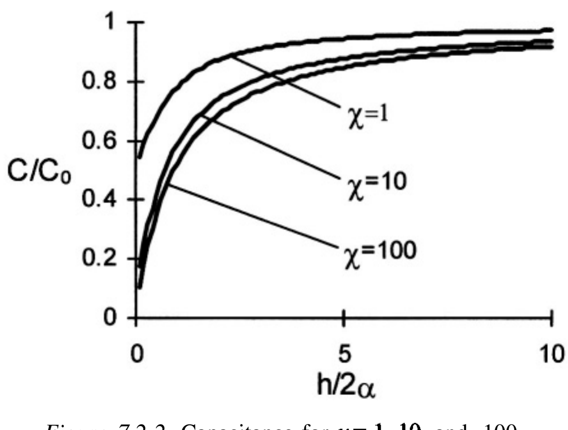

Figure 7.2-2. Capacitance for $\chi = 1$, 10, and 100.

The spatial distribution of the electric field for $\chi = 10$ and for two values of $h/2\alpha = 1$ and 5, respectively, is shown in Figure 7.2-3. It is interesting to see that the field is large near the electrodes compared to the local solution with the fixed value of 1. The curve with $h/2\alpha = 5$ has a larger electric field near the electrodes than the curve with $h/2\alpha = 1$. This is a boundary effect exhibited by the nonlocal theory. Even for a thick capacitor, (7.2-15) still yields

$$\lim_{\frac{h}{\alpha}\to\infty}\frac{E(h)}{E_0}=1+\frac{\chi}{1+\sqrt{1+\chi}}>1\,. \tag{7.2-21}$$

For our case, with $\chi = 10$, Equation (7.2-21) yields a limit value of 3.32. For materials with a large χ the value of (7.2-21) can be large. Since E is larger near the electrodes and D is a constant, P must be smaller near the electrodes than near the center of the plate.

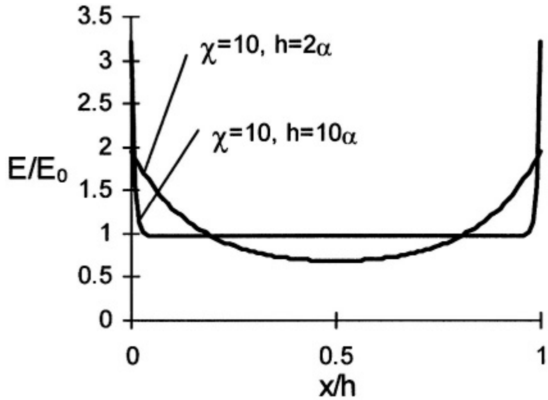

Figure 7.2-3. Electric field distribution for $\chi = 10$, $h/2\alpha = 1$ and 5.

The spatial distribution of the normalized deviation of the electric potential from the local solution for $\chi = 10$ and for two values of $h/2\alpha = 1$ and 5, respectively, are shown in Figure 7.2-4. The curve with $h/2\alpha = 5$ shows a smaller deviation.

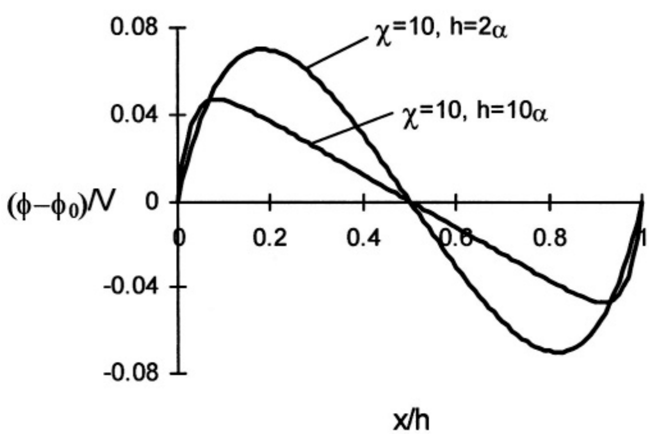

Figure 7.2-4. Electric potential deviation for $\chi = 10$, $h/2\alpha = 1$ and 5.

Finally, we note that in (7.2-12) the small parameter α appears as the coefficient of the term with the highest derivative. Hence when α tends to zero we have a singular perturbation problem of boundary layer type of a differential equation. For this type of problem, when the small parameter is set to zero, certain boundary conditions have to be dropped because the order of the differential equation is lowered. Equation (7.2-12) is a consequence of an integral-differential equation of ϕ defined by (7.2-2), which only needs two boundary conditions. In the solution procedure, two of the integration constants in the general solution to (7.2-12) were determined by the integral equation (7.2-10). However, if we take (7.2-12) as our starting point, we need two more boundary conditions. This is because (7.2-12) is a fourth-order differential equation for ϕ (considering it has already been integrated once with an integration constant σ_e). Then when α is set to zero, two boundary conditions have to be dropped.

3. GRADIENT EFFECTS

3.1 Gradient Effect as a Weak Nonlocal Effect

Gradient effects in constitutive relations can be shown to be related to weak nonlocal effects. For example, consider a one-dimensional nonlocal constitutive relation between Y and X in a homogeneous, unbounded medium. We have

$$\begin{aligned}
Y(x) &= \int_{-\infty}^{+\infty} K(x'-x)X(x')dx' \\
&= \int_{-\infty}^{+\infty} K(x'-x)X[x+(x'-x)]dx' \\
&= \int_{-\infty}^{+\infty} K(x'-x)[X(x)+X'(x)(x'-x)+\cdots]dx' \\
&\cong \int_{-\infty}^{+\infty} K(x'-x)[X(x)+X'(x)(x'-x)]d(x'-x) \\
&= \int_{-\infty}^{+\infty} K(x'-x)X(x)d(x'-x)+\int_{-\infty}^{+\infty} K(x'-x)X'(x)(x'-x)d(x'-x) \\
&= X(x)\int_{-\infty}^{+\infty} K(x'-x)d(x'-x)+X'(x)\int_{-\infty}^{+\infty} K(x'-x)(x'-x)d(x'-x) \\
&= aX(x)+bX'(x),
\end{aligned} \tag{7.3-1}$$

where

$$a = \int_{-\infty}^{+\infty} K(x'-x)d(x'-x),$$
$$b = \int_{-\infty}^{+\infty} K(x'-x)(x'-x)d(x'-x). \qquad (7.3\text{-}2)$$

Therefore, to the lowest order of approximation, the nonlocal relation reduces to a local one, and to the next order a gradient term arises.

3.2 Gradient Effect and Lattice Dynamics

Gradient terms can also be introduced in the following procedure. Consider the extensional motion of a one-dimensional spring-mass system (see Figure 7.3-1).

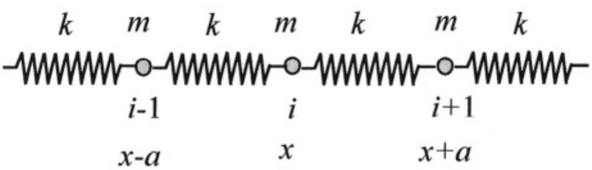

Figure 7.3-1. A spring-mass system.

The motion of the i-th particle is governed by the finite difference equation

$$m\ddot{u}(i) = k[u(i+1) - u(i)] - k[u(i) - u(i-1)], \qquad (7.3\text{-}3)$$

or, with the introduction of x

$$\begin{aligned} m\ddot{u}(x) &= k[u(x+a) + u(x-a) - 2u(x)] \\ &= k\Big[u(x) + u'(x)a + \frac{1}{2}u''(x)a^2 + \frac{1}{6}u'''(x)a^3 + \frac{1}{24}u''''(x)a^4 + \cdots \\ &\quad + u(x) - u'(x)a + \frac{1}{2}u''(x)a^2 - \frac{1}{6}u'''(x)a^3 \\ &\quad + \frac{1}{24}u''''(x)a^4 + \cdots - 2u(x)\Big] \\ &\cong k\left[u''(x)a^2 + \frac{1}{12}u''''(x)a^4\right] = T'(x), \end{aligned} \qquad (7.3\text{-}4)$$

where the extensional force T is given by the following constitutive relation

$$T = ka^2 u'(x) + \frac{ka^4}{12} u'''(x), \qquad (7.3\text{-}5)$$

which depends on the strain u' and its second gradient. It should be noted that, according to Mindlin [44], a continuum theory with the first strain gradient is fundamentally flawed in that it is qualitatively inconsistent with lattice dynamics and the second strain gradient needs to be included to correct the inconsistency.

3.3 Polarization Gradient

Mindlin [45] generalized the theory of piezoelectricity by allowing the stored energy density to depend on the polarization gradient $P_{j,i}$

$$\Pi(u_i, P_i, \phi) = \int_V \left[W(S_{ij}, P_i, P_{j,i}) - \frac{1}{2}\varepsilon_0 \phi_{,i}\phi_{,i} + \phi_{,i} P_i \right] dV, \qquad (7.3\text{-}6)$$

where boundary terms are dropped for simplicity. The stationary conditions of the above functional for independent variations of u_i, ϕ and P_i are

$$\begin{aligned} &\left(\frac{\partial W}{\partial S_{ij}}\right)_{,i} = 0, \\ &-\varepsilon_0 \phi_{,ii} + P_{i,i} = 0, \\ &-\frac{\partial W}{\partial P_i} + \left(\frac{\partial W}{\partial P_{j,i}}\right)_{,j} - \phi_{,i} = 0. \end{aligned} \qquad (7.3\text{-}7)$$

Equations (7.3-7) represent seven equations for u_i, P_i and ϕ. If the dependence of W on the polarization gradient is dropped, Equations (7.3-7) reduce to the theory of linear piezoelectricity. The inclusion of polarization gradient is supported by lattice dynamics [46,47]. The polarization gradient theory and lattice dynamics both predict the thin film capacitance to be smaller than the classical result [47], as shown in Figure 7.2-2.

3.4 Electric Field Gradient and Electric Quadrupole

3.4.1 Governing Equations

Electric field gradient can also be included in constitutive relations [48]. Electric field gradient theory is equivalent to the theory of dielectrics with electric quadrupoles [1], because electric quadrupole is the thermodynamic conjugate of the electric field gradient. Consider the following functional [49]

$$\Pi(u_i,\phi) = \int_V \left[W(S_{ij},E_i,E_{i,j}) - \frac{1}{2}\varepsilon_0 E_i E_i - f_i u_i + \rho_e \phi\right]dV$$
$$- \int_S \left(\bar{t}_i u_i + \bar{d}\phi + \bar{\pi}\frac{\partial\phi}{\partial\mathbf{n}}\right)dS, \tag{7.3-8}$$

where $\bar{d}$ is related to surface free charge. The presence of the $\bar{\pi}$ term is variationally consistent. We choose

$$W(S_{ij},E_i,E_{i,j}) - \frac{1}{2}\varepsilon_0 E_i E_i$$
$$= H(S_{ij},E_i) - \varepsilon_0 \gamma_{ijk} E_i E_{j,k} - \frac{1}{2}\varepsilon_0 \alpha_{ijkl} E_{i,j} E_{k,l}, \tag{7.3-9}$$

where H is the usual electric enthalpy function of piezoelectric materials given in (2.1-9), which is repeated below:

$$H(S_{ij},E_i) = \frac{1}{2}c_{ijkl}S_{ij}S_{kl} - e_{ikl}E_i S_{kl} - \frac{1}{2}\varepsilon_{ij}E_i E_j. \tag{7.3-10}$$

γ_{ijk} and α_{ijkl} are new material constants due to the introduction of the electric field gradient into the energy density function. γ_{ijk} has the dimension of length. α_{ijkl} has the dimension of (length)^2. Physically they may be related to characteristic lengths of microstructural interactions of the material. Since $E_{i,j} = E_{j,i}$, α_{ijkl} has the same structure as c_{ijkl} as required by crystal symmetry, and γ_{ijk} has the same structure as e_{ijk}. For W to be negative definite in the case of pure electric phenomena without mechanical fields, we require α_{ijkl} to be positive definite like ε_{ij}.

With the following constraints

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \tag{7.3-11}$$

from the variational functional in (7.3-8), for independent variations of u_i and ϕ in V, we have

$$T_{ji,j} + f_i = 0,$$
$$D_{i,i} = \rho_e, \tag{7.3-12}$$

where we have denoted

$$
\begin{aligned}
T_{ij} &= \frac{\partial W}{\partial S_{ij}} = c_{ijkl}S_{kl} - e_{kij}E_k, \\
D_i &= \varepsilon_0 E_i + P_i = \varepsilon_{ij}E_j + e_{ikl}S_{kl} \\
&\quad - \varepsilon_0(\gamma_{kij} - \gamma_{ijk})E_{j,k} - \varepsilon_0\alpha_{ijkl}E_{k,lj}, \\
P_i &= \Pi_i - \Pi_{ij,j} = \varepsilon_0\chi_{ij}E_j + e_{ikl}S_{kl} \\
&\quad - \varepsilon_0(\gamma_{kij} - \gamma_{ijk})E_{j,k} - \varepsilon_0\alpha_{ijkl}E_{k,lj}, \\
\Pi_i &= -\frac{\partial W}{\partial E_i} = e_{ikl}S_{kl} + \varepsilon_0\chi_{ij}E_j + \varepsilon_0\gamma_{ijk}E_{j,k}, \\
\Pi_{ij} &= -\frac{\partial W}{\partial E_{i,j}} = \varepsilon_0\gamma_{kij}E_k + \varepsilon_0\alpha_{ijkl}E_{k,l},
\end{aligned}
\tag{7.3-13}
$$

and $\varepsilon_{ij} = \varepsilon_0(\delta_{ij}+\chi_{ij})$. χ_{ij} is the relative electric susceptibility. When the energy density does not depend on the electric field gradient, the equations reduce to the linear theory of piezoelectricity. The first variation of the functional in (7.3-8) also implies the following as possible forms of boundary conditions on S

$$
\begin{aligned}
&T_{ji}n_j = \bar{t}_i \quad \text{or} \quad \delta u_i = 0, \\
&\int_S \left[(D_i n_i - \bar{d})\delta\phi + \Pi_{ij}n_j(\nabla_s\delta\phi)_i\right] dS = 0, \\
&\Pi_{ij}n_j n_i = \bar{\pi} \quad \text{or} \quad \delta\left(\frac{\partial\phi}{\partial \mathbf{n}}\right) = 0,
\end{aligned}
\tag{7.3-14}
$$

where ∇_s is the surface gradient operator. One obvious possibility of Equation **(7.3-14)**$_2$ is $\delta\phi = 0$ on S. With substitutions from (7.3-13) and (7.3-11), Equation (7.3-12) can be written as four equations for u_i and ϕ:

$$
\begin{aligned}
c_{ijkl}u_{k,lj} + e_{kij}\phi_{,kj} + f_i &= \rho\ddot{u}_i, \\
e_{ikl}u_{k,li} - \varepsilon_{ij}\phi_{,ij} + \varepsilon_0\alpha_{ijkl}\phi_{,ijkl} &= \rho_e,
\end{aligned}
\tag{7.3-15}
$$

where we have added the acceleration term.

3.4.2 Anti-Plane Problems of Ceramics

For anti-plane motions of polarized ceramics, Equations (7.3-15) reduce to a much simpler form. Consider

$$u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2, t),$$
$$\phi = \phi(x_1, x_2, t). \tag{7.3-16}$$

The non-vanishing strain and electric field components are

$$\begin{Bmatrix} S_5 \\ S_4 \end{Bmatrix} = \nabla u, \quad \begin{Bmatrix} E_1 \\ E_2 \end{Bmatrix} = -\nabla \phi. \tag{7.3-17}$$

For ceramics poled in the x_3 direction, the nontrivial components of T_{ij} and D_i are

$$\begin{Bmatrix} T_5 \\ T_4 \end{Bmatrix} = c\nabla u + e\nabla \phi,$$
$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = e\nabla u - \varepsilon \nabla \phi + \varepsilon_0 \alpha \nabla(\nabla^2 \phi), \tag{7.3-18}$$
$$D_3 = -\varepsilon_0 (\gamma_{31} - \gamma_{15}) \nabla^2 \phi,$$

where ∇^2 is the two-dimensional Laplacian, $c = c_{44}$, $e = e_{15}$, $\varepsilon = \varepsilon_{11}$, and $\alpha = \alpha_{11}$. The nontrivial ones of (7.3-15) take the form

$$c\nabla^2 u + e\nabla^2 \phi + f = \rho \ddot{u},$$
$$e\nabla^2 u - \varepsilon \nabla^2 \phi + \varepsilon_0 \alpha \nabla^2 \nabla^2 \phi = \rho_e, \tag{7.3-19}$$

where $f = f_3$.

3.4.3 Thin Film Capacitance

To see the most basic effects of the electric field gradient, consider the infinite plate capacitor shown in Figure 7.3-1.

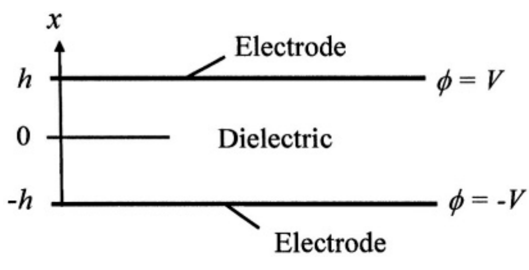

Figure 7.3-1. A thin dielectric plate.

The problem is one-dimensional. We assume that the material is isotropic so that there is no piezoelectric coupling. The equations and boundary conditions from the electric field gradient theory are

$$\begin{aligned} &\frac{dD}{dx} = 0, \quad -h < x < h, \\ &D = \varepsilon_0 E + P, \quad -h < x < h, \\ &P = \varepsilon_0 \chi E - \varepsilon_0 \alpha \frac{d^2 E}{dx^2}, \quad -h < x < h, \\ &E = -\frac{d\phi}{dx}, \quad -h < x < h, \\ &\phi(-h) = -V, \quad \phi(h) = V. \end{aligned} \tag{7.3-20}$$

From (7.3-20) the following equation for ϕ can be obtained:

$$\frac{d^4\phi}{dx^4} - k^2 \frac{d^2\phi}{dx^2} = 0, \tag{7.3-21}$$

where

$$k^2 = \frac{1+\chi}{\alpha}, \tag{7.3-22}$$

The general solution to (7.3-22) can be obtained in a straightforward manner. The anti-symmetric solution for ϕ is

$$\phi = C_1 x + C_2 \sinh kx, \tag{7.3-23}$$

where C_1 and C_2 are integration constants. Due to the introduction of the electric field gradient, the order of the equation for ϕ is now higher than the Laplace equation in the classical theory. Therefore more boundary conditions than in the classical theory are needed. Following Mindlin [47], we prescribe

$$P(x = h) = -\lambda \varepsilon_0 \chi \frac{V}{h}, \tag{7.3-24}$$

where $0 \le \lambda \le 1$ is a parameter. $\lambda = 1$ represents the classical solution. Equation (7.3-24) is for Mindlin's polarization theory. When it is directly introduced here for the electric field gradient theory, it is not variationally consistent. This can be resolved by translating it into a different form mathematically while still keeping its physical interpretation, which is left as an exercise. With the solution in (7.3-23), the boundary conditions in (7.3-20) and (7.3-24), and the identification of the relation between an integration constant and the surface charge on the electrode at $x = h$, we obtain the capacitance C per unit area, the potential ϕ and the electric field E as

$$\frac{C}{C_0} = \frac{1+\lambda\chi\dfrac{\tanh kh}{kh}}{1+\chi\dfrac{\tanh kh}{kh}}, \tag{7.3-25}$$

$$\frac{\phi}{V} = \frac{1+\lambda\chi\dfrac{\tanh kh}{kh}}{1+\chi\dfrac{\tanh kh}{kh}}\frac{x}{h} + \left(1-\frac{1+\lambda\chi\dfrac{\tanh kh}{kh}}{1+\chi\dfrac{\tanh kh}{kh}}\right)\frac{\sinh kx}{\sinh kh}, \tag{7.3-26}$$

$$\frac{E}{E_0} = \frac{1+\lambda\chi\dfrac{\tanh kh}{kh}}{1+\chi\dfrac{\tanh kh}{kh}} + \left(1-\frac{1+\lambda\chi\dfrac{\tanh kh}{kh}}{1+\chi\dfrac{\tanh kh}{kh}}\right)\frac{\cosh kx}{\sinh kh}kh, \tag{7.3-27}$$

where

$$C_0 = \frac{\varepsilon}{2h}, \quad E_0 = -\frac{V}{h}, \tag{7.3-28}$$

are the capacitance and electric field from the classical theory, and $\varepsilon = \varepsilon_0(1+\chi)$ is the electric permittivity. Equation (7.3-25) is exactly the same as the result of the polarization gradient theory [47], and its behavior is qualitatively the same as what is shown in Figure 7.2-2.

3.4.4 A Line Source

Consider the potential field of a line charge Q_e at the origin [50]. We need to solve Equation (7.3-19) with a concentrated electric source. Eliminating u we obtain

$$\begin{aligned} &-\bar{\varepsilon}\nabla^2\phi + \varepsilon_0\alpha\nabla^2\nabla^2\phi = Q_e\delta(x_1,x_2), \\ &\bar{\varepsilon} = \varepsilon(1+k^2), \quad k^2 = e^2/(\varepsilon c). \end{aligned} \tag{7.3-29}$$

Equation (7.3-29) can be rewritten as

$$(-\bar{\varepsilon} + \varepsilon_0\alpha\nabla^2)\nabla^2\phi = Q_e\delta(x_1,x_2). \tag{7.3-30}$$

Therefore $\nabla^2\phi$ is the fundamental solution of the differential operator in Equation (7.3-30), which is known. Hence

$$\nabla^2\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\frac{Q_e}{2\pi\varepsilon_0\alpha}K_0(\beta r), \tag{7.3-31}$$

where K_0 is the zero order modified Bessel function of the second-kind. Since

$$xK_0(x) = -\frac{d}{dx}[xK_1(x)], \quad K_1(x) = -\frac{d}{dx}[K_0(x)], \tag{7.3-32}$$

integrating Equation (7.3-31) twice we obtain

$$\phi = -\frac{Q_e}{2\pi\bar{\varepsilon}}[\ln r + K_0(\beta r)], \qquad \beta^2 = \bar{\varepsilon}/(\varepsilon_0 \alpha), \tag{7.3-33}$$

where the ln$r$ term is the classical solution. Since

$$K_0(x) \to -\ln x, \quad x \to 0, \qquad K_0(x) \to \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}, \quad x \to \infty, \tag{7.3-34}$$

we have

$$\phi \to \frac{Q_e}{2\pi\bar{\varepsilon}}\ln\beta = \frac{Q_e}{4\pi\bar{\varepsilon}}\ln\frac{\bar{\varepsilon}}{\varepsilon_0\alpha}, \quad r \to 0, \qquad \phi \to -\frac{Q_e}{2\pi\bar{\varepsilon}}\ln r, \quad r \to \infty. \tag{7.3-35}$$

The potential field is plotted in Figure 7.3-2.

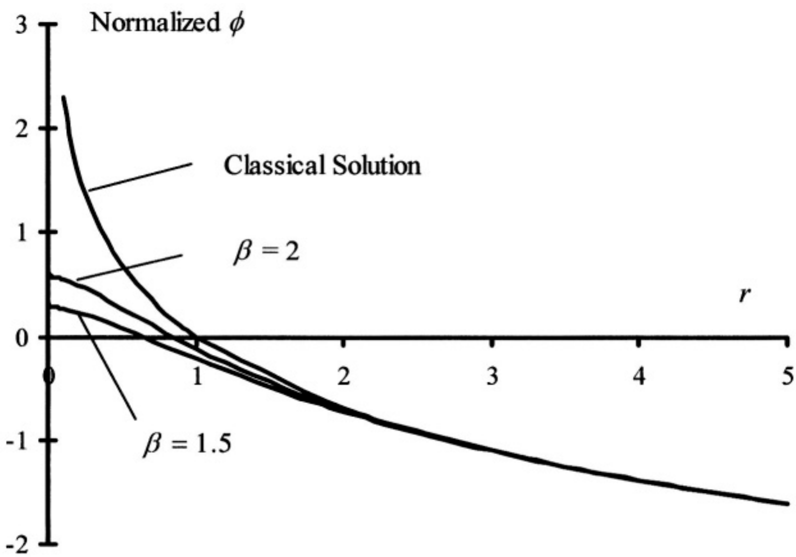

Figure 7.3-2. Normalized potential field ($-2\pi\bar{\varepsilon}\phi/Q_e$) of a line source.

For far field ϕ approaches the classical solution. At the source point ϕ is not singular. This is fundamentally different from the classical solution. When α approaches zero, Equation (7.3-33) reduces to the classical result. The curve with the larger value of β is closer to the classical solution. These qualitative behaviors are as expected.

3.4.5 Dispersion of Plane waves

In the source-free case, eliminating ϕ from (7.3-19) we obtain

$$\begin{aligned} &\bar{c}\nabla^2 u + \frac{\varepsilon_0}{\varepsilon}\alpha\nabla^2(\rho\ddot{u} - c\nabla^2 u) = \rho\ddot{u}, \\ &\bar{c} = c(1+k^2). \end{aligned} \tag{7.3-36}$$

Consider the propagation of the following plane wave

$$u = \exp[i(\xi x_1 - \omega t)]. \tag{7.3-37}$$

Substitution of Equation (7.3-37) into the homogeneous form of Equation (7.3-36) yields the following dispersion relation [50]

$$\omega^2 = \frac{c}{\rho}\xi^2 \frac{1 + k^2 + \frac{\varepsilon_0}{\varepsilon}\alpha\xi^2}{1 + \frac{\varepsilon_0}{\varepsilon}\alpha\xi^2}. \tag{7.3-38}$$

Different from the plane waves in linear piezoelectricity, Equation (7.3-38) shows that the waves are dispersive, and the dispersion is caused by the electric field gradient through electromechanical coupling. The dispersion disappears when $k = 0$, or when there is no electromechanical coupling. We note that the dispersion is more pronounced when $\xi\sqrt{\alpha}$ is not small, or when the wavelength $2\pi/\xi$ is not large when compared to the microscopic characteristic length $\sqrt{\alpha}$. When $\xi\sqrt{\alpha}$ just begins to show its effect, Equation (7.3-38) can be approximated by

$$\omega^2 \cong \frac{\bar{c}}{\rho}\xi^2\left[1 - \frac{k^2}{1+k^2}\frac{\varepsilon_0}{\varepsilon}\alpha\xi^2\right]. \tag{7.3-39}$$

As a numerical example we consider polarized ceramics PZT-7A. For polarized ferroelectric ceramics the grain size, which may be taken as the microscopic characteristic length $\sqrt{\alpha}$, is at sub-micron range. We plot

Equation (7.3-39) in Figure 7.3-3 for different values of $\sqrt{\alpha}$. It can be seen that larger values of $\sqrt{\alpha}$ yields more dispersion, as expected.

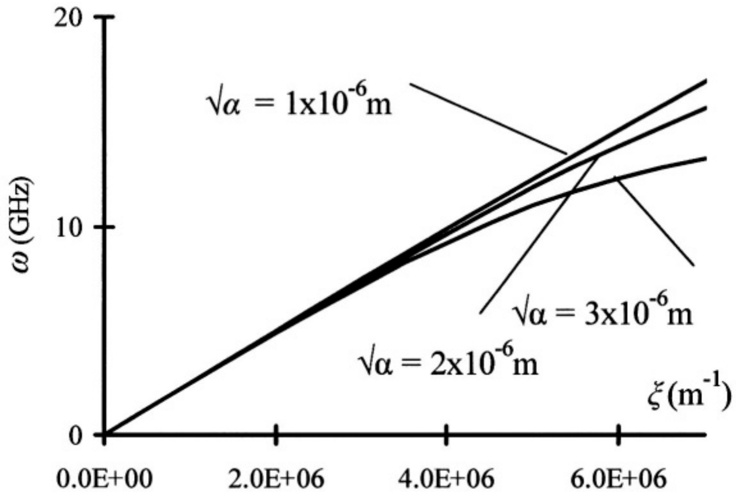

Figure 7.3-3. Dispersion curves of plane waves.

Problem

7.3-1. Study the capacitance of the dielectric plate in Figure 7.3-1 using the electric field gradient theory with the following additional boundary condition instead of (7.3-24)

$$E(x = h) = -\lambda \frac{V}{h}, \tag{7.3-40}$$

where $1 \leq \lambda$ is aparameter. $\lambda = 1$ represents the classical solution.

4. THERMAL AND VISCOUS EFFECTS

4.1 Equations in Spatial Form

Thermal and viscous effects often appear together and are treated in this section. The energy equation in the global balance laws in (1.2-3) needs to be extended to include thermal effects, and the second law of thermodynamics needs to be added as follows:

$$\frac{D}{Dt}\int_v \rho(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+e)dv$$
$$=\int_v [(\rho\mathbf{f}+\mathbf{F}^E)\cdot\mathbf{v}+w^E+\rho\gamma]dv+\int_s (\mathbf{t}\cdot\mathbf{v}-\mathbf{n}\cdot\mathbf{q})ds, \qquad (7.4\text{-}1)$$
$$\frac{D}{Dt}\int_v \rho\eta dv \geq \int_v \frac{\rho\gamma}{\theta}dv - \int_s \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}ds,$$

where $\mathbf{q}$ is the heat flux vector, η is the entropy per unit mass, γ is the body heat source per unit mass, and θ is the absolute temperature. The above integral balance laws can be localized to yield

$$\rho\dot{e} = \tau_{ij}v_{j,i} + \rho\gamma - q_{i,i} + w^E,$$
$$\rho\dot{\eta} \geq \frac{\rho\gamma}{\theta} - (\frac{q_i}{\theta})_{,i}. \qquad (7.4\text{-}2)$$

Eliminating γ in (7.4-2), we obtain the Clausius-Duhem inequality as

$$\rho(\theta\dot{\eta}-\dot{e}) + \tau_{ij}v_{j,i} + \rho E_i\dot{\pi}_i - \frac{q_i\theta_{,i}}{\theta} \geq 0. \qquad (7.4\text{-}3)$$

The free energy ψ can be introduced through the following Legendre transform:

$$\psi = e - \theta\eta - E_i\pi_i, \qquad (7.4\text{-}4)$$

then the energy equation $(7.4\text{-}2)_1$ and the C–D inequality (7.4-3) become

$$\rho(\dot{\psi}+\eta\dot{\theta}+\dot{\eta}\theta) = \tau_{ij}v_{j,i} - P_i\dot{E}_i + \rho\gamma - q_{i,i}, \qquad (7.4\text{-}5)$$

and

$$-\rho(\dot{\psi}+\eta\dot{\theta}) + \tau_{ij}v_{j,i} - P_i\dot{E}_i - \frac{q_i\theta_{,i}}{\theta} \geq 0. \qquad (7.4\text{-}6)$$

7.2 Equations in Material Form

Introducing the material heat flux and temperature gradient

$$Q_K = JX_{K,k}q_k, \quad \Theta_K = \theta_{,K} = \theta_{,k}y_{k,K}, \qquad (7.4\text{-}7)$$

the energy equation and the C–D inequality can be written as

$$\rho_0(\dot\psi + \eta\dot\theta + \dot\eta\theta) = T^S_{KL} E\dot S_{KL} - \mathcal{P}_K\dot{\mathcal{E}}_K + \rho_0\gamma - Q_{K,K},$$
$$-\rho_0(\dot\psi + \eta\dot\theta) + T^S_{KL}\dot S_{KL} - \mathcal{P}_K\dot{\mathcal{E}}_K - \frac{Q_K\Theta_K}{\theta} \geq 0. \tag{7.4-8}$$

7.3 Constitutive Relations

For constitutive relations we start with the following:

$$\begin{aligned}
\psi &= \psi(S_{KL}, \mathcal{E}_K, \theta, \Theta_K),\\
T^S_{KL} &= T^S_{KL}(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot S_{KL}, \dot{\mathcal{E}}_K),\\
\mathcal{P}_K &= \mathcal{P}_K(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot S_{KL}, \dot{\mathcal{E}}_K),\\
Q_K &= Q_K(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot S_{KL}, \dot{\mathcal{E}}_K).
\end{aligned} \tag{7.4-9}$$

Substitution of (7.4-9) into the C–D inequality $(7.4\text{-}8)_2$ yields

$$\begin{aligned}
&-\rho_0\frac{\partial\psi}{\partial\Theta_K}\dot\Theta_K - \rho_0\left(\eta + \frac{\partial\psi}{\partial\theta}\right)\dot\theta\\
&+\left(T^S_{KL} - \rho_0\frac{\partial\psi}{\partial S_{KL}}\right)S_{KL} - \left(\mathcal{P}_K + \rho_0\frac{\partial\psi}{\partial\mathcal{E}_K}\right)\dot{\mathcal{E}}_K - \frac{1}{\theta}Q_K\Theta_K \geq 0.
\end{aligned} \tag{7.4-10}$$

Since (7.4-10) is linear in $\dot\Theta_K$ and $\dot\theta$, for the inequality to hold ψ cannot depend on Θ_K, and η is related to ψ by

$$\eta = -\frac{\partial\psi}{\partial\theta}. \tag{7.4-11}$$

We break T^S_{KL} and $\mathcal{P}_K$ into reversible and dissipative parts as follows:

$$\begin{aligned}
&T^S_{KL} = T^R_{KL} + T^D_{KL}, \quad \mathcal{P}_K = \mathcal{P}^R_K + \mathcal{P}^D_K,\\
&T^R_{KL} = \rho_0\frac{\partial\psi}{\partial S_{KL}}, \quad \mathcal{P}^R_K = -\rho_0\frac{\partial\psi}{\partial\mathcal{E}_K},\\
&T^D_{KL} = T^D_{KL}(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot S_{KL}, \dot{\mathcal{E}}_K),\\
&\mathcal{P}^D_K = \mathcal{P}^D_K(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot S_{KL}, \dot{\mathcal{E}}_K).
\end{aligned} \tag{7.4-12}$$

Then what is left for the C–D inequality (7.4-10) is

$$T^D_{KL}\dot S_{KL} - \mathcal{P}^D_K\dot{\mathcal{E}}_K - \frac{1}{\theta}Q_K\Theta_K \geq 0. \tag{7.4-13}$$

From $(7.4\text{-}8)_1$ and (7.4-11) we obtain the heat equation

$$\rho_0\theta\dot\eta = T^D_{KL}\dot S_{KL} - \mathcal{P}^D_K\dot{\mathcal{E}}_K + \rho_0\gamma - Q_{K,K}. \tag{7.4-14}$$

4.4 Boundary-Value Problem

In summary, the nonlinear equations for thermoviscoelectroelasticity are

$$\begin{aligned} &\rho_0 = \rho J, \\ &K_{Lk,L} + \rho_0 f_k = \rho_0 \ddot{y}_k, \\ &\mathcal{D}_{K,K} = \rho_E, \\ &\rho_0 \theta \dot{\eta} = T^D_{KL} \dot{S}_{KL} - \mathcal{P}^D_K \dot{\mathcal{E}}_K + \rho_0 \gamma - Q_{K,K}, \end{aligned} \tag{7.4-15}$$

with constitutive relations

$$\begin{aligned} &\psi = \psi(S_{KL}, \mathcal{E}_K, \theta), \quad \eta = -\frac{\partial \psi}{\partial \theta}, \\ &T^S_{KL} = T^R_{KL} + T^D_{KL}, \quad \mathcal{P}_K = \mathcal{P}^R_K + \mathcal{P}^D_K, \\ &T^R_{KL} = \rho_0 \frac{\partial \psi}{\partial S_{KL}}, \quad \mathcal{P}^R_K = -\rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}, \\ &T^D_{KL} = T^D_{KL}(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot{S}_{KL}, \dot{\mathcal{E}}_K), \\ &\mathcal{P}^D_K = \mathcal{P}^D_K(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot{S}_{KL}, \dot{\mathcal{E}}_K). \\ &Q_K = Q_K(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot{S}_{KL}, \dot{\mathcal{E}}_K), \end{aligned} \tag{7.4-16}$$

which are restricted by

$$T^D_{KL} \dot{S}_{KL} - \mathcal{P}^D_K \dot{\mathcal{E}}_K - \frac{1}{\theta} Q_K \Theta_K \geq 0. \tag{7.4-17}$$

The equation for the conservation of mass in $(7.4\text{-}15)_1$ can be used to determine ρ separately from the other equations in (7.4-15). Equations $(7.4\text{-}15)_{2,3,4}$ can be written as five equations for $y_i(X_L, t)$, $\phi(X_L, t)$ and $\theta(X_L, t)$. On the boundary surface *S*, the thermal boundary conditions may be either prescribed temperature or heat flux

$$N_L Q_L = \overline{Q}. \tag{7.4-18}$$

4.5 Linear Equations

For small deformations and weak electric fields

$$\begin{aligned} &D_{i,i} = \rho_e, \\ &T_{ji,j} + \rho_0 f_i = \rho_0 \ddot{u}_i, \\ &\rho_0 \theta \dot{\eta} = T^D_{ij} \dot{S}_{ij} - P^D_i \dot{E}_i - q_{i,i}. \end{aligned} \tag{7.4-19}$$

The reversible part of the constitutive equations for small deformations and weak electric fields are determined by $\psi = \psi(S_{ij}, E_i, \theta)$ and

$$\eta = -\frac{\partial \psi}{\partial \theta}, \quad T_{ij}^R = \rho_0 \frac{\partial \psi}{\partial S_{ij}}, \quad P_i^R = -\rho_0 \frac{\partial \psi}{\partial E_i}. \tag{7.4-20}$$

In order to linearize the constitutive relations we expand ψ into a power series about $\theta = T_0$, $S_{ij} = 0$, and $E_i = 0$, where T_0 is a reference temperature. Denoting $T = \theta - T_0$, assuming $|T/T_0| << 1$, and keeping quadratic terms only, we can write

$$\begin{aligned} \rho_0 \psi = \frac{1}{2} c_{ijkl} S_{ij} S_{kl} - e_{kij} E_k S_{ij} - \frac{1}{2} \chi_{ij} E_i E_j \\ -\frac{1}{2}\frac{\alpha}{T_0} T^2 - \beta_{kl} S_{kl} T - \lambda_k E_k T, \end{aligned} \tag{7.4-21}$$

where β_{ij} are the thermoelastic constants, λ_k are the pyroelectric constants and α is related to the specific heat. Equations (7.4-20) and (7.4-21) yield

$$\begin{aligned} T_{ij}^R &= c_{ijkl} S_{kl} - e_{kij} E_k - \beta_{ij} T, \\ D_i^R &= \varepsilon_0 E_i + P_i^R = e_{ijk} S_{jk} + \varepsilon_{ij} E_j + \lambda_k T, \\ \rho_0 \eta &= \frac{\alpha}{T_0} T + \beta_{kl} S_{kl} + \lambda_k E_k, \end{aligned} \tag{7.4-22}$$

which are the equations for linear thermopiezoelectricity given by Mindlin [51]. For the dissipative part of the constitutive relations we choose the linear relations

$$\begin{aligned} T_{ij}^D &= \mu_{ijkl} \dot{S}_{kl} - \alpha_{kij} \dot{E}_k, \\ D_i^D &= P_i^D = \beta_{ijk} \dot{S}_{jk} + \zeta_{ij} \dot{E}_j, \\ q_k &= -\kappa_{kl} \theta_{,l}. \end{aligned} \tag{7.4-23}$$

In the following we will assume $\beta_{ijk} = \alpha_{ijk}$. Equations (7.4-23) are restricted by

$$T_{ij}^D \dot{S}_{ij} - P_i^D \dot{E}_i - \frac{q_i \theta_{,i}}{\theta} \geq 0. \tag{7.4-24}$$

A dissipation function can be introduced as follows:

$$\chi(\dot{S}_{kl}, \dot{E}_j) = \frac{1}{2} \mu_{ijkl} \dot{S}_{kl} \dot{S}_{ij} - \alpha_{kij} \dot{E}_k \dot{S}_{ij} - \frac{1}{2} \zeta_{ij} \dot{E}_i \dot{E}_j, \tag{7.4-25}$$

whereby Equations $(7.4\text{-}19)_3$, $(7.4\text{-}23)_{1,2}$ and (7.4-24) can be written as

$$\rho_0 T_0 \dot{\eta} = 2\chi(\dot{S}_{ij}, \dot{E}_i) - q_{i,i},$$
$$T_{ij}^D = \frac{\partial\chi}{\partial\dot{S}_{ij}}, \quad P_i^D = -\frac{\partial\chi}{\partial\dot{E}_i}, \tag{7.4-26}$$
$$2\chi(\dot{S}_{ij}, \dot{E}_i) + \frac{\kappa_{ij}\theta_{,i}\theta_{,j}}{\theta} \geq 0.$$

Equation **(7.4-26)**$_4$ implies that

$$\chi(\dot{S}_{kl}, \dot{E}_j) \geq 0, \quad \kappa_{ij}\theta_{,i}\theta_{,j} \geq 0, \tag{7.4-27}$$

which further implies that μ_{ijkl}, $-\zeta_{ij}$, and κ_{ij} are positive definite. The formal similarity between (7.4-25) and the first three terms on the right-hand side of (7.4-21) suggests that the structures of μ_{ijkl}, ζ_{ij}, and α_{ijk} are the same as those of c_{ijkl}, χ_{ij}, and e_{ijk}, which are known for various crystal classes.

When the thermoelastic and pyroelectric effects are small, they can be neglected. Then the above equations for the linear theory reduce to two one-way coupled systems of equations, where one represents the problem of viscopiezoelectricity with the following constitutive relations

$$T_{ij}^R = c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i^R = e_{ijk}S_{jk} + \varepsilon_{ij}E_j,$$
$$T_{ij}^D = \mu_{ijkl}\dot{S}_{kl} - \alpha_{kij}\dot{E}_k, \quad D_i^D = \alpha_{ijk}\dot{S}_{jk} + \zeta_{ij}\dot{E}_j, \tag{7.4-28}$$

and the other governs the temperature field

$$\rho_0 T_0 \dot{\eta} = T_{ij}^D \dot{S}_{ij} - D_i^D \dot{E}_i - q_{i,i},$$
$$\rho_0 \eta = \frac{\alpha}{T_0} T. \tag{7.4-29}$$

Equations (7.4-28) can be substituted into **(7.4-19)**$_{1,2}$ for four equations for y_i and ϕ. Once the mechanical and electric fields are found, they can be substituted into (7.4-29) to solve for the temperature field T.

Under harmonic excitation with an $e^{i\omega t}$ factor, the linear constitutive relations in (7.4-28) can be written as

$$T_{ij} = c_{ijkl}S_{kl} + \mu_{ijkl}\dot{S}_{kl} - e_{kuj}E_k - \alpha_{kij}\dot{E}_k$$
$$= (c_{ijkl} + i\omega\mu_{ijkl})S_{kl} - (e_{kuj} + i\omega\alpha_{kij})E_k,$$
$$D_i = e_{ijk}S_{jk} + \alpha_{ijk}\dot{S}_{jk} + \varepsilon_{ij}E_j + \zeta_{ij}\dot{E}_j \tag{7.4-30}$$
$$= (e_{ijk} + i\omega\alpha_{ijk})S_{jk} + (\varepsilon_{ij} + i\omega\zeta_{ij})E_j,$$

Formally, the material constants become complex and frequency-dependent.

5. SEMICONDUCTION

Piezoelectric materials are either dielectrics or semiconductors. Mechanical fields and mobile charges in piezoelectric semiconductors can interact, and this is called the acoustoelectric effect. An acoustic wave traveling in a piezoelectric semiconductor can be amplified by application of a dc electric field. The acoustoelectric effect and the acoustoelectric amplification of acoustic waves have led to piezoelectric semiconductor devices. The basic behavior of piezoelectric semiconductors can be described by a simple extension of the theory of piezoelectricity.

5.1 Governing Equations

Consider a homogeneous, one-carrier piezoelectric semiconductor under a uniform dc electric field $\overline{E}_j$. The steady state current is $\overline{J}_i = q\overline{n}\mu_{ij}\overline{E}_j$, where q is the carrier charge which may be the electronic charge or its opposite, $\overline{n}$ is the steady state carrier density which produces electrical neutrality,and μ_{ij} is the carrier mobility. When an acoustic wave propagates through the material, perturbations of the electric field, the carrier density and the current are denoted by E_j, n and J_i. The linear theory for small signals consists of the equations of motion, Gauss's law, and conservation of charge [52]

$$
\begin{aligned}
&T_{ji,j} = \rho \ddot{u}_i, \\
&D_{i,i} = qn, \\
&q\dot{n} + J_{i,i} = 0.
\end{aligned}
\tag{7.5-1}
$$

The above equations are accompanied by the following constitutive relations:

$$
\begin{aligned}
&T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \\
&D_i = e_{ijk}S_{jk} + \varepsilon_{ij}E_j, \\
&J_i = q\overline{n}\mu_{ij}E_j + qn\mu_{ij}\overline{E}_j - qd_{ij}n_{,j},
\end{aligned}
\tag{7.5-2}
$$

where d_{ij} are the carrier diffusion constants. Equations (7.5-1) can be written as five equations for $\mathbf{u}$, ϕ and n

$$
\begin{aligned}
&c_{ijkl}u_{k,lj} + e_{kij}\phi_{,kj} + f_i = \rho \ddot{u}_i, \\
&e_{ikl}u_{k,li} - \varepsilon_{ij}\phi_{,ij} = qn, \\
&\dot{n} - \overline{n}\mu_{ij}\phi_{,ij} + \mu_{ij}\overline{E}_j n_{,i} - d_{ij}n_{,ij} = 0.
\end{aligned}
\tag{7.5-3}
$$

On the boundary of a finite body with a unit outward normal n_i, the mechanical displacement u_i or the traction vector $T_{ij}n_i$, the electric potential ϕ or the normal component of the electric displacement vector D_in_i, and the carrier density n or the normal current J_in_i may be prescribed.

The Acoustoelectric effect and amplification of acoustic waves can also be achieved through composite structures of piezoelectric dielectrics and nonpiezoelectric semiconductors. In these composites the acoustoelectric effect is due to the combination of the piezoelectric effect and semiconduction in each component phase.

5.2 Surface Waves

As an example, consider the propagation of anti-plane surface waves in a piezoelectric dielectric half-space carrying a thin, nonpiezoelectric semiconductor film of silicon (see Figure 6.5-1) [53].

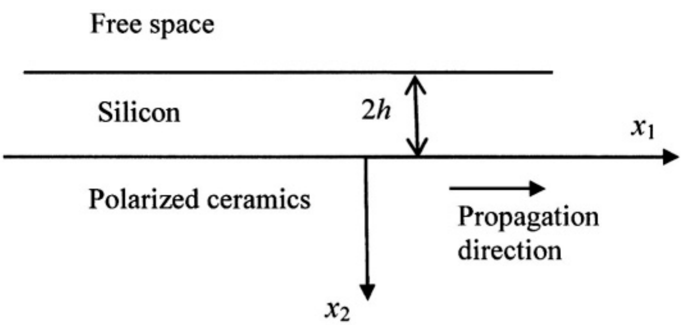

Figure 6.5-1. A ceramic half-space with a silicon film.

5.2.1 Equations for a Thin Film

The film is assumed to be very thin in the sense that its thickness is much smaller than the wavelength of the waves we are interested in. For thin films the following stress components can be approximately taken to vanish

$$T_{2j} = 0, \quad j = 1,2,3. \tag{7.5-4}$$

According to the compact matrix notation, with the range of p, q as 1,2, ... and 6, Equation (7.5-4) can be written as

$$T_q = 0, \quad q = 2,4,6. \tag{7.5-5}$$

For convenience we introduce a convention that subscripts u, v, w take the values 2, 4, 6 while subscripts r, s, t take the remaining values 1, 3, 5. Then Equation $(7.5\text{-}2)_{1,2}$ can be written as

$$\begin{aligned} T_r &= c_{rs}S_s + c_{ru}S_u - e_{kr}E_k, \\ T_v &= c_{vs}S_s + c_{vw}S_w - e_{kv}E_k = 0, \\ D_i &= e_{is}S_s + e_{iu}S_u + \varepsilon_{ij}E_j, \end{aligned} \tag{7.5-6}$$

where (7.5-5) has been used. From $(7.5\text{-}6)_2$ we have

$$S_u = -c_{uv}^{-1}c_{vs}S_s + c_{uv}^{-1}e_{kv}E_k. \tag{7.5-7}$$

Substitution of (7.5-7) into $(7.5\text{-}6)_{1,3}$ gives the constitutive relations for the film

$$\begin{aligned} T_r &= c_{rs}^p S_s - e_{kr}^p E_k, \\ D_i &= e_{is}^p S_s + \varepsilon_{ij}^p E_j, \end{aligned} \tag{7.5-8}$$

where the film material constants are

$$\begin{aligned} c_{rs}^p &= c_{rs} - c_{rv}c_{vw}^{-1}c_{ws}, \quad e_{ks}^p = e_{ks} - e_{kw}c_{wv}^{-1}c_{vs}, \\ \varepsilon_{kj}^p &= \varepsilon_{kj} + e_{kv}c_{vw}^{-1}e_{jw}. \end{aligned} \tag{7.5-9}$$

We now introduce another convention that subscripts a, b, c and d assume 1 and 3 but not 2. Then Equation (9.5-8) can be written as

$$\begin{aligned} T_{ab} &= c_{abcd}^p S_{cd} - e_{kab}^p E_k, \\ D_i &= e_{iab}^b S_{ab} + \varepsilon_{ij}^p E_j. \end{aligned} \tag{7.5-10}$$

Integrating the equations in $(7.5\text{-}1)_1$ for $i = 1,3$ and $(7.5\text{-}1)_{2,3}$ with respect to x_2 through the film thickness, we obtain the following two-dimensional equations of motion, Gauss's law and conservation of charge:

$$\begin{aligned} & T_{ab,a} + \frac{1}{2h}[T_{2b}(x_2 = h) - T_{2b}(x_2 = -h)] = \rho\ddot{u}_b, \\ & D_{a,a} + \frac{1}{2h}[D_2(x_2 = h) - D_2(x_2 = -h)] = qn, \\ & q\dot{n} + J_{a,a} + \frac{1}{2h}[J_2(x_2 = h) - J_2(x_2 = -h)] = 0, \end{aligned} \tag{7.5-11}$$

where u_a, T_{ab}, D_a, J_a and n are averages of the corresponding quantities along the film thickness.

5.2.2 Fields in the Ceramic Half-Space

From the equations in Section 6 of Chapter 3, the equations for the ceramic half-space are

$$\bar{c}_{44}\nabla^2 u_3 = \rho \ddot{u}_3, \qquad \nabla^2\psi = 0, \tag{7.5-12}$$

$$\psi = \phi - \frac{e_{15}}{\varepsilon_{11}} u_3, \tag{7.5-13}$$

and

$$\begin{aligned} T_{23} &= \bar{c}_{44}u_{3,2} + e_{15}\psi_{,2}, \\ T_{31} &= \bar{c}_{44}u_{3,1} + e_{15}\psi_{,1}, \\ D_1 &= -\varepsilon_{11}\psi_{,1}, \\ D_2 &= -\varepsilon_{11}\psi_{,2}, \end{aligned} \tag{7.5-14}$$

where

$$\bar{c}_{44} = c_{44} + \frac{e_{15}^2}{\varepsilon_{11}} = c_{44}(1 + k_{15}^2), \quad k_{15}^2 = \frac{e_{15}^2}{\varepsilon_{11}c_{44}}. \tag{7.5-15}$$

For a surface wave solution we must have

$$u_3, \phi \to 0, \quad x_2 \to +\infty. \tag{7.5-16}$$

Consider the possibility of solutions in the following form:

$$\begin{aligned} u_3 &= A\exp(-\xi_2 x_2)\exp[i(\xi_1 x_1 - \omega t)], \\ \psi &= B\exp(-\xi_1 x)\exp[i(\xi_1 x_1 - \omega t)], \end{aligned} \tag{7.5-17}$$

where A and B are undetermined constants, and ξ_2 should be positive for decaying behavior away from the surface. Equation $(7.5\text{-}17)_2$ already satisfies $(7.5\text{-}12)_2$. For $(7.5\text{-}17)_1$ to satisfy $(7.5\text{-}12)_1$ we must have

$$\bar{c}_{44}(\xi_1^2 - \xi_2^2) = \rho\omega^2, \tag{7.5-18}$$

which leads to the following expression for ξ_2

$$\xi_2^2 = \xi_1^2 - \frac{\rho\omega^2}{\bar{c}_{44}} = \xi_1^2(1 - \frac{v^2}{v_T^2}) > 0, \tag{7.5-19}$$

where

$$v^2 = \frac{\omega^2}{\xi_1^2}, \quad v_T^2 = \frac{\bar{c}_{44}}{\rho}. \tag{7.5-20}$$

The following are needed for prescribing boundary and continuity conditions:

$$\phi = [B\exp(-\xi_1 x_2) + \frac{e_{15}}{\varepsilon_{11}} A\exp(-\xi_2 x_2)]\exp[i(\xi_1 x_1 - \omega t)],$$

$$T_{23} = -[A\bar{c}_{44}\xi_2 \exp(-\xi_2 x_2) + e_{15} B\xi_1 \exp(-\xi_1 x_2)]\exp[i(\xi_1 x_1 - \omega t)], \qquad (7.5\text{-}21)$$

$$D_2 = \varepsilon_{11} B\xi_1 \exp(-\xi_1 x_2)\exp[i(\xi_1 x_1 - \omega t)].$$

5.2.3 Fields in the Free Space

Electric fields can also exist in the free space of $x_2 < 0$, which is governed by

$$\nabla^2\phi = 0, \quad x_2 < 0, \qquad \phi \to 0, \quad x_2 \to -\infty. \qquad (7.5\text{-}22)$$

A surface wave solution to (7.5-22) is

$$\phi = C\exp(\xi_1 x_2)\exp[i(\xi_1 x_1 - \omega t)], \qquad (7.5\text{-}23)$$

where C is an undetermined constant. From (7.5-23), in the free space

$$D_2 = -\varepsilon_0 \xi_1 C\exp(\xi_1 x_2)\exp[i(\xi_1 x_1 - \omega t)]. \qquad (7.5\text{-}24)$$

5.2.4 Fields in the Semiconductor Film

The semiconductor film is one-dimensional with $n = n(x_1, t)$. Consider the case when the dc biasing electric field is in the x_1 direction. Let

$$u_3 = A\exp[i(\xi_1 x_1 - \omega t)], \quad \phi = C\exp[i(\xi_1 x_1 - \omega t)], \qquad n = N\exp[i(\xi_1 x_1 - \omega t)], \qquad (7.5\text{-}25)$$

where N is an undetermined constant. Equation (7.5-25) already satisfies the continuity of displacement between the film and the ceramic half-space, and the continuity of electric potential between the film and the free space. We use a prime to indicate the elastic and dielectric constants as well as the mass density of the film. Silicon is a cubic crystal with m3m symmetry. The elastic and dielectric constants are given by

$$\begin{pmatrix} c'_{11} & c'_{12} & c'_{12} & 0 & 0 & 0 \\ c'_{12} & c'_{11} & c_{12} & 0 & 0 & 0 \\ c'_{12} & c'_{12} & c'_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c'_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c'_{44} \end{pmatrix}, \begin{pmatrix} \varepsilon'_{11} & 0 & 0 \\ 0 & \varepsilon'_{11} & 0 \\ 0 & 0 & \varepsilon'_{11} \end{pmatrix}. \qquad (7.5\text{-}26)$$

From (7.5-10) and (7.5-2)$_3$ we obtain:

$$
\begin{aligned}
T_{13} &= c_{55}^{p} S_{13} = c_{44}^{p} u_{3,1} = c_{44}^{p} i\xi_1 A \exp[i(\xi_1 x_1 - \omega t)], \\
D_1 &= \varepsilon_{11}^{p} E_1 = -\varepsilon_{11}^{p} \phi_{,1} = -\varepsilon_{11}^{p} i\xi_1 C \exp[i(\xi_1 x_1 - \omega t)], \\
J_1 &= -q\bar{n}\mu_{11}\phi_{,1} + qn\mu_{11}\bar{E}_1 - qd_{11}n_{,1} \\
&= (-q\bar{n}\mu_{11} i\xi_1 C + qN\mu_{11}\bar{E}_1 - qd_{11} i\xi_1 N)\exp[i(\xi_1 x_1 - \omega t)].
\end{aligned}
\tag{7.5-27}
$$

5.2.5 Continuity Conditions and Dispersion Relation

Substitution of (7.5-21), (7.5-23), (7.5-24), (7.5-25) and (7.5-27) into the continuity condition of the electric potential between the ceramic half-space and the film, (7.5-11)$_1$ for $b = 3$, and (7.5-11)$_{2,3}$ yields

$$
\begin{aligned}
& B + \frac{e_{15}}{\varepsilon_{11}} A = C, \\
& -c_{44}^{p}\xi_1^2 A - \frac{1}{2h}(A\bar{c}_{44}\xi_2 + e_{15}B\xi_1) = -\rho'\omega^2 A, \\
& \varepsilon_{11}^{p}\xi_1^2 C + \frac{1}{2h}(\varepsilon_{11}B\xi_1 + \varepsilon_0\xi_1 C) = qN, \\
& -qi\omega N + i\xi_1(-q\bar{n}\mu_{11} i\xi_1 C + qN\mu_{11}\bar{E}_1 - qd_{11} i\xi_1 N) = 0,
\end{aligned}
\tag{7.5-28}
$$

which is a system of linear, homogeneous equations for *A, B, C* and *N.* For nontrivial solutions the determinant of the coefficient matrix has to vanish

$$
\begin{vmatrix}
\dfrac{e_{15}}{\varepsilon_{11}} & 1 & -1 & 0 \\
\rho'\omega^2 - c_{44}^{p}\xi_1^2 - \dfrac{\bar{c}_{44}\xi_2}{2h} & -\dfrac{e_{15}\xi_1}{2h} & 0 & 0 \\
0 & \dfrac{\varepsilon_{11}\xi_1}{2h} & \dfrac{\varepsilon_0\xi_1}{2h} + \varepsilon_{11}^{p}\xi_1^2 & -q \\
0 & 0 & q\bar{n}\mu_{11}\xi_1^2 & \begin{array}{c} -qi\omega + i\xi_1 q\mu_{11}\bar{E}_1 \\ + qd_{11}\xi_1^2 \end{array}
\end{vmatrix} = 0,
\tag{7.5-29}
$$

which determines the dispersion relation, a relation between ω and ξ_1, of the surface wave. In terms of the surface wave speed $v = \omega / \xi_1$, Equation (7.5-29) can be written in the following form:

$$\left(\frac{v^2}{v_T'^2}-1\right)\frac{c_{44}^p}{\bar{c}_{44}}2h\xi_1-\sqrt{1-\frac{v^2}{v_T^2}}+\bar{k}_{15}^2$$
$$=\frac{\bar{k}_{15}^2}{1+\dfrac{\varepsilon_0}{\varepsilon_{11}}+\dfrac{\varepsilon_{11}^p}{\varepsilon_{11}}\xi_1\,2h+\dfrac{q\bar{n}\mu_{11}2h}{\varepsilon_{11}[d_{11}\xi_1\;+i(\mu_{11}\bar{E}_1-v)]}}, \tag{7.5-30}$$

where (7.5-19) has been used, and

$$v_T'^2=\frac{c_{44}^p}{\rho'},\quad \bar{k}_{15}^2=\frac{e_{15}^2}{\varepsilon_{11}\bar{c}_{44}}. \tag{7.5-31}$$

When $h = 0$, i.e., the semiconductor film does not exist, (7.5-30) reduces to

$$v^2=v_T^2\left[1-\frac{\bar{k}_{15}^4}{(1+\varepsilon_{11}/\varepsilon_0)^2}\right]=v_{B-G}^2, \tag{7.5-32}$$

which is the speed of the Bleustein-Gulyaev wave in (5.3-20).

When $\bar{k}_{15}^2=0$, i.e., the half-space is non-piezoelectric, electromechanical coupling disappears and the wave is purely elastic. In this case Equation (7.5-30) reduces to

$$\left(\frac{v^2}{v_T'^2}-1\right)\frac{c_{44}^p}{\bar{c}_{44}}2h\xi_1-\sqrt{1-\frac{v^2}{v_T^2}}=0, \tag{7.5-33}$$

which is the equation that determines the speed of Love wave (an anti-plane surface wave in an elastic half-space carrying an elastic layer) in the limit when the film is very thin compared to the wavelength ($\xi_1 h \ll 1$). Love waves are known to exist when the elastic stiffness of the layer is smaller that that of the half-space.

The denominator of the right hand side of (7.5-30) indicates that a complex wave speed may be expected and the imaginary part of the complex wave speed may change its sign (transition from a damped wave to a growing wave) when $\mu_{11}\bar{E}_1\xi_1-\omega$ changes sign or

$$v=\frac{\omega}{\xi_1}=\mu_{11}\bar{E}_1, \tag{7.5-34}$$

i.e., the acoustic wave speed is equal to the carrier drift speed [52].

When semiconduction is small, Equation (7.5-30) can be solved by an iteration or perturbation procedure. As the lowest (zero) order of approximation, we neglect the small semiconduction and denote the zero-order solution by $v_{(0)}$. Then from (7.5-30),

$$\left(\frac{v_{(0)}^2}{v_T'^2}-1\right)\frac{c_{44}^p}{\bar{c}_{44}}2h\xi_1-\sqrt{1-\frac{v_{(0)}^2}{v_T^2}+\bar{k}_{15}^2} = \frac{\bar{k}_{15}^2}{1+\dfrac{\varepsilon_0}{\varepsilon_{11}}+\dfrac{\varepsilon_{11}^p}{\varepsilon_{11}}\xi_1\,2h}, \tag{7.5-35}$$

which is dispersive. For the next order, we substitute $v_{(0)}$ into the right-hand side of (7.5-30) and obtain the following equation for $v_{(1)}$

$$\left(\frac{v_{(1)}^2}{v_T'^2}-1\right)\frac{c_{44}^p}{\bar{c}_{44}}2h\xi_1-\sqrt{1-\frac{v_{(1)}^2}{v_T^2}+\bar{k}_{15}^2} = \frac{\bar{k}_{15}^2}{1+\dfrac{\varepsilon_0}{\varepsilon_{11}}+\dfrac{\varepsilon_{11}^p}{\varepsilon_{11}}\xi_1\,2h+\dfrac{q\bar{n}\mu_{11}2h}{\varepsilon_{11}[d_{11}\xi_1\ +i(\mu_{11}\bar{E}_1-v_{(0)})]}}, \tag{7.5-36}$$

which suggests a wave that is both dispersive and dissipative.

For numerical results consider PZT-5H. Since $c_{44}' > c_{44}$, the counterpart of the elastic Love wave does not exist, but a modified Bleustein-Gulyaev wave is expected. We plot the real parts of $v_{(0)}$ and $v_{(1)}$ versus ξ_1 in Figure 7.5-2. The dimensionless wave number X and the dimensionless wave speed Y of different orders are defined by

$$\begin{aligned} X &= \xi_1\Big/\frac{\pi}{2h}, \\ Y_{(0)} &= v_{(0)}\big/v_{B-G}, \\ Y_{(1)} &= \mathrm{Re}\{v_{(1)}\}/v_{B-G}. \end{aligned} \tag{7.5-37}$$

γ is a dimensionless number given by

$$\gamma = \mu_{11}\bar{E}_1\big/v_{B-G}, \tag{7.5-38}$$

which may be considered as a normalized electric field. It represents the ratio of the carrier drift velocity and the speed of the Bleustein-Gulyave wave. Because of the use of thin film equations for the semiconductor film, the solution is valid only when the wavelength is much larger than the film thickness ($X << 1$). It can be seen that semiconduction causes additional dispersion. This conduction induced dispersion varies according to the dc biasing electric field.

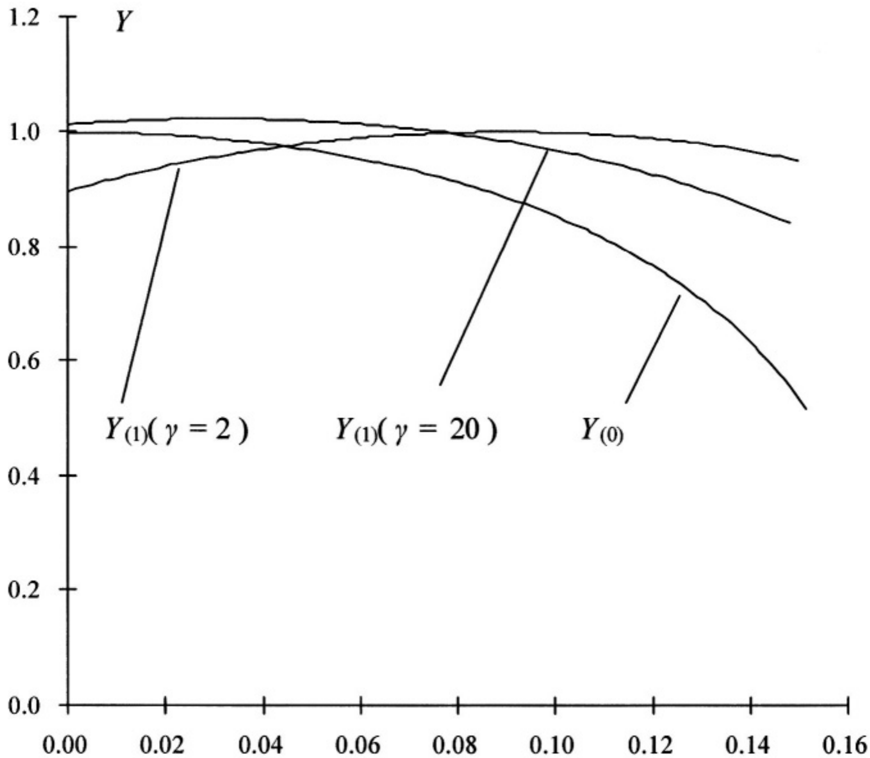

Figure 7.5-2. Dispersion relations.

Figure 7.5-3 shows the imaginary part of $v_{(1)}$ versus γ. The dimensionless number describing the decaying behavior of the waves is defined by

$$Y = \mathrm{Im}\{v_{(1)}\}/v_{B-G}\ . \tag{7.5-39}$$

When the dc bias is large enough (approximately $\gamma > 1$) the decay constant becomes negative indicating wave amplification. The transition from damped waves to growing waves indeed occurs when (7.5-34) is true for $v_{(0)}$.

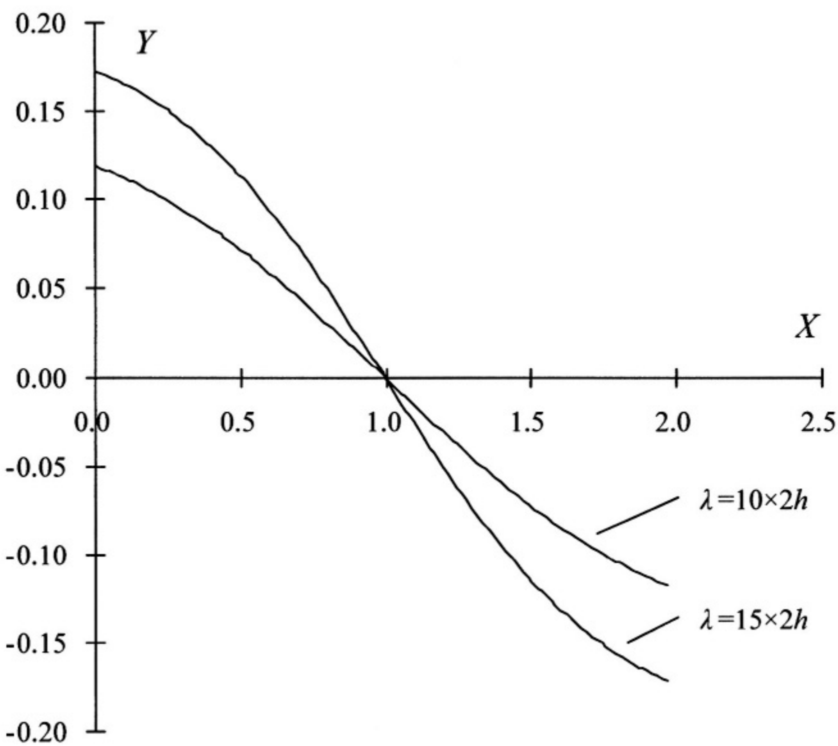

Figure 7.5-3. Dissipation as a function of the dc bias.

6. DYNAMIC THEORY

The theory of linear piezoelectricity is based on the quasistatic approximation. In piezoelectricity theory, the mechanical equations are dynamic but the electromagnetic equations appear to be static. The electric field and the magnetic field are not directly coupled in Maxwell's equations. When the complete set of Maxwell's equations is included, the fully dynamic theory is called piezoelectromagnetism [54].

6.1 Governing Equations

For a piezoelectric but nonmagnetizable dielectric body, the three-dimensional equations of linear piezoelectromagnetism consist of the equations of motion and Maxwell's equations, as shown by

$$\begin{aligned} &T_{ji,j} + \rho f_i = \rho \ddot{u}_i, \\ &\varepsilon_{ijk} E_{k,j} = -\dot{B}_i, \quad \varepsilon_{ijk} H_{k,j} = \dot{D}_i, \\ &B_{i,i} = 0, \quad D_{i,i} = 0, \end{aligned} \tag{7.6-1}$$

as well as the following constitutive relations

$$\begin{aligned} &T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \\ &D_i = e_{ijk} S_{jk} + \varepsilon_{ij} E_j, \\ &B_i = \mu_0 H_i, \end{aligned} \tag{7.6-2}$$

where B_i is the magnetic induction, H_i is the magnetic field, and μ_0 is the magnetic permeability of free space. With Equation (7.6-2), Equation (7.6-1) becomes

$$\begin{aligned} &c_{ijkl} u_{k,li} - e_{kij} E_{k,i} = \rho \ddot{u}_j, \\ &\varepsilon_{ijk} E_{k,j} = -\dot{B}_i, \\ &\frac{1}{\mu_0} \varepsilon_{ijk} B_{k,j} = e_{ikl} \dot{u}_{k,l} + \varepsilon_{ik} \dot{E}_k. \end{aligned} \tag{7.6-3}$$

6.2 Quasistatic Approximation

The quasistatic approximation made in Section 2 of Chapter 1 can be considered as the lowest order approximation of the dynamic theory given by (7.6-3) through the following perturbation procedure [5]. Consider an acoustic wave with frequency ω in a piezoelectric crystal of size L. We scale the various independent and dependent variables with respect to characteristic quantities

$$\begin{aligned} &\xi_i = \frac{x_i}{L}, \quad \tau = \omega t, \\ &U_i = \frac{u_i}{L}, \quad b_i = cB_i, \end{aligned} \tag{7.6-4}$$

where

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}, \tag{7.6-5}$$

is the speed of light in free space and the scaling yields a **b** in the same units as **E**. Then Equation (7.6-3) takes the following form:

$$\frac{1}{L}c_{ijkl}\frac{\partial^2 U_k}{\partial\xi_l\partial\xi_i}-\frac{1}{L}e_{kij}\frac{\partial E_k}{\partial\xi_i}=\rho\omega^2 L\frac{\partial^2 U_j}{\partial\tau^2},$$

$$\frac{1}{L}\varepsilon_{ijk}\frac{\partial E_k}{\partial\xi_j}=-\frac{\omega}{c}\frac{\partial b_i}{\partial\tau}, \tag{7.6-6}$$

$$\frac{1}{cL\mu_0}\varepsilon_{ijk}\frac{\partial b_k}{\partial\xi_j}=\omega\varepsilon_0\frac{e_{ikl}}{\varepsilon_0}\frac{\partial^2 U_k}{\partial\xi_l\partial\tau}+\omega\varepsilon_0\frac{\varepsilon_{ik}}{\varepsilon_0}\frac{\partial E_k}{\partial\tau},$$

or

$$c_{ijkl}\frac{\partial^2 U_k}{\partial\xi_l\partial\xi_i}-e_{kij}\frac{\partial E_k}{\partial\xi_i}=\rho\omega^2 L^2\frac{\partial^2 U_j}{\partial\tau^2},$$

$$\varepsilon_{ijk}\frac{\partial E_k}{\partial\xi_j}=-\eta\frac{\partial b_i}{\partial\tau}, \tag{7.6-7}$$

$$\varepsilon_{ijk}\frac{\partial b_k}{\partial\xi_j}=\eta\left(\frac{e_{ikl}}{\varepsilon_0}\frac{\partial^2 U_k}{\partial\xi_l\partial\tau}+\frac{\varepsilon_{ik}}{\varepsilon_0}\frac{\partial E_k}{\partial\tau}\right),$$

where

$$\eta=\frac{\omega L}{c}<<1. \tag{7.6-8}$$

To the lowest order

$$c_{ijkl}\frac{\partial^2 U_k}{\partial\xi_l\partial\xi_i}-e_{kij}\frac{\partial E_k}{\partial\xi_i}=\rho\omega^2 L^2\frac{\partial^2 U_j}{\partial\tau^2},$$

$$\varepsilon_{ijk}\frac{\partial E_k}{\partial\xi_j}=0, \tag{7.6-9}$$

$$\varepsilon_{ijk}\frac{\partial b_k}{\partial\xi_j}=0,$$

or

$$c_{ijkl}u_{k,li}-e_{kij}E_{k,i}=\rho\ddot{u}_j,$$

$$\varepsilon_{ijk}E_{k,j}=0, \tag{7.6-10}$$

$$\varepsilon_{ijk}H_{k,j}=0.$$

6.3 Anti-Plane Problems of Ceramics

For anti-plane motions of polarized ceramics we have [55]

$$
\begin{aligned}
&u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2, t),\\
&E_1 = E_1(x_1, x_2, t), \quad E_2 = E_2(x_1, x_2, t), \quad E_3 = 0,\\
&H_1 = H_2 = 0, \quad H_3 = H_3(x_1, x_2, t).
\end{aligned}
\tag{7.6-11}
$$

The non-vanishing components of S_{ij}, T_{ij}, D_i and B_i are

$$
\begin{aligned}
&S_4 = u_{3,2}, \quad S_5 = u_{3,1},\\
&T_4 = c_{44}u_{3,2} - e_{15}E_2, \quad T_5 = c_{44}u_{3,1} - e_{15}E_1,\\
&D_1 = e_{15}u_{3,1} + \varepsilon_{11}E_1, \quad D_2 = e_{15}u_{3,2} + \varepsilon_{11}E_2,\\
&B_3 = \mu_0 H_3.
\end{aligned}
\tag{7.6-12}
$$

The nontrivial ones of the equations of motion and Maxwell's equations in (7.6-1) take the following form:

$$
\begin{aligned}
&c_{44}(u_{3,11} + u_{3,22}) - e_{15}(E_{1,1} + E_{2,2}) = \rho\ddot{u}_3,\\
&e_{15}(u_{3,11} + u_{3,22}) + \varepsilon_{11}(E_{1,1} + E_{2,2}) = 0,\\
&E_{2,1} - E_{1,2} = -\mu_0\dot{H}_3,\\
&H_{3,2} = e_{15}\dot{u}_{3,1} + \varepsilon_{11}\dot{E}_1, \quad -H_{3,1} = e_{15}\dot{u}_{3,2} + \varepsilon_{11}\dot{E}_2.
\end{aligned}
\tag{7.6-13}
$$

Eliminating the electric field components from $(7.6\text{-}13)_{1,2}$,

$$
\bar{c}_{44}(u_{3,11} + u_{3,22}) = \rho\ddot{u}_3, \tag{7.6-14}
$$

where $\bar{c}_{44} = c_{44} + e_{15}^2/\varepsilon_{11}$. Differentiating $(7.6\text{-}13)_3$ with respect to time once and substituting from $(7.6\text{-}13)_{4,5}$, we have

$$
H_{3,11} + H_{3,22} = \varepsilon_{11}\mu_0\ddot{H}_3. \tag{7.6-15}
$$

The above equations can be written in coordinate independent forms as

$$
\begin{aligned}
&\bar{c}_{44}\nabla^2 u_3 = \rho\ddot{u}_3, \quad \nabla^2 H_3 = \varepsilon_0\mu_0\ddot{H}_3,\\
&\dot{\mathbf{D}} = -\mathbf{i}_3 \times \nabla H_3,
\end{aligned}
\tag{7.6-16}
$$

where ∇ and ∇^2 are the two-dimensional gradient operator and Laplacian, respectively. $\mathbf{D}$ is the electric displacement in the (x_1, x_2) plane. $\mathbf{i}_3$ is the unit vector in the x_3 direction. Equations $(7.6\text{-}16)_{1,2}$ govern the displacement and magnetic fields. Once u_3 and H_3 are determined, D_1 and D_2 can be obtained from Equation $(7.6\text{-}16)_3$. Then the electric field and the stress components can be obtained from constitutive relations. From Equations $(7.6\text{-}16)_3$ and

$(3.6\text{-}9)_{3,4}$, it can be seen that physically the ψ introduced by Bleustein [18] is related to H_3.

6.4 Surface Waves

To see the dynamic effects more specifically, we study the propagation of surfaces waves in a ceramic half-space [55]. The corresponding quasistatic problem was analyzed in Section 3 of Chapter 5. Consider a ceramic half-space poled in the x_3 direction (see Figure 7.6-1).

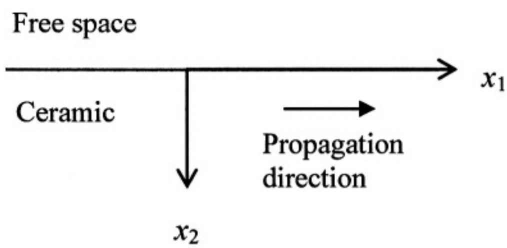

Figure 7.6-1. A ceramic half-space.

Consider surface waves propagating in the x_1 direction with

$$\begin{aligned} u_3 &= U \exp(-\xi_2 x_2) \cos(\xi_1 x_1 - \omega t), \\ H_3 &= H \exp(-\eta_2 x_2) \cos(\xi_1 x_1 - \omega t), \end{aligned} \tag{7.6-17}$$

where U, H, ξ_1, ξ_2, η_2 and ω are undetermined constants. Substitution of Equations (7.6-17) into (7.6-14) and (7.6-15) results in

$$\begin{aligned} \xi_2^2 &= \xi_1^2 - \rho\omega^2 / \bar{c}_{44} > 0, \\ \eta_2^2 &= \xi_1^2 - \varepsilon_{11}\mu_0\omega^2 > 0, \end{aligned} \tag{7.6-18}$$

where the inequalities are for decaying behavior from the surface. From Equations $(7.6\text{-}13)_{4,5}$ and (7.6-17) we obtain

$$\begin{aligned} E_1 &= \frac{1}{\varepsilon_{11}\omega}[e_{15}\omega\xi_1 U \exp(-\xi_2 x_2) \\ &\quad + \eta_2 H \exp(-\eta_2 x_2)] \sin(\xi_1 x_1 - \omega t), \\ E_2 &= \frac{1}{\varepsilon_{11}\omega}[e_{15}\omega\xi_2 U \exp(-\xi_2 x_2) \\ &\quad + \xi_1 H \exp(-\eta_2 x_2)] \cos(\xi_1 x_1 - \omega t). \end{aligned} \tag{7.6-19}$$

6.4.1 A Half-Space with an Electroded Surface

First consider the case when the surface at $x_2 = 0$ is electroded with a perfect conductor for which we have $E_1 = 0$. The electrode is assumed to be very thin with negligible mass. Hence we have the traction-free condition $T_4 = 0$ on the surface. Then from (7.6-12) and $(7.6\text{-}19)_1$ we can write

$$\begin{aligned} &e_{15}\omega\xi_1 U + \eta_2 H = 0, \\ &\varepsilon_{11}\bar{c}_{44}\omega\xi_2 U + e_{15}\xi_1 H = 0. \end{aligned} \tag{7.6-20}$$

For nontrivial solutions of U and H, the determinant of the coefficient matrix has to vanish which, with (7.6-18), leads to

$$\sqrt{1-\frac{v^2}{v_T^2}}\sqrt{1-\alpha n^2\frac{v^2}{v_T^2}} = \bar{k}_{15}^2, \tag{7.6-21}$$

where

$$\begin{aligned} &v^2 = \frac{\omega^2}{\xi_1^2}, \quad v_T^2 = \frac{\bar{c}_{44}}{\rho}, \quad \alpha = \frac{v_T^2}{c^2}, \\ &c^2 = \frac{1}{\varepsilon_0\mu_0}, \quad n^2 = \frac{\varepsilon_{11}}{\varepsilon_0}, \quad \bar{k}_{15}^2 = \frac{e_{15}^2}{\varepsilon_{11}\bar{c}_{44}}. \end{aligned} \tag{7.6-22}$$

In Equations (7.6-22), v is the surface wave speed, v_T is the speed of plane shear waves propagating in the x_1 direction, α is the ratio of acoustic and light wave speeds which is normally a very small number, c is the speed of light in a vacuum, and n is the refractive index in the x_1 direction. Equation (7.6-21) is an equation for the surface wave speed v. Waves with speed determined by (7.6-21) are clearly nondispersive. Since α is very small, it is simpler and more revealing to examine the following perturbation solution of (7.6-21) for small α

$$v^2 \cong v_T^2(1-\bar{k}_{15}^4)(1-\alpha n^2\bar{k}_{15}^4). \tag{7.6-23}$$

It is seen that the effect of electromagnetic coupling on the wave speed of Bleustein-Gulyaev waves is of the order of $\alpha n^2\bar{k}_{15}^4$. As a numerical example we consider PZT-7A. Calculation shows that

$$\begin{aligned} &\bar{k}_{15} = 0.671, \quad n^2 = 460, \\ &\alpha = 6.85\times10^{-9}, \quad \alpha n^2\bar{k}_{15}^4 = 6.38\times10^{-7}. \end{aligned} \tag{7.6-24}$$

Hence the modification on the wave speed is very small and is negligible in most applications. When α is set to zero, or when the speed of light approaches infinity, Equation (7.6-23) reduces to the speed of the Bleustein-Gulyaev waves in Section 3 of Chapter 5. The above solution serves as a good example for illustrating the quasistatic approximation, which can only be done from the dynamic theory.

6.4.2 A Half-Space with an Unelectroded Surface

When the surface of the half-space at $x_2 = 0$ is unelectroded, electromagnetic waves also exist in the free space of $x_2 < 0$. The solution for the free space $x_2 < 0$ can be written as:

$$H_3 = \overline{H}\exp(\overline{\eta}_2 x_2)\cos(\xi_1 x_1 - \omega t), \tag{7.6-25}$$

where $\overline{H}$ and $\overline{\eta}_2$ are undetermined constants. Substitution of (7.6-25) into (7.6-15) with ε_{11} replaced by ε_0 for free space, we obtain

$$\overline{\eta}_2^2 = \xi_1^2 - \varepsilon_0\mu_0\omega^2 > 0\,. \tag{7.6-26}$$

The electric field generated by H_3 in (7.6-25) through (7.6-13) with u_3 dropped and ε_{11} replaced by ε_0 for fee space, is given by

$$\begin{aligned} E_1 &= -\frac{1}{\varepsilon_0\omega}\overline{\eta}_2\overline{H}\exp(\overline{\eta}_2 x_2)\sin(\xi_1 x_1 - \omega t), \\ E_2 &= \frac{1}{\varepsilon_0\omega}\xi_1\overline{H}\exp(\overline{\eta}_2 x_2)\cos(\xi_1 x_1 - \omega t). \end{aligned} \tag{7.6-27}$$

We require the continuity of E_1 and H_3 at $x_2 = 0$ as well as the vanishing of shear stress T_4. This implies that

$$\begin{aligned} &\frac{1}{\varepsilon_{11}\omega}(e_{15}\omega\xi_1 U + \eta_2 H) + \frac{1}{\varepsilon_0\omega}\overline{\eta}_2\overline{H} = 0, \\ &H - \overline{H} = 0, \\ &\varepsilon_{11}\overline{c}_{44}\omega\xi_2 U + e_{15}\xi_1 H = 0. \end{aligned} \tag{7.6-28}$$

Vanishing of the determinant of the coefficient matrix leads to

$$\sqrt{1-\frac{v^2}{v_T^2}}\left(\sqrt{1-\alpha n^2\frac{v^2}{v_T^2}} + n^2\sqrt{1-\alpha\frac{v^2}{v_T^2}}\right) = k_{15}^2, \tag{7.6-29}$$

which is an equation for v. Again, the waves are nondispersive. When α is set to zero the result of Section 3 of Chapter 5 will be obtained. A perturbation solution of (7.6-29) to the first order in α is

$$v^2 \cong v_T^2\left[1-\frac{k_{15}^4}{(1+n^2)^2}\right]\left[1-\alpha 2n^2\frac{k_{15}^4}{(1+n^2)^3}\right], \tag{7.6-30}$$

and calculation shows that, for PZT-7A,

$$\alpha 2n^2\frac{k_{15}^4}{(1+n^2)^3}=1.30\times 10^{-16}. \tag{7.6-31}$$

6.5 Electromagnetic Radiation

Next we consider electromagnetic radiation from a vibrating circular cylinder of ceramics poled in the x_3 direction as shown in Figure 7.6-2 [56].

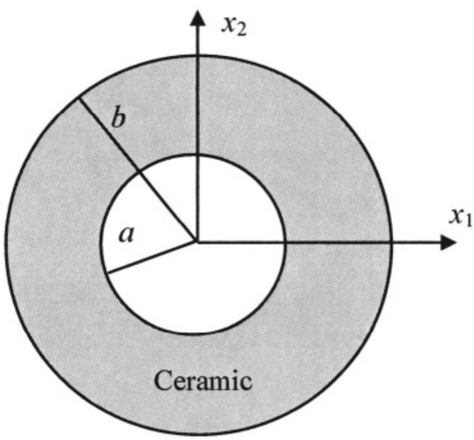

Figure 7.6-2. A Circular cylinder of ceramics poled in the x_3 direction.

The cylinder is mechanically driven at $r = b$. The surface at $r = b$ is unelectroded. Electromagnetic waves propagate away from the cylinder (radiation).

6.5.1 Boundary-Value Problem

For the special case of a solid cylinder ($a = 0$), from Equation (7.6-16) the boundary-value problem is:

$$\begin{aligned}
&v_T^2\nabla^2 u_3 = \ddot{u}_3, \quad c^2\nabla^2 H_3 = \ddot{H}_3, \quad r < a,\\
&c_0^2\nabla^2 H_3 = \ddot{H}_3, \quad r > a,\\
&u_3, H_3 \quad \text{finite}, \quad r = 0,\\
&H_3 \quad \text{outgoing}, \quad r \to \infty,\\
&T_{r3} = \tau \sin \nu\theta \exp(-i\omega t), \quad r = b,\\
&H_3, E_\theta \quad \text{continuous}, \quad r = b.
\end{aligned} \tag{7.6-32}$$

6.5.2 Interior Fields

For fields inside the cylinder, in polar coordinates, from Equation (7.6-16) we have

$$\begin{aligned}
&v_T^2\left(\frac{\partial^2 u_3}{\partial r^2} + \frac{1}{r}\frac{\partial u_3}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_3}{\partial \theta^2}\right) = \ddot{u}_3,\\
&c^2\left(\frac{\partial^2 H_3}{\partial r^2} + \frac{1}{r}\frac{\partial H_3}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_3}{\partial \theta^2}\right) = \ddot{H}_3,
\end{aligned} \tag{7.6-33}$$

and

$$\begin{aligned}
\varepsilon_{11}\dot{E}_r &= \frac{1}{r}H_{3,\theta} - e_{15}\dot{u}_{3,r},\\
\varepsilon_{11}\dot{E}_\theta &= -H_{3,r} - e_{15}\frac{1}{r}\dot{u}_{3,\theta}.
\end{aligned} \tag{7.6-34}$$

Consider the possibility of

$$\begin{aligned}
u_3(r,\theta,t) &= u(r)\sin \nu\theta \exp(-i\omega t),\\
H_3(r,\theta,t) &= H(r)\cos \nu\theta \exp(-i\omega t),
\end{aligned} \tag{7.6-35}$$

where ν is allowed to assume any real, positive value for the moment (for solutions periodic in θ, ν has to be an integer). Other values of ν may also be physically meaningful. For example, ν= 1/2 with $-\pi \le \theta \le \pi$ represents a crack at $\theta = \pi$. Substitution of (7.6-35) into (7.6-33)results in

$$\begin{aligned}
&\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \left(\alpha^2 - \frac{\nu^2}{r^2}\right)u = 0,\\
&\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} + \left(\beta^2 - \frac{\nu^2}{r^2}\right)H = 0,
\end{aligned} \tag{7.6-36}$$

where we have denoted

$$\alpha = \frac{\omega}{v_T}, \quad \beta = \frac{\omega}{c}. \tag{7.6-37}$$

Equation (7.6-36) can be written as Bessel's equations of order ν. Then general solutions for u_3 and H_3 can be written as

$$\begin{aligned} u_3 &= [C_1 J_\nu(\alpha r) + C_2 Y_\nu(\alpha r)] \sin \nu\theta \exp(-i\omega t), \\ H_3 &= [C_3 J_\nu(\beta r) + C_4 Y_\nu(\beta r)] \cos \nu\theta \exp(-i\omega t), \end{aligned} \tag{7.6-38}$$

where J_ν and Y_ν are the ν-*th* order Bessel functions of the first and second kind. $C_1 - C_4$ are undetermined constants. From (7.6-38) we obtain the following expressions that are useful for boundary and/or continuity conditions:

$$\begin{aligned} D_r &= \frac{\nu}{i\omega r}[C_3 J_\nu(\beta r) + C_4 Y_\nu(\beta r)] \sin \nu\theta \exp(-i\omega t), \\ D_\theta &= \frac{\beta}{i\omega}[C_3 J'_\nu(\beta r) + C_4 Y'_\nu(\beta r)] \cos \nu\theta \exp(-i\omega t), \end{aligned} \tag{7.6-39}$$

$$\begin{aligned} E_r &= \{\frac{\nu}{\varepsilon_{11}\omega r}[C_3 J_\nu(\beta r) + C_4 Y_\nu(\beta r)] \\ &\quad - \frac{e_{15}\alpha}{\varepsilon_{11}}[C_1 J'_\nu(\alpha r) + C_2 Y'_\nu(\alpha r)]\} \sin \nu\theta \exp(-i\omega t), \\ E_\theta &= \{\frac{\beta}{\varepsilon_{11} i\omega}[C_3 J'_\nu(\beta r) + C_4 Y'_\nu(\beta r)] \\ &\quad - \frac{e_{15}\nu}{\varepsilon_{11} r}[C_1 J_\nu(\alpha r) + C_2 Y_\nu(\alpha r)]\} \cos \nu\theta \exp(-i\omega t), \end{aligned} \tag{7.6-40}$$

$$\begin{aligned} T_{rz} &= \{\bar{c}_{44}\alpha[C_1 J'_\nu(\alpha r) + C_2 Y'_\nu(\alpha r)] \\ &\quad - \frac{e_{15}\nu}{\varepsilon_{11} i\omega r}[C_3 J_\nu(\beta r) + C_4 Y_\nu(\beta r)]\} \sin \nu\theta \exp(-i\omega t), \\ T_{\theta z} &= \{\frac{\bar{c}_{44}\nu}{r}[C_1 J_\nu(\alpha r) + C_2 Y_\nu(\alpha r)] \\ &\quad - \frac{e_{15}}{i\omega\varepsilon_{11}}[C_3 J'_\nu(\beta r) + C_4 Y'_\nu(\beta r)]\} \cos \nu\theta \exp(-i\omega t), \end{aligned} \tag{7.6-41}$$

where a superimposed prime indicates differentiation with respect to the whole argument of a function.

6.5.3 Exterior Fields

In the free space of $r > b$, the electromagnetic fields are given by

$$H_3 = [C_5 H_\nu^{(1)}(\gamma r) + C_6 H_\nu^{(2)}(\gamma r)]\cos\nu\theta \exp(-i\omega t),$$

$$D_r = \frac{\nu}{i\omega r}[C_5 H_\nu^{(1)}(\gamma r) + C_6 H_\nu^{(2)}(\gamma r)]\sin\nu\theta \exp(-i\omega t),$$

$$D_\theta = \frac{\gamma}{i\omega}[C_5 H_\nu^{(1)\prime}(\gamma r) + C_6 H_\nu^{(2)\prime}(\gamma r)]\cos\nu\theta \exp(-i\omega t), \quad (7.6\text{-}42)$$

$$E_r = \frac{\nu}{i\omega r\varepsilon_0}[C_5 H_\nu^{(1)}(\gamma r) + C_6 H_\nu^{(2)}(\gamma r)]\sin\nu\theta \exp(-i\omega t),$$

$$E_\theta = \frac{\gamma}{i\omega\varepsilon_0}[C_5 H_\nu^{(1)\prime}(\gamma r) + C_6 H_\nu^{(2)\prime}(\gamma r)]\cos\nu\theta \exp(-i\omega t),$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the v-th order Hankel function of the first and second kind, and

$$\gamma = \frac{\omega}{c_0}. \quad (7.6\text{-}43)$$

6.5.4 Boundary and Continuity Conditions

Since Y_ν is singular at the origin, terms associated with C_2 and C_4 have to be dropped. To satisfy the radiation condition at $r \to \infty$ we must have $C_6 = 0$. What need to be satisfied at $r = b$ are

$$T_{rz}(b) = \bar{c}_{44}\alpha C_1 J_\nu'(\alpha b) - \frac{e_{15}\nu}{\varepsilon_{11} i\omega b} C_3 J_\nu(\beta b) = \tau,$$

$$H_3(b^-) = C_3 J_\nu(\beta b) = C_5 H_\nu^{(1)}(\gamma b) = H_3(b^+),$$

$$E_\theta(b^-) = \frac{\beta}{\varepsilon_{11} i\omega} C_3 J_\nu'(\beta b) - \frac{e_{15}\nu}{\varepsilon_{11} b} C_1 J_\nu(\alpha b) \quad (7.6\text{-}44)$$

$$= \frac{\gamma}{i\omega\varepsilon_0} C_5 H_\nu^{(1)\prime}(\gamma b) = E_\theta(b^+).$$

Note that when $\nu = 0$ (axi-symmetric), Equation $(7.6\text{-}44)_1$ becomes uncoupled to $(7.6\text{-}44)_{2,3}$. In this case H_3 cannot be excited by τ. Hence there is no radiation. In the following we consider the case of $\nu \neq 0$. From Equation (7.6-44)

$$C_1 = b\frac{\beta b J_\nu'(\beta b)H_\nu^{(1)}(\gamma b) - \frac{\varepsilon_{11}}{\varepsilon_0}\gamma b J_\nu(\beta b)H_\nu^{(1)'}(\gamma b)}{\Delta}\frac{\tau}{\bar{c}_{44}},$$

$$C_3 = i\omega e_{15} b\frac{\nu J_\nu(\alpha b)H_\nu^{(1)}(\gamma b)}{\Delta}\frac{\tau}{\bar{c}_{44}}, \tag{7.6-45}$$

$$C_5 = i\omega e_{15} b\frac{\nu J_\nu(\alpha b)J_\nu(\beta b)}{\Delta}\frac{\tau}{\bar{c}_{44}},$$

where

$$\begin{aligned}\Delta &= \alpha b\beta b J_\nu'(\alpha b)J_\nu'(\beta b)H_\nu^{(1)}(\gamma b)\\ &\quad - \bar{k}_{15}^2\nu^2 J_\nu(\alpha b)J_\nu(\beta b)H_\nu^{(1)}(\gamma b)\\ &\quad - \frac{\varepsilon_{11}}{\varepsilon_0}\alpha b\gamma b J_\nu'(\alpha b)J_\nu(\beta b)H_\nu^{(1)'}(\gamma b).\end{aligned} \tag{7.6-46}$$

$\Delta = 0$ yields a frequency equation. The corresponding modes are coupled acousto-electromagnetic modes.

6.5.5 Electromagnetic Radiation

We calculate the radiation at far fields with large r using the following asymptotic expressions of Bessel functions with large arguments

$$\begin{aligned}H_\nu^{(1)}(x) &\cong \sqrt{\frac{2}{\pi x}}\exp i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right),\\ H_\nu^{(1)'}(x) &\cong i\sqrt{\frac{2}{\pi x}}\exp i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).\end{aligned} \tag{7.6-47}$$

Then

$$\begin{aligned}H_3 &\cong C_5\sqrt{\frac{2}{\pi\gamma r}}\exp i\left(\gamma r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\cos\nu\theta\exp(-i\omega t),\\ E_\theta &\cong \frac{\gamma}{\omega\varepsilon_0}C_5\sqrt{\frac{2}{\pi\gamma r}}\exp i\left(\gamma r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\cos\nu\theta\exp(-i\omega t),\end{aligned} \tag{7.6-48}$$

which are clearly outgoing. To calculate radiated power we need the radial component of the Poynting vector which, when averaged over a period of time, with the complex notation, is given by

$$S_r = \frac{1}{2}(\mathbf{E}^* \times \mathbf{H})_r = \frac{1}{2}\mathrm{Re}\{E_\theta^* H_3\} = \frac{C_5 C_5^*}{\pi\omega\varepsilon_0 r}\cos^2 \nu\theta\,, \qquad (7.6\text{-}49)$$

where an asterisk represents complex conjugate. Equation (7.6-49) shows that the energy flux is inversely proportional to r. It also shows the angular distribution of the power radiation. The radiated power per unit length of the cylinder is

$$S = \int_0^{2\pi} S_r r d\theta = \frac{C_5 C_5^*}{2\pi\omega\varepsilon_0}(2\pi + \frac{1}{2\nu}\sin 4\nu\pi)\,. \qquad (7.6\text{-}50)$$

We are interested in the frequency range of acoustic waves. Therefore αb is finite, $\beta b << 1$, and $\gamma b <<1$. For small arguments we have

$$J_\nu(x) \cong \frac{x^\nu}{2^\nu \Gamma(1+\nu)}, \quad H_\nu^{(1)}(x) \cong -i\frac{2^\nu \Gamma(\nu)}{\pi x^\nu},$$
$$\frac{xJ_\nu'(x)}{J_\nu(x)} \cong \nu, \quad \frac{xH_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)} \cong -\nu. \qquad (7.6\text{-}51)$$

Then, approximately,

$$C_5 = \frac{i\omega e_{15} b J_\nu(\alpha b)}{(1+\frac{\varepsilon_{11}}{\varepsilon_0})\alpha b J_\nu'(\alpha b) - \bar{k}_{15}^2 \nu J_\nu(\alpha b)} \frac{\tau}{\bar{c}_{44}} \frac{1}{H_\nu^{(1)}(\gamma b)}\,. \qquad (7.6\text{-}52)$$

In this approximate form, the denominator of the first factor of (7.6-52) represents the frequency equation for quasistatic electromechanical resonances in piezoelectricity. With Equation (7.6-52), the radiated power can be written as

$$S = \frac{\omega e_{15}^2 b^2}{2\pi\varepsilon_0}\left|\frac{J_\nu(\alpha b)}{(1+\frac{\varepsilon_{11}}{\varepsilon_0})\alpha b J_\nu'(\alpha b) - \bar{k}_{15}^2 \nu J_\nu(\alpha b)} \frac{\tau}{\bar{c}_{44}}\right|^2 \frac{2\pi + \frac{\sin 4\nu\pi}{2\nu}}{H_\nu^{(1)}(\gamma b)[H_\nu^{(1)}(\gamma b)]^*}$$

(7.6-53)

From Equation (7.6-53) we make the following observations:

(i) S is large near resonance frequencies. It is singular at these frequencies unless some damping is present.

(ii) In the limit of $\omega \to 0$, α, β, and γ all $\to 0$. In this case $S \to 0$ as expected.

(iii) S is proportional to the square of a piezoelectric constant. For materials with strong piezoelectric coupling, the radiated power is much more than that of a material with weak coupling.

Problems

7.6-1. Study piezoelectromagnetic SH waves in a ceramic plate [57].

7.6-2. Study piezoelectromagnetic SH surface waves in a ceramic half-space carrying a thin layer of isotropic conductor or dielectric [32].

7.6-3. Study piezoelectromagnetic SH gap waves between two ceramic half-spaces [58].